Cyclic Compositions and Trisotopies[1]

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Dedicated to Michel L. Racine on the occasion of his 60^{th} *birthday*

Abstract

Cyclic compositions in the sense of Springer [18] and Knus-Merkurjev-Rost-Tignol [5] are investigated by means of cyclic trisotopies, a concept originally due to Albert [1]. Using the quadrupling of composition algebras, we enumerate cyclic trisotopies and compositions in a rational manner, i.e., without extending the base field. We relate cyclic trisotopies explicitly to simple associative algebras of degree 3 with involution and to the Tits process of cubic Jordan algebras.

0. Introduction

Cyclic compositions were introduced by Springer [18] under the name of twisted compositions (verschränkte Kompositionsalgebren) in order to understand what are known today as Albert algebras; more specifically, Springer was able to set up a two-to-one correspondence between cyclic compositions of rank 8 and Albert division algebras containing a given cubic Galois extension of the base field. A systematic up-to-date account of cyclic compositions, emphasizing their connections with algebraic groups and Galois cohomology, may be found in Knus-Merkurjev-Rost-Tignol [5] and Springer-Veldkamp [19], the latter calling them normal twisted composition algebras instead; see also Engelberger [2] for additional information on the subject. Cyclic trisotopies, on the other hand, were introduced by Albert [1] basically for the same purpose but under no name at all[2]. They have never been treated in book form and seem to be largely forgotten. Yet, since both concepts pretty much succeeded in the objectives they were designed for, they must be in some sense equivalent.

Our principal aim in the present paper will be to describe a new approach to cyclic compositions by working out this equivalence in full detail and taking advantage of it as much as possible. This allows us to view cyclic trisotopies as a tool of co-ordinatizing cyclic compositions by means of base points. We then develop a structure theory for cyclic trisotopies that is quite different from the one originally due to Albert and has many applications. For example, we will be able to enumerate "free" cyclic compositions in a rational manner, i.e., without the need of extending the base field. In particular, contrary to the approach adopted in [19, Chapter 4] or [2, Chapter 2], we do not have to distinguish between isotropic and anisotropic cyclic compositions in the process. For a summary of our enumeration results, the reader is referred to Theorem 1.8 below. As another application, we will be able to work out the connection with the Tits process of Petersson-Racine [15], generalizing the second Tits construction as presented by McCrimmon [7], one of the most powerful tools in the

[1] *Date*: December 6, 2005. Supported by Deutsche Forschungsgemeinschaft

[2] For reasons that will be explained in 2.2 below, the nomenclature adopted here arises from the term "trialitarian isotopy" through amalgamation.

2000 *Mathematics Subject Classification.* Primary 17A75, 17C40; Secondary 11E04, 16K20, 16W10

structure theory of Albert algebras. These applications underscore the fact that cyclic trisotopies, as compared to cyclic compositions, have the technical advantage of being more intimately tied up with ordinary composition algebras, whose structure theory, well understood as it is, will play a crucial role in our investigation. There are irrefutable reasons, however, to be explained in 10.1 below, why the Cayley-Dickson doubling process, the standard tool for describing composition algebras, is not appropriate in the present context. Instead, a certain quadrupling procedure, producing composition algebras of rank $4r$ out of composition algebras of rank $r \leq 2$, will take its place.

The content of the paper, as it unfolds along the preceding lines, may be summarized as follows. Cyclic compositions and trisotopies are introduced in the first two sections. These concepts are then brought together by the notion of a pointed cyclic composition in section 3. The connection with cubic norm structures and their associated (quadratic) Jordan algebras is briefly recalled in section 4 in a characteristic-free manner, allowing us to put isotopes of cyclic compositions and trisotopies into context. Changing base points leads to hybrids and weak homomorphisms of cyclic trisotopies in section 5. Using these concepts, we are able to set up an equivalence of categories between cyclic trisotopies with weak homomorphisms and free cyclic compositions. Cyclic trisotopies and compositions of rank at most 2 are then enumerated in section 6. We also relate their associated cubic norm structures to simple associative algebras of degree 3 with involution by a slight generalization of the étale Tits process, going back to Petersson-Racine [14], the terminology being due to Petersson-Thakur [16]. The significance of these results derives from the fact that every cyclic trisotopy of rank > 2 up to weak isomorphism contains a cyclic sub-trisotopy of rank equal to 2. After having investigated the core of a cyclic trisotopy as a useful technical tool in section 7, we turn to reduced cyclic compositions and trisotopies in section 8. In particular, we relate these notions to the corresponding ones for Jordan algebras of degree 3 and give a self-contained proof for the fact that all cyclic trisotopies of rank 4 are reduced, allowing us as an immediate application to enumerate free cyclic compositions of rank 4. Finally, following a brief sketch of the quadrupling procedure for composition algebras in section 9, cyclic trisotopies of rank 8 are enumerated in section 10. More generally, we will present an explicit construction that yields all cyclic trisotopies of rank $4r$ containing a given cyclic sub-trisotopy of rank $r \leq 2$. It is no accident that the ingredients required for this construction are exactly the same as the ones needed for the Tits process. Indeed, using these ingredients and the simple associative algebra of degree 3 with involution attached to the given cyclic sub-trisotopy of rank r, the Tits process yields a Jordan algebra of degree 3 whose corresponding cubic norm structure will be seen to agree with the one belonging to the ambient cyclic trisotopy of rank $4r$.

After appropriate modifications, the techniques developed in this paper can also be used for a new approach to the structure theory of symmetric (rather than cyclic) compositions [5, § 34]. The reader is referred to a forthcoming paper by Stenger for details. As yet another application, we intend to investigate twisted compositions [5, § 36] by means of unitary involutions on cyclic trisotopies and to describe the obstructions to the validity of the Skolem-Noether theorem for cubic étale subalgebras of Albert algebras. Throughout this article, we fix a base field k of arbitrary characteristic. We will occasionally use elementary facts about alternative and Jordan algebras. Standard references are Jacobson [4] and McCrimmon [9].

1. Cyclic compositions

In this section, we adopt the terminology of [5, § 36.B] to define cyclic compositions and to recall their most important elementary properties.

1.1 Cyclic cubic étale algebras. By a *cyclic cubic étale k-algebra* we mean a pair (E, ρ) consisting of an étale algebra E of dimension 3 over k and a k-automorphism ρ of E having order 3, so $\rho^3 = \mathbf{1}_E \neq \rho$. A *homomorphism* $\varphi : (E, \rho) \to (E', \rho')$ of cyclic cubic étale k-algebras is a (unital) homomorphism $\varphi : E \to E'$ of k-algebras satisfying $\varphi \circ \rho = \rho' \circ \varphi$. A cyclic cubic étale k-algebra (E, ρ) is either indecomposable, i.e., E/k is a cubic Galois extension, with ρ being one of the two generators of the corresponding Galois group, or it splits, i.e., it is isomorphic to the 3-*shift*

$$(k \oplus k \oplus k, \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \longmapsto \alpha_2 \oplus \alpha_3 \oplus \alpha_1) .$$

It follows that all cyclic cubic étale algebras are central simple as algebras with automorphism, so there are no ρ-stable ideals other than the trivial ones and the only elements of E remaining fixed under ρ are scalars. In particular, every homomorphism of cyclic cubic étale algebras is an isomorphism.

