# WEIGHTED BERGMAN KERNELS AND VIRTUAL BERGMAN KERNELS

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Abstract. We introduce the notion of *virtual Bergman kernel* and study some of its applications.

Contents



## Introduction

Let $\Omega \subset \mathbb{C}^n$ be a domain and $p : \Omega \to ]0, +\infty[$ a weight function on $\Omega$. Consider the "inflated domains"

$$\widehat{\Omega}_1 = \left\{ (z, \zeta) \in \Omega \times \mathbb{C} \mid |\zeta|^2 < p(z) \right\}, \tag{0.1}$$

$$\widehat{\Omega}_m = \left\{ (z, Z) \in \Omega \times \mathbb{C}^m \mid \|Z\|^2 < p(z) \right\}, \tag{0.2}$$

where $\| \ \|$ is the standard Hermitian norm on $\mathbb{C}^m$.

In our joint work [1] with Yin Weiping, we computed explicitly the Bergman kernel of some "egg domains"; among them $\widehat{\Omega}_1$, when $\Omega$ is a bounded symmetric domain and $p$ a real power of the generic norm of $\Omega$. We then obtained the Bergman kernel of the corresponding $\widehat{\Omega}_m$ by using the *"inflation principle"* of [2], which allows to deduce (for any weight function $p$) the Bergman kernel of $\widehat{\Omega}_m$ from the Bergman kernel of $\widehat{\Omega}_1$. The "inflation principle" says that if the Bergman kernel of $\widehat{\Omega}_1$ is

$$\widehat{\mathcal{K}}_1(z, \zeta) = \mathcal{L}_1\left(z, |\zeta|^2\right),$$

then the Bergman kernel of $\widehat{\Omega}_m$ is

$$\widehat{\mathcal{K}}_m(z, Z) = \frac{1}{m!} \frac{\partial^{m-1}}{\partial r^{m-1}} \mathcal{L}_1(z, r) \Big|_{r=\|Z\|^2}. \tag{0.3}$$

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It appears that the two previous steps can be unified in the following way. There exists a function $\mathcal{L}_0(z,r)$, defined in a neighborhood of $\Omega \times \{0\}$ in $\Omega \times [0,+\infty[$, such that for all $m \geq 1$, the Bergman kernel of $\widehat{\Omega}_m$ is

$$\widehat{\mathcal{K}}_m(z,Z) = \frac{1}{m!}\frac{\partial^m}{\partial r^m}\mathcal{L}_0(z,r)\bigg|_{r=\|Z\|^2}. \tag{0.4}$$

We call $\mathcal{L}_0(z,r)$ the *virtual Bergman kernel* of $(\Omega,p)$. Its existence is closely related to the "Forelli-Rudin construction" ([3], [4], [5]).

In this talk, we investigate the properties of this virtual Bergman kernel. We then show how it can be explicitly computed on bounded symmetric domains, for a special but natural choice of the weight function $p$: $p$ is taken to be a real power of the "generic norm" of the bounded symmetric domain. The explicit computation of the virtual Bergman kernel is then related to properties of the Hua integral.

## 1. Virtual Bergman kernels

1.1. **Notations.** Let $V \cong \mathbb{C}^n$ be a Hermitian vector space, with Hermitian norm $\| \, \|_V$ and volume form $\omega_V(z) = \left(\frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\|z\|^2\right)^n$. Let $\Omega$ be a domain in $V$ and $p : \Omega \to ]0,+\infty[$ a continuous function on $\Omega$. The space of holomorphic functions on $\Omega$ is denoted by $\mathrm{Hol}(\Omega)$. We denote by $H(\Omega)$ the Bergman space

$$H(\Omega) = H(\Omega,\omega_V) = \left\{ f \in \mathrm{Hol}(\Omega) \mid \|f\|^2_\Omega = \int_\Omega |f(z)|^2 \omega_V(z) < \infty \right\}$$

and by $H(\Omega,p)$ the weighted Bergman space

$$H(\Omega,p) = H(\Omega,p\omega_V) = \left\{ f \in \mathrm{Hol}(\Omega) \mid \|f\|^2_{\Omega,p} = \int_\Omega |f(z)|^2 p(z)\omega_V(z) < \infty \right\}.$$

