The socle of a nondegenerate Lie algebra

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ABSTRACT: In this paper we give a definition of socle for nondegenerate Lie algebras which is only based on minimal inner ideals. The socle turns out to be an ideal of the whole algebra, and it is sum of simple components. All the minimal inner ideals contained in a simple component are conjugated under elementary automorphisms, which allows us to associate a division Jordan algebra to any of the simple component containing abelian minimal inner ideals. All Classical Lie algebras coincide with their socles, while relevant examples of infinite dimensional simple Lie algebras with socle can be found within the class of finitary Lie algebras. The notion of socle is compatible with the associative and Jordan definitions of socle, and satisfies the descending chain condition on principal inner ideals. Furthermore, we give a structure theory for nondegenerate Lie algebras containing abelian minimal inner ideals, and show that a simple Lie algebra over an algebraically closed field of characteristic zero is finitary if and only if it is nondegenerate and contains nonzero reduced elements. i.e., contains one-dimensional inner ideals.

Introduction

One of the great early achievements in Lie theory is the classification of finite dimensional simple complex Lie algebras by W. Killing and E. Cartan at the end of 19th century. A decade later, E. Cartan classified simple infinite dimensional Lie algebras of vectors fields on a finite dimensional space. But the study of infinite dimensional Lie algebras disappears until the mid-sixties, with works of Guillemin, Sternberg, Singer and others.

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Our work studies (mainly infinite dimensional) Lie algebras from a "classical" point of view in the sense of associative or Jordan algebras, because we are interested in Lie algebras that satisfy the descending chain condition on (principal) inner ideals. We remark that this is not a "classical" approach when dealing with Lie algebras, where one normally tries to reproduce properties of finite dimensional Lie algebras such as having root systems.

Nevertheless, this approach we adopt to study Lie algebras with chain conditions is by no means a novelty; on the contrary, the notion of Artinian Lie algebra appears in a work [**3**] of G. Benkart in 1977, where she studied nondegenerate Artinian Lie algebras with the added hypothesis that they contain nonzero ad-nilpotent elements. G. Benkart was in fact interested in characterizing classical Lie algebras, obtaining that a finite dimensional Lie algebra L over an algebraically closed field of characteristic $p > 5$ is classical if and only if it is nondegenerate and contains an ad-nilpotent element (the last restriction would be removed later by A.A. Premet). Moreover, in the mentioned paper, she wrote: "It is hoped that inner ideals will play a role analogous to Jordan inner ideals in the development of an Artinian theory for Lie algebras".

The main goals of this paper are to develop a socle theory for nondegenerate Lie algebras and describe simple nondegenerate Lie algebras containing abelian minimal inner ideals.

The paper is organized as follows. After a section of preliminaries, we record in Section 2 the structure of minimal inner ideals in a nondegenerate Lie algebra (any minimal inner ideal is either abelian or an ideal which is simple as a Lie algebra and without proper inner ideals), and the existence of a 5-grading in any Lie algebra containing a von Neumann regular element. We introduce in this section a notion of idempotent for Lie algebras which parallels the corresponding notion for Jordan pairs, and show how to construct new idempotents starting from a given one.

In Section 3, we prove that the socle of a nondegenerate Lie algebra L, defined as the sum of all its minimal inner ideals, is an ideal of L which is a direct sum of simple ideals. We also prove that any two abelian minimal inner ideals of a simple L are conjugate under an elementary automorphism of L, which allows us to associate a division Jordan pair (and hence also a division Jordan algebra, uniquely determined up to isotopy) with any simple nondegenerate Lie algebra containing an abelian minimal inner ideal. Nondegenerate finite dimensional and, more generally, Artinian Lie algebras have essential socles, while relevant examples of infinite dimensional Lie algebras coinciding with their socles can be found within the class of finitary Lie algebras.

We see in Section 4 that the notion of socle we have introduced extends a previous one defined by means of the Jordan socles of the Jordan pairs associated to 3-graded ideals, and prove the compatibility of the Lie socle with the Jordan socles of the Jordan pairs defined by short gradings. As a consequence, we prove that the socle of a nondegenerate Lie algebra satisfies the descending chain condition on its principal inner ideals.

The notion of Lie socle is related to the associative one in Section 5. Let R be a simple associative algebra. Then

(i) $\overline{R}' = [R, R]/Z(R) \cap [R, R]$ is a simple nondegenerate Lie algebra. Moreover, $\overline{R}'$ contains abelian minimal inner ideals if and only if R coincides with its socle and is not a division algebra.

(ii) If R has an involution $*$ and either $Z(R) = 0$ or $\dim_{Z(R)} R$ is greater than 16, then $\overline{K}' = [K, K]/Z(R) \cap [K, K]$, where $K = \mathrm{Skew}(R, *)$, is a simple nondegenerate Lie algebra. Moreover, $\overline{K}'$ contains abelian minimal inner ideals if and only if R coincides with its socle and $*$ is isotropic.

We end up our work giving in Section 6 a structure theorem for simple nondegenerate Lie algebras containing abelian minimal inner ideals. Indeed, we describe how such algebras are and obtain that, up to the exceptional cases, they are closely related to associative algebras coinciding with their socles. Among them, those which are finitary central simple over a field of characteristic 0 are characterized by the property that the division Jordan algebras associated with them are PI. As a consequence, we obtain that a simple Lie algebra over an algebraically closed field is finitary if and only it is nondegenerate and contains a minimal inner ideal of dimension one, i.e., contains a reduced element.

1. **Preliminaries**

1.1 Throughout this paper, we will be dealing with Lie algebras, associative algebras, and Jordan pairs over a ring of scalars Φ such that $\frac{1}{2}, \frac{1}{3} \in \Phi$ (sometimes we will also require that $\frac{1}{5} \in \Phi$). Notice that the centroid of a simple Lie algebra L is a field, and we can take it as a ring of scalars for L, so there will be no loss of generality in our arguments if we sometimes consider simple Lie algebras over fields of characteristic greater than 3 (or greater than 5). As usual, $[x, y]$ will denote the Lie bracket, with ad_x the adjoint map determined by x; associative products will be written by juxtaposition; Jordan products of a Jordan pair $V = (V^+, V^-)$ will be denoted by $Q_x y$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z}y = \{x, y, z\} = D_{x,y}z$. The reader is referred to [**14**, **15**, **21**] for basic results, notation and terminology. Nevertheless, we will stress some notions and basic properties for both Lie algebras and Jordan pairs.

1.2 Any associative algebra R gives rise to:

(i) a Lie algebra $R^{(-)}$ with Lie bracket $[x, y] := xy - yx$, for all $x, y \in R$,

(ii) a Jordan algebra $R^{(+)}$ with Jordan product $x \cdot y := \frac{1}{2}(xy + yx)$,

(iii) a Jordan pair $V = (R, R)$ with Jordan quadratic operator $Q_x y := xyx$.

1.3 Let $V = (V^+, V^-)$ be a Jordan pair. An element $x \in V^\sigma$, $\sigma = \pm$, is called an *absolute zero divisor* if $Q_x = 0$. Thus V is said to be *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $Q_{B^\pm} B^\mp = 0$ implies $B = 0$, and *prime* if $Q_{B^\pm} C^\mp = 0$ implies $B = 0$ or $C = 0$, for ideals $B = (B^+, B^-)$, $C = (C^+, C^-)$ of V. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[I, I] = 0$ implies $I = 0$, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for ideals I, J of L. A Jordan pair or Lie algebra is *strongly prime* if it is prime and nondegenerate. A Lie algebra is *simple* if it is nonabelian and contains no proper ideals.

1.4 Ideals of nondegenerate (respectively, strongly prime) Jordan pairs inherit nondegeneracy (respectively, strong primeness) [**15**], JP3, and [**18**]. The same is true for Lie algebras: every ideal of a nondegenerate (strongly prime) Lie algebra is nondegenerate (respectively, strongly prime) [**25**], Lemma 4, and [**9**], 0.4, 1.5.

1.5 Given a subset S of L, the *annihilator* or *centralizer* of S in L, $\mathrm{Ann}_L(S)$, consists of the elements $x \in L$ such that $[x, S] = 0$. By the Jacobi identity, $\mathrm{Ann}_L(S)$ is a subalgebra of L, and also an ideal whenever S is so. Notice that $\mathrm{Ann}_L(L)$ is precisely $Z(L)$, the center of L. If L is semiprime, then

$$I \cap \mathrm{Ann}_L(I) = 0 \qquad (1)$$

for any ideal I of L. Hence I is an essential ideal of L if and only if $\mathrm{Ann}_L(I) = 0$. We also have (see [**6**], (2.5)) that the annihilator of a nondegenerate ideal I of L

has the following nice expression:

$$\mathrm{Ann}_L(I) = \{a \in L \mid [a, [a, I]] = 0\}. \tag{2}$$

1.6 A $(2n+1)$-*grading* of a Lie algebra L is a decomposition

$$L = L_{-n} \oplus \ldots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_n,$$

where each L_i is a submodule of L satisfying $[L_i, L_j] \subset L_{i+j}$, where $L_{i+j} = 0$ if $i + j \neq 0, \pm 1, \ldots \pm n$, and where $L_n + L_{-n} \neq 0$.

1.7 Let $L = L_{-n} \oplus \ldots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_n$ be a Lie algebra with a $(2n+1)$-grading. Then $V := (L_n, L_{-n})$ is a Jordan pair for the triple product defined by $\{x, y, z\} := [[x, y], z]$ for all $x, z \in L_\sigma$, $y \in L_{-\sigma}$, $\sigma = \pm n$, and it is called the *associated Jordan pair* relative to the grading [**26**], p.351. Moreover, if L is nondegenerate, so is V [**26**], Lemma 1.8.

A standard example of a Lie algebra with a 3-grading is that given by the TKK-algebra of a Jordan pair:

1.8 For any Jordan pair V, there exists a Lie algebra with a 3-grading $\mathrm{TKK}(V) = L_{-1} \oplus L_0 \oplus L_1$, the *Tits-Kantor-Koecher algebra of* V, uniquely determined by the following conditions (cf. [**21**], 1.5(6)):

(TKK1) The associated Jordan pair (L_1, L_{-1}) is isomorphic to V.

(TKK2) $[L_1, L_{-1}] = L_0$.

(TKK3) $[x_0, L_1 \oplus L_{-1}] = 0$ implies $x_0 = 0$, for any $x_0 \in L_0$.

In general, by a *TKK-algebra* we mean a Lie algebra of the form $\mathrm{TKK}(V)$ for some Jordan pair V.

2. Abelian inner ideals, regular elements and $\mathfrak{sl}(2)$-triples

2.1 Let L be a Lie algebra. A submodule B of L is an *inner ideal* if $[B, [B, L]] \subset B$. Clearly, any ideal I of L is an inner ideal. Even more, subideals of L, that is, ideals of ideals, are also inner ideals. An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$. Relevant examples of abelian inner ideals can be found in the Lie inner structure of an associative ring (cf. [**2**]). Another source of examples of abelian inner ideals is within the class of Lie algebras with short $\mathbb{Z}$-gradings.

2.2 Recall that an *inner ideal* of a Jordan pair V is a submodule K of V^σ, $\sigma = \pm$, such that $Q_K V^{-\sigma} \subset K$. If $L = L_{-n} \oplus \ldots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_n$ is a Lie algebra with a $(2n+1)$-grading and associated Jordan pair $V = (L_n, L_{-n})$, then the (Jordan) inner ideals of V (for example, $K \subset L_n$) are abelian inner ideals of L. Indeed, because of the grading, $[[K, L], K] = [[K, L_{-n}], K] = \{K, L_{-n}, K\} \subset K$, and $[K, K] \subset [L_n, L_n] = 0$.

Clearly, every element b of an abelian inner ideal B of L satisfies $\mathrm{ad}_b^3 = 0$. Conversely, by [**3**], Lemma 1.8, every element $b \in L$ such that $\mathrm{ad}_b^3 = 0$ generates an abelian inner ideal, namely, $B = \mathrm{ad}_b^2 L$. In fact, the same proof of [**3**], Lemma 1.8, shows the following slightly more general statement:

2.3 Lemma. *Let L be a Lie algebra, B an inner ideal of L, and $c \in L$ such that* $\mathrm{ad}_c^3 = 0$. *Then* $\mathrm{ad}_c^2 B$ *is an abelian inner ideal of L.*

An inner ideal B of L is said to be *minimal* if $B \neq 0$ and for any inner ideal $C \subset B$, either $C = 0$ or $C = B$. By [**3**], 1.12, we have

2.4 Let L be a nondegenerate Lie algebra and let B be a minimal inner ideal of L. Then B is either abelian, or an ideal of L which is simple as a Lie algebra and has no proper inner ideals. Examples of the latter situation are the following:

(i) the finitary orthogonal Lie algebra $\mathfrak{fo}(X, q)$ where q is anisotropic, [**2**], Corollary 4.24,

(ii) $[\Delta, \Delta]/Z(\Delta) \cap [\Delta, \Delta]$, where Δ is a division associative algebra whose dimension over its center is greater than 16, [**2**], Corollary 4.27.