1.2 The concept of a cyclic composition. Following [5, § 36.B] or [19, Definition 4.1.1], a quintuple

$$\mathcal{S} = (E, \rho, M, Q, *)$$

is said to be a *cyclic composition* over k if the following conditions hold.

CC1 (E, ρ) is a cyclic cubic étale k-algebra.

CC2 (M, Q) is a quadratic space over E, so M is a finitely generated (automatically projective) E-module and $Q : M \to E$ is a quadratic form over E which is *non-singular* in the sense that it induces an isomorphism from M onto its dual in the usual way.

CC3 $(M, *)$ is a non-associative k-algebra.

CC4 Q permits *twisted composition*, so

(1.2.1) $$Q(x * y) = \rho\big(Q(x)\big)\rho^2\big(Q(y)\big). \qquad (x, y \in M)$$

CC5 Q is *twisted associative*, so if the bilinearization of Q is again denoted by

$$Q : M \times M \longrightarrow E\,,\ (x, y) \longmapsto Q(x, y) = Q(x + y) - Q(x) - Q(y)\,,$$

we have

(1.2.2) $$Q(x * y, z) = \rho\big(Q(y * z, x)\big). \qquad (x, y, z \in M)$$

Given another cyclic composition $\mathcal{S}' = (E', \rho', M', Q', *')$ over k, a homomorphism $(\varphi, \phi) : \mathcal{S} \to \mathcal{S}'$ consists of a homomorphism $\varphi : (E, \rho) \to (E', \rho')$ of cyclic cubic étale k-algebras and a φ-semi-linear isometry $\phi : (M, Q) \to (M', Q')$ such that $\phi(x * y) = \phi(x) *' \phi(y)$ for all $x, y \in V$.

In the following lemma, the first part follows immediately from the non-degeneracy of Q combined with (1.2.2), while the second and third part are due to Springer (cf. [19, Lemma 4.1.3]).

1.3 Lemma. *Let $\mathcal{S} = (E, \rho, M, Q, *)$ be a cyclic composition over k.*

a) *The k-bilinear product $*$ on M is ρ-semi-linear in the first variable, ρ^2-semi-linear in the second.*

b) *The relations*

$$x * (y * x) = \rho\big(Q(x)\big)y\,, \tag{1.3.1}$$

$$(x * y) * x = \rho^2\big(Q(x)\big)y \tag{1.3.2}$$

hold for all $x, y \in M$.

c) *There is a unique cubic form $N_{\mathcal{S}} : M \to k$ over k such that*

$$Q(x, x * x) = N_{\mathcal{S}}(x)1_E\,. \qquad (x \in M) \tag{1.3.3}$$

Moreover, the relation

$$(x * x) * (x * x) = N_{\mathcal{S}}(x)x - Q(x)(x * x) \tag{1.3.4}$$

holds for all $x \in M$.

□

1.4 Composition algebras. Let R be a commutative associative ring of scalars. Following Petersson [11, 1.4], a *composition algebra* over R is a non-associative R-algebra C that is finitely generated projective and has full support as an R-module (so, in particular, $C_{\mathfrak{p}} \neq \{0\}$ for all $\mathfrak{p} \in \operatorname{Spec} R$), contains a unit element and admits a quadratic form $N_C : C \to R$ uniquely determined by the following conditions: N_C is non-singular and permits composition, so $N_C(xy) = N_C(x)N_C(y)$ for all $x, y \in C$. We call N_C the norm and $T_C = N_C(1_C, -)$ the trace of C. They are related by the universal quadratic equation

$$x^2 - T_C(x)x + N_C(x)1_C = 0\,. \qquad (x \in C) \tag{1.4.1}$$

We also recall the conjugation of C,

$$\iota = \iota_C : C \longrightarrow C\,,\ x \longmapsto \iota(x) = \overline{x} = T_C(x)1_C - x\,, \tag{1.4.2}$$

which is an algebra involution in the usual sense. Recall from loc. cit. that composition algebras are alternative of rank 1,2,4 or 8, and that an element $x \in C$ is invertible in C if and only if $N_C(x)$ is invertible in R, in which case $x^{-1} = N_C(x)^{-1}\overline{x}$. Also, R itself is a composition algebra over R (with norm $N_R(\alpha) = \alpha^2$) if and only if $\frac{1}{2} \in R$. Other standard properties of composition algebras will be used without further comment from now on.

1.5 Cyclic compositions and composition algebras. Let $\mathcal{S} = (E, \rho, M, Q, *)$ be a cyclic composition over k and suppose $a, b \in M$ are *strongly anisotropic* relative to Q, so $Q(a)$ and $Q(b)$ are both units in E. Following Springer (cf. [19, p. 72]), the E-module M becomes a composition algebra *over* E under the multiplication

$$xy = (a * x) * (y * b)\,, \qquad (x, y \in M) \tag{1.5.1}$$

dependence of the left-hand side on a, b being understood. This composition algebra will be denoted by $\mathcal{S}^{(a,b)}$. Its unit element and norm are given by

$$1_{\mathcal{S}^{(a,b)}} = \big(Q(b)^{-1}b\big) * \big(Q(a)^{-1}a\big), \quad N_{\mathcal{S}^{(a,b)}} = \langle \rho^2\big(Q(a)\big)\rho\big(Q(b)\big)\rangle\,.\,Q\,. \tag{1.5.2}$$

1.6 Free cyclic compositions and base points. A cyclic composition $\mathcal{S} = (E, \rho, M, Q, *)$ over k is said to be *free* if $M \neq \{0\}$ is a free E-module. In this case, the rank of M as a free E-module is called the *rank* of $\mathcal{S}$. By a *base point* for $\mathcal{S}$, we mean an element $a \in M$ that is strongly anisotropic relative to Q, so $Q(a) \in E^\times$.

1.7 Proposition. *A cyclic composition over k is free if and only if it admits a base point.*

Proof. Both conditions are fulfilled for $\mathcal{S}$ as in 1.6 if E is a field. On the other hand, if (E, ρ) is split, their equivalence can be easily verified directly. $\square$

Combining Proposition 1.7 with 1.4, 1.5, we conclude that the rank of a free cyclic composition is $1, 2, 4$ or 8. More precisely, we can now summarize our enumeration results for cyclic compositions in the following theorem. For the proof, the reader is referred to 6.4, 8.7, 10.11 below.

