UNIVERSAL ENVELOPING TROS AND STRUCTURE OF W*-TROS

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ABSTRACT. Calculation of the universal enveloping TROs of continuous JBW*-triples, and application of the techniques used to supplement the structural results of Ruan for W*-TROs.

1. Introduction

In 2004, Ruan [20] presented a classification scheme and proved various structure theorems for weakly closed ternary rings of operators (W*-TROs) of particular types. A W*-TRO of type I, II, or III was defined according to the Murray-von Neumann type of its linking von Neumann algebra. W*-TROs of type II were further designated as either of type $II_{1,1}, II_{1,\infty}, II_{\infty,1}$ or $II_{\infty,\infty}$. Representation theorems for W*-TROs of various types were given in Ruan's paper (see Theorem 2.1 below), but with the possible exception of type $II_{1,1}$ (however, see the end of subsection 2.1).

The purpose of this paper is to shed some light on the structure of W*-TROs (Proposition 4.1), and in particular, those of type $II_{1,1}$ (Corollary 4.3), by using ideas from [7], together with the well established structure theory of JBW*-triples (cf. [14, 15]). A W*-TRO is an example of a JBW*-triple.

Let us recall the structure of all JBW*-triples U: there is a surjective linear isometric triple isomorphism

$$U \mapsto \oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha) \oplus pM \oplus H(N,\beta), \tag{1.1}$$

where each C_α is a Cartan factor, M and N are continuous von Neumann algebras, p is a projection in M, and β is a *-antiautomorphism of N of order 2 with fixed points $H(N,\beta)$.

A basic tool in our approach is the universal enveloping TRO $T^*(X)$ of a JC*-triple X as developed in [4] and its sequels [6, 7]. By [6, Theorem 4.9],

$$T^*(L^\infty(\Omega) \otimes C) = L^\infty(\Omega) \otimes T^*(C),$$

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and consequently (see Proposition 2.2 below), identifying $L^\infty(\Omega, C)$ with $L^\infty(\Omega) \otimes C$,

$$T^*(X) \simeq \oplus_\alpha L^\infty(\Omega_\alpha, T^*(C_\alpha)) \oplus T^*(pM) \oplus T^*(H(N,\beta)). \tag{1.2}$$

The TROs $T^*(C)$ where C is a Cartan factor have been determined in [4], and independently and simultaneously in the finite dimensional cases in [3]. Both [4] and [3] make very strong use of [18].

Our main new results are the determination of the TROs $T^*(pM)$ and $T^*(H(N,\beta))$. In Theorem 3.2 it is shown that $T^*(pM) = pM \oplus M^tp^t$, and in Theorem 3.4, that $T^*(H(N,\beta)) = N$.

Only one of these results is needed in the proof of Proposition 4.1 but each is of interest in its own right. In addition, alternate proofs of portions of Proposition 4.1, which use both of these results, are provided in section 5 as an illustration of the power of universal enveloping TROs. It is planned to use this technique in future research.

A representation result, obtained simultaneously and independently by different methods in 2013, and stated in the following theorem, plays a key role in some of our proofs,

Theorem 1.1. (a) (Bunce-Timoney [7, Lemma 5.17]) *A W*-TRO is TRO-isomorphic to the direct sum $eW \oplus Wf$, where W is a von Neumann algebra and e, f are centrally orthogonal projections in W.*

(b) (Kaneda [17, Theorem]) *A W*-TRO X can be decomposed into the direct sum of TROs X_L, X_R, X_T, and there is a complete isometry of X into a von Neumann algebra M which maps X_L (resp. X_R, X_T) into a weak*-closed left ideal (resp. right ideal, two-sided ideal)*

2. Preliminaries

A ternary ring of operators (hereafter TRO) is a norm closed complex subspace of $B(K,H)$ which contains xy^*z whenever it contains x, y, z, where K and H are complex Hilbert spaces. A TRO which is closed in the weak operator topology is called a W*-TRO. A TRO-homomorphism is a linear map φ between two TROs respecting the ternary product: $\varphi(xy^*z) = \varphi(x)\varphi(y)^*\varphi(z)$.

The definition of JB*-triple will not be given here (see for example [4, 8, 14, 15]), since only its concrete realizations, which are called JC*-triples, will be involved, namely, norm closed complex subspaces of $B(K,H)$ which contain $xy^*z + zy^*x$ whenever they contain x, y, z. A JC*-homomorphism is a linear map φ between two JC*-triples respecting the triple product: $\{x, y, z\} := (xy^*z + zy^*x)/2$, that is, $\varphi\{x, y, z\} = \{\varphi(x), \varphi(y), \varphi(z)\}$. Such maps are called triple homomorphisms to distinguish them from TRO-homomorphisms.

A JC-algebra is a norm closed real subspace of $B(H)$ which is stable for the Jordan product $x \circ y = (xy + yx)/2$. A JC*-algebra is a norm closed complex Jordan *-subalgebra of $B(H)$.