The Hilbert products on these spaces are denoted respectively $(\ |\ )_\Omega$ and $(\ |\ )_{\Omega,p}$. The Bergman kernel of $\Omega$ (reproducing kernel of $H(\Omega)$) is denoted by $K_\Omega(z,t)$; it is fully determined by $\mathcal{K}_\Omega$:

$$\mathcal{K}_\Omega(z) = K_\Omega(z,z) \qquad (z \in \Omega),$$

which we call also Bergman kernel of $\Omega$. In the same way, the weighted Bergman kernel of $(\Omega,p)$ (reproducing kernel of $H(\Omega,p)$) is denoted by $K_{\Omega,p}(z,t)$ and is determined by $\mathcal{K}_{\Omega,p}(z) = K_{\Omega,p}(z,z)$.

1.2. Let

$$\widehat{\Omega}_1 = \left\{ (z,\zeta) \in \Omega \times \mathbb{C} \mid |\zeta|^2 < p(z) \right\}.$$

We endow $\widehat{\Omega}_1$ with the volume form

$$\omega_V(z) \wedge \omega_1(\zeta),$$

where $\omega_1(\zeta) = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}|z|^2$. The Bergman space $H\left(\widehat{\Omega}_1\right)$ will be defined with respect to this volume form.

Consider a holomorphic function $f \in \mathrm{Hol}\left(\widehat{\Omega}_1\right)$; such a function may be written

$$f(z,\zeta) = \sum_{k=0}^{\infty} f_k(z)\zeta^k,$$

with $f_k \in \operatorname{Hol}\Omega$. We compute $\|f\|^2_{\widehat{\Omega}_1}$:

$$\begin{aligned}\|f\|^2_{\widehat{\Omega}_1} &= \int_{\widehat{\Omega}_1} |f(z,\zeta)|^2\,\omega_V(z)\wedge\omega_1(\zeta)\\ &= \int_\Omega \omega_V(z)\left(\int_{|\zeta|^2<p(z)} |f(z,\zeta)|^2\,\omega_1(\zeta)\right)\\ &= \int_\Omega \omega_V(z)\left(\sum_{k=0}^{\infty} |f_k(z)|^2\,\frac{p^{k+1}(z)}{k+1}\right),\end{aligned}$$

which gives

$$\|f\|^2_{\widehat{\Omega}_1} = \sum_{k=0}^{\infty}\frac{1}{k+1}\,\|f_k\|^2_{\Omega,p^{k+1}}\,. \tag{1.1}$$

**Proposition 1.1.** *Let $\widehat{\mathcal{K}}_1$ be the Bergman kernel of $\widehat{\Omega}_1$ and $\mathcal{K}_{\Omega,p^k}$ the weighted Bergman kernel of $\Omega$ for the weight function $p^k$. Then*

$$\widehat{\mathcal{K}}_1(z,\zeta) = \sum_{k=0}^{\infty}(k+1)\,\mathcal{K}_{\Omega,p^{k+1}}(z)\,|\zeta|^{2k}\,. \tag{1.2}$$

*Proof.* For each $k\in\mathbb{N}$, let $(\varphi_{jk})_{j\in J_k}$ be a Hilbert basis (complete orthonormal system) of $H\left(\Omega,p^k\right)$. Then it follows from (1.1) that

$$\left((k+1)^{1/2}\,\varphi_{j,k+1}(z)\zeta^k\right)_{k\in\mathbb{N},\ j\in J_{k+1}}$$

is a Hilbert basis of $H\left(\widehat{\Omega}_1\right)$. From the classical properties of Bergman kernels, we get

$$\begin{aligned}\mathcal{K}_{\Omega,p^k}(z) &= \sum_{j\in J_k}|\varphi_{jk}(z)|^2\,,\\ \widehat{\mathcal{K}}_1(z,\zeta) &= \sum_{k\in\mathbb{N},\,j\in J_k}(k+1)\,|\varphi_{j,k+1}(z)|^2\,|\zeta|^{2k}\\ &= \sum_{k=0}^{\infty}(k+1)\,\mathcal{K}_{\Omega,p^{k+1}}(z)\,|\zeta|^{2k}\,.\end{aligned}$$