1.8 Enumeration theorem for free cyclic compositions. *Up to isomorphism, all free cyclic compositions of rank r over k have the form $\mathcal{S} = (E, \rho, M, Q, *)$ where (E, ρ) is a cyclic cubic étale k-algebra and precisely one of the following conditions holds.*

a) *$r \leq 2$ and there exist an r-dimensional composition algebra L_0 over k as well as elements $d \in L_0^\times, v \in E^\times$ such that, setting $\sigma = \mathbf{1}_{L_0} \otimes \rho$ on $L_0 \otimes E$,*

$$M = L_0 \otimes E, \quad \text{(as } E\text{-modules)}$$
$$Q = \langle N_{L_0}(d) v^\sharp \rangle \,.\, (N_{L_0} \otimes E),$$
$$x * y = \sigma(\overline{x})\sigma^2(\overline{y})(d \otimes v)$$

for all $x, y \in M$.

b) *$r = 4$ and there exist a quaternion algebra C_0 over k as well as an element $v \in E^\times$ such that, setting $\varphi = \mathbf{1}_{C_0} \otimes \rho$ on $C_0 \otimes E$,*

$$M = C_0 \otimes E, \quad \text{(as } E\text{-modules)}$$
$$Q = \langle v^\sharp \rangle \,.\, (N_{C_0} \otimes E),$$
$$x * y = v\varphi(\overline{x})\varphi^2(\overline{y})$$

for all $x, y \in M$.

c) *$r = 8$ and there exist a quadratic étale k-algebra L_0 as well as elements $d \in L_0^\times, v \in E^\times, T \in \mathrm{SL}_3(L)$ $(L = L_0 \otimes E)$, $P \in M_3(L_0)$ such that, letting the maps $\sigma = \mathbf{1}_{L_0} \otimes \rho, \iota_{L_0} \otimes \mathbf{1}_E$ act componentwise on matrices of any size over L and writing $\times$ for the ordinary vector product in 3-space,*

$$T = \overline{T}^t \quad \text{is hermitian},$$
$$\text{(1.8.1)} \qquad \det P = \overline{d}d^{-1}, \quad P^3 = \overline{d}d^{-1}\mathbf{1}_3, \quad \overline{P}^t T P = \sigma(T),$$
$$M = L \oplus L^3, \quad \text{(as } E\text{-modules)}$$
$$Q(a \oplus x) = N_{L_0}(d) v^\sharp \big((N_{L_0} \otimes E)(a) + \overline{x}^t T x\big),$$
$$(a \oplus x) * (b \oplus y) = \Big(\sigma(\overline{a})\sigma^2(\overline{b}) - \sigma\big(\overline{x}^t T P \sigma(y)\big)\Big)(d \otimes v) \oplus$$
$$\Big(- P\big(\sigma(x)\sigma^2(\overline{b}) + P\sigma^2(y)\sigma(a)\big)(d \otimes v) + \big(\overline{TP\sigma(x)} \times \overline{TP^2\sigma^2(y)}\big)(\overline{d} \otimes v)\Big)$$

for all $a, b \in L, x, y \in L^3$.

Part c) of this theorem generalizes the construction of isotropic cyclic compositions of rank 8 given in [19, Theorem 4.6.2] to the (possibly) non-isotropic case in a unified manner. The most delicate ingredient of our construction is (1.8.1). As we shall see in Section 10 below, particularly 10.8, this set of relations can be fully understood by the notion of admissible pairs (see 6.9) and their connections with the Tits process.

2. Cyclic trisotopies

Isolating the principal ingredients of Albert's approach [1] to Albert division algebras containing a given cubic Galois extension of the base field from their original surroundings will lead us in this section to the concept of a cyclic trisotopy. We begin with a digression into alternative algebras that was not available to Albert at the time.

2.1 Isotopes of alternative algebras. Let R be a commutative associative ring of scalars and C a unital alternative algebra over R. Given invertible elements $a, b \in C$, we follow McCrimmon [8] to define the (a, b)-*isotope* of C as the R-algebra $C^{(a,b)}$ living on the same R-module as C by the multiplication

(2.1.1) $$x \cdot_{a,b} y = (xa)(by)$$

for all $x, y \in C$. The algebra $C^{(a,b)}$ is again unital alternative with identity element

(2.1.2) $$1^{(a,b)} = (ab)^{-1}\,.$$

Unital alternative algebras C, C' are said to be *isotopic* if $C' \cong C^{(a,b)}$ for some $a, b \in C^{\times}$. If C is a composition algebra as in 1.4 , so is $C^{(a,b)}$, with norm

(2.1.3) $$N_{C^{(a,b)}} = \langle N_C(ab)\rangle\,.\,N_C\,.$$

This formula immediately implies that L_{ab}, the left multiplication by ab in C, is an isometry from $N_{C^{(a,b)}}$ onto N_C, so isotopic composition algebras *over fields*, having isometric norms, are isomorphic.

2.2 The concept of a cyclic trisotopy. By a *cyclic trisotopy* over k, we mean a quintuple

$$\mathcal{A} = (E, \rho, C, g, b)$$

such that the following conditions hold.

CT1 (E, ρ) is a cyclic cubic étale k-algebra in the sense of 1.1.

CT2 C is a composition algebra over E.

CT3 b is an invertible element of C.

CT4 $g : C \to C$ is a ρ-semi-linear map satisfying the functional equation

(2.2.1) $$g(xy) = [g(x)b][b^{-1}g(y)]. \qquad (x, y \in C)$$

CT5 g stabilizes the line through b and

(2.2.2) $$g^3(x) = bxb^{-1}. \qquad (x \in C)$$

Given another cyclic trisotopy $\mathcal{A}' = (E', \rho', C', g', b')$ over k, a *homomorphism* $(\varphi, \phi) : \mathcal{A} \to \mathcal{A}'$ consists of a homomorphism $\varphi : (E, \rho) \to (E', \rho')$ of cyclic cubic étale k-algebras and a φ-semi-linear homomorphism $\phi : C \to C'$ of unital algebras satisfying $\phi \circ g = g' \circ \phi$, $N_{C'} \circ \phi = \varphi \circ N_C$ and $\phi(b) = b'$. In the presence of the remaining conditions, the relation $N_{C'} \circ \phi = \varphi \circ N_C$ is equivalent to ϕ being injective.

Comparing (2.1.1) with (2.2.1), we see that CT4, (2.2.2) amount to $g : C \xrightarrow{\sim} C^{(b,b^{-1})}$ being a ρ-semi-linear isomorphism that cubes to conjugation by b; in particular, (g, b, b^{-1}) belongs to the structure group of C as a k-algebra in the sense of Petersson [12, 2.2, 2.3]. Summing up, g may thus be viewed as a "trialitarian isotopy" of C.

2.3 Lemma. *Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy over k. Then the following statements hold.*

a) $g(1_C) = 1_C$.

b) $N_C \circ g = \rho \circ N_C$, $T_C \circ g = \rho \circ T_C$.

c) $g(\overline{x}) = \overline{g(x)}$ *for all* $x \in C$.

d) $g(xy)b = g(x)\big(g(y)b\big)$ *for all* $x, y \in C$.

e) $g : C^+ \xrightarrow{\sim} C^+$ *is a ρ-semi-linear isomorphism of unital quadratic Jordan algebras over E. In particular, g preserves all powers whenever they make sense.*

Proof. As we have just noted, g may be viewed as a ρ-semi-linear isomorphism from C onto $C^{(b,b^{-1})}$. But using (2.1.2), (2.1.3), $C^{(b,b^{-1})}$ turns out to be a composition algebra over E with unit 1_C and norm N_C. This implies a), b), hence c) by (1.4.2), while d) is a special case of [12, 2.6]. Finally, e) is an immediate consequence of the equation $C^{(b,b^{-1})+} = C^+$ [8, (19)]. □

Remark. By [12, 2.6], Lemma 2.3 d) is actually *equivalent* to (2.2.1).