Corresponding to an orthonormal basis of a complex Hilbert space H, let J be the unique conjugate linear isometry which fixes that basis elementwise. The transpose $x^t \in B(H)$ of an element $x \in B(H)$ is then defined by $x^t = Jx^*J$

2.1. **Ruan Classification Scheme.** If R is a von Neumann algebra and e is a projection in R, then $V := eR(1-e)$ is a W*-TRO. Conversely if $V \subset B(K,H)$ is a W*-TRO, then with $V^* = \{x^* : x \in V\} \subset B(H,K)$, $M(V) = \overline{XX^*}^{sot} \subset B(H)$, $N(V) = \overline{X^*X}^{sot} \subset B(K)$, let

$$R_V = \begin{bmatrix} M(V) & V \\ V^* & N(V) \end{bmatrix} \subset B(H \oplus K)$$

denote the linking von Neumann algebra of V. Then there is a SOT-continuous TRO-isomorphism $V \simeq eRe^{\perp}$, where $e = \begin{bmatrix} 1_H & 0 \\ 0 & 0 \end{bmatrix}$ and $e^{\perp} = \begin{bmatrix} 0 & 0 \\ 0 & 1_K \end{bmatrix}$.

In particular, if $V = pM$ where p is a projection in a von Neumann algebra M, then

$$R_V = \begin{bmatrix} pMp & pM \\ Mp & c(p)M \end{bmatrix} \subset B(H \oplus H),$$

where $c(p)$ denotes the central support of p (see [4, p. 965]).

A W*-TRO V is of type I,II, or III according as R_V is a von Neumann algebra of the corresponding type. A W*-TRO of type II is said to be of type $II_{\epsilon,\delta}$, where $\epsilon, \delta \in \{1, \infty\}$, if $M(V)$ is of type II_ϵ and $N(V)$ is of type II_δ.

Ruan's main representation theorems from [20] are summarized in the following theorem.

Theorem 2.1. (**Ruan** [20]) Let V be a W*-TRO.

- **i:** If V is a W*-TRO of type I, then V is TRO-isomorphic to $\oplus_\alpha L^\infty(\Omega_\alpha, B(K_\alpha, H_\alpha))$. ([20, Theorem 4.1])
- **ii:** If V is a W*-TRO of one of the types $I_{\infty,\infty}, II_{\infty,\infty}$ or III, acting on a separable Hilbert space, then V is a stable W*-TRO, and hence TRO-isomorphic to a von Neumann algebra. ([20, Corollary 4.3])
- **iii:** If V is a W*-TRO of type $II_{1,\infty}$ (respectively $II_{\infty,1}$), then V is TRO-isomorphic to $B(H, \mathbb{C}) \otimes M$ (respectively $B(\mathbb{C}, H) \otimes N$), where M (respectively N) is a von Neumann algebra of type II_1. ([20, Theorem 4.4])

According to Ruan [20, page 862], "The structure of a type $II_{1,1}$ W*-TRO is a little bit more complicated." Nevertheless, using techniques developed for approximately finite dimensional (AFD) von Neumann algebras of type II_1, he is able to prove ([20, Theorem 5.4]) that every injective W*-TRO of type $II_{1,1}$ acting on a separable Hilbert space is rectangularly AFD (approximately finite dimensional). Together with other results from [20, Sections 3,4], he proves that any W*-TRO acting on a separable Hilbert space is injective if and only if it is rectangularly AFD ([20, Theorem 5.5]).

2.2. **Horn-Neher Classification Scheme.**

A complex *JBW*-triple* is a complex JB*-triple which is also a dual Banach space. The structure of JBW*-triples is fairly well understood. Every JBW*-triple is a direct sum of a JBW*-triple of type I and a continuous JBW*-triple (defined below). JBW*-triples of type I have been defined and classified in [14] and continuous JBW*-triples have been classified in [15]. JBW*-triples of type I will not be defined here. Their classification theorem from [14] states: A JBW*-triple of type I is an ℓ^∞-direct sum of JBW*-triples of the form $A \otimes C$, where A is a commutative von Neumann algebra and C is a Cartan factor. (For Cartan factors of types 1-6, see [8, Theorem 2.5.9 and page 168]. A Cartan factor of type 1 is by definition $B(H, K)$, where H and K are complex Hilbert spaces. No other information about Cartan factors is needed in this paper)

A JBW^*-triple $\mathcal{A}$ is said to be *continuous* if it has no type I direct summand. In this case it is known that, up to isometry, $\mathcal{A}$ is a JW^*-triple, that is, a subspace of the bounded operators on a Hilbert space which is closed under the triple product $xy^*z + zy^*x$ and closed in the weak operator topology. More importantly, it has a unique decomposition into weak*-closed triple ideals, $\mathcal{A} = H(W, \alpha) \oplus pV$, where W and V are continuous von Neumann algebras, p is a projection in V, α is a *-antiautomorphism of W order 2 and $H(W, \alpha) = \{x \in W : \alpha(x) = x\}$ (see [15, (1.20) and section 4]). Notice that the triple product in pV is given by $(xy^*z + zy^*x)/2$ and that $H(W, \alpha)$ is a JBW*-algebra with the Jordan product $x \circ y = (xy + yx)/2$.

A continuous JBW*-triple of the form pM (which is a W*-TRO), is said to be of associative type, and is classified into four types in [15] as follows.

- II_1^a if M is of type II_1 and p is (necessarily) finite.
- $II_{\infty,1}^a$ if M is of type II_∞ and p is a finite projection.
- II_∞^a if M is of type II_∞ and p is a properly infinite projection.
- III^a if M is of type III and p is a (necessarily) properly infinite projection.