□

1.3. This leads to the following definition.

**Definition 1.1.** *Let $\Omega$ be a domain in $V$ and $p:\Omega\to]0,+\infty[$ a continuous function on $\Omega$. Denote by $K_{\Omega,p^k}(z,w)$ ($\mathcal{K}_{\Omega,p^k}(z)$) the weighted Bergman kernel of $\Omega$ w.r. to $p^k$. The* virtual Bergman kernel *of $(\Omega,p)$ is defined by*

$$L_{\Omega,p}(z,w;r) = L_0(z,w;r) = \sum_{k=0}^{\infty}K_{\Omega,p^k}(z,w)r^k. \tag{1.3}$$

*The function $\mathcal{L}_0(z,r)=L_0(z,z;r)$, i.e.*

$$\mathcal{L}_{\Omega,p}(z,r) = \mathcal{L}_0(z,r) = \sum_{k=0}^{\infty}\mathcal{K}_{\Omega,p^k}(z)r^k \tag{1.4}$$

*will also be called virtual Bergman kernel of $(\Omega,p)$.*

With these definitions, the relation (1.2) may be rewritten and the reproducing kernel of $\widehat{\Omega}_1$ is given by

$$\widehat{K}_1\left((z,\zeta),(w,\eta)\right) = L_1\left(z,w;\zeta\overline{\eta}\right), \tag{1.5}$$

$$\widehat{\mathcal{K}}_1(z,\zeta) = \mathcal{L}_1\left(z,|\zeta|^2\right), \tag{1.6}$$

with

$$L_1(z,w;r) = \frac{\partial}{\partial r}L_0\left(z,w;r\right),$$
$$\mathcal{L}_1(z,r) = \frac{\partial}{\partial r}\mathcal{L}_0\left(z,r\right).$$

*Remark.* From the virtual Bergman kernel of $(\Omega,p)$, it is easy to recover the weighted Bergman kernels of $\left(\Omega,p^k\right)$ $(k\in\mathbb{N})$:

$$\mathcal{K}_{\Omega,p^k}(z) = \frac{1}{k!}\,\frac{\partial^k}{\partial r^k}\mathcal{L}_{\Omega,p}\left(z,r\right)\Big|_{r=0}. \tag{1.7}$$

1.4. Let us recall some facts about harmonic analysis in the Hermitian unit ball $B_m$. Let $H\left(B_m\right) = H\left(B_m,\omega_m\right)$ be the Bergman space of $B_m$; let $K_{B_m}(Z,T)$, $\mathcal{K}_{B_m}(Z) = K_{B_m}(Z,Z)$ be the Bergman kernel. It is well known (using for instance the automorphisms of $B_m$) that

$$\mathcal{K}_{B_m}(Z) = \frac{1}{\left(1-\|Z\|^2\right)^{m+1}}. \tag{1.8}$$

This may also be written

$$\mathcal{K}_{B_m}(Z) = \frac{1}{m!}\,\frac{\partial^m}{\partial r^m}\left(\frac{1}{1-r}\right)\Big|_{r=\|Z\|^2}. \tag{1.9}$$

Let $f\in H\left(B_m\right)$; then $f$ may be written

$$f(Z) = \sum_{k=0}^{\infty} f_k(Z), \tag{1.10}$$

where the $f_k$ are $k$-homogeneous polynomials, which can be obtained through

$$f_k(Z) = \int_0^1 f\left(\mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\,Z\right)\mathrm{e}^{-2\pi\,\mathrm{i}\,k\theta}\,\mathrm{d}\,\theta.$$

The expansion (1.10) converges uniformly on each compact of $B_m$.

For $k\neq\ell$, $f_k$ and $f_\ell$ are orthogonal in $H\left(B_m\right)$. This implies that $H\left(B_m\right)$ is the Hilbert direct sum

$$H\left(B_m\right) = \widehat{\bigoplus}_{k\geq 0} H_k\left(B_m\right),$$

where $H_k\left(B_m\right)$ is the space of $k$-homogeneous polynomials, endowed with the scalar product of $H\left(B_m\right)$. For each $k\in\mathbb{N}$, let $\left(\phi_{k,j}\right)_{j\in J(k)}$ be an orthonormal basis of $H_k\left(B_m\right)$; then