In the following proposition, we record some characterizations and constructions of abelian minimal inner ideals in nondegenerate Lie algebras.

2.5 Lemma. *Let L be a nondegenerate Lie algebra.*

(i) *A nonzero abelian subalgebra B of L is an abelian minimal inner ideal if and only if $B = \mathrm{ad}_b^2(L)$ for every $0 \neq b \in B$.*

(ii) *If B is a minimal inner ideal and $c \in L$ satisfies $\mathrm{ad}_c^3 = 0$, then $\mathrm{ad}_c^2(B)$ is either zero or an abelian minimal inner ideal. Furthermore, if B is not abelian, $\mathrm{ad}_c^2(B) = 0$.*

(iii) *If I is an ideal of L and B is a submodule of I, then B is a minimal (not necessarily abelian) inner ideal of L if and only if it is a minimal inner ideal of I.*

Proof. (i) Use the same proof as that of [**3**], Lemma 1.8.

(ii) If B is an abelian minimal inner ideal of L, the result follows from [**6**], Proposition 3.8(iii). If B is not abelian, it is a simple ideal of L without proper inner ideals. Now, again by [**6**], Proposition 3.8, $\mathrm{ad}_c^2(B)$ is an abelian inner ideal of L contained in B, which implies that $\mathrm{ad}_c^2(B) = 0$.

(iii) Let $B \subset I$ be a minimal inner ideal of L. If B is not abelian, then B is an ideal of L which is simple as a Lie algebra and without proper inner ideals (2.4). Clearly, B is a minimal inner ideal of I. Suppose then that B is abelian and let $0 \neq b \in B$. Since $\mathrm{ad}_b^3 = 0$, by (2.3) $\mathrm{ad}_b^2 I$ is an abelian inner ideal of L. Moreover, $\mathrm{ad}_b^2 I \neq 0$ by nondegeneracy of I (inherited from that of L). Thus, $B = \mathrm{ad}_b^2 I$ by minimality of B, and hence B is an abelian minimal inner ideal of I by (i).

Let now B be a minimal inner ideal of I, and use again the dichotomy abelian-nonabelian. If B is not abelian, then $B = [B, B]$ by simplicity, and hence it is an ideal of L. Moreover, since B contains no proper inner ideals, it is a minimal inner ideal of L. If B is abelian, $B = \mathrm{ad}_b^2 I$ for any nonzero element $b \in B$ as before, and hence $\mathrm{ad}_b^3 I = 0$. We claim that $\mathrm{ad}_b^3 L = 0$: Otherwise, there exists $a \in L$ such that $0 \neq c = \mathrm{ad}_b^3 a \in B$. But $\mathrm{ad}_b^4 L \subset \mathrm{ad}_b^3 I = 0$ and hence by using a Kostrikin's result (cited in [**3**], Proposition 1.5), $\mathrm{ad}_c^3 L = 0$, giving that $B = \mathrm{ad}_c^2 I$ (by its minimality) is an inner ideal of L by (2.3). But then $\mathrm{ad}_b^3 L = [b, [b, [b, L]]] \subset [b, B] = 0$, since B is abelian, a contradiction. Therefore, $\mathrm{ad}_b^3 L = 0$ for every $b \in B$ and hence, by (2.3), B is an abelian inner ideal of L, which is clearly minimal. ■

2.6 Following [**3**], a Lie algebra L is called a $*$-Lie algebra if there exists an element $0 \neq e \in L$ such that $\mathrm{ad}_e^3 = 0$ and $e \in \mathrm{ad}_e^2(L)$. An element e satisfying these two conditions is called (*von Neumann*) *regular*.

It follows from (2.5)(i) that if L is nondegenerate and B is an abelian minimal inner ideal of L, then any nonzero element $b \in B$ is regular. We also have, by (2.3), that any regular element $e \in L$ generates the principal inner ideal $\mathrm{ad}_e^2(L)$, where an inner ideal B is *principal* if $B = \mathrm{ad}_b^2(L)$ for a regular element b in L. Notice that principal inner ideals are abelian. Moreover, this notion of von Neumann regularity is compatible with the usual one for associative rings (see [**7**], Proposition 2.4).

In the next proposition, we record some well-known results for $*$-Lie algebras which will play a relevant role in our approach. Let us first establish some notation.

2.7 Let L be a Lie algebra. A pair of elements (e, f) of L is said to be an *idempotent* if they satisfy:

$$\mathrm{ad}_e^3 = \mathrm{ad}_f^3 = 0,\ [[e,f],e] = 2e \text{ and } [[e,f],f] = -2f. \tag{3}$$

Notice that the two last conditions imply that $(e,[e,f],f)$ is a $\mathfrak{sl}(2)$-*triple*.

2.8 Following [**15**], 5.1, an *idempotent* of a Jordan pair V is a pair $(x,y) \in V^+ \times V^-$ such that $Q_x y = x$ and $Q_y x = y$. It is a direct consequence of the grading properties that if $L = L_{-n} \oplus \ldots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_n$ is a $(2n+1)$-grading of a Lie algebra L with associated Jordan pair $V = (L_n, L_{-n})$, then every idempotent of V is a idempotent of L.

2.9 Proposition. *Suppose that* $2, 3$ *and* 5 *are invertible in* Φ*, and let* $0 \neq e \in L$ *be regular. Then.*

(i) *For every* $h \in [e,L]$ *such that* $[h,e] = 2e$*, there exists* $f \in L$ *such* $[e,f] = h$ *and* (e,f) *is an idempotent.*

(ii) *Let* (e,f) *be an idempotent and put* $h = [e,f]$*. Then* ad_h *is semisimple and*

$$L = L_{-2}^{(e,f)} \oplus L_{-1}^{(e,f)} \oplus L_0^{(e,f)} \oplus L_1^{(e,f)} \oplus L_2^{(e,f)}$$

is a 5*-grading, where* $L_i = L_i^{(e,f)}$ *is the* i*-eigenspace of* L *relative to* ad_h*, for each* $i \in \{-2,-1,0,1,2\}$*. Moreover,*

(iii) $L_2^{(e,f)} = \mathrm{ad}_e^2(L)$ *and* $L_{-2}^{(e,f)} = \mathrm{ad}_f^2(L)$*, and they are abelian inner ideals of* L*.*

Suppose in addition that L *is nondegenerate. Then*

(iv) $V(e,f) := (L_2^{(e,f)}, L_{-2}^{(e,f)})$ *is a nondegenerate Jordan pair containing an invertible element:* $e \in L_2^{(e,f)}$ *is invertible with inverse* $f \in L_{-2}^{(e,f)}$.

(v) $L_2^{(e,f)}$ *is a minimal inner ideal if and only so is* $L_{-2}^{(e,f)}$*, equivalently,* $V(e,f)$ *is a division Jordan pair. Such an idempotent* (e,f) *will be then called* minimal.

Proof. (i) and (ii). They are a simple adaptation of Seligman's proof [**23**], V.8.2, to the setting of Lie algebras over a ring of scalars Φ in which $2, 3$ and 5 are invertible. Using that $\mathrm{ad}_e^3 = 0$, one proves as in [**14**], p.99, that

$$(\mathrm{ad}_h - 2\,\mathrm{Id})(\mathrm{ad}_h - \mathrm{Id})\,\mathrm{ad}_h\, x = 0 = \mathrm{ad}_e\, \mathrm{ad}_h\, x, \tag{4}$$

for all $x \in L$ such that $[e,x] = 0$. From the right-hand side of (4), it follows that $\ker(\mathrm{ad}_e)$ is invariant under $\mathrm{ad}_h + 2Id$. On the other hand, one can verify that the ideals $(\lambda+2)$ and $(\lambda(\lambda-1)(\lambda-2))$ are comaximal in $\Phi[\lambda]$ (whenever 2 and 3 are invertible in Φ). By (4) it is clear that the restriction of $\mathrm{ad}_h + 2Id$ to $\ker(\mathrm{ad}_e)$ is invertible. Let $h = [e, y]$ for some $y \in L$. As observed in [**23**], V.8.2, $[h, y] + 2y \in \ker(\mathrm{ad}_e)$, so there exists $v \in \ker(\mathrm{ad}_e)$ so that $[h, v] + 2v = [h, y] + 2y$. Letting $f = y - v$, we get that $[e, f] = h$ and $[h, f] = -2f$. Using now [**13**], Lemma 1 (which also works over a ring containing $\frac{1}{2}, \frac{1}{3}$), we get that ad_h satisfies the polynomial $f(\lambda) = \lambda(\lambda-1)(\lambda+1)(\lambda-2)(\lambda+2)$, which is separable (its discriminant equals $2^{10} \times 3^4$ so is invertible in Φ) and therefore it yields the five decomposition $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ of (ii), which is actually a 5-grading in L. (In [**3**], Lemma 2.1, Benkart provides the polynomial equality:

$$1 = \frac{1}{24}p_2(\lambda) - \frac{1}{6}p_1(\lambda) + \frac{1}{4}p_0(\lambda) - \frac{1}{6}p_{-1}(\lambda)\frac{1}{24}p_{-2}(\lambda),$$

where $p_i(\lambda) = p(\lambda)/(\lambda + i)$, which produces directly the 5-decomposition.)

Then ad_h is semisimple with eigenvalues among $0, \pm 1, \pm 2$. From

$$\mathrm{ad}_h \mathrm{ad}_f = [\mathrm{ad}_h, \mathrm{ad}_f] + \mathrm{ad}_f \mathrm{ad}_h = -2\,\mathrm{ad}_f + \mathrm{ad}_f \mathrm{ad}_h,$$

it follows that $\mathrm{ad}_h(\mathrm{ad}_f x_{-2}) = -4\,\mathrm{ad}_f x_{-2}$ for any $x_{-2} \in L_{-2}$. Since 4 is invertible in Φ and it is not an eigenvalue of ad_h, we have that $\mathrm{ad}_f(L_{-2}) = 0$; similarly, $\mathrm{ad}_h(\mathrm{ad}_f^2 x_i) = (-4+i)\,\mathrm{ad}_f^2 x_i$ for $x_i \in L_i$ implies that $\mathrm{ad}_f^2(L_0) = \mathrm{ad}_f^2(L_{-1})$, since $\frac{1}{2}, \frac{1}{5} \in \Phi$; finally, $\mathrm{ad}_h(\mathrm{ad}_f^3 x_i) = (-6+i)\,\mathrm{ad}_f^3 x_i$ for $x_i \in L_i$ yields $\mathrm{ad}_f^3(L_1) = \mathrm{ad}_f^3(L_2) = 0$, again since $\frac{1}{2}, \frac{1}{5} \in \Phi$. (Notice that this is the unique point of the proof where $\frac{1}{5} \in \Phi$ is required. Moreover, as observed in [**22**], Section 3, a peculiarity of characteristic 5 is that the equality $\mathrm{ad}_f^3 = 0$ need not hold.) Thus, $\mathrm{ad}_f^3 = 0$ and therefore (e, f) is an idempotent, as required. Now (iii) is [**3**], Lemma 2.1(3), (iv) follows from (1.7) and [**15**], 5.5, and (v) is an standard result of Jordan theory [**17**], Lemma 1. ■

In the proof of the next lemma we will use the following identity in Lie algebras (see [**3**], 1.7(iii)) which resembles the fundamental Jordan identity [**15**], JP3.