A continuous JBW*-triple of the form $H(W, \alpha)$ (which is a JBW*-algebra), is said to be of hermitian type, and is classified into three types in [15] as follows.

- II_1^{herm} if W is of type II_1.
- II_∞^{herm} if W is of type II_∞.
- III^{herm} if W is of type III.

2.3. **Universal Enveloping TROs.** If E is a JC*-triple, denote by $C^*(E)$ and $T^*(E)$ the universal C*-algebra and the universal TRO of E respectively (see [4, Theorem 3.1,Corollary 3.2, Definition 3.3]). Recall that the former means that $C^*(E)$ is a C*-algebra, there is an injective JC*-homomorphism $\alpha_E :\to C^*(E)$ with the properties that $\alpha_E(E)$ generates $C^*(E)$ as a C*-algebra and for each JC*-homomorphism $\pi : E \to A$, where A is a C*-algebra, there is a unique *-homomorphism $\tilde{\pi} : C^*(E) \to A$ such that $\tilde{\pi} \circ \alpha_E = \pi$. The latter means that $T^*(E)$ is a TRO, there is an injective TRO-homomorphism $\alpha_E :\to T^*(E)$ with the properties that $\alpha_E(E)$ generates $T^*(E)$ as a TRO and for each JC*-homomorphism $\pi : E \to T$, where T is a TRO, there is a unique TRO-homomorphism $\tilde{\pi} : T^*(E) \to T$ such that $\tilde{\pi} \circ \alpha_E = \pi$.

In several places in the papers [4, 6, 7], reference is made to the fact that the universal TRO construction commutes with finite direct sums of JC*-triples. More generally:

Proposition 2.2. *If E_i $(i \in I)$ is a family of JC*-triples, then*

$$T^*(\oplus_i E_i) = \oplus_i T^*(E_i).$$

Proof. Let $E = \oplus_i E_i$. It will be shown that $(R, \beta) := (\oplus_i T^*(E_i), \oplus \alpha_{E_i})$ satisfies the properties enjoyed by $(T^*(E), \alpha_E)$, that is, R is a TRO and $\beta : E \to R$ is an injective triple isomorphism such that

(a) $\beta(E)$ generates R as a TRO;

(b) for each triple homomorphism $\pi : E \to T$, where T is a TRO, there is a (necessarily unique) TRO homomorphism $\tilde{\pi} : R \to T$ such that $\tilde{\pi} \circ \beta = \pi$.

It is clear that R is a TRO, β is an injective triple isomorphism, and $\beta(E)$ generates R as a TRO. Let $\pi : E \to R$ be a triple homomorphism. Then $\pi_i := \pi | E_i$ is a triple homomorphism from E_i to T, so there exists a TRO homomorphism $\tilde{\pi}_i : T^*(E_i) \to T$ such that $\tilde{\pi}_i \circ \alpha_{E_i} = \pi_i$.

Consider the TRO homomorphism $\sigma := \oplus_i \tilde{\pi}_i : R \to \oplus_i \pi_i(E_i)$. Since the E_i are pairwise orthogonal ideals in E, the $\pi(E_i)$ are pairwise orthogonal (triple) ideals in T and $\oplus_i \pi_i(E_i) \subset T$, that is, σ has range in T. Moreover, it is easily verified that $\sigma \circ \beta = \pi$ so that $\tilde{\pi}$ may be taken to be σ. □

The property of being universally reversible (cf. [7]) will be important for our proofs. A JC-algebra $A \subset B(H)_{sa}$ is called *reversible* if

$$a_1, \dots, a_n \in A \Rightarrow a_1 \cdots a_n + a_n \cdots a_1 \in A.$$

A is *universally reversible* if $\pi(A)$ is reversible for each representation (=Jordan homomorphism) $\pi : A \to B(K)_{sa}$. A JC*-algebra $A \subset B(H)$ is called *reversible* if

$$a_1, \dots, a_n \in A \Rightarrow a_1 \cdots a_n + a_n \cdots a_1 \in A.$$

and A is *universally reversible* if $\pi(A)$ is reversible for each representation (=Jordan *-homomorphism) $\pi : A \to B(K)$. Since JC-algebras are exactly the self-adjoint parts of JC*-algebras, a JC*-algebra A is reversible (respectively, universally reversible) if and only if the JC-algebra A_{sa} is reversible (respectively, universally reversible).

A JC*-triple $A \subset B(H,K)$ is called *reversible* if $a_1, \ldots, a_{2n+1} \in A \Rightarrow$

$$a_1 a_2^* a_3 \cdots a_{2n-1} a_{2n}^* a_{2n+1} + a_{2n+1} a_{2n}^* a_{2n-1} \cdots a_3 a_2^* a_1 \in A.$$

and A is *universally reversible* if $\pi(A)$ is reversible for each representation (=triple homomorphism) $\pi : A \to B(H', K')$.

It is easy to check that if a JC*-algebra is universally reversible as a JC*-triple, then it is universally reversible as a JC*-algebra.