2.4 Albert's approach to cyclic trisotopies. Cyclic trisotopies are intimately tied up with Jordan algebras of degree 3 by means of their associated cubic norm structures, see 4.6 below for details. In [1, sections 7 – 10] they arise in a different manner; we merely sketch the details. Let J be an Albert algebra over k, E/k a cubic Galois extension that at the same time is a unital subalgebra of J and ρ a generator of the corresponding Galois group. Then $J' = J \otimes E$ is a reduced Albert algebra over E, containing $E' = E \otimes E \cong E \oplus E \oplus E$ as a subalgebra and acted upon semi-linearly by ρ through the second factor. Co-ordinatizing appropriately, we may thus assume $J' = H_3(C, \Gamma)$ (the Jordan algebra of Γ-hermitian 3-by-3 matrices with entries in C and scalars (in E) down the diagonal), for some octonion algebra C over E and some diagonal matrix $\Gamma \in \mathrm{GL}_3(E)$, in such a way that E' sits diagonally in J'. Inspecting [1, (45), (47), (61), Lemmata 10 – 13], the action of ρ on J' is seen to be completely determined by an element $b \in C^\times$ and a ρ-semi-linear bijection $g : C \to C$ satisfying $g(b) \in Eb$, Lemma 2.3 d) and 2.2 CT5, thus making (E, ρ, C, g, b) a cyclic trisotopy over k by the remark following Lemma 2.3. In [1, Lemma 12], matters are normalized still further by the condition $g(b) = b$. This, however, may be assumed only up to isotopy, see 4.8 below.

2.5 Proposition. *Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy over k. Then there is a unique element $u \in E^\times$ such that the quantities*

(2.5.1) $$b_0 := \rho(u)b \in C^\times, \quad u_0 := \rho(u)u^{-1} \in E^\times$$

satisfy the relations

(2.5.2) $$N_E(u) = N_C(b_0)\,,$$

(2.5.3) $$g(b) = \rho(u_0)^{-1}b\,.$$

Furthermore, the element b_0 remains fixed under g.

Proof. By 2.2 CT5, we find a unique element $u_0 \in E$ satisfying (2.5.3), and (2.2.2) combined with the ρ-semi-linearity of g (2.2 CT4) implies $N_E(u_0) = 1$. Hence Hilbert's Theorem 90 yields an element $u \in E^\times$ satisfying the second equation of (2.5.1). Notice that u is unique only up to a non-zero factor in k. On the other hand, $g(b_0) = \rho^2(u)g(b) = \rho^2(u)\rho(u_0)^{-1}b = \rho(u)b = b_0$, proving the final assertion of the proposition. Also, $N_C(b_0)$ by Lemma 2.3 b) remains fixed under ρ and hence belongs to k. But since the first (resp. second) term of the expression $N_E(u) - N_C(b_0)$ is cubic (resp. quadratic) in u, we find a unique element $\gamma \in k^\times$ such that all of (2.5.2) – (2.5.3) hold with u replaced by γu. □

2.6 The multiplier of a cyclic trisotopy. Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy over k. Then the unique element $u \in E^\times$ exhibited in Proposition 2.5 is called the *multiplier* of $\mathcal{A}$. It is preserved by homomorphisms of cyclic trisotopies in the obvious way.

3. Connecting pointed cyclic compositions with cyclic trisotopies

In this section, cyclic compositions and cyclic trisotopies will be brought together by the following concept.

3.1 Pointed cyclic compositions. A *pointed cyclic composition* is a pair $(\mathcal{S}, a)$ consisting of a cyclic composition $\mathcal{S}$ and a base point a for $\mathcal{S}$. *Homomorphisms* of pointed cyclic compositions are homomorphisms of the underlying cyclic compositions preserving base points.

In what follows, we wish to set up an equivalence (actually, an isomorphism) of categories between pointed cyclic compositions and cyclic trisotopies. We begin with the easier direction from cyclic trisotopies to cyclic compositions.

3.2 Theorem. *Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy over k. Writing u for the multiplier of $\mathcal{A}$ and setting $u_0 = \rho(u)u^{-1}$ as in (2.5.1), we put $M = C$ as an E-module and*

(3.2.1) $$Q = \langle u \rangle \,.\, N_C\,,$$

(3.2.2) $$x * y = u_0 g(\overline{x})[g^2(\overline{y})b], \qquad (x, y \in C)$$

(3.2.3) $$\mathcal{A}^0 = (E, \rho, M, Q, *)\,,$$

(3.2.4) $$a^0 = u^{-1}1_C\,.$$

Then $\mathrm{Spr}(\mathcal{A}) = (\mathcal{A}^0, a^0)$ *is a pointed cyclic composition over k.*

Proof. Among the defining conditions 1.2 CC1 – CC5 for a cyclic composition, only the last two are not obvious, so we must show that Q permits twisted composition and is twisted associative. While the former may be derived from Lemma 2.3 and Proposition 2.5 by a straightforward computation, twisted associativity is more troublesome. Using the formulae

(3.2.5) $$T_C(x, y) := T_C(xy) = N_C(x, \overline{y})\,,$$

(3.2.6) $$T_C(x, y) = T_C(y, x), \quad T_C(xy, z) = T_C(x, yz)\,,$$

(3.2.7) $$T_C\big(g(x), g(y)\big) = \rho\big(T_C(x, y)\big)\,,$$

the first two being valid in arbitrary composition algebras, the last one following from Lemma 2.3 b), we let $x, y, z \in C$ and compute

$$\begin{aligned}
Q(x * y, z) &= uN_C\big(u_0 g(\overline{x})[g^2(\overline{y})b], z\big) && \text{(by (3.2.1), (3.2.2))} \\
&= uu_0 N_C\Big(g\big(\overline{x}g(\overline{y})\big)b, z\Big) && \text{(by Lemma 2.3 d))} \\
&= \rho(u)T_C\Big(g\big(\overline{x}g(\overline{y})\big)b, \overline{z}\Big) && \text{(by (2.5.1), (3.2.5))} \\
&= \rho(u)T_C\Big(g\big(\overline{x}g(\overline{y})\big), (b\overline{z}b^{-1})b\Big) && \text{(by (3.2.6))} \\
&= \rho(u)T_C\Big(g\big(\overline{x}g(\overline{y})\big)\big(g^3(\overline{z})b\big)\Big) && \text{(by (2.2.2), (3.2.5))} \\
&= \rho(u)T_C\Big(g\big([\overline{x}g(\overline{y})]g^2(\overline{z})\big), b\Big) && \text{(by Lemma 2.3 d), (3.2.5))} \\
&= \rho(u)T_C\Big(\rho(u_0)g\big([\overline{x}g(\overline{y})]g^2(\overline{z})\big), g(b)\Big) && \text{(by (2.5.3))} \\
&= \rho(u)T_C\Big(g\big(u_0[\overline{x}g(\overline{y})]g^2(\overline{z})\big), g(b)\Big) && \\
&= \rho\Big(uT_C\big(u_0[\overline{x}g(\overline{y})]g^2(\overline{z}), b\big)\Big) && \text{(by (3.2.7))} \\
&= \rho\Big(uT_C\big(\overline{x}, u_0 g(\overline{y})[g^2(\overline{z})b]\big)\Big) && \text{(by (3.2.6))} \\
&= \rho\big(uN_C(y * z, x)\big) && \text{(by (3.2.2), (3.2.5))} \\
&= \rho\big(Q(y * z, x)\big). && \text{(by (3.2.1))}
\end{aligned}$$

Hence Q is twisted associative, and we have shown that $\mathcal{A}^0$ as defined in (3.2.3) is a cyclic composition. Since the base points for $\mathcal{A}^0$ by (3.2.1) agree with the invertible elements of C, this completes the proof. $\square$

3.3 The rank of a cyclic trisotopy. Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy over k. Applying Proposition 1.7 to $(\mathcal{A}^0, a^0)$, the pointed cyclic composition attached to $\mathcal{A}$ via Theorem 3.2, we see that C is free as an E-module, allowing us to call

$$\operatorname{rank}\mathcal{A} := \operatorname{rank}_E C \in \{1, 2, 4, 8\}$$

the *rank* of $\mathcal{A}$. By 1.4, $\operatorname{rank}\mathcal{A} = 1$ can occur only for $\operatorname{char} k \neq 2$.