2.10 Let $x \in L$ be such that $\mathrm{ad}_x^3 = 0$. For any $y \in L$, we have that

$$\mathrm{ad}^2_{\mathrm{ad}_x^2 y} = \mathrm{ad}_x^2 \mathrm{ad}_y^2 \mathrm{ad}_x^2 .$$

2.11 Lemma. *Let L be a nondegenerate Lie algebra and let B, C be abelian inner ideals of L such that C is minimal. Let $c \in C$ be such that $\mathrm{ad}_c^2(B) \neq 0$.*

Then $C = \operatorname{ad}_c^2(B)$*. Moreover, for any* $x \in B$ *such that* $\operatorname{ad}_c^2 x = 2c$*, it holds that* $(c, -\frac{1}{2}\operatorname{ad}_x^2 c)$ *is a minimal idempotent.*

Proof. That $C = \operatorname{ad}_c^2(B)$ follows from (2.3), since C is an abelian minimal inner ideal. Put $b := -\frac{1}{2}\operatorname{ad}_x^2 c \in B$. Since B, C are abelian inner ideals, $\operatorname{ad}_b^3 = \operatorname{ad}_c^3 = 0$. Therefore we only need to show that $[[c,b],c] = 2c$ and $[[c,b],b] = -2b$, which is in fact a standard application of (2.10). Indeed,

$$[[c,b],c] = -\operatorname{ad}_c^2 b = \frac{1}{2}\operatorname{ad}_c^2\operatorname{ad}_x^2 c = \frac{1}{4}\operatorname{ad}_c^2\operatorname{ad}_x^2\operatorname{ad}_c^2 x = \frac{1}{4}\operatorname{ad}_{\operatorname{ad}_c^2 x}^2 x = \frac{1}{4}\operatorname{ad}_{2c}^2 x = \operatorname{ad}_c^2 x = 2c.$$

Similarly,

$$\begin{aligned}[[c,b],b] &= \operatorname{ad}_b^2 c = \operatorname{ad}_{(-\frac{1}{2}\operatorname{ad}_x^2 c)}^2 c = \frac{1}{4}\operatorname{ad}_{\operatorname{ad}_x^2 c}^2 c = \frac{1}{4}\operatorname{ad}_x^2\operatorname{ad}_c^2\operatorname{ad}_x^2 c = \\ &= \frac{1}{8}\operatorname{ad}_x^2(\operatorname{ad}_c^2\operatorname{ad}_x^2\operatorname{ad}_c^2 x) = \frac{1}{8}\operatorname{ad}_x^2\operatorname{ad}_{\operatorname{ad}_c^2 x}^2 x = \frac{1}{8}\operatorname{ad}_x^2\operatorname{ad}_{2c}^2 x = \\ &\frac{1}{2}\operatorname{ad}_x^2\operatorname{ad}_c^2 x = \operatorname{ad}_x^2 c = -2b,\end{aligned}$$

which completes the proof. ■

2.12 Let $a \in L$ be such that ad_a is nilpotent of index 3. Then

$$\exp(\operatorname{ad}_a) = \operatorname{Id} + \operatorname{ad}_a + \frac{1}{2}\operatorname{ad}_a^2$$

is an automorphism of L. The subgroup of $\operatorname{Aut}(L)$ generated by these automorphisms will be called the *group of elementary automorphisms* of L and denoted by $\operatorname{Elem}(L)$.

2.13 Lemma. *Let* (e,f) *be an idempotent of a nondegenerate Lie algebra* L*, and set* $h = [e,f]$*. Then*

(i) $(e+h-f, f)$ *is an idempotent of* L*.*

(ii) *The principal inner ideals* $\operatorname{ad}_e^2(L)$*,* $\operatorname{ad}_f^2(L)$*,* $\operatorname{ad}_{e+h-f}^2(L)$ *are conjugate under elementary automorphisms of* L*.*

(iii) (e,f) *is minimal if and only if so is* $(e+h-f, f)$*.*

Proof. Since ad_f is nilpotent of index 3, it makes sense to consider the elementary automorphism $\exp(-\operatorname{ad}_f)$. Apply this automorphism to both e and f. We get $\exp(-\operatorname{ad}_f)(f) = f$ and $\exp(-\operatorname{ad}_f)(e) = e+h-f$. This proves that $(e+h-f, f)$ is an idempotent, that the principal inner ideals $\operatorname{ad}_e^2(L)$ and $\operatorname{ad}_{e+h-f}^2(L)$ are conjugate, and (iii). Finally, the automorphism $\exp(\operatorname{ad}_e)\exp(-\operatorname{ad}_f)$ satisfies

$$\exp(\operatorname{ad}_e)\exp(-\operatorname{ad}_f)(e) = \exp(\operatorname{ad}_e)(e+h-f) = -f$$

and hence maps $\operatorname{ad}_e^2(L)$ onto $\operatorname{ad}_f^2(L)$. ■

2.14 Example. To illustrate the above construction, take, in the Lie algebra $L = \mathfrak{sl}(2, F)$ where F is a field,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$h = [e, f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } e' = e + h - f = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Notice that $L = Fe + Ff + Fe'$ is a sum of abelian minimal inner ideals. This is by no means a privilege reserved for $\mathfrak{sl}(2, F)$. On the contrary, as it will be seen in the next section, any simple nondegenerate Lie algebra containing an abelian minimal inner ideal is the sum of its abelian minimal inner ideals.

3. The socle of a nondegenerate Lie algebra

In this section we develop a socle theory for nondegenerate Lie algebras based solely on the notion of minimal inner ideal. The socle turns out to be an ideal which is a direct sum of simple components. Nondegenerate Artinian Lie algebras have essential socle. Relevant examples of infinite dimensional Lie algebras coinciding with their socles can be found within the class of finitary Lie algebras. The relationship between this Lie socle and a previous one defined in terms of the socles of the Jordan pairs associated with ideals with 3-gradings, as well as its connection with the associative socle, will be dealt with in the following two sections.

3.1 Proposition. *Let L be a nondegenerate Lie algebra. Then any minimal inner ideal of L generates an ideal which is simple as a Lie algebra.*

Proof. Given a minimal inner ideal B of L, denote by $\operatorname{Id}_L(B)$ the ideal of L generated by B. We must show that $I = \operatorname{Id}_L(B)$ is simple as a Lie algebra. By (2.4), we may assume that B is abelian. Then we observe:

(i) For any ideal J of I and any $b \in B$, $B = \mathrm{ad}_b^2(J)$, whenever $\mathrm{ad}_b^2(J) \neq 0$.

This follows from (2.3) since J is an inner ideal of L and $\mathrm{ad}_b^3 = 0$. Let now $K = \mathrm{Id}_I(B)$ be the ideal of I generated by B. We claim that

(ii) $B = \mathrm{ad}_b^2(K)$ for any $0 \neq b \in B$.

By (i) it suffices to show that $\mathrm{ad}_b^2(K) \neq 0$. If $\mathrm{ad}_b^2(K) = 0$, then $b \in \mathrm{Ann}_I(K)$ by (2), and hence $b \in K \cap \mathrm{Ann}_I(K) = 0$ by (1), a contradiction.

It follows from (ii) that $B \subset [K, K]$, and hence $K = [K, K]$. Thus, K is actually an ideal of L, so that $K = I$. Now let J be a nonzero ideal of I and pick $0 \neq b \in B$. We have that $\mathrm{ad}_b^2(J) \neq 0$, since otherwise $b \in \mathrm{Ann}_I(J)$ and hence $I = K$ would be contained in $\mathrm{Ann}_I(J)$, which again yields a contradiction. Consequently, $B = \mathrm{ad}_b^2(J)$ by (i), and hence $B \subset J$, so $I = K \subset J$, which proves that I is simple. ∎

3.2 Theorem. *A nondegenerate simple Lie algebra containing a minimal inner ideal is the sum of all its minimal inner ideals. In fact, for any minimal inner ideal B of L, we have $L = \sum_{\phi \in \mathrm{Elem}(L)} \phi(B)$, where ϕ ranges over all elementary automorphisms of L.*

Proof. We may assume that B is abelian (otherwise $B = L$ by (2.4), and there is nothing to prove). In this case, L contains a minimal idempotent (e, f) such that the induced 5-grading (see (2.9))

$$L = L_{-2}^{(e,f)} \oplus L_{-1}^{(e,f)} \oplus L_0^{(e,f)} \oplus L_1^{(e,f)} \oplus L_2^{(e,f)},$$

satisfies $L_2^{(e,f)} = \mathrm{ad}_e^2 L = B$. Put $S = \sum \phi(B)$, where ϕ ranges over all elementary automorphisms of L. We will show in several steps that $L = S$. Our strategy will be constructing minimal inner ideals conjugate to B, via (2.13).

(I) $L_{-2}^{(e,f)} \subset S$.

This follows from (2.13)(ii).

(II) $[L_2^{(e,f)}, L_{-2}^{(e,f)}] \subset S$.

Let $0 \neq x \in [L_2^{(e,f)}, L_{-2}^{(e,f)}]$. We may assume that $x = [y, f_2]$, with $y \in L_2^{(e,f)}$, $f_2 \in L_{-2}^{(e,f)}$. Since $V(e, f)$ is a division Jordan pair, by [**15**], 5.1, there exists $e_2 \in L_2^{(e,f)}$ such that (e_2, f_2) is a division idempotent of $V(e, f)$, equivalently, a minimal idempotent of L. Then (e_2, f_2) is a new idempotent with associated 5-grading on L

$$L = L_{-2}^{(e_2,f_2)} \oplus L_{-1}^{(e_2,f_2)} \oplus L_0^{(e_2,f_2)} \oplus L_1^{(e_2,f_2)} \oplus L_2^{(e_2,f_2)},$$

and where it holds that $L_2^{(e_2,f_2)} = L_2^{(e,f)} = B$ and $L_{-2}^{(e_2,f_2)} = L_{-2}^{(e,f)}$. Put

$$h_2 := [e_2, f_2],\ e_3 := e_2 + h_2 - f_2 \ \text{ and }\ f_3 := f_2.$$

Then (e_3, f_3) is a new minimal idempotent of L by (2.13)(i),(iii). Let us now see that $y' := y + [y, f_2] + \frac{1}{2}[[y, f_2], f_2]$ belongs to $L_2^{(e_3,f_3)}$. Indeed,

$$\begin{aligned}[h_3, y'] &= [h_2, y + [y, f_2] + \frac{1}{2}[[y, f_2], f_2]] - 2[f_2, y] - 2[f_2, [y, f_2]] \\ &= 2y - [[y, f_2], f_2] + 2[y, f_2] + 2[[y, f_2], f_2] \\ &= 2(y + [y, f_2] + \frac{1}{2}[[y, f_2], f_2]) = 2y'.\end{aligned}$$

Hence, $[y, f_2] = y' - y - \frac{1}{2}[[y, f_2], f_2] \in L_2^{(e_3,f_3)} + L_2^{(e,f)} + L_{-2}^{(e,f)} \subset S$, since $L_2^{(e_3,f_3)}$, $L_2^{(e,f)} = B$ and $L_{-2}^{(e,f)}$ are conjugate by (2.13).

(III) For every $x_1 \in L_1^{(e,f)}$, there exists $f' \in L$ such that (e, f') is a minimal idempotent of L and $[e, f'] = [e, f] + x_1$.

Indeed, write

$$x_1 = [h, x_1] = [[e, f], x_1] = [e, [f, x_1]], \tag{5}$$

since $[e, x_1] \in [L_2^{(e,f)}, L_1^{(e,f)}] = 0$, and set

$$x_{-1} := [f, x_1] \in L_{-1}^{(e,f)}. \tag{6}$$

Then $[e, [e, f + x_{-1}]] = [e, [e, f]] = -2e$, since $[e, [e, x_{-1}]] = 0$. Hence, by (2.9)(i), there exists $f' \in L$ such that (e, f') is a minimal idempotent of L and $[e, f'] = [e, f + x_{-1}] = [e, f] + [e, x_{-1}] = [e, f] + x_1$, by (5) and (6).

(IV) $L_{-1}^{(e,f)}$ and $L_1^{(e,f)}$ are contained in S.

By symmetry, (2.13)(ii), it suffices to see that $L_1^{(e,f)} \subset S$. But this follows from (II) and (III): $x_1 = [e, f'] - [e, f] \in S$.

(V) $[L_{-1}^{(e,f)}, L_1^{(e,f)}] \subset S$.

Let $x_1 \in L_1^{(e,f)}$ and take $f' \in L$ as in (III). Then $h' := [e, f'] = [e, f] + x_1 = h + x_1$, and hence, for any $x_{-1} \in L_{-1}^{(e,f)}$, we have

$$[x_1, x_{-1}] = [h', x_{-1}] - [h, x_{-1}] = [h', x_{-1}] + x_{-1},$$

but $[h', x_{-1}] \in L_{-2}^{(e,f')} \oplus L_{-1}^{(e,f')} \oplus L_1^{(e,f')} \oplus L_2^{(e,f')}$ by (2.9)(ii), and hence $[x_1, x_{-1}] \in S$ by (I) and (IV).