Given a JC-algebra A , there is a universal C*-algebra B of A, analogous to the definition of $C^*(E)$ given above for JC*-triples E, with the following properties: there is a Jordan homomorphism π from A into B_{sa} such that B is the C*-algebra generated by $\pi(A)$ and for every Jordan homomorphism π_1 from A into C_{sa} for some C*-algebra C, there is a *-homomorphism $\pi_2 : B \to C$ such that $\pi_1 = \pi_2 \circ \pi$. (see [13, section 4]). It is clear that $B = C^*(E)$ where E is the complexification of A.

For the convenience of the reader, the following theorem is stated.

Theorem 2.3. ([13, Theorem 4.4]) Let A be a universally reversible JC-algebra, B a C*-algebra , and $\theta : A \to B_{sa}$ an injective homomorphism such that B is the C*-algebra generated by $\theta(A)$. If B admits an antiautomorphism φ such that $\varphi \circ \theta = \theta$, then θ extends to a *-isomorphism of $C^*(A)$ onto B.

3. The universal enveloping TROs of pM and of $H(N, \beta)$

The proofs of the theorems in this section are very short since several results from [7] are used, as well as one each from [4] and [12].

3.1. The universal enveloping TRO of pM.

Lemma 3.1. *Let W be a continuous von Neumann algebra, and let e be a projection in W. Then the TRO eW does not admit a nonzero TRO homomorphism onto $\mathbb{C}$.*

Proof. Suppose, by way of contradiction, that f is a nonzero TRO homomorphism of eW onto $\mathbb{C}$. Since $f(e) = f(ee^*e) = f(e)|f(e)|^2$, either $f(e) = 0$ or $|f(e)| = 1$. The former case can be ruled out since for $x \in W$, $f(ex) = f((e1)(e1)^*(ex)) = |f(e)|^2 f(ex)$ and f would be zero. If then $f(e) = \lambda$ with $|\lambda| = 1$, then replacing f by $\overline{\lambda} f$ it can be assumed that $f(e) = 1$.

For $x, y \in W$, $f((exe)(eye)) = f(exee^*eye) = f(exe)\overline{f(e)}f(eye) = f(eye)f(eye)$ and $f((exe)^*) = f(ex^*e) = f(e(exe)^*e) = \overline{f(exe)}$ so that $f|eWe$ is a *-homomorphism onto $\mathbb{C}$ and since $f(e) = 1 = \|f\|$, $f|eWe$ is a state of eWe. Moreover $f|eWe$, being a *-homomorphism is order preserving and has the value 0 or 1 on each projection of eWe. It follows trivially that f is completely additive on projections and is therefore a normal functional by a theorem of Dixmier [21, 1.13.2, and page 30]. Now apply the theorem of Plymen ([19]) to the effect that a continuous von Neumann algebra admits no dispersion-free normal state. (A state is dispersion-free if it preserves squares of self-adjoint elements.) □

Theorem 3.2. *Let $W \subset B(H)$ be a continuous von Neumann algebra, and let e be a projection in W. Then $T^*(eW) = eW \oplus W^t e^t$, where x^t be any transposition on $B(H)$.*

Proof. By [7, Proposition 3.9], eW is universally reversible and so by [7, Theorem 4.11], it does not admit a TRO homomorphism onto a Hilbert space of dimension greater than 2. The proof is completed by applying Lemma 3.1 and [7, Theorem 5.4]. □

3.2. **The universal enveloping TRO of $H(N,\beta)$.** Let E be a JC*-algebra. Similar to the construction of $C^*(E)$ when E is considered as a JC*-triple, there is a C*-algebra $C^*_J(E)$ and a Jordan *-homomorphism $\beta_E : E \to C^*_J(E)$ such that $C^*_J(E)$ is the C*-algebra generated by $\beta_E(E)$ and every Jordan *-homomorphism $\pi : E \to B$, where B is a C*-algebra, extends to a *-homomorphism of $C^*_J(E)$ into B. (see [4, Remark 3.4])

Lemma 3.3. *If E is a JC*-algebra, then $C^*_J(E)$ is *-isomorphic to $C^*(E)$.*

Proof. By definition of $C^*_J(E)$, there exists a *-homomorphism $\widetilde{\alpha}_E : C^*_J(E) \to C^*(E)$ such that $\widetilde{\alpha}_E \circ \beta_E = \alpha_E$. By definition of $C^*(E)$, there exists a *-homomorphism $\widetilde{\beta}_E : C^*(E) \to C^*_J(E)$ such that $\widetilde{\beta}_E \circ \alpha_E = \beta_E$.

By definition of $C^*_J(E)$, there exists a *-homomorphism $(\widetilde{\beta}_E \circ \alpha_E)\tilde{} : C^*_J(E) \to C^*_J(E)$ such that $(\tilde{\beta}_E \circ \alpha_E)\tilde{} \circ \beta_E = \tilde{\beta}_E \circ \alpha_E$. By definition of $C^*(E)$, there exists a *-homomorphism $(\widetilde{\alpha}_E \circ \beta_E)\tilde{} : C^*(E) \to C^*(E)$ such that $(\tilde{\alpha}_E \circ \beta_E)\tilde{} \circ \alpha_E = \tilde{\alpha}_E \circ \beta_E$.