3.4 Example. Let C_0 be a composition algebra over k. Then $\mathcal{A} = (E, \rho, C, g, 1_C)$, where $C = C_0 \otimes E, g = \mathbf{1}_{C_0} \otimes \rho$, is a cyclic trisotopy over k with multiplier 1_E, and (3.2.2) collapses to $x * y = g(\overline{x})g^2(\overline{y})$ for $x, y \in C$, which agrees with [19, (4.9)] and hence relates to Springer's notion of reduced cyclic compositions [19, Definition 4.1.8]. This topic will be taken up more systematically in Section 8 below.

Turning to the converse of the preceding discussion, we wish to attach a cyclic trisotopy to any pointed cyclic composition in an explicit manner. This will require considerably more effort. We begin with a straightforward consequence of Lemma 1.3 and the definitions.

3.5 Lemma. *Let $\mathcal{S} = (E, \rho, M, Q, *)$ be a cyclic composition over k. If $a \in M$ is a base point for $\mathcal{S}$, so is $a * a$ and the map*

$$L_{\mathcal{S}}(a) : M \longrightarrow M\,,\ x \longmapsto a * x, \quad (\text{resp. } R_{\mathcal{S}}(a) : M \longrightarrow M\,,\ x \longmapsto x * a\,)$$

is ρ^2- (resp. ρ-) *semi-linear bijective. Moreover, we have*

$$L_{\mathcal{S}}(a)^{-1} = R_{\mathcal{S}}\big(Q(a)^{-1}a\big), \quad (\text{resp. } R_{\mathcal{S}}(a)^{-1} = L_{\mathcal{S}}\big(Q(a)^{-1}a\big))\,.$$

□

3.6 Theorem. *Let* $(\mathcal{S}, a)$ *be a pointed cyclic composition over* k. *We set*[3]

$$\mathcal{S} = (E, \rho, M, Q, *)\,,$$

$$d = \rho^2\big(Q(a)\big)^{-1}(a * a)\,, \tag{3.6.1}$$

$$C = \mathcal{S}^{(d,a)}\,,$$

$$b = \rho^2\big(Q(a)\big)^{-1}Q(a)^{-1}(a * a) = Q(a)^{-1}d\,, \tag{3.6.2}$$

$$g = Q(a)^{-1}L_{\mathcal{S}}(a)^2\,, \tag{3.6.3}$$

so $g : C \to C$ *is given by*

$$g(x) = Q(a)^{-1}[a * (a * x)]\,. \qquad (x \in C) \tag{3.6.4}$$

Then $\mathrm{Alb}(\mathcal{S}, a) := (E, \rho, C, g, b)$ *is a cyclic trisotopy over* k *with multiplier* $u = Q(a)^{-1}$.

As a key ingredient of the proof, we first establish a formula allowing us to show later on that the pointed cyclic composition attached to $\mathrm{Alb}(\mathcal{S}, a)$ via Theorem 3.2 is $(\mathcal{S}, a)$ itself. As in (1.5.1), we write

$$xy = (d * x) * (y * a) \qquad (x, y \in M) \tag{3.6.5}$$

for the product in C and recall the well known relations

$$N_C(xy, z) = N_C(x, z\overline{y}) = N_C(y, \overline{x}z)\,. \qquad (x, y, z \in C) \tag{3.6.6}$$

3.7 Some useful identities. Notations being as above, setting

$$c := a * a\,, \tag{3.7.1}$$

and using (1.4.2), (1.5.2) as well as Lemma 1.3, the following formulae may be derived by routine computations, for all $x, y \in C$.

$$Q(c) = \rho\big(Q(a)\big)\rho^2\big(Q(a)\big)\,, \tag{3.7.2}$$

$$Q(d) = \rho\big(Q(a)\big)\rho^2\big(Q(a)\big)^{-1}\,,$$

$$1_C = Q(a)^{-1}a\,, \tag{3.7.3}$$

$$N_C = \langle Q(a)\rangle\,.\,Q\,, \tag{3.7.4}$$

$$g(1_C) = 1_C\,, \tag{3.7.5}$$

$$g(c) = c\,, \tag{3.7.6}$$

$$T_C(x) = Q(a, x)\,, \tag{3.7.7}$$

$$\overline{x} = Q(a)^{-1}Q(a, x)a - x\,, \tag{3.7.8}$$

$$g(x) = Q(a)^{-1}(\,\overline{x} * c)\,, \tag{3.7.9}$$

$$N_C\big(g(x)\big) = \rho\big(N_C(x)\big)\,, \tag{3.7.10}$$

$$g(\overline{x}) = \overline{g(x)}\,, \tag{3.7.11}$$

$$xc = (c * x) * a\,. \tag{3.7.12}$$

---

[3] The simple expression for the map g in (3.6.3), (3.6.4) below is due to U. Stenger; the one I had originally proposed was more complicated.

Noting that g is ρ-semi-linear bijective by Lemma 3.5, we are now prepared to derive the key formula.

3.8 Lemma. *Keeping the previous notations, the formula*

$$\big(Q(a)x\big) * \big(Q(a)y\big) = g(\overline{x})[g^2(\overline{y})c]$$

holds for all $x, y \in M$.

Proof. We first show

$$g(\overline{x})c = \rho\big(Q(a)\big)(x * a) \tag{3.8.1}$$

for all $x \in M$. Indeed,

$$\begin{aligned}
g(\overline{x})c &= \big(c * g(\overline{x})\big) * a && \text{(by (3.7.12))}\\
&= [c * \big(Q(a)^{-1}(x * c)\big)] * a && \text{(by (3.7.9))}\\
&= Q(a)^{-1}\big([c * (x * c)] * a\big) &&\\
&= Q(a)^{-1}\rho^2\big(Q(c)\big)(x * a) && \text{(by (1.3.1))}\\
&= \rho\big(Q(a)\big)(x * a). && \text{(by (3.7.2))}
\end{aligned}$$

Next we show

$$g^2(\overline{y})c = \rho^2\big(Q(a)\big)(a * y) \tag{3.8.2}$$

for all $y \in M$. Indeed,

$$\begin{aligned}
g^2(\overline{y})c &= \rho\big(Q(a)\big)\big(g(y) * a\big) && \text{(by (3.7.11), (3.8.1))}\\
&= \rho\big(Q(a)\big)\Big(\big(Q(a)^{-1}(\,\overline{y} * c)\big) * a\Big) && \text{(by (3.7.9))}\\
&= (\overline{y} * c) * a &&\\
&= \rho^2\big(Q(a,\,\overline{y}\,)\big)c - (a * c) * \overline{y} && \text{(by (1.3.2) linearized)}\\
&= \rho^2\big(Q(a,\,\overline{y}\,)\big)c - \rho^2\big(Q(a)\big)(a * \overline{y}\,). && \text{(by (1.3.1), (3.7.1))}
\end{aligned}$$