Finally, $L_{-2}^{(e,f)} \oplus L_{-1}^{(e,f)} \oplus ([L_{-2}^{(e,f)}, L_2^{(e,f)}] + [L_{-1}^{(e,f)}, L_1^{(e,f)}]) \oplus L_1^{(e,f)} \oplus L_2^{(e,f)}$ is an ideal of L (contained in S by the above), which coincides with L by simplicity of L. ∎

The following corollary answers a question posed by Ottmar Loos about the relationship among the minimal inner ideals of a simple nondegenerate Lie algebra (see [**16**] for a similar question for Jordan systems).

3.3 Corollary. *Let L be a nondegenerate simple Lie algebra. Then*

(i) *any two abelian minimal inner ideals of L are conjugate under an elementary automorphism of L.*

(ii) *If (e_1, f_1), (e_2, f_2) are minimal idempotents of L, the corresponding division Jordan pairs $V(e_1, f_1)$, $V(e_2, f_2)$ are isomorphic.*

Proof. (i) Suppose that B, C are abelian minimal inner ideals. By (3.2), $L = \sum_{\phi \in \mathrm{Elem}(L)} \phi(B)$. By nondegeneracy of L, for any $0 \neq c \in C$, there exists $\phi \in \mathrm{Elem}(L)$ such that $\mathrm{ad}_c^2(\phi(B)) \neq 0$. Then, by (2.11), $C = \mathrm{ad}_c^2(\phi(B))$ and for any $x \in \phi(B)$ such that $\mathrm{ad}_c^2 x = 2c$, $(c, -\frac{1}{2}\mathrm{ad}_x^2 c)$ is a minimal idempotent. Hence $C = \mathrm{ad}_c^2(L)$ and $\phi(B) = \mathrm{ad}^2_{-\frac{1}{2}\mathrm{ad}_x^2 c}(L)$ are conjugate under an elementary automorphism of L by (2.13)(ii) and, therefore, B and C are also conjugate.

(ii) By (i), there exists $\phi \in \mathrm{Elem}(L)$ mapping $\mathrm{ad}_{e_1}^2(L)$ onto $\mathrm{ad}_{e_2}^2(L)$, and hence ϕ induces a Jordan pair isomorphism of $V(e_1, f_1)$ onto $V(\phi(e_1), \phi(f_1))$. Moreover, $V(\phi(e_1), \phi(f_1))$ and $V(e_2, f_2)$ have the same first component: $\mathrm{ad}_{e_2}^2(L) = \mathrm{ad}_{\phi(e_1)}^2(L)$ by minimality of $\mathrm{ad}_{e_2}^2(L)$. But two division Jordan pairs sharing a component are isomorphic, as follows from [**15**], 1.11. ∎

3.4 Notice that, by (3.3)(ii), with any simple nondegenerate Lie algebra L containing abelian minimal inner ideals, we can associate an invariant, namely, the isomorphism class of the division Jordan pairs defined by its minimal idempotents, equivalently (see [**15**], 1.12), the isotopism class of its division Jordan algebras: for any minimal idempotent (e, f) of L, $L_2^{(e,f)}$ becomes a division Jordan algebra for the product defined by $x \cdot y := \frac{1}{2}[[x, f], y]$.

We will write $\mathrm{DJP}(L)$ to denote the division Jordan pair defined by any minimal idempotent of L, and $\mathrm{DJA}(L)$ for the corresponding division Jordan algebra. Since $\mathrm{DJP}(L)$ and $\mathrm{DJA}(L)$ are uniquely determined by isomorphism and isotopism respectively, these notations make sense.

3.5 Definition. Let L be a nondegenerate Lie algebra. We define the *socle* of L, denoted by $\mathrm{Soc}(L)$, as the sum of all minimal inner ideals of L.

By [**3**], 1.12, a minimal inner ideal of a nondegenerate Lie algebra L is either abelian or a simple ideal which contains no proper inner ideals. This result implies that there are two different types of nondegenerate simple Lie algebras which coincide with their socles:

(a) those having abelian minimal inner ideals (these algebras have a 5-grading, or even a 3-grading, and are the sum of their abelian minimal inner ideals), and

(b) those containing no proper inner ideals, and therefore no short $\mathbb{Z}$-gradings.

3.6 Theorem. *Let L be a nondegenerate Lie algebra containing minimal inner ideals. Then,*

(i) $\mathrm{Soc}(L)$ *is a direct sum of simple ideals each of which is a simple nondegenerate Lie algebra equal to its socle.*

(ii) *For any ideal I of L, I is nondegenerate and* $\mathrm{Soc}(I) = \mathrm{Soc}(L) \cap I$.

(iii) $\mathrm{Soc}(L) = [\mathrm{Soc}(L), \mathrm{Soc}(L)] = \mathrm{Soc}([L, L])$.

(iv) *If B is an abelian inner ideal of L, then either B contains a minimal inner ideal of L, or $B \subset \mathrm{Ann}_L(\mathrm{Soc}(L))$.*

(v) $\mathrm{Soc}(L)$ *is an essential ideal if and only if any nonzero ideal of L contains a minimal inner ideal.*

Proof. (i) If B be a minimal inner ideal of L, B generates an ideal which is simple as a Lie algebra (3.1). Moreover, since by (2.5)(iii) the minimal inner ideals of an ideal I are those minimal inner ideals of L contained in I, it follows from (3.2) that $\mathrm{Soc}(L) = \sum \mathrm{Id}_L(B_\alpha)$, where B_α ranges over all minimal inner ideals of L. Define two minimal inner ideals to be *equivalent* if they generate the same ideal, and fix a minimal ideal B_λ for each class of equivalence. Then $\mathrm{Soc}(L) = \oplus \, \mathrm{Id}_L(B_\lambda)$ is a direct sum of simple nondegenerate Lie algebras equal to their socles.

(ii) By (1.4), I is nondegenerate. Now, since the minimal inner ideals of I are precisely the minimal inner ideals of L contained in I (2.5)(iii), we have that $\mathrm{Soc}(I) \subset \mathrm{Soc}(L) \cap I$. For the reverse inclusion note that, by (i), $\mathrm{Soc}(L) \cap I = \oplus M_\alpha$, where the M_α are the simple ideals of $\mathrm{Soc}(L)$ contained in I, and hence

$\mathrm{Soc}(L) \cap I \subset \mathrm{Soc}(I)$, again by (2.5)(iii).

(iii) Note first that $\mathrm{Soc}(L) = [\mathrm{Soc}(L), \mathrm{Soc}(L)]$ by (i). Now we have by (ii),

$$\mathrm{Soc}([L,L]) = [L,L] \cap \mathrm{Soc}(L) = [L,L] \cap [\mathrm{Soc}(L), \mathrm{Soc}(L)] = [\mathrm{Soc}(L), \mathrm{Soc}(L)].$$

(iv) Let B be an abelian inner ideal of L. If $\mathrm{ad}_b^2(\mathrm{Soc}(L)) = 0$ for every $b \in B$, then $B \subset \mathrm{Ann}_L(\mathrm{Soc}(L))$ by formula (2). Suppose otherwise that $\mathrm{ad}_b^2(M) \neq 0$ for some simple component M of $\mathrm{Soc}(L)$ and some $b \in B$. By (2.3), $\mathrm{ad}_b^2(M) \subset B \cap M$ is an abelian inner ideal, and since M is not abelian, $\mathrm{ad}_b^2(M)$ is a *proper* inner ideal of M, so M is a sum of abelian minimal inner ideals, by (3.2). Then, $\mathrm{ad}_b^2(C) \neq 0$ for some abelian minimal inner ideal C of L, and by (2.5), $\mathrm{ad}_b^2(C)$ is an abelian minimal inner contained in B.

(v) Let L be nondegenerate. By (1.5), $\mathrm{Soc}(L)$ is an essential ideal of L if and only if $\mathrm{Ann}_L(\mathrm{Soc}(L)) = 0$. Moreover, by (ii), for any ideal I of L, $\mathrm{Soc}(I) = \mathrm{Soc}(L) \cap I$, and hence either I contains a minimal inner ideal or $I \subset \mathrm{Ann}_L(\mathrm{Soc}(L))$. ■

Recall that a Lie algebra L is said to be *Artinian* (cf. [**3**]) if it satisfies the descending chain condition on inner ideals.

3.7 Corollary. *Let L be a nondegenerate Artinian Lie algebra. Then L has essential socle, and* $\mathrm{Soc}(L)$ *is a direct sum of a finite number of simple ideals.*

Proof. Since L is Artinian, any nonzero ideal of L contains a minimal inner ideal. Hence $\mathrm{Soc}(L)$ is essential, by (3.6)(v). Suppose now that $\mathrm{Soc}(L)$ contains an infinite number of simple ideals, $\{M_i\}_{i=1}^{\infty}$. Then we have the strictly descending chain of ideals

$$\bigoplus_{i=1}^{\infty} M_i \supset \bigoplus_{i=2}^{\infty} M_i \supset \bigoplus_{i=3}^{\infty} M_i \supset \dots,$$

which yields a contradiction. ■

3.8 Remark. Nondegenerate Artinian Lie algebras do not need to coincide with their socles, even if they are finite dimensional (see [**22**], p.152). In fact, by [**22**], Theorem 3, a finite dimensional nondegenerate Lie algebra over an algebraically closed field F of characteristic $p > 5$ is a classical semisimple Lie algebra (and therefore, it coincides with its socle) if and only if it is perfect, i.e., $L = [L, L]$. On the other hand, if the field is of characteristic 0, nondegenerate finite dimensional Lie algebras are semisimple and hence they do coincide with their socles.

A natural question is to know what type of the socle does a nondegenerate Lie algebra L have, that is, if L has or does not have abelian minimal inner ideals (see (3.5)). For a finite dimensional (or, more generally, Artinian) Lie algebra, the existence of abelian minimal inner ideals is equivalent to the existence of nonzero ad-nilpotent elements (this is in fact a direct consequence of the Kostrikin's result already cited.) In the following examples, we consider the question about the existence of abelian minimal inner ideals in some well-known types of Lie algebras.

3.9 Examples. (i) Let L be a simple Lie algebra which is finite dimensional over an an algebraically closed field F. If F is of characteristic zero, then the abelian minimal inner ideals are exactly the root spaces corresponding to long roots relative to some Cartan subalgebra (therefore, they have dimension one). Indeed, if $L = H \oplus (\oplus_{\alpha\in\phi} L_\alpha)$ is the root space decomposition relative to a Cartan subalgebra H, it is easy to see that L_α is an abelian minimal inner ideal of L for any long root of $\alpha \in \phi$. Now, given such an abelian minimal inner ideal L_α, if I is another abelian minimal inner ideal of L, by (3.3) there is an automorphism φ of L such that $\varphi(L_\alpha) = I$. Hence $L = \varphi(L) = \varphi(H) \oplus (\oplus_{\alpha\in\phi}\varphi(L_\alpha))$, and $I = \varphi(L_\alpha)$ is a root space (corresponding to a long root).

(ii) Let F be a field of characteristic 0, and let L be a locally finite split simple Lie algebra (recall that L is split if there is a maximal abelian subalgebra H such that the endomorphisms $\operatorname{ad} h$ for $h \in H$ are simultaneously diagonalizable; and L is locally finite if every finite dimensional subspace of L generates a finite dimensional subalgebra of L). As in case (i), the abelian minimal inner ideals are exactly the root spaces corresponding to long roots of some root system ϕ (we have to take care because one can have different types of locally finite root systems for the same algebra. Moreover, if the dimension of L is countable, there exists a root base and an attached Dynkin diagram, but isomorphic root systems can have different Dynkin diagrams).

Indeed, take a root decomposition $L = H \oplus (\oplus_{\alpha\in\phi} L_\alpha)$ of L. Let Δ be a generalized base of ϕ (its elements are linearly independent and $\phi \subset \operatorname{span}_{\mathbb{Z}} \Delta$). According to [**20**], the root system ϕ is the directed union of the finite irreducible root subsystems ϕ_M of simple type, where M is any finite subset of Δ and $\phi_M = (\operatorname{span}_F M) \cap \phi$, therefore L is the directed union of the subalgebras L_{ϕ_M} (whose roots systems are ϕ_M). There are four types of locally finite root systems up to isomorphism for each infinite cardinality, and we can have roots of at most two different lengths (see [**20**]), which are called short or long depending on how long they are.