By diagram chasing $(\tilde{\alpha}_E \circ \beta_E)\tilde{} = \tilde{\alpha}_E \circ \tilde{\beta}_E$ and $(\tilde{\beta}_E \circ \alpha_E)\tilde{} = \tilde{\beta}_E \circ \tilde{\alpha}_E$. (It is enough to check this on the generating sets $\alpha_E(E)$ and $\beta_E(E)$.) It follows that $\tilde{\alpha}_E \circ \tilde{\beta}_E = \mathrm{id}_{C^*(E)}$ and $\tilde{\beta}_E \circ \tilde{\alpha}_E = \mathrm{id}_{C^*_J(E)}$ so that $\tilde{\alpha}_E$ is a *-isomorphism with inverse $\widetilde{\beta}_E$. □

Theorem 3.4. *If N is a continuous von Neumann algebra, then*

$$T^*(H(N,\beta)) = N.$$

Proof. Let $E = H(N,\beta)$. By [4, Proposition 3.7], $T^*(E) = C^*_J(E)$. By Lemma 3.3, $C^*_J(E) \simeq C^*(E)$. By [7, Proposition 2.2], E is universally reversible. In Theorem 2.3, let $A = E_{sa}$, $B = N$, $\alpha = \beta$ and $\theta(x) = x$ for $x \in A$. By [12, Corollary 2.9], N is the C*-algebra generated by $\theta(A)$, so that Theorem 2.3 applies to finish the proof. □

Remark 3.5. [12, Corollary 2.9], which was used in the proof of Theorem 3.4, is a corollary to [12, Theorem 2.8], which states that if N is a von Neumann algebra admitting a *-antiautomorphism α and if $H(N,\alpha)_{sa}$ has no type I_1 part, then N is generated as a von Neumann algebra by $H(N,\alpha)_{sa}$. The author of [12] was apparently unaware that [12, Corollary 2.9] was proved in the case of a continuous factor by Ayupov in 1985 [1], and the theorem in this case appeared as Theorem 1.5.2 in the book [2] in 1997.

4. Structure of W*-TROs via JC*-triples

Now suppose that X is a W*-TRO, and consider the space X with the JC*-triple structure given by $\{xyz\} = (xy^*z + zy^*x)/2$, so that X becomes a JBW*-triple. As noted in (1.1), there is a surjective linear isometry

$$X \to \oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha) \oplus pM \oplus H(N,\beta), \tag{4.1}$$

where each C_α is a Cartan factor, M and N are continuous von Neumann algebras, p is a projection in M, β is a *-antiautomorphism of N of order 2 with fixed points $H(N,\beta)$.

The author acknowledges that in the following proposition, (a) is only a mild improvement of the results of Theorem 1.1, and Corollary 4.2 was proved by Ruan [20] without the separability assumption. However, the approach is different and has promise for future research (see section 5).

Proposition 4.1. *Let V be a W*-TRO.*

(a) If V has no type I part, then it is TRO-isomorphic to $eA \oplus Af$, where A is a continuous von Neumann algebra.

(b) If V acts on a separable Hilbert space, then it is TRO-isomorphic to

$$\oplus_\alpha L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha)) \oplus eA \oplus Af$$

where A is a continuous von Neumann algebra.

Proof. For any W*-TRO, by (4.1), write $V = V_1 \oplus V_2 \oplus V_3$, where V_i are weak*-closed orthogonal triple ideals of V with V_1 triple isomorphic to a JBW*-triple $\oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha)$ of type I, V_2 triple isomorphic to a right ideal pM in a continuous von Neumann algebra M, and V_3 triple isomorphic to $H(N,\beta)$ for some continuous von Neumann algebra N admitting a *-antiautomorphism β of order 2.

Since the triple ideals coincide with the TRO ideals (see [4, Lemma 2.1]), in particular each V_i is a sub-W*-TRO of V.

Consider first V_2. By Theorem 1.1(a), V_2 is TRO-isomorphic to $eA \oplus Af$, for some von Neumann algebra A. In particular, V_2 is triple isomorphic to $eA \oplus f^tA^t = (e \oplus f^t)(A \oplus A^t)$ and to pM, so by [15], $A \oplus A^t$ has the same type as M. It follows that A is a continuous von Neumann algebra.

Next it is shown that $V_3 = 0$. V_3 is triple isomorphic to $H(N,\beta)$ and TRO-isomorphic to $eA \oplus Af$, for a von Neumann algebra A.

Thus the continuous JBW*-triple $H(N,\beta)$ of hermitian type is triple isomorphic to the JBW*-triple $(e \oplus f^t)(A \oplus A^t)$, which is necessarily continuous and hence of associative type. By the uniqueness of the representation theorem for continuous JBW*-triples ([15, Section 4]), $H(N,\beta) = 0$. (For alternate proofs of the descriptions of V_2 and V_3 just given, using techniques from the theories of Jordan triples and universal enveloping TROs, see section 5.)

Finally, consider V_1. It will be shown that if V has no type I part, then $V_1 = 0$, which would prove (a); and if V acts on a separable Hilbert space, then V_1 is of the form $\oplus_\alpha L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha))$, up to TRO-isomorphism, which would prove (b) and complete the proof of the theorem.

