Train algebras of rank 3 with finiteness conditions

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Dedicated to Professor El Amin Kaidi
on the occasion of his 60th birthday

Abstract

In this paper we prove that a train algebra of rank 3 which is finitely generated, Nœtherian or Artinian is finite-dimensional.

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0 Introduction

Train algebras were initiated by Etherington [10] as an algebraic framework for treating problems in population genetics. Train algebras of rank 3 have been studied from several points of view (see, for instance, [1],[7–11],[17]). In particular, Abraham [1] proved that a finite-dimensional train algebra of rank not greater than 3 is a special train algebra (that is, the barideal is nilpotent and its principal powers are ideals), and that 3 is the best possible rank. In parallel to train algebras, there is another important class of algebras with genetical

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significance, called Bernstein algebras, and which have been the subject of intense researches (see, for instance, [12], [18], [19]).

On the other hand, it is well known that one of the most satisfactory segments of the theory of associative, Jordan and alternative algebras is the structure theory of algebras with various finiteness conditions, and there is at present a substantial bibliography in this direction. Concerning Bernstein algebras, Peresi [21] and Krapivin [16] have shown independently that the barideal of a finitely generated nuclear algebra is nilpotent. Consequently, such algebras must be finite-dimensional, as was directly established by Suazo [23] using a different approach. Recently, Boudi and the present author [5] have undertaken a systematic study of Bernstein algebras satisfying chain conditions. Among many other results in that paper, it was especially proved that for a Bernstein algebra which is Jordan or nuclear, each of the Nœtherian and Artinian hypotheses implies finite-dimensionality of the algebra and so nilpotency of the barideal. Thus, in the light of those results, it is quite legitimate to investigate the counterpart of train algebras of rank 3 with finiteness conditions. In this context, we will prove that train algebras of rank 3 which are finitely generated, Nœtherian or Artinian are finite-dimensional, and therefore special train algebras.

1 Preliminaries

In order to keep the paper reasonably self-contained, we summarize in this section the basic notions that will be used in this work. Let K be an infinite field of characteristic different from 2 and 3, and let A be an algebra over K, not necessarily associative, commutative or finite-dimensional. If A has a non-zero algebra homomorphism $\omega : A \longrightarrow K$, then the ordered pair (A, ω) is called a *baric algebra* and ω is its *weight function*. For each $e \in A$ with $\omega(e) \neq 0$, we have $A = Ke \oplus N$, where $N = \ker(\omega)$ is a two-sided ideal of A, called the *barideal* of A. By a *baric ideal* of A we mean a two-sided ideal I of A with $I \subseteq N$, while a *baric subalgebra* of A is a subalgebra B of A with $B \nsubseteq N$.

A *train algebra of rank r* is a commutative baric algebra (A, ω) satisfying the equation

$$x^r + \gamma_1\omega(x)x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x = 0 \tag{1.1}$$

for all $x \in A$, where $\gamma_1, \dots \gamma_{r-1} \in K$, r is the minimal integer for which such an equation holds and $x^1 = x, \dots, x^i = x^{i-1}x$ are the *principal powers* of x. Equation (1.1) is called the *train equation* of A. Applying ω to (1.1) provides $1 + \gamma_1 + \cdots + \gamma_{n-1} = 0$.
For $r = 3$, the train equation is

$$x^3 - (1+\gamma)\omega(x)x^2 + \gamma\omega(x)^2x = 0, \tag{1.2}$$

with $\gamma \in K$. A well-known result of Etherington [11] guarantees the existence of at least a non-zero idempotent in a train algebra of rank 3 with $\gamma \neq \frac{1}{2}$, and this produces a Peirce decomposition of the algebra [7]. This property is not in general true when $\gamma = \frac{1}{2}$.

A *Bernstein algebra* is a commutative baric algebra satisfying the identity $(x^2)^2 = \omega(x)^2 x^2$. Recall that a commutative algebra is a *Jordan algebra* if the identity $x(x^2y) = x^2(xy)$ holds in A. A remarkable result of Walcher [24] says that a Bernstein algebra is Jordan if and only if $x^3 = \omega(x)x^2$. Thus, Bernstein-Jordan algebras are special instances of train algebras of rank 3 with $\gamma = 0$.