But $Q(a, \overline{y}) = Q(a, y)$ by (3.7.7), so

$$\begin{aligned}
g^2(\overline{y})c &= \rho^2\big(Q(a,y)\big)c - \rho^2\big(Q(a)\big)\Big(a * \big(Q(a)^{-1}Q(a,y)a - y\big)\Big) && \text{(by (3.7.8))}\\
&= \rho^2\big(Q(a)\big)(a * y), && \text{(by (3.7.1))}
\end{aligned}$$

giving (3.8.2). Hence

$$\begin{aligned}
g(\overline{x})[g^2(\overline{y})c] &= \big(Q(a)^{-1}(x * c)\big)\big(\rho^2\big(Q(a)\big)(a * y)\big) && \text{(by (3.7.9), (3.8.2))}\\
&= Q(a)^{-1}\rho^2\big(Q(a)\big)\big(d * (x * c)\big) * \big((a * y) * a\big) && \text{(by (3.6.5))}\\
&= Q(a)^{-1}\rho^2\big(Q(a)\big)\rho\big(Q(a)\big)^{-1}\big(c * (x * c)\big) * \big((a * y) * a\big) && \text{(by (3.6.1))}\\
&= Q(a)^{-1}\rho\big(Q(a)\big)^{-1}\rho^2\big(Q(a)\big)\rho^2\big(Q(c)\big)\rho\big(Q(a)\big)(x * y) && \text{(by (1.3.1), (1.3.2))}\\
&= \rho\big(Q(a)\big)\rho^2\big(Q(a)\big)(x * y), && \text{(by (3.7.2))}
\end{aligned}$$

which proves the lemma. $\square$

3.9 We are now ready to enter into the proof of Theorem 3.6. The idea is to express twisted associativity of Q in terms of g, using Lemma 3.8. Indeed, given $x, y, z \in M$, Lemma 3.8 implies

$$\begin{aligned} Q\Big(\big(Q(a)x\big) * \big(Q(a)y\big), Q(a)z\Big) &= Q(a)^{-1} N_C\big(g(\overline{x})[g^2(\overline{y})c], Q(a)z\big) && \text{(by (3.7.4))} \\ &= N_C\big(g(\overline{x})[g^2(\overline{y})c], z\big) \\ &= N_C\big(g(\overline{x}), z[\overline{c}g^2(y)]\big). && \text{(by (3.6.6))} \end{aligned}$$

Here (1.2.2) tells us that, at the expense of an application of ρ, the expression

$$N_C\big(g(\overline{x})[g^2(\overline{y})c], z\big) = N_C\big(g(\overline{x}), z[\overline{c}g^2(y)]\big)$$

remains unaffected by a cyclic change of variables. Hence we obtain

$$\begin{aligned} N_C\big(g(\overline{x}), z[\overline{c}g^2(y)]\big) &= \rho\Big(N_C\big(g(\overline{y})[g^2(\overline{z})c], x\big)\Big) \\ &= N_C\Big(g\big(g(\overline{y})[g^2(\overline{z})c]\big), g(x)\Big) && \text{(by (3.7.10))} \\ &= N_C\Big(g\big([\overline{c}g^2(z)]g(y)\big), g(\overline{x})\Big). && \text{(by (3.7.11))} \end{aligned}$$

Since N_C is non-singular, this implies

(3.9.1) $$g\Big(\big(\overline{c}g^2(x)\big)g(y)\Big) = x\big(\overline{c}g^2(y)\big)$$

for all $x, y \in C$. Observing (3.7.11), applying the canonical involution to (3.9.1) and replacing x by $\overline{y}$, y by $g^{-1}(\overline{x})$, we obtain

(3.9.2) $$g\Big(x\big(g^2(y)c\big)\Big) = \big(g(x)c\big)y$$

for all $x, y \in C$. Setting $y = 1$, (3.7.5) yields

(3.9.3) $$g(xc) = g(x)c.$$

Setting $x = 1$ in (3.9.2) therefore implies $g^3(y)c = cy$, hence

(3.9.4) $$g^3(y) = cyc^{-1},$$

and we have (2.2.2) (by (3.6.2), (3.7.1)). Finally, given $x, y \in C$, we determine $y' \in C$ such that $g^2(y')c = y$ and obtain

$$\begin{aligned} [g(x)b][b^{-1}g(y)] &= \big(g(x)c\big)[c^{-1}g\big(g^2(y')c\big)] \\ &= \big(g(x)c\big)\big(c^{-1}g^3(y')c\big) && \text{(by (3.9.3))} \\ &= \big(g(x)c\big)y' && \text{(by (3.9.4))} \\ &= g(xy)\,. && \text{(by (3.9.2))} \end{aligned}$$

This gives (2.2.1), and we have shown that $\mathrm{Alb}(\mathcal{S}, a)$ as defined in Theorem 3.6 is indeed a cyclic trisotopy over k. To prove that its multiplier is $u = Q(a)^{-1}$, i.e., that this element satisfies the conditions of Proposition 2.5, one uses (3.7.2), (3.7.4), (3.7.6). Details are left to the reader. ☐

Our final aim in this section will be to show that the two constructions presented in the preceding theorems are inverses of one another. As indicated earlier, the hardest part of the job has been done already in Lemma 3.8.

3.10 Theorem. *Keeping the previous notations, the following statements hold.*

a) *Let* $(\mathcal{S}, a)$ *be a pointed cyclic composition over* k*. Then* $\mathrm{Spr}\big(\mathrm{Alb}(\mathcal{S}, a)\big) = (\mathcal{S}, a)$.

b) *Let* $\mathcal{A}$ *be a cyclic trisotopy over* k*. Then* $\mathrm{Alb}\big(\mathrm{Spr}(\mathcal{A})\big) = \mathcal{A}$.

Proof. a) We write

$$\mathcal{S} = (E, \rho, M, Q, *)\,,\ \mathcal{A} := \mathrm{Alb}(\mathcal{S}, a) = (E, \rho, C, g, b)$$

and must show that

$$(\mathcal{A}^0, a^0) = \mathrm{Spr}(\mathcal{A})\,,\ \mathcal{A}^0 =: (E, \rho, M^0, Q^0, *^0)$$

agrees with $(\mathcal{S}, a)$. First of all, $M^0 = C = M$ as E-modules. Furthermore, by Theorem 3.6, $u = Q(a)^{-1}$ is the multiplier of $\mathcal{A}$, whence $Q^0 = Q$ by (3.2.1),(3.7.4), $a^0 = a$ by (3.2.4), (3.7.3), and it remains to prove $*^0 = *$. To this end, we set $c = a * a$ as in (3.7.1) and $u_0 = \rho(u)u^{-1} = Q(a)\rho(Q(a))^{-1}$ as in (2.5.1) to obtain, for $x, y \in M$,

$$\begin{aligned}
x * y &= g\big(Q(a)^{-1}\overline{x}\big)[g^2\big(Q(a)^{-1}\overline{y}\big)c] && \text{(by Lemma 3.8)}\\
&= \rho\big(Q(a)\big)^{-1}\rho^2\big(Q(a)\big)^{-1}\rho^2\big(Q(a)\big)Q(a)g(\overline{x})[g^2(\overline{y})b] && \text{(by (3.6.2))}\\
&= u_0 g(\overline{x})[g^2(\overline{y})b]\\
&= x *^0 y. && \text{(by (3.2.2))}
\end{aligned}$$