If $\tilde{\alpha}$ is a long root of ϕ, $\tilde{\alpha}$ is a long root of ϕ_M for every finite subset $M \subset \Delta$ such that $\tilde{\alpha} \in \phi_M$, so that $[L_{\tilde{\alpha}}, [L_{\tilde{\alpha}}, L_{\phi_M}]] \subset L_{\tilde{\alpha}}$ by (i), and obviously $L_{\tilde{\alpha}}$ is an abelian minimal inner ideal of L (notice that every $x \in L$ is in some L_{ϕ_M}). Conversely, for any abelian minimal inner ideal I of L, we can argue as in (i) and use (3.3) to show that I is a root space L_α (corresponding to a long root).

(iii) Among the real simple finite dimensional Lie algebras, those which contain abelian minimal inner ideals are exactly the non-compact ones (see [**10**] for definitions and basic properties).

If L has an abelian minimal inner ideal and we take a minimal idempotent (e, f), and $h = [e, f]$, L is 5-graded with L_i the eigenspace of $\operatorname{ad} h$ with eigenvalue i, $i = 0, \pm 1, \pm 2$, hence $K(h, h) = \operatorname{tr} \operatorname{ad}^2 h = 8 \dim L_2 + 2 \dim L_1 > 0$ (K the Killing form) and K is not negative definite (L is non-compact).

Conversely, let L be a non-compact real simple Lie algebra. Let $= \mathcal{R} + \mathcal{M}$ be a Cartan decomposition of L ($K|_{\mathcal{R}}$ is negative definite and $K|_{\mathcal{M}}$ is positive definite). Let $\mathcal{U}_{\mathcal{M}}$ denote a maximal abelian subalgebra of $\mathcal{M}$ and $\mathcal{U}$ a maximal abelian subalgebra of L containing $\mathcal{U}_{\mathcal{M}}$. Since $\mathcal{U}$ is a Cartan subalgebra of L, $\mathcal{U}^{\mathbb{C}} = \mathcal{U} \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $L_{\mathbb{C}} = L \otimes_{\mathbb{R}} \mathbb{C}$. Let ϕ be the root system of $L^{\mathbb{C}}$ relative to $\mathcal{U}^{\mathbb{C}}$, and let $\phi_{\mathcal{M}} = \{\alpha \in \phi \mid \alpha(\mathcal{U}_{\mathcal{M}}) \neq 0\}$ be the roots which do not vanish identically on $\mathcal{U}_{\mathcal{M}}$. The set $\phi_{\mathcal{M}}$ is non-empty because L is non-compact ($\mathcal{M} \neq 0$). Now, for any $\alpha \in \phi_{\mathcal{M}}$ (recall that also $\bar{\alpha} \in \phi_{\mathcal{M}}$), it is not difficult to check that $h := h_\alpha + h_{\bar{\alpha}} \in L$ ($h_\alpha \in L^{\mathbb{C}}$, but not necessarily belongs to L) and that h diagonalices L and all its eigenvalues are integers, hence L is short $\mathbb{Z}$-graded and L_n (where n is the greatest eigenvalue) is a proper inner ideal ($h \in L_0$).

Notice that the abelian minimal inner ideals can have dimension greater than 1, in contrast to cases (i) or (ii) (for instance, in $\mathrm{sl}_n(\mathbb{H})$ the minimal inner ideals are four-dimensional). In general, L contains a reduced nonzero element if and only if (see [**4**]) there is a long root α in $\phi_{\mathcal{M}}$ such that the multiplicity of the restricted root is equal to 1 (this last condition is very easy to check from its corresponding Satake diagram).

Relevant examples of infinite dimensional Lie algebras coinciding with their socles are found within the class of finitary Lie algebras:

3.10 Recall that a Lie algebra over a field F is said to be *finitary* if it is isomorphic to a subalgebra of the Lie algebra $\mathfrak{fgl}(X)$ of all finite rank operators on a vector space X over F.

3.11 Proposition. *Let L be a finitary central simple Lie algebra over a field F of characteristic zero. Then L is nondegenerate and coincides with its socle.*

Proof. If L is finite dimensional, then L is nondegenerate by [**3**], p.64, and clearly contains minimal inner ideals. Suppose that L has infinite dimension. By [**1**], Theorem 4.4, L is a direct limit of finite dimensional simple Lie algebras L_α. Since the L_α are nondegenerate, so is the whole L. The existence of (not necessarily abelian) minimal inner ideals in L can be verified by using Baranov's classification of infinite dimensional finitary simple Lie algebras over a field of characteristic 0 [**1**], Theorem 1.1, and the description of their proper inner ideals given in [**7**]. ■

4. **Lie socle versus Jordan socle**

We see in this section that the notion of socle we have just introduced extends a previous one [**6**] defined by means of the Jordan socles of the Jordan pairs associated with ideals with 3-gradings. We begin by recalling the notion of socle of a nondegenerate Jordan pair V.

4.1 Following [**17**], the *socle* of a nondegenerate Jordan pair V is defined as $\mathrm{Soc}(V) = (\mathrm{Soc}(V^+), \mathrm{Soc}(V^-))$, where $\mathrm{Soc}(V^\sigma)$ is the sum of all the minimal inner ideals of V contained in V^σ. Among other properties, $\mathrm{Soc}(V)$ is an ideal which is a direct sum of simple ideals, and it satisfies the descending chain condition on principal inner ideals.

4.2 Proposition. *Let $L = L_{-n} \oplus \ldots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_n$ be a nondegenerate Lie algebra with a $(2n+1)$-grading, and let $V = (L_n, L_{-n})$ be its associated Jordan pair.*

(i) *If B is a minimal inner ideal of L and π_i denotes the projection onto L_i, then $\pi_{\pm n}(B)$ is either zero or a minimal inner ideal of L contained in $L_{\pm n}$.*

(ii) $\mathrm{Soc}(V^\pm) = \mathrm{Soc}(L) \cap V^\pm$.

Proof. (i) As it was mentioned in (1.7), V is nondegenerate. Moreover, because of the grading, both L_n and L_{-n} are inner ideals of L. Let us suppose that $\pi_n(B) \neq 0$ and let $x \in B$ be such that $\pi_n(x) \neq 0$. Then, $0 \neq [\pi_n(x), [\pi_n(x), L]] = [\pi_n(x), [\pi_n(x), L_{-n}]] = \pi_n[x, [x, L_{-n}]] = \pi_n(B)$, because $[x, [x, L_{-n}]]$ is a nonzero inner ideal of L contained in B so it is equal to B by minimality of B, hence by

(2.5(i)), $\pi_n(B)$ is a minimal inner ideal of L contained in L_n. Similarly, $\pi_{-n}(B)$ is either zero or a minimal inner ideal of L contained in L_{-n}.

(ii) By (2.2), any minimal inner ideal $B \subset L_{\pm n}$ of V is an (abelian) minimal inner ideal of L, so $\mathrm{Soc}(V^{\pm}) \subset \mathrm{Soc}(L) \cap V^{\pm}$. Conversely, if $x \in \mathrm{Soc}(L) \cap V^{\sigma}$, x can be expressed as a sum of elements $x_1 + \ldots + x_m$, where each x_i belongs to a minimal inner ideal B_i of L. Therefore, $x = \pi_n(x) = \pi_n(x_1) + \ldots + \pi_n(x_n)$ where each $\pi_n(x_i)$ is either zero or belongs to the minimal inner ideal $\pi_n(B_i)$ of L and therefore of V. We have shown that $\mathrm{Soc}(L) \cap V^{\sigma} \subset \mathrm{Soc}\, V$. ■

4.3 Corollary. *Let L be a nondegenerate Lie algebra. Then* $\mathrm{Soc}(L)$ *satisfies the descending chain condition on principal inner ideals. Indeed, a von Neumann regular element $e \in L$ belongs to* $\mathrm{Soc}(L)$ *if and only if L satisfies the descending chain condition for all inner ideals* $\mathrm{ad}_x^2(L)$*, $x \in \mathrm{ad}_e^2(L)$.*

Proof. Let $e \in L$ be von Neumann regular. Extend e to an idempotent (e, f) as in (2.9)(i). If $e \in \mathrm{Soc}(L)$, then $V = V(e, f)$ is a nondegenerate Jordan pair coinciding with its socle by (4.2)(ii), and hence V satisfies the descending chain condition on principal inner ideals by [**17**], Corollary 1. Therefore, $\mathrm{Soc}(L)$ satisfies the descending chain condition for all inner ideals $\mathrm{ad}_x^2(L)$, $x \in \mathrm{ad}_e^2(L)$.

Assume, conversely, that L satisfies the descending chain condition for all inner ideals $\mathrm{ad}_x^2(L)$, $x \in \mathrm{ad}_e^2(L) = V^+$. By [**17**], Corollary 1, $x \in \mathrm{Soc}(V^+) \subset \mathrm{Soc}(L)$, by (4.2)(ii). ■

4.4 Let $(L, \pi) = L_{-1} \oplus L_0 \oplus L_1$ be a nondegenerate Lie algebra with a 3-grading, where the $\pi = (\pi_1, \pi_0, \pi_{-1})$ denote the projections onto the subspaces L_1, L_0, L_{-1}. Following [**5**], the *socle* of (L, π) is defined as the ideal of L generated by $\mathrm{Soc}(L_1) + \mathrm{Soc}(L_{-1})$, where $(\mathrm{Soc}(L_1), \mathrm{Soc}(L_{-1}))$ is the socle of the Jordan pair $\pi(L) = (L_1, L_{-1})$, and it is denoted by $\mathrm{Soc}_\pi(L)$ to show which grading we are taking. We have that $\mathrm{Soc}_\pi(L) = \mathrm{Soc}(\pi_1(L)) \oplus [\mathrm{Soc}(\pi_1(L)), \mathrm{Soc}(\pi_{-1}(L))] \oplus \mathrm{Soc}(\pi_{-1}(L))$, [**5**], 4.3. Moreover, $\mathrm{Soc}_\pi(L)$ can be decomposed as a direct sum of simple ideals,

$$\mathrm{Soc}_\pi(L) = \bigoplus S^{(i)} = \bigoplus \mathrm{TKK}(\pi(S^{(i)})),$$

where the $\pi(S^{(i)})$ are the simple components of $\mathrm{Soc}(\pi(L))$.

In general, the definition of the socle of a nondegenerate Lie algebra with a 3-grading depends on the 3-grading, as can be seen in the example given in [**6**], 3.3. Nevertheless, it is independent of the grading of L when this is *effective* in the sense that there is no nonzero ideal contained in the zero part of L. Motivated by this fact, a notion of socle was introduced in [**6**] for nondegenerate Lie algebras which do not necessarily have a 3-grading:

4.5 Given a nondegenerate Lie algebra L, the *Jordan socle* of L, denoted by $\mathrm{JSoc}(L)$, is defined as the sum of the socles of (I, π), where I is any ideal of L having a 3-grading and π denotes any of its possible 3-gradings:

$$\mathrm{JSoc}(L) = \sum_{(I,\pi)} \mathrm{Soc}_\pi(I).$$

The Jordan socle of a nondegenerate Lie algebra L is an ideal of L. If $\mathrm{JSoc}(L) \neq 0$ then it is a direct sum of simple ideals each of which is the TKK-algebra of a simple Jordan pair with minimal inner ideals. Therefore, $\mathrm{JSoc}(L) \cong \mathrm{TKK}(V)$, where V is a nondegenerate Jordan pair coinciding with its socle.

The relationship between the socle and the Jordan socle of a nondegenerate Lie algebra is shown in the following proposition.

4.6 Proposition. *Let L be a nondegenerate Lie algebra. Then the Jordan socle of L is equal to the sum of the simple components of* $\mathrm{Soc}(L)$ *which have a* 3*-grading. Hence, if L has an effective* 3*-grading π, then* $\mathrm{Soc}(L) = \mathrm{Soc}_\pi(L) = \mathrm{JSoc}(L)$.