We let now A be an arbitrary algebra over K. We say that A is *nilpotent* (respectively, *right nilpotent* if the descending chain of ideals (respectively, right ideals) defined recursively by $A^1 = A$ and $A^n = \sum_{i+j=n} A^i A^j$ (respectively, $A^{<1>} = A$ and $A^{<i>} = A^{<i-1>}A$) ends up in zero. Clearly, if A is nilpotent, then A is right nilpotent. Conversely, if A is commutative, then $A^{2^n} \subseteq A^{<n>}$ by [27, Proposition 1]. Therefore, if A is a right nilpotent commutative algebra, it is nilpotent too. We define the *plenary powers* of A by $A^{(1)} = A^2$ and $A^{(n)} = (A^{(n-1)})^2$. The algebra A is said to be *solvable* when $A^{(n)} = 0$ for some n, and the smallest such n is the *index of solvability* of A.

Let $\mathtt{End}_K(A)$ be the algebra of endomorphisms of the linear space A. The linear mappings $L_a : A \longrightarrow A$ and $R_a : A \longrightarrow A$ defined by $L_a(x) = ax$ and $R_a(x) = xa$ generate a subalgebra of $\mathtt{End}_K(A)$, denoted by $\mathcal{M}_*(A)$. The subalgebra of $\mathtt{End}_K(A)$ generated by those operators and the identity endomorphism $\mathtt{id}_A$ will be denoted by $\mathcal{M}(A)$. If B is a subalgebra of A, we design by $\mathcal{M}_*^A(B)$ the subalgebra of $\mathcal{M}_*(A)$ generated by all operators L_b and R_b, where $b \in B$.
For any subset $X \subseteq A$, we write $< X >$ for the subspace of A spanned by X.

Returning to baric algebras, recall that a *special train algebra* is a commutative baric algebra for which the barideal $N = \ker(\omega)$ is nilpotent and its principal powers $N^{<i>}$ are ideals of A. It is well known that in a train algebra (A, ω) of rank 3, the $N^{<i>}$ are ideals of A (see [1]). Thus, a train algebra of rank 3 is a special train algebra if and only N is nilpotent. Since N satisfies the equation $x^3 = 0$, it follows from [22] (see also [6, Proposition] and [27, page 114]) that N is *locally nilpotent*, namely, every finitely generated subalgebra of N is nilpotent. Other characterizations of baric, train and Bernstein algebras can be found in [18] and [25].

A commutative baric algebra (A, ω) is said to be Nœtherian (Artinian) if it satisfies the ascending chain condition a.c.c. (descending chain condition d.c.c.) on ideals, that is, every ascending (descending) sequence of ideals becomes ultimately stationary. Before passing to the next section, we should like to point out that if A is a train algebra of rank 2, its train equation $x^2 = \omega(x)x$ shows that A is a Bernstein-Jordan algebra. Hence, in view of [23, Corollary 1] and [5, Theorem 2.3], A is finite-dimensional whenever it is finitely generated, Nœtherian or Artinian.

2 Finitely generated train algebras of rank 3

Let (A, ω) be a train algebra of rank 3 satisfying the train equation (1.2), and set $N = \ker(\omega)$. Choose an element $e \in A$ with $\omega(e) = 1$, so that $c_0 := e^2 - e \in N$. Using the linearization process, one obtain from (1.2) the following relations

$$(ex)y + (ey)x + e(xy) - (1+\gamma)xy = 0, \tag{2.1}$$

$$c_0x + 2e(ex) - (1+2\gamma)ex + \gamma x = 0, \tag{2.2}$$

$$ec_0 = \gamma c_0 \tag{2.3}$$

for all x and y in N (see [1, page 56] or [25, page 64] for details).
The ideal N satisfies the identity $x^3 = 0$ and so also the Jacobi identity

$$(xy)z + (yz)x + (zx)y = 0. \tag{2.4}$$

Let us verify by induction that the plenary powers $N^{(k)}$ of N are also ideals of A, like the principal powers $N^{<k>}$. The case $k = 1$ is clear. If $x, y \in N^{(k)}$ and $z \in N$, then $x(zy), y(zx), (ex)y$ and $(ey)x$ belong to $N^{(k)}N^{(k)}$ by the induction hypothesis. It follows from (2.4) and (2.1) that $z(xy)$ and $e(xy)$ are in $N^{(k+1)}$, as desired.