Hence $(\mathcal{A}^0, a^0) = (\mathcal{S}, a)$, as claimed.

b) By a) it suffices to show that $\mathrm{Spr}(\mathcal{A}) = \big((E, \rho, M, Q, *), a^0\big)$ determines $\mathcal{A} = (E, \rho, C, g, b)$ uniquely. Since $Q(a^0) = uN_C(u^{-1}1_C)$ (by (3.2.1),(3.2.4)) $= u^{-1}$, u being the multiplier of $\mathcal{A}$, we obtain

$$\begin{aligned}
u &= Q(a^0)^{-1},\\
1_C &= ua^0, && \text{(by (3.2.4))}\\
N_C &= \langle u^{-1}\rangle\,.\,Q, && \text{(by (3.2.1))}\\
T_C &= N_C(1_C, -),\\
\overline{x} &= T_C(x)1_C - x, \quad (x \in C)\\
u_0 &= \rho(u)u^{-1}, && \text{(by (2.5.1))}\\
b &= u_0^{-1}(1_C * 1_C)\,, && \text{(by (3.2.2) and Lemma 2.3 a))}\\
g(x) &= u_0^{-2}N_C(b)^{-1}(\overline{x} * b), \quad (x \in C) && \text{(by (3.2.2) and since } N_E(u_0) = 1)\\
xb &= u_0^{-1}\big(g^{-1}(\overline{x}) * 1_C\big). \quad (x \in C)
\end{aligned}$$

Thanks to the last equation, R_b, the right multiplication by b in C, only depends on $\mathrm{Spr}(\mathcal{A})$. Hence so does its inverse $R_b^{-1} = R_{b^{-1}}$. But now the relation

$$xy = u_0^{-1}\big(g^{-1}(\overline{x}) * g^{-2}(\overline{yb^{-1}})\big) \qquad (x, y \in C)$$

shows that also the product of C is completely determined by $\mathrm{Spr}(\mathcal{A})$, and we are done. $\square$

4. Isotopes

This section serves as a brief digression into isotopes, giving us the opportunity to recall the well known connection of cyclic compositions with cubic norm structures and their associated Jordan algebras in a characteristic-free manner. We then proceed to define isotopes of pointed cyclic compositions and to translate this concept into the setting of cyclic trisotopies by means of the correspondence set up in the previous section.

4.1 Cubic norm structures and their associated Jordan algebras. Let R be an arbitrary commutative associative ring of scalars. Following McCrimmon [7], adopting the terminology of [15] at the same time, a *cubic norm structure* over R is a quadruple $(W, N, \sharp, 1)$ consisting of an R-module W, a cubic form $N : W \to R$, called the *norm*, a quadratic map $\sharp : W \to W$, called the *adjoint*, and a distinguished element $1 \in W$, called the *base point*, such that the relations $x^{\sharp\sharp} = N(x)x$ (the *adjoint identity*), $N(1) = 1$ (the *base point identity*), $T(x^\sharp, y) = (DN)(x)y$ (the *gradient identity*), $1^\sharp = 1$, $1 \times y = T(y)1 - y$ hold under all scalar extensions. Here $T = -(D^2 \log N)(1) : W \times W \longrightarrow W$ stands for the *associated trace form*, $x \times y = (x+y)^\sharp - x^\sharp - y^\sharp$ for the bilinearization of the adjoint and $T(y) = T(y, 1)$. A *homomorphism* of cubic norm structures is a linear map of the underlying R-modules preserving norms, adjoints and base points in the obvious sense. The argument given in [7, p. 507] shows that, given two cubic norm structures over R whose associated trace forms are non-degenerate, every linear surjection of the underlying modules preserving norms and units is, in fact, an isomorphism of cubic norm structures.

If $(W, N, \sharp, 1)$ is a cubic norm structure over R, then the U-operator defined by $U_x y = T(x, y)x - x^\sharp \times y$ for all $x, y \in W$ and the base point 1 give W the structure of a unital quadratic Jordan algebra, which we denote by $J = J(W, N, \sharp, 1)$ and call the *Jordan algebra associated with* $(W, N, \sharp, 1)$. Assigning to a cubic norm structure its associated Jordan algebra is clearly functorial.

As an easy example, let (E, ρ) be a cyclic cubic étale k-algebra. Viewing E merely as a vector space over k, $N_E : E \to k, v \mapsto N_E(v) = v\rho(v)\rho^2(v), \sharp : E \to E, v \mapsto v^\sharp = \rho(v)\rho^2(v)$, and the unit element of E define a cubic norm structure $(E, N, \sharp, 1)$ over k satisfying $J(E, N, \sharp, 1) = E^+$. In particular, the associated trace form is given by $(v, w) \mapsto T_E(v, w) := T_E(vw)$.

4.2 The cubic norm structure of a cyclic composition. Let $\mathcal{S} = (E, \rho, M, Q, *)$ be a cyclic composition over k. Then the cubic form $N_{\mathcal{S}}$ defined in (1.3.3) lies at the core of the cubic norm structure that may be attached to $\mathcal{S}$ as follows (see [5, (38.6)] or, in characteristic not two, [19, 6.3] for a generalization). Regarding $W = E \oplus M$ merely as a vector space over k, we define a cubic form $N : W \to k$, an adjoint $\sharp : W \to W$ and a base point $1 \in W$ by the formulae

$$N(v \oplus x) = N_E(v) - T_E\big(v, Q(x)\big) + N_{\mathcal{S}}(x)\,, \tag{4.2.1}$$

$$(v \oplus x)^\sharp = \big(v^\sharp - Q(x)\big) \oplus (-vx + x * x)\,, \tag{4.2.2}$$

$$1 = 1_E \oplus 0$$

for $v \in E, x \in M$ to obtain a cubic norm structure $\mathrm{Cube}(\mathcal{S}) = (W, N, \sharp, 1)$ over k. The corresponding Jordan algebra of degree 3 will be denoted by $J(\mathcal{S}) = J\big(\mathrm{Cube}(\mathcal{S})\big) = J(W, N, \sharp, 1)$. Clearly, E^+ identifies canonically with a subalgebra of $J(\mathcal{S})$ through the first factor. The trace form associated with $\mathrm{Cube}(\mathcal{S})$, which agrees with the generic trace of $J(\mathcal{S})$, is given by

$$T(v \oplus x, w \oplus y) = T_E(v, w) + T_E\big(Q(x, y)\big) \tag{4.2.3}$$

for all $v, w \in E, x, y \in M$. It follows that $J(\mathcal{S})$ is *nonsingular* as a Jordan algebra of degree 3 in the sense that its generic trace is a nonsingular symmetric bilinear form over k.

4.3 Isotopes of cyclic compositions. Let $\mathcal{S} = (E, \rho, M, Q, *)$ be a cyclic composition over k and $w \in E$ an invertible element. Following [19, p. 70] or, in a slightly different context, [5, (36.1)], we define

$$Q^{(w)} = \langle w^\sharp \rangle \,. \, Q$$

as a quadratic form over E and a k-bilinear product $*^{(w)}$ on M by the formula

$$x *^{(w)} y = w(x * y)$$

for all $x, y \in M$. Then it follows easily that

$$\mathcal{S}^{(w)} = (E, \rho, M, Q^{(w)}, *^{(w)})$$

is again a cyclic composition over k, called the *w-isotope* of $\mathcal{S}$. It is also straightforward to check that the cubic form attached to $\mathcal{S}^{(w)}$ via (1.3.3) is related to the one attached to $\mathcal{S}$ by the formula

$$N_{\mathcal{S}^{(w)}}(x) = N_E(w) N_{\mathcal{S}}(x) . \qquad (x \in M)$$

Using this formula, the connection between isotopes of cyclic compositions and those of their associated Jordan algebras (cf. 4.2) can be described immediately as follows.