There are weak*-closed TRO ideals V_α such that $V_1 = \oplus_\alpha V_\alpha$ with V_α triple isomorphic to $L^\infty(\Omega_\alpha, C_\alpha)$ provided that $V_\alpha \neq 0$, which is assumed henceforth. It is shown in [16, Lemma 2.4 and Proof of Theorem 1.1] that no Cartan factor of type 2,3,4,5,6 can be isometric to a TRO. It follows easily that $L^\infty(\Omega_\alpha, C_\alpha)$ cannot be isometric to a TRO unless C_α is a Cartan factor of type 1. Therefore each C_α is a Cartan factor of type 1, and therefore V_α is either zero, or triple isomorphic to $L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha))$ for suitable Hilbert spaces H_α and K_α.

Next consider V_α for a fixed α. To simplify notation let U denote V_α and W denote $L^\infty(\Omega, B(H,K))$. By [6, Theorem 4.9],

$$\begin{aligned} T^*(W) &= L^\infty(\Omega, B(H,K) \oplus B(H,K)^t) \\ &= L^\infty(\Omega, B(H,K)) \oplus L^\infty(\Omega, B(H,K)^t) \end{aligned}$$

and $\alpha_W(x)(\omega) = x(\omega) \oplus x(\omega)^t$, for $x \in T^*(W)$ and $\omega \in \Omega$.

By Theorem 1.1(a), U is TRO-isomorphic to $eA \oplus Af$, for some von Neumann algebra A. Since $T^*(U)$ is TRO-isomorphic to $T^*(W)$, by Theorem 3.2,

$$eA \oplus A^t e^t \oplus Af \oplus f^t A^t \overset{TRO}{\simeq} L^\infty(\Omega, B(H,K)) \oplus L^\infty(\Omega, B(H,K)^t). \tag{4.2}$$

The right side of (4.2) is a JBW*-triple of type I and thus by [9, Theorem 5.2] or [5, Theorem 4.2], eA is a JBW*-triple of type I, which implies that A is a von Neumann algebra of type I.

Summarizing up to this point, V is arbitrary, and $V = V_1 \oplus V_2 + V_3$, where

$$\begin{aligned} V_1 &\overset{TRO}{\simeq} \oplus_\alpha e_\alpha A_\alpha \oplus A_\alpha f_\alpha, \\ V_2 &\overset{TRO}{\simeq} eA \oplus Af, \quad V_3 = 0, \end{aligned} \tag{4.3}$$

where each A_α is a von Neumann algebra of type I, and A is a continuous von Neumann algebra.

Now suppose that V has no type I part. Then $M(V)$ has no type I part and the same holds for $M(V_\alpha)$. But $M(V_\alpha)$ is *-isomorphic to

$e_\alpha A_\alpha e_\alpha \oplus c(f_\alpha)A_\alpha$, which is a von Neumann algebra of type I, hence $V_\alpha = 0$. But it was assumed that $V_\alpha \neq 0$ so this contradiction shows that $V_1 = 0$ and (a) is proved.

To prove (b) consider again V_1, and focus on a component on the right side of (4.3) for a fixed α, which is denoted, again for notation's sake, by $eB \oplus Bf$ where B is a von Neumann algebra of type I. Write $B = \oplus_{\gamma\in\Gamma} L^\infty(\Sigma_\gamma, B(H_\gamma))$, $e = \oplus_\gamma e_\gamma$, and $f = \oplus_\gamma f_\gamma$ so that

$$eB = \oplus_{\gamma\in\Gamma} e_\gamma L^\infty(\Sigma_\gamma, B(H_\gamma)),$$

$$Bf = \oplus_{\gamma\in\Gamma} L^\infty(\Sigma_\gamma, B(H_\gamma)) f_\gamma.$$

The reduction theory of von Neumann algebras ([10, Part II]) will now be used to conclude this proof, so assume that B acts on a separable Hilbert space. For a fixed $\gamma \in \Gamma$,

$$L^\infty(\Sigma_\gamma, B(H_\gamma)) = \int_{\Sigma_\gamma}^{\oplus} B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma),$$

$$L^2(\Sigma_\gamma, H_\gamma) = \int_{\Sigma_\gamma}^{\oplus} H_\gamma\, d\mu_\gamma(\sigma_\gamma),$$

$$B = \textstyle\sum_{\gamma\in\Gamma}^{\oplus} \int_{\Sigma_\gamma}^{\oplus} B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma),$$

$$e_\gamma = \int_{\Sigma_\gamma}^{\oplus} e_\gamma(\sigma_\gamma)\, d\mu_\gamma(\sigma_\gamma),$$

and

$$eB = \textstyle\sum_{\gamma\in\Gamma}^{\oplus} \int_{\Sigma_\gamma}^{\oplus} e_\gamma(\sigma_\gamma) B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma).$$

For notation's sake, for a fixed $\gamma \in \Gamma$, let $\sigma = \sigma_\gamma$, $\mu = \mu_\gamma$, $e = e_\gamma$, $\Sigma = \Sigma_\gamma$, $H = H_\gamma$, and suppose H is a separable Hilbert space. For each $n \leq \aleph_0$, let $\Sigma_n = \{\sigma \in \Sigma : e(\sigma) \text{ has rank } n\}$, $e_n = e|_{\Sigma_n}$, and let K_n be a Hilbert space of dimension n. Then

$$\textstyle\int_{\Sigma}^{\oplus} e(\sigma)B(H)\, d\mu(\sigma) = \sum_{n\leq\aleph_0}^{\oplus} \int_{\Sigma_n}^{\oplus} e_n(\sigma)B(H)\, d\mu(\sigma).$$