On the other hand, by [11, (7.8)] (see also [25, (4.13)]), A satisfies the following *plenary train equation*

$$(x^2)^2 - (1+2\gamma)\omega(x)^2x^2 + 2\gamma\omega(x)^3x = 0. \tag{2.5}$$

A powerful ingredient to use here is the closed relationship between train algebras of rank 3 with $\gamma \neq \frac{1}{2}$ and Bernstein-Jordan algebras. Given a train algebra (A, ω) of rank 3 with $\gamma \neq \frac{1}{2}$, the vector space A equipped with the new multiplication

$$x * y = (1-2\gamma)^{-1}\Big(xy - \gamma(\omega(y)x + \omega(x)y)\Big) \tag{2.6}$$

and with the same weight function ω is a Bernstein-Jordan algebra, denoted by $(\widetilde{A}, \omega)$. This construction was first introduced by Guzzo and Vicente [13] and subsequently explored by Mallol, Benavides and Varro [3] in the more general setting of baric algebras. The advantage of employing this tool lies in the fact that the algebras A and $\widetilde{A}$ have many properties in common. In this spirit, train algebras of rank 3 with $\gamma \neq \frac{1}{2}$ are, in some way, the 'dual' of Bernstein-Jordan algebras.

In connection with the goal of this section, Correa [6, Theorem] proved that the barideal $\ker(\omega)$ of a finitely generated Bernstein-Jordan algebra (A, ω) is nilpotent. An equivalent formulation obtained separately by Suazo [23, Corollary 1] says that, under the above circumstances, such an algebra is finite-dimensional. The following theorem extends this result to arbitrary train algebras of rank 3.

Theorem 2.1 *Every finitely generated train algebra of rank 3 is finite-dimensional.*

Proof. In case $\gamma \neq \frac{1}{2}$, it is readily seen from (2.6) that the subalgebras of the Bernstein-Jordan algebra $\widetilde{A}$ are precisely those of A. Therefore, $\widetilde{A}$ is also finitely generated, and so finite-dimensional by [23, Corollary 1].

Let us now focus our attention on the case $\gamma = \frac{1}{2}$. First, we are going to show that N is finitely generated as an algebra. Let $\{a_1, \ldots, a_n\}$ be a system of generators of A and write $a_i = \alpha_i e + b_i$, where $\alpha_i \in K$ and $b_i \in N$. Let y in N; then y can be expressed as $y = f(a_1, \ldots, a_n)$, where f is a non-associative polynomial. It is a simple matter to check that y takes the form

$$y = f(\alpha_1 e, \ldots, \alpha_n e) + \sum_{i=1}^{s} g_i(e, b_1, \ldots, b_n), \tag{2.7}$$

where each g_i is a non-associative monomial such that $g_i(e, b_1, \ldots, b_n)$ admits at least one factor among $b_1, \ldots, b_n$.
According to (1.2) and (2.5), we have $e^3 = \frac{3}{2}e^2 - \frac{1}{2}e$ and $(e^2)^2 = 2e^2 - e$. Hence, the subspace $< e, e^2 >$ is the subalgebra of A generated by e. As a consequence, $f(\alpha_1 e, \ldots, \alpha_n e) = \lambda e + \mu e^2$ for some $\lambda, \mu \in K$. Now, since $\omega(y) = 0$ and $\omega(g_i(e, b_1, \ldots, b_n)) = 0$, we have $\lambda + \mu = 0$, so that