$$\left(\phi_{k,j}\right)_{k\in\mathbb{N},j\in J(k)}$$

is a Hilbert basis of $H\left(B_m\right)$ and

$$\mathcal{K}_{B_m}(Z) = \sum_{k\in\mathbb{N},j\in J(k)}\left|\phi_{k,j}(Z)\right|^2 = \sum_{k\geq 0}\mathcal{K}_{B_m,k}(Z), \tag{1.11}$$

where

$$\mathcal{K}_{B_m,k}(Z) = \sum_{j\in J(k)} |\phi_{k,j}(Z)|^2$$

is the reproducing kernel of $H_k(B_m)$. The expansions (1.11) also converge uniformly on compact subsets of $B_m$. Clearly, $\mathcal{K}_{B_m,k}$ is a real polynomial, homogeneous of bidegree $(k,k)$. Comparing with the expansion of (1.9)

$$\mathcal{K}_{B_m}(Z) = \sum_{k=0}^{\infty} \binom{k+m}{m} \|Z\|^{2k},$$

we conclude that

$$\mathcal{K}_{B_m,k}(Z) = \binom{k+m}{m} \|Z\|^{2k}. \tag{1.12}$$

More generally, consider the Hermitian ball $B_m(\rho)$ of radius $\rho$. Its Bergman kernel w.r. to *the same* $\omega_m$ is

$$\mathcal{K}_{B_m(\rho)}(Z) = \frac{1}{\rho^{2m}}\mathcal{K}_{B_m}\left(\frac{Z}{\rho}\right);$$

the component of bidegree $(k,k)$ is

$$\mathcal{K}_{B_m(\rho),k}(Z) = \frac{1}{\rho^{2m+2k}}\binom{k+m}{m} \|Z\|^{2k}. \tag{1.13}$$

If $f \in H(B_m(\rho))$, its component of degree $k$ is then given by

$$f_k(Z) = \int_{B_m(\rho)} \frac{1}{\rho^{2m+2k}}\binom{k+m}{m} \langle Z,W\rangle^k f(W)\omega_V(W). \tag{1.14}$$

1.5. Now we show that the virtual Bergman kernel of $(\Omega, p)$ allows us to compute the Bergman kernel of any inflated domain

$$\widehat{\Omega}_m = \left\{(z,Z)\in\Omega\times\mathbb{C}^m \mid \|Z\|^2 < p(z)\right\}.$$

Here $\widehat{\Omega}_m$ is endowed with the volume form

$$\omega_V(z)\wedge\omega_m(Z),$$

where $\omega_m(Z) = \left(\frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\|Z\|^2\right)^m$.

**Theorem 1.2.** *The Bergman kernel* $\widehat{K}_m$ $(\widehat{\mathcal{K}}_m)$ *of* $\widehat{\Omega}_m$ *is*

$$\widehat{K}_m((z,Z),(w,W)) = L_m(z,w;\langle Z,W\rangle), \tag{1.15}$$

$$\widehat{\mathcal{K}}_m(z,Z) = \mathcal{L}_m\left(z,\|Z\|^2\right), \tag{1.16}$$

*where*

$$L_m(z,w;r) = \frac{1}{m!}\frac{\partial^m}{\partial r^m}L_0(z,w;r), \tag{1.17}$$

$$\mathcal{L}_m(z,r) = \frac{1}{m!}\frac{\partial^m}{\partial r^m}\mathcal{L}_0(z,r). \tag{1.18}$$

Note that the relation between $\widehat{\mathcal{K}}_1$ and $\widehat{\mathcal{K}}_m$, deduced from (1.16) and (1.18), is nothing else that the inflation principle (0.3).

*Proof.* We have

$$L_m(z,w;r)=\sum_{k=0}^{\infty}\binom{k+m}{m}K_{\Omega,p^{k+m}}(z,w)r^k.$$

So we want to prove that the Bergman kernel of $\widehat{\Omega}_m$ is

$$\widehat{K}_m((z,Z),(w,W))=\sum_{k=0}^{\infty}\binom{k+m}{m}K_{\Omega,p^{k+m}}(z,w)\langle Z,W\rangle^k.$$