For each $\sigma \in \Sigma_n$, let $G_\sigma = \{\text{all unitaries } U : e_n(\sigma)H \to K_n\}$, let $G = \cup_{\sigma\in\Sigma_n} G_\sigma$, and then set

$$E = \{(\sigma, U) \in \Sigma_n \times G : U \in G_\sigma\}.$$

By the measurable selection theorem [10, Appendix V], there exists a μ-measurable subset $\Sigma_n' \subset \Sigma_n$ of full measure and a μ-measurable mapping η of Σ_n' into G, such that $\eta(\sigma) \in G_\sigma$ for every $\sigma \in \Sigma_n'$.

It is easy to verify that for each $\sigma \in \Sigma_n'$, $T_{n,\sigma} : e_n(\sigma)x \mapsto \eta(\sigma)e_n(\sigma)x$ is a TRO-isomorphism of $e_n(\sigma)B(H)$ onto $B(H, K_n)$ and that $\{T_{n,\sigma} : \sigma \in \Sigma_n'\}$ is a μ-measurable field of TRO-isomorphisms.

Hence $\int_{\Sigma_n}^{\oplus} T_{n,\sigma}\, d\mu(\sigma)$ is a TRO-isomorphism of $\int_{\Sigma_n}^{\oplus} e_n(\sigma)B(H)\, d\mu(\sigma)$ onto $\int_{\Sigma_n}^{\oplus} B(H,K_n)\, d\mu(\sigma)$, that is

$$\int_{\Sigma_n}^{\oplus} e_n(\sigma)B(H)\, d\mu(\sigma) \overset{TRO}{\simeq} L^\infty(\Sigma_n, B(H,K_n)).$$

Going back to the earlier notation, since

$$eB = \textstyle\sum_{\gamma\in\Gamma}^{\oplus} \int_{\Sigma_\gamma}^{\oplus} e_\gamma(\sigma_\gamma)B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma).$$

it follows that

$$eB \overset{TRO}{\simeq} \textstyle\sum_{\gamma\in\Gamma}^{\oplus} \sum_{n\le\aleph_0} L^\infty(\Sigma_{\gamma,n}, B(H_\gamma,K_n)).$$

By the same arguments, it is clear that also

$$Bf \overset{TRO}{\simeq} \textstyle\sum_{\gamma\in\Gamma'}^{\oplus} \sum_{n\le\aleph_0} L^\infty(\Sigma'_{\gamma,n}, B(K_n,H'_\gamma)).$$

Recalling that B was one of the A_α in (4.3), this completes the proof of (b). □

Corollary 4.2 (Ruan)**.** *A W*-TRO of type I, acting on a separable Hilbert space, is TRO-isomorphic to* $\oplus_\alpha L^\infty(\Omega_\alpha, B(H_\alpha,K_\alpha))$.

Corollary 4.3. *A W*-TRO of type* $II_{1,1}$ *is TRO-isomorphic to* $eA \oplus Af$*, where* e, f *are centrally orthogonal projections in a von Neumann algebra* A *of type* II_1.

5. Alternate proofs

Presented here are alternate approaches to the proofs of the assertions concerning V_2 and V_3 in the proof of Proposition 4.1(a), along the lines of the proof of the assertion concerning V_1. The purpose for doing this is that, despite the fact that the proofs are longer, they illustrate the power of the techniques used from [4] and [7].

Consider first V_2. In what follows, it is assumed that V has no type I part. Recall that V_2 is triple isomorphic to a right ideal pM in a continuous von Neumann algebra M. For notation's sake, denote V_2 by V and pM by W. By Theorem 3.2, $T^*(W) = W \oplus W^t$ and $\alpha_W(x) = x \oplus x^t$. By [7, Proposition 3.9]), and [7, Theorem 4.11]) W does not admit a triple homomorphism onto a Hilbert space of dimension greater than 2, and therefore the same holds for V.

Next it is shown that W does not admit a triple homomorphism onto $\mathbb{C}$, and it follows that V does not admit a triple homomorphism, and *a priori*, a TRO-homomorphism onto $\mathbb{C}$, thus guaranteeing, by [7, Theorem 5.4]), that $T^*(V) = V \oplus V^t$ and $\alpha_V(x) = x \oplus x^t$.

Suppose then, that $f : pM \to \mathbb{C}$ is a nonzero triple homomorphism, that is, for $x, y, z \in M$,

$$f\{px,py,pz\} = f\left(\frac{pxy^*pz + pzy^*px}{2}\right) = f(px)\overline{f(py)}f(pz). \tag{5.1}$$

Putting $x = y = z = 1$ in (5.1) yields $f(p) = |f(p)|^2 f(p)$, so either $f(p) = 0$ or $|f(p)| = 1$. Suppose $f(p) = 0$. Then setting $y = 1$ in (5.1) yields

$$f(pxpz + pzpx) = 0, \quad (x, z \in M)$$

and setting $z = 1$ in (5.1) yields

$$f(pxy^*p + py^*px) = 0, \quad (x, y \in M),$$

which implies

$$f(pxp + px) = 0 \quad (x \in M).$$

Thus

$$0 = f(pxy^*p) + f(py^*px) = -f(pxy^*) - f(pxy^*p)$$

and in particular

$$0 = f(py^*p) + f(py^*p) = -f(py^*) - f(py^*p)$$

so that $f(py^*) = 0$ for $y \in M$, that is, $f = 0$.