$$y = \mu c_0 + \sum_{i=1}^{s} g_i(e, b_1, \ldots, b_n). \tag{2.8}$$

We denote by B the subalgebra of A generated by the set $\{c_0, b_1, \ldots, b_n, eb_1, \ldots, eb_n\}$. We claim that $N = B$. Indeed, in view of (2.8), it is enough to show that for any monomial $g = g(X_0, X_1, \ldots, X_n)$, the element $\overline{g} := g(e, b_1, \ldots, b_n)$ lies in B whenever some b_k appears in its expression, equivalently, whenever the monomial g does not depend only on the indeterminate X_0. For abbreviation, such elements $\overline{g}$ with the above condition will be called b-elements. We shall carry out an induction on the length $d(g)$ of the monomial g. If $d(g) = 1$, there is nothing to verify, since $\overline{g} = b_k$ for some $k \in \{1, \ldots, n\}$. Now let $d(g) = l \geq 2$, and assume our assertion valid for all b-elements $\overline{h}$ with length $d(h) < l$. Decompose $g = g_1 g_2$, and so $\overline{g} = \overline{g}_1 \overline{g}_2$, where g_1 and g_2 are monomials of lesser lengths. If $\overline{g}_1$ and $\overline{g}_2$ are b-elements, then by the induction assumption $\overline{g}_1$, $\overline{g}_2 \in B$ and hence $\overline{g} \in B$. In the opposite situation, we may suppose that $\overline{g}_2$ is a b-element and $\overline{g}_1$ is not. Then $\overline{g}_1 \in < e, e^2 > = < e, c_0 >$, and $\overline{g}_2 \in B$ by the induction hypothesis. Accordingly, to establish that $\overline{g}_1 \overline{g}_2 \in B$, it remains to prove that $e\overline{g}_2 \in B$, since $c_0 \overline{g}_2 \in B$. For this purpose, if $d(g_2) = 1$, we have $\overline{g}_2 = b_k$ for some $k \in \{1, \ldots, n\}$, and so $e\overline{g}_2 = eb_k \in B$. If $d(g_2) \geq 2$, we have $g_2 = h_1 h_2$, where the h_i are monomials with $d(h_i) < d(g_2) < l$ $(i = 1, 2)$, and hence $e\overline{g}_2 = e(\overline{h}_1 \overline{h}_2)$. Once again, two possibilities are presented:

(a) If $\overline{h}_1$ and $\overline{h}_2$ are b-elements, they belong to B by the induction hypothesis, and therefore $\overline{h}_1 \overline{h}_2 \in B$. Furthermore, since the $e\overline{h}_i$ are also b-elements with lengths $< l$, then by induction $e\overline{h}_i \in B$, which yields $(e\overline{h}_1)\overline{h}_2 \in B$ and $(e\overline{h}_2)\overline{h}_1 \in B$. From this and (2.1) we find that

$$e\overline{g}_2 = e(\overline{h}_1\overline{h}_2) = \frac{3}{2}(\overline{h}_1\overline{h}_2) - (e\overline{h}_1)\overline{h}_2 - (e\overline{h}_2)\overline{h}_1 \in B.$$

(b) If $\overline{h}_2$ is a b-element and $\overline{h}_1$ is not, then $\overline{h}_1 = e + \alpha c_0$ for some $\alpha \in K$, so that,

$$e\overline{g}_2 = e(\overline{h}_1\overline{h}_2) = e(e\overline{h}_2) + \alpha e(c_0\overline{h}_2). \tag{2.9}$$

As above, the induction hypothesis implies that $\overline{h}_2 \in B$ and $e\overline{h}_2 \in B$, which furnishes via (2.1),

$$e(c_0\overline{h}_2) = \frac{3}{2}(c_0\overline{h}_2) - (ec_0)\overline{h}_2 - (e\overline{h}_2)c_0 \in B, \tag{2.10}$$

because $ec_0 = \frac{1}{2}c_0 \in B$. Also, in virtue of (2.2), we get

$$e(e\overline{h}_2) = \frac{1}{2}(2e\overline{h}_2 - \frac{1}{2}\overline{h}_2 - c_0\overline{h}_2) \in B. \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we conclude that $e\overline{g}_2 \in B$. This completes the induction proof, and in consequence, $N = B$, as claimed. It follows that N is finitely generated as an algebra, and since N is locally nilpotent, it is nilpotent. Finally, a standard argument concludes that A finite-dimensional. ■

Applying the well-known result of Abraham [1], we can formulate the following immediate consequence.

Corollary 2.2 *Every finitely generated train algebra of rank* 3 *is a special train algebra.*

The next example is devoted to discussing free train algebras defined by Holgate [15] with regard to our Theorem 2.1.