Let $H_k\left(\widehat{\Omega}_m\right)$ be the subspace of functions $f(z,Z)$ in $H\left(\widehat{\Omega}_m\right)$, which are $k$-homogeneous polynomial w.r. to the variable $Z$. For $k\neq\ell$, $f\in H_k\left(\widehat{\Omega}_m\right)$, $g\in H_\ell\left(\widehat{\Omega}_m\right)$, we have

$$\begin{aligned}(f\mid g)_{\widehat{\Omega}_m}&=\int_{\widehat{\Omega}_m}f(w,W)\overline{g(w,W)}\omega_V(w)\wedge\omega_m(W)\\&=\int_{w\in\Omega}(f(w,\ )\mid g(w,\ ))_{B_m\left(p(w)^{1/2}\right)}\omega_V(w)=0.\end{aligned}$$

This implies that $H\left(\widehat{\Omega}_m\right)$ is the Hilbert direct sum

$$H\left(\widehat{\Omega}_m\right)=\widehat{\bigoplus_{k\geq 0}}H_k\left(\widehat{\Omega}_m\right). \tag{1.19}$$

Fix $k\in\mathbb{N}$. Let $f\in H_k\left(\widehat{\Omega}_m\right)$. For almost all $w\in\Omega$, the function $W\mapsto f(w,W)$ belongs to $H\left(B_m\left(p(w)^{1/2}\right)\right)$; by (1.14),

$$p(w)^{m+k}f(w,Z)=\int_{\|W\|^2<p(w)}\binom{k+m}{m}\langle Z,W\rangle^k f(w,W)\omega_m(W).$$

By the reproducing property of $K_{\Omega,p^{k+m}}$, we have

$$f(z,Z)=\int_{w\in\Omega}K_{\Omega,p^{k+m}}(z,w)p(w)^{m+k}f(w,Z)\omega_V(w).$$

These relations imply

$$f(z,Z)=\int_{\widehat{\Omega}_m}K_{\Omega,p^{k+m}}(z,w)\binom{k+m}{m}\langle Z,W\rangle^k f(w,W)\omega_V(w)\wedge\omega_m(W),$$

which means that

$$K_{\Omega,p^{k+m}}(z,w)\binom{k+m}{m}\langle Z,W\rangle^k$$

is the reproducing kernel of $H_k\left(\widehat{\Omega}_m\right)$.

From (1.19), we deduce that

$$\sum_{k=0}^{\infty}K_{\Omega,p^{k+m}}(z,w)\binom{k+m}{m}\langle Z,W\rangle^k$$

is the reproducing kernel of $H\left(\widehat{\Omega}_m\right)$. □

## 2. Virtual Bergman kernels for bounded symmetric domains

In this section, we compute the virtual Bergman kernel $\mathcal{L}_{\Omega,p}(z,r) = \mathcal{L}_0(z,r)$ when $\Omega$ is an irreducible bounded circled symmetric domain and $p$ is a power of the generic norm of $\Omega$.

2.1. Let $V$ be a complex finite-dimensional vector space and $\Omega \subset V$ an irreducible bounded circled symmetric domain. Then $V$ is endowed with a canonical structure of *positive Hermitian Jordan triple*. The numerical invariants of $V$ (or of $\Omega$) are the *rank* $r$ and the *multiplicities* $a$ and $b$ ($b = 0$ iff the domain is of tube type); the *genus* is defined by

$$g = 2 + a(r-1) + b.$$

The *generic minimal polynomial* $m(T,x,y)$ and the *generic norm* $N(x,y)$ of $\Omega$ are written as

$$\begin{aligned} m(T,x,y) &= T^r - T^{r-1}m_1(x,y) + \cdots + (-1)^r m_r(x,y), \\ N(x,y) &= m(1,x,y) = 1 - m_1(x,y) + \cdots + (-1)^r m_r(x,y), \end{aligned}$$

where $m_1,\ldots,m_r$ are polynomials on $V \times \overline{V}$, homogeneous of bidegrees $(1,1),\ldots,(r,r)$ respectively. In particular, $m_1$ is an Hermitian inner product on $V$; we endow $V$ with the Kähler form

$$\alpha(z) = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial} m_1(z,z)$$

and with the volume form

$$\omega = \alpha^n,$$

where $n$ is the complex dimension of $V$.

(See [1] for a review of the above properties).