Proof. Let $I = I_{-1} \oplus I_0 \oplus I_1$ be an ideal of L with a 3-grading, with associated Jordan pair $V = (I_1, I_{-1})$. By (4.2)(i), $\mathrm{Soc}(I_1) + \mathrm{Soc}(I_{-1}) \subset \mathrm{Soc}(I) \subset \mathrm{Soc}(L)$. Hence, $\mathrm{JSoc}(L) \subset \mathrm{Soc}(L)$. Conversely, let $M = M_{-1} \oplus M_0 \oplus M_1$ be a simple component of $\mathrm{Soc}(L)$ with a 3-grading. Since M contains proper inner ideals (M_1, M_{-1} are proper inner ideals because M has a *nontrivial* 3-grading), it follows from (3.2) and the structure of minimal inner ideals (2.4), that M is a sum of *abelian* minimal inner ideals. Let $0 \neq x \in M_1$. Then $\mathrm{ad}_x^3 = 0$ and there exists an abelian minimal inner ideal B of L contained in M such that $\mathrm{ad}_x^2(B) \neq 0$. Hence, by (2.5)(ii), $\mathrm{ad}_x^2(B) \subset \mathrm{ad}_x^2(M_{-1})$ is a minimal inner ideal of V. Then, $M = \mathrm{JSoc}(M) \subset \mathrm{JSoc}(L)$. ∎

5. Relationship between Lie socle and associative socle

In this section we relate the socle of a semiprime associative algebra R, with or without involution, to the socles of the related nondegenerate Lie algebras.

5.1 Let R be a (not necessarily unital) associative algebra with associated Lie algebra $R^{(-)}$. We will also consider the Lie algebras $R' = [R, R]$, $\overline{R} = R^{(-)}/Z(R)$

and $\overline{R}' = R'/R' \cap Z(R)$, where $Z(R)$ stands for the center of R. Note that $\overline{R}'$ can be regarded as an ideal of $\overline{R}$. The following result is a generalization of [**2**], 2.2, or [**11**], p.5.

5.2 Lemma. *If R is semiprime and $a \in R$ is such that $[a,[a,R]] \subset Z(R)$, then $a \in Z(R)$. Therefore, the Lie algebras $\overline{R}$ and $\overline{R}'$ are nondegenerate.*

Proof. Let us see that $[a,[a,R]] = 0$, and then apply [**11**], Sublemma p.5, to get that $a \in Z(R)$. Indeed, $\mathrm{ad}_a^3 = 0$, so for any $x \in R$,

$$0 = \mathrm{ad}_a^3(x[a,x]) = \mathrm{ad}_a^2([a,x][a,x]) + \mathrm{ad}_a^2(x[a,[a,x]]).$$

Since $[a,[a,x]] \in Z(R)$, $\mathrm{ad}_a^2(x[a,[a,x]]) = (\mathrm{ad}_a^2(x))^2$. On the other hand, we have $\mathrm{ad}_a^2([a,x][a,x]) = \mathrm{ad}_a([a,[a,x]][a,x]) + \mathrm{ad}_a([a,x][a,[a,x]]) = 2(\mathrm{ad}_a^2(x))^2$, using again that $[a,[a,x]] \in Z(R)$. Therefore,

$$0 = \mathrm{ad}_a^3(x[a,x]) = 3(\mathrm{ad}_a^2(x))^2,$$

so $\mathrm{ad}_a^2(x)$ is an element in $Z(R)$ whose square is zero, so that $\mathrm{ad}_a^2(x) = 0$ by semiprimeness of R. This proves that $\overline{R}$ is nondegenerate, and also that $\overline{R}'$ is nondegenerate because it is isomorphic to an ideal of $\overline{R}$ (1.4). ■

5.3 If R is semiprime, the sum of all minimal right ideals of R is equal to the sum of all its minimal left ideals. This set is an ideal which is called the *socle* of R and denoted by $\mathrm{Soc}(R)$. In order to relate $\mathrm{Soc}(R)$ to the socles of the nondegenerate Lie algebras $\overline{R}$ and $\overline{R}'$, it is very useful the following characterization of the rank-one elements of R, i.e., those elements generating minimal right (equivalently, left) ideals. We begin by recalling the definition of local algebra at an element.

5.4 Let $a \in R$. The *a-homotope of R* , which is denoted by $R^{(a)}$, is the (associative) algebra defined by the same linear structure as R and the new product $x \cdot_a y = xay$, for all $x, y \in R$. The set $\ker(a) = \{x \in R : axa = 0\}$ is an ideal of $R^{(a)}$, and the factor algebra $R^{(a)}/\ker(a)$ is called the *local algebra of R at a* and is be denoted by R_a.

5.5 Lemma. [**8**], Proposition 2.1. *Let R be semiprime. For any nonzero element x in R the following conditions are equivalent:*

(i) *xR is a minimal right ideal of R.*

(ii) *xRx is a minimal inner ideal of the Jordan algebra $R^{(+)}$.*

(iii) *R_x is a division algebra.*

5.6 Let $x \in R$ be an element such that $x^2 = 0$ and set $B = xRx$. For any $b \in B$ and $a \in R$, we have

$$[[b, a], b] = 2bab, \tag{7}$$

hence B is an inner ideal of $R^{(-)}$, clearly abelian. Moreover, since $B = [xR, x]$, B is also an inner ideal of R'.

Suppose now that R is semiprime. It follows from (7) that B is minimal as a *Jordan* inner ideal if and only if it is minimal as a *Lie* inner ideal. Moreover, $B \cap Z(R) = 0$ and hence $B = B/B \cap Z(R) \cong (B + Z(R))/Z(R)$ can be regarded as an inner ideal of $\overline{R}$, contained in $\overline{R}'$. Therefore we have

5.7 Proposition. *Let R be semiprime, and let $0 \neq x \in R$ be an element such that $x^2 = 0$. Then the following conditions are equivalent:*

(i) *x is a rank-one element of R.*

(ii) *xRx is an (abelian) minimal inner ideal of $R^{(-)}$.*

(iii) *xRx is a minimal inner ideal of $\overline{R}$.*

(iv) *xRx is a minimal inner ideal of $\overline{R}'$.*

Moreover, if y is another nonzero element of R of square zero, then

(v) *(x, y) is a minimal idempotent of the Jordan pair (R, R) if and only if it is a minimal idempotent of $\overline{R}'$.*

5.8 Theorem. *If R is simple and R' is not contained in $Z(R)$, then $\overline{R}'$ is a simple nondegenerate Lie algebra. Moreover, $\overline{R}'$ contains an abelian minimal inner ideal if and only if R coincides with its socle and it is not a division algebra.*

Proof. $\overline{R}'$ is nondegenerate by (5.2), and simple by [**11**], 1.12. Suppose now that R has minimal right ideals and it is not a division algebra, i.e., R has a capacity greater than one. Then R contains a rank-one element x such that $x^2 = 0$, and hence, by (5.7), $\overline{R}'$ contains an abelian minimal inner ideal.

Suppose conversely that $\overline{R}'$ contains an abelian minimal inner ideal, say $\overline{V} = V/R' \cap Z(R)$, where V is a *proper* inner ideal (otherwise, $\overline{V} = \overline{R}'$ would not be abelian, by simplicity of $\overline{R}'$). Then V is abelian by [**2**], 3.13, and for any $v \in V$, there exists $z \in Z(R)$ such that $(v - z)^2 = 0$ by [**2**], 3.14. Since $\overline{V} \neq 0$, V is not contained in $Z(R)$, so there exists a nonzero element $a \in R$ ($a = v - z$) such that $a^2 = 0$. Put $W = aRa$. We have by (5.6) that W is a nonzero inner ideal of $\overline{R}'$.

But $\overline{R}'$ coincides with its socle, and hence, by (3.6)(iv), W contains a minimal inner ideal of $\overline{R}'$, necessarily of the form xRx with $x^2 = 0$. It follows from (5.7) that R coincides with its socle, and it is not a division algebra. ∎

5.9 Let R be a simple associative algebra coinciding with its socle. By [**12**], 1.2.1, we can regard R as the algebra $\mathcal{F}_Y(X)$ of the continuous linear operators of finite rank $a : X \to X$ relative to a dual pair $\mathcal{P} = (X, Y, g)$ over a division algebra Δ, this division algebra Δ being uniquely determined by R, in fact, $\Delta \cong R_a$ for any rank-one element a in R (cf. (5.5)). In this way, we can describe $\overline{R}'$ as the *special linear algebra* (see [**6**], 5.9) of finite rank continuous operators $\mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$, where Z stands for the center of the associative algebra $\mathcal{F}_Y(X)$. Suppose that R is not a division algebra, equivalently, $\overline{R}' = \mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$ contains abelian minimal inner ideals.

5.10 Proposition. *Let* $\overline{R}' = \mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$ *be as in (5.9). We have*

(i) $\mathrm{DJP}(\overline{R}')$ *is isomorphic to the division Jordan pair* $V = (\Delta, \Delta)$.

(ii) The centroid is $\Gamma(\overline{R}') = Z(\Delta)1_{\overline{R}'}$.

Proof. (i) Take $x_1, x_2 \in X$, $y_1, y_2 \in Y$ such that $g(x_i, y_j) = \delta_{ij}$ (which is possible since R is not a division algebra) and set $e = y_1^* x_2$, $f = y_2^* x_1$ (where $y^* x$ is the operator of X defined by $y^* x(x') = g(x', y)x$ for all $x' \in X$). It is routine to see that (e, f) is a minimal idempotent of the Jordan pair (R, R), in fact,

$$(eRe, fRf) = (y_1^* \Delta x_2,\, y_2^* \Delta x_1) \cong (\Delta, \Delta)$$

via the Jordan pair isomorphism $(\alpha, \beta) \mapsto (y_1^* \alpha x_2,\, y_2^* \beta x_1)$, for all $\alpha, \beta \in \Delta$. Since $e^2 = f^2 = 0$, (e, f) is also a minimal idempotent of $\overline{R}'$, by (5.7)(v).

(ii) We first observe that there is a natural imbedding of $Z(\Delta)$ into $\Gamma(\overline{R}')$, and that the mapping $\gamma \to (\gamma, \gamma)$ defines, by restriction, an isomorphism of $\Gamma(\overline{R}')$ into the centroid $\Gamma(V(e, f))$, for any minimal idempotent (e, f). Finally, by [**19**], 3.5 and 5.8, $\Gamma(\Delta, \Delta) = \{(l_\alpha, l_\alpha) : \alpha \in Z(\Delta)\}$, where l_α denotes the dilatation of ratio α. ∎

Given an ideal I of R, set $I' = [I, I]$, $\overline{I} = I/Z(I)$ and $\overline{I}' = I'/I' \cap Z(I)$ as in (5.1). Note that if R is semiprime, $Z(I) = Z(R) \cap I$ and hence $\overline{I}$ can be regarded as an ideal of $\overline{R}$, and $\overline{I}'$ as an ideal of $\overline{R}'$.

5.11 Corollary. *If R is semiprime, then* $\overline{\mathrm{Soc}(R)}$ *is contained in* $\mathrm{Soc}(\overline{R}')$.

Proof. Since $\overline{R}'$ is nondegenerate by (5.2), $\mathrm{Soc}(\overline{R}')$ makes sense. We have $\mathrm{Soc}(R) = \bigoplus M_i$ where each M_i is a simple ideal coinciding with its socle. Therefore, we only need to verify that $\overline{M}_i' \subset \mathrm{Soc}(\overline{R}')$ for each index i. There are two possibilities:

(a) M_i is a division algebra. Then, by [**2**], 3.15, $\overline{M}_i'$ is either zero or a simple ideal not containing non trivial inner ideals. In any case, $\overline{M}_i' \subset \mathrm{Soc}(\overline{R}')$.

(b) M_i contains a rank-one element of square zero. Then we have by (5.8) that $\overline{M}_i'$ is a simple nondegenerate algebra containing an abelian minimal inner ideal, so that $\overline{M}_i' \subset \mathrm{Soc}(\overline{R}')$. ■

5.12 Remark. There exist simple associative algebras R with zero socle such that $\overline{R}'$ coincides with its socle. Indeed, if we consider any simple algebra R without zero divisors which is not a division algebra, $\mathrm{Soc}(R) = 0$, but $\overline{R}'$ is inner simple by [**2**], 3.13 and 3.14, so $\mathrm{Soc}(\overline{R}') = \overline{R}'$.

Assume now that R has an involution $*$ and denote by $K = \mathrm{Skew}(R, *)$ the Lie algebra of the skew-symmetric elements of R. We also consider the Lie algebras $K' = [K, K]$, $\overline{K} = K/K \cap Z(R)$ and $\overline{K}' = K'/K' \cap Z(R)$. Notice that $\overline{K}' \cong [\overline{K}, \overline{K}]$.