Example 2.3 In [15] Holgate constructed free train algebras in the following manner. Let $\mathcal{F}_k$ be the free commutative non-associative algebra with k generators $a_1, \ldots, a_k$ and without unity. We can endow $\mathcal{F}_k$ with a weight function ω by assigning weight 1 to each symbol a_i. Let $\mathcal{R}_{k,r}(\gamma_1, \ldots, \gamma_{r-1})$ be the ideal of $\mathcal{F}_k$ generated by all elements $x^r + \gamma_1\omega(x)x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x$, $x \in \mathcal{F}_k$, where $\gamma_1, \ldots, \gamma_{r-1}$ are some fixed scalars with $1 + \gamma_1 + \cdots + \gamma_{r-1} = 0$. Then the factor algebra $\mathcal{F}_k/\mathcal{R}_{k,r}(\gamma_1, \ldots, \gamma_{r-1})$, denoted by $\mathcal{F}_{k,r}(\gamma_1, \ldots, \gamma_{r-1})$, or simply, $\mathcal{F}_{k,r}$, is a train algebra of rank r satisfying the train equation (1.1). Furthermore, every train algebra with train equation (1.1) is a homomorphism image of $\mathcal{F}_{k,r}$. This algebra $\mathcal{F}_{k,r}$ is called the *free train algebra with k generators satisfying the train equation* (1.1). As special cases, it is proved in [15, Theorem 2] that $\mathcal{F}_{k,r}$ is finite-dimensional if $k = 1$ and $r = 2$ or 3. Actually, in view of our Theorem 2.1 and the observation closing Section 1, we may generalize Holgate's result to $\mathcal{F}_{k,r}$ for an arbitrary number k of generators (where $r = 2$ or 3).

On the other hand, it is also established in [15, page 317] that the free train algebra $\mathcal{F}_{1,4}$ with one generator and satisfying the train equation $x^4 - \omega(x)x^3 = 0$ is infinite-dimensional. Hence, the rank 3 in Theorem 2.1 is now shown to be best possible.

3 Nœtherian and Artinian train algebras of rank 3

Our objective in this section is to investigate train algebras of rank 3 that are Nœtherian or Artinian. As already seen in the latter section, the case $\gamma \neq \frac{1}{2}$ will be deduced from the corresponding Bernstein-Jordan version, while the case $\gamma = \frac{1}{2}$ will require more efforts. Before embarking on the main result, we need some preparation. Let N be an arbitrary commutative algebra satisfying the identity $x^3 = 0$. It is well known that N is a Jordan algebra (see [2, Lemma 1] or [27, page 114]). In addition, Zel'manov and Skosyrskii [26, Corollary, page 448] proved that if N has no elements of order ≤ 5 in its additive group, then N is solvable; but the index of solvability is not known. Later, Hentzel, Jacobs, Peresi and Sverchkov established in [14, Theorem 6] that the barideal $\ker(\omega)$ of any Bernstein algebra (A, ω) is solvable and $(\ker(\omega))^{(4)} = 0$ in characteristic different from 2 and 3. Bernad, Gonzalez and Martinez [4, Theorem 2.11] improved subsequently the above result by showing that $(\ker(\omega))^{(3)} = 0$. On the other hand, as already mentioned, the barideal of a Bernstein-Jordan algebra satisfies the identity $x^3 = 0$. But it is not known if an arbitrary commutative algebra N satisfying this identity can be embedded as the barideal of a Bernstein-Jordan algebra. However, with the help of [14], one may state the following lemma which indicates that N is in fact solvable of index at most 4.

Lemma 3.1 *Let N be a commutative nil-algebra of index* 3. *Then N is solvable and $N^{(4)} =$ 0.*

Proof. From [14, Lemma 3] we know that $(N^2)^{<5>} = 0$. Given $x, y, z, t \in N^2$, the Jacobi identity yields

$$(xy)(zt) = -z\,(t(xy)) - t\,(z(xy)) \in (N^2)^{<4>}.$$

Hence, $N^{(3)} = (N^2N^2)(N^2N^2) \subseteq (N^2)^{<4>}$. Now, since $N^{(3)} \subseteq N^{(2)} \subseteq N^{(1)} = N^2$, we get $N^{(4)} = N^{(3)}N^{(3)} \subseteq (N^2)^{<4>}N^2 = (N^2)^{<5>} = 0$. ∎

Another crucial instrument we require here before approaching Theorem 3.3 is the following auxiliary result, which is of some interest in its own right.