2.2. The *Bergman kernel* of $\Omega$ is then

$$\mathcal{K}(z) = \frac{1}{\operatorname{vol}\Omega}\frac{1}{N(z,z)^g}, \tag{2.1}$$

with $\operatorname{vol}\Omega = \int_\Omega \omega$.

More generally, consider the *weighted Bergman space* of $\Omega$ with respect to a power of the generic norm:

$$H^{(\mu)}(\Omega) = \left\{ f \in \operatorname{Hol}\Omega \mid \int_\Omega |f(z)|^2 N(z,z)^\mu \omega(z) < \infty \right\}.$$

For $\mu > -1$, the space $H^{(\mu)}(\Omega)$ is non-zero and is a Hilbert space of holomorphic functions. Its reproducing kernel is

$$\mathcal{K}^{(\mu)}(z) = \frac{1}{\int_\Omega N(z,z)^\mu \omega(z)} N(z,z)^{-g-\mu}. \tag{2.2}$$

The denominator $\int_\Omega N(z,z)^\mu \omega(z)$ of the previous formula is called the *Hua integral*. It has been computed for the four series of classical domains (with different normalizations of the volume element) by Hua L.K. [6].

**Theorem 2.1.** [1]*Let $\Omega$ be an irreducible bounded circled symmetric domain. The value of the Hua integral is given by*

$$\int_\Omega N(z,z)^s \omega(z) = \frac{\chi(0)}{\chi(s)} \int_\Omega \omega, \tag{2.3}$$

*where $\chi$ is the polynomial of degree $n = \dim_{\mathbb{C}} \Omega$, related to the numerical invariants of $\Omega$ by*

$$\chi(s) = \prod_{j=1}^{r} \left( s + 1 + (j-1)\frac{a}{2} \right)_{1+b+(r-j)a}. \tag{2.4}$$

Here $(s)_k$ denotes the *raising factorial*

$$(s)_k = s(s+1)\cdots(s+k-1) = \frac{\Gamma(s+k)}{\Gamma(s)}.$$

The proof uses the polar decomposition in positive Hermitian Jordan triples (which generalizes the polar decomposition of matrices) and the following generalization, due to Selberg [7], of the Beta integral:

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 \prod_{j=1}^{n} t_j^{x-1} (1-t_j)^{y-1} \prod_{1 \le j < k \le n} |t_j - t_k|^{2z} \, \mathrm{d}\, t_1 \cdots \mathrm{d}\, t_n \\ &= \prod_{j=1}^{n} \frac{\Gamma(x+(j-1)z)\Gamma(y+(j-1)z)\Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z)\Gamma(z+1)}, \end{aligned} \tag{2.5}$$

for $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$, $\operatorname{Re} z > \min\left(\frac{1}{n}, \frac{\operatorname{Re} x}{n-1}, \frac{\operatorname{Re} y}{n-1}\right)$.

**Corollary 2.2.** *The Bergman kernel of $H^{(\mu)}(\Omega)$ is*

$$\mathcal{K}^{(\mu)}(z) = \frac{\chi(\mu)}{\chi(0)} N(z,z)^{-\mu} \mathcal{K}(z), \tag{2.6}$$

*where $\mathcal{K} = \mathcal{K}^{(0)}$ is the Bergman kernel of $\Omega$ .*

2.3.

**Theorem 2.3.** *Let $\Omega$ be an irreducible bounded circled symmetric domain. The virtual Bergman kernel of $(\Omega, N(z,z)^{\mu})$ is*

$$\mathcal{L}^{(\mu)}(z,r) = \mathcal{K}(z) F_{\chi,\mu}\left( \frac{r}{N(z,z)^{\mu}} \right), \tag{2.7}$$

*where $\mathcal{K}$ is the Bergman kernel of $\Omega$, $\chi$ the polynomial defined by (2.4) and $F_{\chi,\mu}$ is the rational function*

$$F_{\chi,\mu}(t) = \frac{1}{\chi(0)} \sum_{k=0}^{\infty} \chi(\mu k) t^k. \tag{2.8}$$

The proof of (2.7) is straightforward, using (1.4) and (2.6):

$$\begin{aligned} \mathcal{L}^{(\mu)}(z,r) &= \sum_{k=0}^{\infty} \mathcal{K}^{(k\mu)}(z) r^k \\ &= \sum_{k=0}^{\infty} \frac{\chi(k\mu)}{\chi(0)} N(z,z)^{-k\mu} \mathcal{K}(z) r^k \\ &= \frac{\mathcal{K}(z)}{\chi(0)} \sum_{k=0}^{\infty} \chi(k\mu) \left( \frac{r}{N(z,z)^{\mu}} \right)^k. \end{aligned}$$