5.13 Lemma. *Let R be a simple algebra endowed with an involution $*$.*

(i) *If $a \in K$ satisfies $[a, [a, K]] \subset Z(R)$, then $a \in Z(R)$.*

(ii) *The Lie algebra $\overline{K}$ is nondegenerate.*

(iii) *If either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16, then $\overline{K}'$ is a simple nondegenerate Lie algebra.*

Proof. (i) Let $a \in K$ be such that $[a, [a, K]] \subset Z(R)$. If $[a, [a, K]]$ were nonzero, we could take $0 \neq k \in K \cap Z(R)$ and write $R = K \oplus kK$, getting $[a, [a, R]] \subset Z(R)$, which would imply that $a \in Z(R)$ by (5.2), and hence that $[a, [a, K]] = 0$, a contradiction. So $[a, [a, K]] = 0$. If the involution $*$ is of the first kind, it follows from [**2**], 2.10, that $a = 0$, while if $*$ is of the second kind, we have by [**2**], 2.13, that $a \in Z(R)$. Now (ii) is a direct consequence of (i), and (iii) follows from (ii) and [**2**, 4.2]. ■

5.14 Let R be an associative algebra with involution $*$. An element $a \in R$ is called *isotropic* if $a^*a = 0$. The involution $*$ is *isotropic* if R contains nonzero isotropic elements.

By the classification of simple associative algebras with nonzero socle and their involutions (see, for example, [**12**]), such an algebra R is isomorphic to the algebra $\mathcal{F}_X(X)$ of all continuous linear operators of finite rank of a left vector space X endowed with a nonsingular Hermitian or skew-Hermitian form h over a division algebra Δ with involution, the involution $*$ being then the adjoint involution with respect to h. Note that

(i) $a \in (\mathcal{F}_X(X), *)$ is isotropic if and only if its image $a(X)$ is a totally isotropic subspace.

(ii) We may assume, without loss of generality, that either h is symmetric (in this case Δ is a field with the identity as involution), or h is skew-Hermitian. In the first case we say that $*$ is *orthogonal*, and *skew-Hermitian* in the second case.

5.15 Lemma. *Let R be an associative algebra with involution $*$, and let a be an isotropic element of R. Then aKa^* is an abelian inner ideal of K. Moreover, if R is semiprime and a is a rank-one element of R, then aKa^* is either zero or an abelian minimal inner ideal of $\overline{K}'$.*

Proof. It is routine to verify that if a is an isotropic element, then aKa^* is an abelian inner ideal of K. Assume now that R is semiprime and a is an isotropic rank-one element of R. If b is a nonzero element of aKa^*, we have by minimality of aR that given $x \in aKa^*$ there exists $c \in R$ such that $x = bc$. Since $x^* = -x$, we also have that $x = -(bc)^* = -c^*b^* = c^*b$. Moreover, since the socle of R is a von Neumann regular ideal, there exists $y \in K$ such that $x = xyx$ (indeed, there exists $y' \in R$ such that $x = xy'x$, so consider $y = \frac{1}{2}(y' - y'^*) \in K$, which satisfies $x = xyx$). Therefore, $x = xyx = bcyc^*b = \frac{1}{2}[b, [cyc^*, b]] \in \mathrm{ad}_b^2(K)$, because $b^2 = 0$ and $(cyc^*)^* = cy^*c^* = -cyc^* \in K$. This proves that aKa^* is either zero or a minimal inner ideal of K. Since $Z(R)$ contains no nonzero nilpotent elements (by semiprimeness), $aKa^* \cap Z(R) = 0$, and hence, if $aKa^* \neq 0$ then it can be identified with an abelian minimal inner ideal of $\overline{K}$. Finally, $\overline{K}$ and $\overline{K}'$ share the same minimal inner ideals, by (2.5)(iii) and (3.6)(iii). ∎

5.16 Theorem. *Let R be a simple associative algebra with involution $*$. Suppose that either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16. Then $\overline{K}'$ is a simple nondegenerate Lie algebra. Moreover, $\overline{K}'$ contains an abelian minimal inner ideal if and only if R coincides with its socle and $*$ is isotropic.*

Proof. That $\overline{K}'$ is simple and nondegenerate was already commented in (5.13)(iii). Suppose now that $R = \mathrm{Soc}(R)$ and it has nonzero isotropic elements.

As pointed out in (5.14), we may assume that $R = \mathcal{F}_X(X)$, where X is a left vector space endowed with a nonsingular symmetric or skew-Hermitian form h over a division algebra Δ with involution, and $*$ is the adjoint involution with respect to h. Moreover, X has nonzero isotropic vectors. We deal separately with the two cases, the skew-Hermitian and the orthogonal one.

(a) If $*$ is skew-Hermitian, then $aKa^* \neq 0$ for any rank-one element $a \in \mathcal{F}_X(X)$. Let x be a nonzero isotropic vector of X. Then $a = x^*x$ (defined by $ay = h(y,x)x$ for all $y \in X$) belongs to $K = \mathrm{Skew}(\mathcal{F}_X(X), *)$ and satisfies $a^2 = 0$. Hence, by (5.15), aKa^* is an abelian minimal inner ideal of $\overline{K}'$.

(b) If $*$ is orthogonal, then $aKa^* = 0$ for any rank-one element $a \in \mathcal{F}_X(X)$, so we have by [**7**], (3.6) and (3.7), that $K = K' = \overline{K}'$ contains abelian minimal inner ideals.

Suppose, conversely, that $L := \overline{K}'$ contains an abelian minimal inner ideal, i.e., L contains a minimal idempotent (u, v) that induces a 5-grading in L with $L_2^{(u,v)} = \mathrm{ad}_u^2 L$ and $L_{-2}^{(u,v)} = \mathrm{ad}_v^2 L$, the Jordan pair $V(u,v) = (L_2^{(u,v)}, L_{-2}^{(u,v)})$ being a division Jordan pair (2.9). By [**24**], 4.5, the grading of L is induced by a unique grading of R. But the length of the grading in L may be less than the length of the grading in R; for example, if $R = \mathcal{F}_X(X)$ with orthogonal involution and X contains nonzero isotropic vectors, then L, which is equal to the finitary orthogonal Lie algebra $\mathfrak{fo}(X, h)$, has a 3-grading (see [**5**], (5.8)(2)), but this grading is induced by a 5-grading of R. Thus, again, we have to consider two possibilities.

(i) If the length of the gradings in R and L do not coincide, the example above is the only possible one as stated [**24**], Theorem 5.4: R has nonzero socle with orthogonal involution.

(ii) If R and L have gradings of the same length, we have that $L_2^{(u,v)} = \mathrm{Skew}(R_2, *)$ and $L_{-2}^{(u,v)} = \mathrm{Skew}(R_{-2}, *)$, because $Z(R) \cap R_2 = Z(R) \cap R_{-2} = 0$ ($Z(R)$ does not contain nonzero nilpotent elements). The Jordan division pair $V(u,v)$ coincides with $(\mathrm{Skew}(R_2,*), \mathrm{Skew}(R_{-2},*)) = (H(R_2,-*), H(R_{-2},-*))$, hence $\mathrm{Soc}(H(R_2,-*), H(R_{-2},-*)) \neq 0$, which implies that the associative pair (R_2, R_{-2}) has nonzero socle [**8**], (4.1)(i). Because of the grading, for any $x \in R_2$, $xRx = xR_{-2}x$, and hence, by the local characterization of rank-one elements (5.5), R coincides with its socle.

Therefore, R coincides with its socle in both cases. Let us now show that $*$ is isotropic.

Let $\overline{V} = V/Z(R) \cap [K,K]$ be an abelian minimal inner ideal of L. Then V is a proper inner ideal of $[K,K]$ and hence $[V,V] = 0$ by [**2**], 4.21 and 4.26. If the involution $*$ is of the first kind (over its centroid), $v^3 = 0$ for every $v \in V$ [**2**], 4.23. Taking a nonzero vector $v \in V$, we have that either v or v^2 is a nonzero isotropic vector of R. Suppose then that $*$ is of the second kind. We have by [**2**], 4.26, that for any $v \in V$ there exists $\alpha \in Z(R)$ such that $(v-\alpha)^2 = 0$. Write $\alpha = \alpha_s + \alpha_k$, where $\alpha_s \in \mathrm{Sym}(Z(R),*)$ and $\alpha_k \in \mathrm{Skew}(Z(R),*)$. Then $(v - (\alpha_s + \alpha_k))^2 = 0$ and $((v-(\alpha_s+\alpha_k))^2)^* = (-v-(\alpha_s-\alpha_k))^2 = 0$. Hence

$$0 = (v-(\alpha_s+\alpha_k))^2 - (v+(\alpha_s-\alpha_k))^2 = -4\alpha_s v + 4\alpha_s\alpha_k = -4\alpha_s(v-\alpha_k),$$

which implies $\alpha_s = 0$ or $v = \alpha_k$. If the latter holds for any $v \in V$, then $V \subset Z(R)$, and hence $\overline{V} = 0$, which is a contradiction. Thus there exist $v \in V$ and $\alpha \in Z(R) \cap K$ such that $v - \alpha$ is a nonzero isotropic element of R. ■

5.17 Let R be as in (5.16). As already commented, we may assume that $R = \mathcal{F}_X(X)$, where X is a left vector space endowed with a nonsingular symmetric or skew-Hermitian form h over $(\Delta, -)$, a division algebra with involution, and where $*$ is the adjoint involution with respect to h. Moreover, $\overline{K}'$ contains abelian minimal inner ideals if and only if $*$ is isotropic, equivalently, (X,h) has nonzero isotropic vectors. Let us now compute the Jordan division pair $\mathrm{DJP}(\overline{K}')$ and the centroid of $\overline{K}'$ in each one of the cases: the skew-Hermitian and the orthogonal one.

5.18 Proposition. *Let R and K be as in (5.17). Then.*

(i) If h is skew-Hermitian, then $\mathrm{DJP}(\overline{K}') \cong (\mathrm{Sym}(\Delta,-), \mathrm{Sym}(\Delta,-))$, *and the centroid* $\Gamma(\overline{K}') = \mathrm{Sym}(Z(\Delta),-)1_{\overline{K}'}$.

(ii) If h is symmetric (Δ is a field, say F), then $\mathrm{DJP}(\overline{K}')$ *is the division Clifford pair (see [**6**], 5.7, for definition) defined by an anisotropic symmetric bilinear form on a vector space over F, and* $\Gamma(\overline{K}') = F1_{\overline{K}'}$.

Proof. (i) Let (x,y) be a hyperbolic pair in X, i.e., $h(x,x) = h(y,y) = 0$, $h(x,y) = 1$, and consider the operators $e := x^*x$, $f := y^*y \in K'$. It is easy to see that $(x^*x,\, y^*y)$ is a minimal idempotent of the Jordan pair (K,K), in fact,

$$(eKe, fKf) = (x^*\,\mathrm{Sym}(\Delta,-)x,\; y^*\,\mathrm{Sym}(\Delta,-)y) \cong (\mathrm{Sym}(\Delta,-), \mathrm{Sym}(\Delta,-))$$

via the Jordan pair isomorphism $(\alpha,\beta) \mapsto (x^*\alpha x,\; y^*\beta y)$, for all $\alpha,\beta \in \mathrm{Sym}(\Delta,-)$. Since $e^2 = -e^*e = 0$ and $f^2 = -f^*f = 0$, it follows from (7) and (5.15) that there is a (e,f) remaining a minimal idempotent in $\overline{K}'$. To prove that $\Gamma(\overline{K}') = \mathrm{Sym}(Z(\Delta),-)1_{\overline{K}'}$ we follow the path sketched in the proof of (5.9)(ii): observe that there is a natural imbedding of $\mathrm{Sym}(Z(\Delta),-)$ into $\Gamma(\overline{K}')$, and that, for any minimal idempotent (e,f) of $\overline{K}'$, the mapping $\gamma \to (\gamma,\gamma)$ defines, by restriction, an isomorphism of $\Gamma(\overline{K}')$ into the centroid $\Gamma(V(e,f)) = \Gamma(\mathrm{Sym}(\Delta,-),\mathrm{Sym}(\Delta,-))$, by (i). Finally, apply [**19**], 3.6, to get $\Gamma(\mathrm{Sym}(\Delta,-),\mathrm{Sym}(\Delta,-)) = \{(l_\alpha,l_\alpha) : \alpha \in \mathrm{Sym}(Z(\Delta)\}$.