Lemma 3.2 *Let N be a commutative nil-algebra of index* 3. *If N/N^2 is finite-dimensional, then so is N.*

Proof (Sketch). It is similar to the proof of [5, Theorem 2.2], where N was the barideal of a Bernstein-Jordan algebra. There is no point to repeat the details here, but we are going to explain briefly the basic ideas of the proof. First, let G be a subspace of N such that $N = N^2 \oplus G$, and let F be the subalgebra of N generated by G. Since F is a finitely generated algebra satisfying the identity $x^3 = 0$, F is nilpotent and so finite-dimensional. The second step is to show by induction on $i \geq 0$ that

$$N^{(i)} = N^{(i+1)} + F^{(i)}. \tag{3.1}$$

For this induction, we utilize some technical arguments involving the fact that the powers $N^{(i)}$ are ideals of N and a theorem of Zhevlakov which states that the multiplication algebra $\mathcal{M}_*^N(F)$ is nilpotent [27, Theorem 1, page 87] (see details in [5, Theorem 2.2]). Now, since $N = N^{(1)} + F$, it follows from (3.1) that $N = N^{(i)} + F$ for all $i \geq 1$. Putting $i = 4$, we infer from Lemma 3.1 that $N = F$. ■

Having in our disposal enough machinery, we are now ready to prove the next result.

Theorem 3.3 *Let A be a train algebra of rank 3. Then the following conditions are equivalent:*
(i) *A is Nœtherian (resp. Artinian);*
(ii) *A satisfies a.c.c. (resp. d.c.c.) on baric ideals;*
(iii) *A is finite-dimensional.*

Proof. It is sufficient to prove the implication (ii) $\Rightarrow$ (iii), for which we begin by the case $\gamma \neq \frac{1}{2}$. In account of (2.6), if x and y are in $N = \ker(\omega)$, their product in the attached Bernstein-Jordan algebra $(\widetilde{A}, \omega)$ is $x * y = (1-2\gamma)^{-1}\big(xy - \gamma\omega(y)x\big)$. So, a subspace I of A is a baric ideal of (A, ω) if and only if I is a baric ideal of $(\widetilde{A}, \omega)$. It follows that $(\widetilde{A}, \omega)$ satisfies again a. c. c. (d. c. c.) on baric ideals. By [5, Proposition 2.1, Theorem 2.3], we conclude that A is finite-dimensional.
Let now treat the situation where $\gamma = \frac{1}{2}$, and fix an element $e \in A$ of weight 1. We first consider the case when $N^2 = 0$. In this case, the identity (2.2) is reduced to $2e(ex) - 2ex + \frac{1}{2}x = 0$ for all $x \in N$, which means $(L_e - \frac{1}{2}\mathtt{id}_N)^2 = 0$. Obviously, the subspace $M := \ker(L_e - \frac{1}{2}\mathtt{id}_N)$ is a baric ideal of A, and every subspace S of M is also a baric ideal of A. Therefore, M must be finite-dimensional by hypothesis. On the other hand, the baric algebra A/M satisfies clearly a. c. c. (d. c. c.) on baric ideals. In addition, since $(L_e - \frac{1}{2}\mathtt{id}_N)^2 = 0$, we have $(L_e - \frac{1}{2}\mathtt{id}_N)(N) \subseteq M$, that is, $L_e(x) = \frac{1}{2}x \ (\mathtt{modulo}\, M)$ for all $x \in N$. Hence, each subspace of N/M is a baric ideal of A/M. This entails that N/M is finite-dimensional, and so also is N.
We now turn to the case $N^2 \neq 0$. Since N^2 is a baric ideal of A, then the baric algebra A/N^2 satisfies a. c. c. (d. c. c.), and hence finite-dimensional by the first case. Finally, we make appeal to Lemma 3.2 which ends the proof that A is finite-dimensional. ■

In closing this article, we have:

Corollary 3.4 *Every train algebra of rank 3 that is Nœtherian or Artinian is a special train algebra.*

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