Assume now without loss of generality, that $f(p) = 1$. Writing $(pxp)(pyp) = (pxp)p^*(pyp)$, then for $x, y \in M$,

$$f((pxp) \circ (pyp)) = f\{pxp, p, pyp\} = f(pxp)f(pyp)$$

so that f is a Jordan *-homomorphism of pMp onto $\mathbb{C}$. It follows that f is a normal dispersion-free state on a continuous von Neumann algebra, and hence must be zero (see the proof of Lemma 3.1).

Thus $T^*(V) = V \oplus V^t$, $\alpha_V(x) = x \oplus x^t$ and there is a weak*-continuous TRO-isomorphism of $T^*(V)$ onto $T^*(W)$, by [11, Proposition 2.4]. Thus V is TRO-isomorphic to a weak*-closed ideal I in $W \oplus W^t$. Writing $I = (I \cap W) \oplus (I \cap W^t)$, then $I \cap W$ is a weak*-closed ideal in W, let's call it I_1, and $I \cap W^t$ is a weak*-closed ideal in W^t, let's call it I_2. As noted in [15], there are projections $p_1 \le p, p_2 \le p^t$ such that $I_1 = p_1 M$ and $I_2 = M^t p_2$.

More precisely,

$$I = I_1 \oplus I_2 = (p_1 \oplus 0)(M \oplus M^t) \oplus (M \oplus M^t)(0 \oplus p_2) = eA \oplus Af, \tag{5.2}$$

where $A = M \oplus M^t$ is a continuous von Neumann algebra, $e = p_1 \oplus 0$ and $f = 0 \oplus p_2$.

With regard to Corollary 4.3, suppose now that V is of type $II_{1,1}$. It will be shown that A can be chosen to be of type II_1. Since

$$R_V \overset{*-iso.}{\simeq} R_{I_1} \oplus R_{I_2} = \begin{bmatrix} eAe & eA \\ Ae & c(e)A \end{bmatrix} \oplus \begin{bmatrix} c(f)A & Af \\ fA & fAf \end{bmatrix},$$

it follows that $c(f)A$ and $c(e)A$ are each of type II_1.

Since $p_1 M = p_1(c(p_1)M)$ and $M^t p_2 = (M^t c(p_2))p_2$, if $A = M \oplus M^t$ is replaced by $\tilde{A} = c(p_1)M \oplus c(p_2)M^t$, then $\tilde{A}$ is a continous von Neumann algebra, $eA \oplus Af = e\tilde{A} \oplus \tilde{A}f$, and

$$R_V \overset{*-iso.}{\simeq} \begin{bmatrix} e\tilde{A}e & e\tilde{A} \\ \tilde{A}e & \tilde{A} \end{bmatrix} \oplus \begin{bmatrix} \tilde{A} & \tilde{A}f \\ f\tilde{A} & f\tilde{A}f \end{bmatrix},$$

so that $\tilde{A}$ is of type II_1.

Consider next V_3. V_3 is triple isomorphic to $H(N, \beta)$ for some continuous von Neumann algebra N which admits a *-anti-automorphism β of order 2. For notation's sake, denote V_3 by V and $H(N, \beta)$ by W.

Note first that V is a universally reversible TRO. This follows by the same arguments which were used in the discussion of V_2 in this subsection. Indeed, by [7, Proposition 2.2] and the paragraph preceding it, W is a universally reversible JC*-triple, and therefore so is V. As before, V does not admit a triple homomorphism onto a Hilbert space of dimension different from 2.

On the other hand, V has no nonzero TRO-homomorphism onto $\mathbb{C}$, since such a homomorphism would extend to a *-homomorphism of the linking von Neumann algebra R_V of V onto $M_2(\mathbb{C})$, whose restriction ρ to the upper left corner of R_V would be a dispersion-free state on a continuous von Neumann algebra. It is easily seen that ρ is completely additive on projections, hence normal and hence cannot exist (see the proof of Lemma 3.1).

So $T^*(V) = V \oplus V^t$, $\alpha_V(x) = x \oplus x^t$, and $V \oplus V^t$ is TRO-isomorphic to $T^*(W) = N$, by Theorem 3.4. By [11, Proposition 2.4], the TRO-isomorphism is weak*-continuous. Hence the weak*-closed TRO ideal V in $V \oplus V^t$ is mapped onto a weak*-closed TRO ideal in N, which is necessarily a two-sided ideal in N, say zN for some central projection z in N. From (5.2) it follows that $V_2 \oplus V_3$ is TRO-isomorphic to

$$[(e \oplus 0)(A \oplus N)] \oplus [(A \oplus N)(f \oplus 0) \oplus [(0 \oplus z)(A \oplus N)],$$

so that $V_2 \oplus V_3$ is the direct sum of a weakly closed left ideal and a weakly closed right ideal in a continuous von Neumann algebra, which is tantamount to proving that $V_3 = 0$.

This last argument shows that (a) implies (b) in Theorem 1.1.

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