If the polynomial $k \mapsto \chi(k\mu)$ is decomposed as

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{n} c_{\mu,j} \frac{(k+1)_j}{j!}, \tag{2.9}$$

the function defined by (2.8) is

$$F_{\chi,\mu}(t) = \sum_{j=0}^{n} c_{\mu,j} \left(\frac{1}{1-t}\right)^j .$$

## 3. Tables for bounded symmetric domains

Hereunder we give the results for each type of irreducible bounded symmetric domains, which allow the reader to apply Theorems 1.2 and 2.3 to special cases.

3.1. **Classification of irreducible circled bounded symmetric domains.** Here is the complete list of irreducible circled bounded symmetric domains, up to linear isomorphisms. There is some overlapping between the four infinite families, due to a finite number of isomorphisms in low dimemsions.

*Type* $I_{m,n}$ $(1 \leq m \leq n)$. $V = \mathcal{M}_{m,n}(\mathbb{C})$ (space of $m \times n$ matrices with complex entries).

$$\Omega = \left\{ x \in V \mid I_m - x^t\overline{x} \gg 0 \right\} .$$

*Type* $II_n$ $(n \geq 2)$. $V = \mathcal{A}_n(\mathbb{C})$ (space of $n \times n$ alternating matrices).

$$\Omega = \{ x \in V \mid I_n + x\overline{x} \gg 0 \} .$$

*Type* $III_n$ $(n \geq 1)$. $V = \mathcal{S}_n(\mathbb{C})$ (space of $n \times n$ symmetric matrices).

$$\Omega = \{ x \in V \mid I_n - x\overline{x} \gg 0 \} .$$

*Type* $IV_n$ $(n \neq 2)$. $V = \mathbb{C}^n$, $q(x) = \sum x_i^2$, $q(x,y) = 2\sum x_i y_i$. The domain $\Omega$ is defined by

$$1 - q(x,\overline{x}) + |q(x)|^2 > 0, \quad 2 - q(x,\overline{x}) > 0.$$

*Type V.* $V = \mathcal{M}_{2,1}(\mathbb{O}_\mathbb{C}) \simeq \mathbb{C}^{16}$, exceptional type.

*Type VI.* $V = \mathcal{H}_3(\mathbb{O}_\mathbb{C}) \simeq \mathbb{C}^{27}$, exceptional type.

3.2. **Numerical invariants.**

*Type* $I_{m,n}$ $(1 \leq m \leq n)$.

$$r = m, \quad a = 2, \quad b = n - m, \quad g = m + n.$$

*Type* $II_{2p}$ $(p \geq 1)$.

$$r = \frac{n}{2} = p, \quad a = 4, \quad b = 0, \quad g = 2\,(n-1)\,.$$

*Type* $II_{2p+1}$ $(p \geq 1)$.

$$r = \left[\frac{n}{2}\right] = p, \quad a = 4, \quad b = 2, \quad g = 2(n-1).$$

*Type* $III_n$ $(n \geq 1)$.

$$r = n, \quad a = 1, \quad b = 0, \quad g = n + 1.$$

*Type* $IV_n$ $(n \neq 2)$.

$$r = 2, \quad a = n - 2, \quad b = 0, \quad g = n.$$

*Type V.*

$$r = 2, \quad a = 6, \quad b = 4, \quad g = 12.$$

*Type VI.*

$$r = 3, \quad a = 8, \quad b = 0, \quad g = 18.$$

3.3. **Generic norm.**

*Type* $I_{m,n}$ $(1 \leq m \leq n)$. $V = \mathcal{M}_{m,n}(\mathbb{C})$ (space of $m \times n$ matrices with complex entries).

$$N(x, y) = \operatorname{Det}(I_m - x^t\overline{y}).$$

*Type* $II_n$ $(n \geq 2)$. $V = \mathcal{A}_{2p}(\mathbb{C})$ (space of $n \times n$ alternating matrices).