(ii) If h is symmetric (over a field F), then $\overline{K}' = K' = K$ is the so-called *finitary orthogonal algebra* $\mathfrak{fo}(X,h)$. Since X contains nonzero isotropic vectors, we can decompose $X = H \oplus H^\perp$, where $H = Fx \oplus Fy$ is the hyperbolic plane defined by a hyperbolic pair (x,y). There are two possibilities:

(1) $H^\perp$ is anisotropic. In this case, we have by [**7**], 3.7(iv), that $[x,H^\perp] := \{x^*z - z^*x : z \in H^\perp\}$ is an abelian minimal inner ideal of $\mathfrak{fo}(X,h)$. Moreover, for $z \in H^\perp$ and $[x,z] := x^*z - z^*x$, we have that $([x,z], -2h(z,z)^{-1}[y,z])$ is a minimal idempotent of $\mathfrak{fo}(X,h)$ with associated division Jordan pair

$$\mathrm{DJP}(\mathfrak{fo}(X,h)) = ([x,H^\perp],[y,H^\perp]) \cong \mathcal{C}(H^\perp,h)$$

(where $\mathcal{C}(H^\perp,h)$ denotes the division Clifford Jordan pair defined by h on $H^\perp$) via the Jordan pair isomorphism given by $([x,z],[y,v]) \mapsto (z,-v)$ $(z,v \in H^\perp)$, which can be verified using the identity (cf. [**7**], (12))

$$\mathrm{ad}^2_{[x,z]}(a) = [[x,z],[[x,z],a]] = 2h(ax,z)[x,z] - h(z,z)[x,ax], \tag{8}$$

for all $a \in \mathfrak{fo}(X,h)$. Since $\Gamma((H^\perp,h)) = \{(l_\alpha,l_\alpha) : \alpha \in F\}$ by [**19**], 3.2, we obtain as in the previous cases that $\Gamma(\mathfrak{fo}(X,h)) \cong F$.

(2) $H^\perp$ is isotropic. Then $H^\perp$ contains a hyperbolic pair (v,z). It is easy to see that $([x,z],[y,v])$ is a minimal idempotent of $\mathfrak{fo}(X,h)$, with associated division Jordan pair isomorphic to (F,F) (equal to the Clifford Jordan pair defined by the one-dimensional vector space F with the quadratic form defined by the product). As before, $\Gamma(\mathfrak{fo}(X,h)) \cong F$. ■

6. Simple nondegenerate Lie algebras with abelian minimal inner ideals

Simple nondegenerate Lie algebras containing abelian minimal inner ideals are described in this section. Among them, those which are finitary central simple over a field of characteristic 0 are characterized by the property that the division

Jordan algebras associated with them are PI. Recall that a Jordan algebra J is called a PI-algebra if it satisfies an identity that is not satisfied by all special Jordan algebras. Note that the PI-property is invariant under isotopism.

6.1 Theorem. *Let L be a Lie algebra over a field of characteristic zero or greater than 7. Then L is simple, nondegenerate and contains abelian minimal inner ideals if and only if it is isomorphic to one of the following algebras:*

(i) *A (finite dimensional over its centroid) simple exceptional Lie algebra of type G_2, F_4, E_6, E_7 or E_8 containing a nonzero ad-nilpotent element.*

(ii) *A Lie algebra of the form $\overline{R}' = [R,R]/Z(R) \cap [R,R]$, where R is a simple associative algebra coinciding with its socle which is not a division algebra, and where $[R,R]$ is not contained in $Z(R)$.*

(iii) *A Lie algebra of the form $\overline{K}' = [K,K]/Z(R) \cap [K,K]$, for $K = \mathrm{Skew}(R,*)$ where R is a simple associative algebra with isotropic involution $*$ which coincides with its socle, and where either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than* 16.

Proof. We begin by checking case by case that any of the Lie algebras listed above is simple, nondegenerate and contains abelian minimal inner ideals: (i) any simple exceptional Lie algebra L is finite dimensional over its centroid C and for the algebraic closure $\overline{C}$ of C, $\overline{C} \otimes L$ is nondegenerate (see, for instance, [**22**], Theorem 3). Since we are assuming that L contains nonzero ad-nilpotent elements, we can find one of index 3, say $b \in L$ (Kostrikin's result, [**3**], 1.5), and hence, by (2.3), L contains the nonzero abelian inner ideal $B = \mathrm{ad}_b^2(L)$. Since B is invariant under the centroid, it contains an (abelian) minimal inner ideal of L, by finite dimensionality of L. The cases (ii) and (iii) follow from (5.8), and (5.16), respectively.

Suppose, conversely, that L is a simple nondegenerate Lie algebra containing an abelian minimal inner ideal, equivalently a minimal idempotent. Then L has a 5-grading by (2.9)(ii). Hence, by [**26**], Theorem 1, L is one of the following: (i) a simple exceptional Lie algebra, (ii) $L = \overline{R}' = [R,R]/Z(R) \cap [R,R]$, where R is a simple associative algebra such that $[R,R]$ is not contained in $Z(R)$, or (iii) $L = \overline{K}' = [K,K]/Z(R) \cap [K,K]$, where $K = \mathrm{Skew}(R,*)$ and R is a simple associative algebra, $*$ is an involution of R, and either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16. (Actually, the list of simple Lie algebras with gradings given in [**26**], Theorem 1, contains two additional algebras: the

Tits-Kantor-Koecher algebra of a nondegenerate symmetric bilinear form and D_4. However, because of we are not interested in describing the gradings, both algebras can be included in case (iii): $\overline{K}' = K' = K = \mathrm{Skew}(R, *)$, where R is a simple algebra with orthogonal involution.) Returning to our list, use the same references as in the previous paragraph to get the existence of nonzero ad-nilpotent elements in (i), the coincidence of R with its socle in (ii) and (iii), and the fact that $*$ is isotropic in (iii). ■

Baranov's classification of infinite dimensional finitary central simple Lie algebras over a field F of characteristic 0 (see [**1**], Theorem 1.1) can be reformulated as follows.

6.2 Let F be a field of characteristic 0. Then any infinite dimensional finitary central simple Lie algebra over F is isomorphic to one of the following algebras:

(i) a finitary special linear algebra $\mathfrak{fsl}(\mathcal{P})$, where $\mathcal{P}$ is an infinite dimensional pair of dual vector spaces over a finite dimensional division F-algebra Δ.

(ii) $[\mathrm{Skew}(\mathcal{F}_X(X), *), \mathrm{Skew}(\mathcal{F}_X(X), *)]$, where X is an infinite dimensional vector space with a nonsingular (skew-Hermitian or symmetric) form h over a division algebra with involution $(\Delta, -)$ which is finite dimensional over $F = \mathrm{Sym}(Z(\Delta), -)$.

We have seen in (3.11) that a finitary central simple Lie algebra L over a field of characteristic 0 is nondegenerate and coincides with its socle. Moreover, if $L = \mathfrak{fsl}(\mathcal{P})$ as in (i), then L actually contains abelian minimal inner ideals (5.8), and $\mathrm{DJA}(L)$ is isomorphic to the Jordan algebra Δ^+ (5.9). If $L = [\mathrm{Skew}(\mathcal{F}_X(X), *), \mathrm{Skew}(\mathcal{F}_X(X), *)]$ as in (ii), then L contains abelian minimal inner ideals if and only if (X, h) is isotropic (5.16). If this is the case, we have by (5.17) that $\mathrm{DJA}(L)$ is either isomorphic to the Jordan algebra $\mathrm{Sym}(\Delta, -)$ or to a Jordan algebra of Clifford type. Since finite dimensional algebras and Clifford algebras are PI, we have shown that the division Jordan algebra associated to a finitary central simple Lie algebra over a field of characteristic 0 containing abelian minimal inner ideals is PI. The converse is also true as it is proved in the following theorem.

6.3 Theorem. *Let L be a central simple nondegenerate Lie algebra over a field F of characteristic* 0 *containing abelian minimal inner ideals. Then the division Jordan algebra* $\mathrm{DJA}(L)$ *associated to L is PI if and only if L is finitary over F.*

Proof. It only remains to prove that as soon as DJA(L) is PI, L is finitary over F. Without loss of generality we may assume that L is infinite dimensional over its centroid $\Gamma(L) = F$. Then, by (6.1), L is isomorphic to one of the following algebras:

(1) $\overline{R}' = [R,R]/Z(R) \cap [R,R]$, where R is a simple associative algebra coinciding with its socle and which is not a division algebra, and where $[R,R]$ is not contained in $Z(R)$.

(2) $\overline{K}' = [K,K]/Z(R) \cap [K,K]$, where $K = \mathrm{Skew}(R,*)$ and R is a simple associative algebra with isotropic involution $*$ which coincides with its socle, and where either $Z(R) = 0$ or the dimension of R over $\mathrm{Z}(R)$ is greater than 16.

Let $L = \overline{R}'$ be as in (1). By (5.9), $L = \mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$, where $\mathcal{P} = (X,Y,g)$ is a pair of dual vector spaces over a central division F-algebra Δ, and DJA($\overline{R}'$) is the Jordan algebra $\Delta^{(+)}$. Since $\Delta^{(+)}$ is PI by hypothesis, Δ is a PI associative algebra and hence Δ is finite dimensional over its center F by Kaplansky's theorem. Then the dual pair $\mathcal{P}$ is necessarily infinite dimensional over Δ (because L is infinite dimensional over F), and hence $\mathfrak{fsl}(\mathcal{P}) \cap Z = 0$. So L is the special finitary linear algebra $\mathfrak{fsl}(\mathcal{P})$.

Suppose now that $L = \overline{K}'$ as in (2) ($R = \mathcal{F}_X(X)$, where X is a left vector space X endowed with a nonsingular isotropic symmetric or skew-Hermitian form h over a division algebra with involution $(\Delta, -)$, and where $*$ is the adjoint involution with respect to h). If h is skew-Hermitian, we have by (5.17)(i) that DJA(L) is the Jordan algebra $\mathrm{Sym}(\Delta,-)$ and $\Gamma(L) = \mathrm{Sym}(Z(\Delta),-)1_L$. Since $\mathrm{Sym}(\Delta,-)$ is PI, the division algebra Δ is also PI by Amitsur's theorem [**12**], 6.5.1, and hence Δ is finite dimensional over F by Kaplansky's theorem. Then X is infinite dimensional over Δ and L is the finitary Lie algebra $[\mathrm{Skew}(\mathcal{F}_X(X),*), \mathrm{Skew}(\mathcal{F}_X(X),*)]$. If h is symmetric, then $\overline{K}' = K' = K$ and L is the finitary orthogonal algebra $\mathfrak{fo}(X,h)$. ■

Let L be a Lie algebra over a field F. Following [**6**], an element $a \in L$ is said to be *reduced* (over F) if $\mathrm{ad}_a^2(L) = Fa$. Note that any nonzero reduced element $a \in L$ determines the (abelian) minimal inner ideal Fa.

6.4 Lemma. *Let L be a simple Lie algebra over a field F. If L contains a reduced nonzero element, then L is central.*

Proof. Since L is simple, every nonzero map $\gamma \in \Gamma(L)$ is one-to-one. Let $\gamma \in \Gamma(L)$ and take a nonzero reduced element $a \in L$. Then $\mathrm{ad}_a^2(L) = Fa$ implies that $\gamma(a) = \alpha a$ for some $\alpha \in F$. Hence $\gamma(x) = \alpha x$ for all $x \in L$, because $0 \neq a \in \ker(\gamma - l_\alpha)$, with $\gamma - l_\alpha \in \Gamma(L)$. ■

As a consequence of (6.3) we obtain the following characterization of the finitary simple Lie algebras over an algebraically closed field of characteristic 0, thus positively answering a question posed in [**6**].

6.5 Corollary. *Let F be an algebraically closed field of characteristic* 0*. For a simple Lie algebra L over F the following conditions are equivalent:*

(i) *L is finitary over F.*

(ii) *L is nondegenerate and contains a nonzero reduced element over F.*

Proof. If L is finitary, then L is nondegenerate by (3.11) and contains a nonzero reduced element by (3.9)(i) (finite dimensional case) and [**6**], 4.6 (infinite dimensional case). Suppose then that L is a simple nondegenerate Lie algebra containing a nonzero reduced element of F. It follows from (6.4) that L is central, and since DJA(L) is isomorphic to F, we have by (6.3) that L is finitary. ■

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