$$N(x, y)^2 = \operatorname{Det}(I_n + x\overline{y}).$$

*Type* $III_n$ $(n \geq 1)$. $V = \mathcal{S}_n(\mathbb{C})$ (space of $n \times n$ symmetric matrices).

$$N(x, y) = \operatorname{Det}(I_n - x\overline{y}).$$

*Type* $IV_n$ $(n \neq 2)$. $V = \mathbb{C}^n$.

$$N(x, y) = 1 - q(x, \overline{y}) + q(x)q(\overline{y}).$$

*Type V.* $V = \mathcal{M}_{2,1}(\mathbb{O}_\mathbb{C})$.

$$N(x, y) = 1 - (x|y) + (x^\sharp|y^\sharp).$$

*Type VI.* $V = \mathcal{H}_3(\mathbb{O}_\mathbb{C})$.

$$N(x, y) = 1 - (x|y) + (x^\sharp|y^\sharp) - \det x \det \overline{y}.$$

3.4. **The polynomial $\chi$ for the Hua integral.** Recall that for $s > -1$,

$$\chi(s) \int_\Omega N(x, x)^s \omega = \chi(0) \int_\Omega \omega.$$

*Type* $I_{m,n}$.

$$\chi(s) = \prod_{j=1}^{m} (s + j)_n.$$

*Type* $II_{2p}$.

$$\chi(s) = \prod_{j=1}^{p} (s + 2j - 1)_{2p-1}.$$

*Type* $II_{2p+1}$.

$$\chi(s) = \prod_{j=1}^{p} (s + 2j - 1)_{2p+1}.$$

*Type* $III_n$.

$$\chi(s) = \prod_{j=1}^{n} \left( s + \frac{j+1}{2} \right)_{1+n-j}.$$

*Type $IV_n$.*

$$\chi(s) = (s+1)_{n-1}\left(s+\frac{n}{2}\right).$$

*Type V.*

$$\chi(s) = (s+1)_8(s+4)_8.$$

*Type VI.*

$$\chi(s) = (s+1)_9(s+5)_9(s+9)_9.$$

## 4. Open problems

4.1. Understand the rational function

$$F_{\chi,\mu}(t) = \frac{1}{\chi(0)}\sum_{k=0}^{\infty}\chi(\mu k)t^k = \sum_{j=0}^{n} c_{\mu,j}\left(\frac{1}{1-t}\right)^j,$$

when $\chi$ is a polynomial from the list of subsection 3.4. Recall that the coefficients $c_{\mu,j}$ are given by

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{n} c_{\mu,j}\frac{(k+1)_j}{j!}.$$

4.2. The virtual Bergman kernel

$$\mathcal{L}_{\Omega,p}(z,r) = \sum_{k=0}^{\infty}\mathcal{K}_{\Omega,p^k}(z)r^k$$

is suitable for the computation of the Bergman kernel of "inflated domains by Hermitian balls"

$$\widehat{\Omega}_m = \left\{(z,Z)\in\Omega\times\mathbb{C}^m \mid \|Z\|^2 < p(z)\right\},$$

which may also be written

$$\widehat{\Omega}_m = \left\{(z,Z)\in\Omega\times\mathbb{C}^m \mid Z\in(p(z))^{1/2}B_m\right\}.$$

If $F$ is any circled domain in $\mathbb{C}^m$, what can be said about the Bergman kernel of

$$\widehat{\Omega}_F = \left\{(z,Z)\in\Omega\times\mathbb{C}^m \mid Z\in(p(z))^{1/2}F\right\},$$

for example when $\Omega$ is a bounded symmetric domain, $p = N_\Omega(z,z)^\mu$ and $F$ another bounded symmetric domain?

For some families $\{F\}$ other than the family $\{B_m\}$ of Hermitian balls, is it possible to define an analogous of the virtual Bergman kernel $\mathcal{L}_{\Omega,p}$ and obtain an analogous of Theorem 1.2?

4.3. It would also be interesting to replace the weighted Bergman space $H(\Omega,p)$ by more general Hilbert spaces of holomorphic functions; for example, when $\Omega$ is a bounded symmetric domain, the spaces with reproducing kernel $N(z,z)^\mu$, where $\mu$ is in the "Berezin-Wallach set" of $\Omega$ (see [8]).

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