4.4 Proposition. *Let $\mathcal{S} = (E, \rho, M, Q, *)$ be a cyclic composition over k and $w \in E^\times$. Then the map*

$$\varphi_w : J(\mathcal{S})^{(w)} \xrightarrow{\sim} J(\mathcal{S}^{(w)}), \quad v \oplus x \longmapsto \varphi_w(v \oplus x) := (wv) \oplus x,$$

for $v \in E, x \in M$ is an isomorphism of Jordan algebras. □

4.5 Isotopes of pointed cyclic compositions. Let $(\mathcal{S}, a)$ be a pointed cyclic composition over k, $\mathcal{S} = (E, \rho, M, Q, *)$ as usual, and $w \in E^\times$. Then

$$(\mathcal{S}, a)^{(w)} := (\mathcal{S}^{(w)}, a^{(w)}), \quad a^{(w)} := w^{\sharp - 1} a ,$$

is again a pointed cyclic composition, called the *w-isotope* of $(\mathcal{S}, a)$.

Thanks to the equivalence of pointed cyclic compositions and cyclic trisotopies, it should be possible to transfer the preceding concepts to cyclic trisotopies. In fact, this turns out to be so easy that we state the results mostly without proof.

4.6 The cubic norm structure of a cyclic trisotopy. Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy with multiplier u over k and, as in Theorem 3.2, write $\mathrm{Spr}(\mathcal{A}) = (\mathcal{A}^0, a^0)$ for its associated pointed cyclic composition. Then we put

(4.6.1) $$N_{\mathcal{A}} := N_{\mathcal{A}^0}, \quad \mathrm{Cube}(\mathcal{A}) := \mathrm{Cube}(\mathcal{A}^0), \quad J(\mathcal{A}) := J(\mathcal{A}^0) .$$

The cubic form $N_{\mathcal{A}}$ can be described quite easily in terms of $\mathcal{A}$ alone because, observing (2.5.1), we have

(4.6.2) $$N_{\mathcal{A}}(x) 1_E = N_C\big(g^2(x)[g(x)x], b_0\big) . \qquad (x \in C)$$

Indeed, since g commutes with the canonical involution of C by Lemma 2.3 c), we may use (1.3.3),(4.6.1) to compute

$$\begin{aligned} N_{\mathcal{A}}(x) 1_E &= u N_C(x, u_0 \overline{g(x)} [\, \overline{g^2(x)} b]) && \text{(by (3.2.1), (3.2.2))} \\ &= u u_0 N_C(g^2(x)[g(x)x], b) && \text{(by (3.6.6))} \end{aligned}$$

and (2.5.1) yields (4.6.2).

4.7 Proposition. *Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy with multiplier u over k. Then, for all $w \in E^\times$,*

$$\mathcal{A}^{(w)} := (E, \rho, C, g, b^{(w)}), \quad b^{(w)} := \rho(w)b\,, \tag{4.7.1}$$

*is a cyclic trisotopy over k with multiplier $u^{(w)} := w^\sharp u$. We call $\mathcal{A}^{(w)}$ the w-*isotope *of $\mathcal{A}$ and have the relation* $\mathrm{Spr}(\mathcal{A}^{(w)}) = \mathrm{Spr}(\mathcal{A})^{(w)}$. *Also, extending the notations of* (2.5.1) *to the present set-up in the obvious way, $(b^{(w)})_0 = N_E(w)b_0, (u^{(w)})_0 = w\rho(w)^{-1}u_0$.* □

4.8 Corollary. *Notations being as in* 4.7, *$\mathcal{A}^{(u)} = (E, \rho, C, g, b^{(u)})$, the u-isotope of $\mathcal{A}$, satisfies the relation $g(b^{(u)}) = b^{(u)}$.*

Proof. Since $b^{(u)} = \rho(u)b$ by (4.7.1), this follows immediately from Propositions 2.5, 4.7. □

Remark. Returning to Albert's approach to cyclic trisotopies (2.4), we conclude from Corollary 4.8 that passing to the u-isotope of $\mathcal{A}$ allows us with Albert to make the additional assumption $g(b) = b$.

4.9 Corollary. *Let $(\mathcal{S}, a)$, $\mathcal{S} = (E, \rho, M, Q, *)$, be a pointed cyclic composition over k and $w \in E^\times$. Then* $\mathrm{Alb}\big((\mathcal{S}, a)^{(w)}\big) = \big(\mathrm{Alb}(\mathcal{S}, a)\big)^{(w)}$. □

5. Hybrids and weak homomorphisms

We now describe in detail how changing base points of pointed cyclic compositions affects their associated cyclic trisotopies. This will enable us to introduce the concept of a weak homomorphism and to set up an equivalence of categories between cyclic trisotopies with weak homomorphisms and cyclic compositions admitting a base point.

5.1 Changing base points. Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy with multiplier u over k and write

$$\mathrm{Spr}(\mathcal{A}) = (\mathcal{A}^0, a^0) \tag{5.1.1}$$

for its associated pointed cyclic composition. It follows from (3.2.1) that the base points of $\mathcal{A}^0$ agree with the invertible elements of C. Hence as t runs through the base points of $\mathcal{A}^0$, so does $u^{-1}\bar{t}$, and we are allowed to define a new cyclic trisotopy

$$\mathcal{A}^{[t]} := \mathrm{Alb}(\mathcal{A}^0, u^{-1}\bar{t}\,) \tag{5.1.2}$$

over k, depending on t and called the *t-hybrid* of $\mathcal{A}$. Notice that (5.1.2) and Theorem 3.10 a) imply

$$\mathrm{Spr}(\mathcal{A}^{[t]}) = (\mathcal{A}^0, u^{-1}\bar{t}\,)\,,\ \mathcal{A}^{[t]0} = \mathcal{A}^0\,. \tag{5.1.3}$$

For technical reasons, it will be important to describe the t-hybrid of $\mathcal{A}$ explicitly in terms of $\mathcal{A}$ and t.

5.2 Theorem. *Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy with multiplier u over k and $t \in C^\times$. We write $\mathcal{A}^{[t]} = (E, \rho, C^{[t]}, g^{[t]}, b^{[t]})$ for the t-hybrid of $\mathcal{A}$ and put $s := tg(\bar{t})^{-1}$. Then*

a) $C^{[t]} = C^{(s,g(\bar{t}))}$ *is the* $(s, g(\bar{t}))$*-isotope of* C*. In particular, unit element and norm of* $C^{[t]}$ *are given by*

$$1_{C^{[t]}} = t^{-1}\,, \tag{5.2.1}$$

$$N_{C^{[t]}} = \langle N_C(t)\rangle \,.\, N_C\,. \tag{5.2.2}$$

b) $b^{[t]} \in C^{[t]\times}$ and $g^{[t]} : C^{[t]} \to C^{[t]}$ *are given by the formulae*

$$b^{[t]} = N_C(t)^{-1}g(t)[g^2(\bar{t})^{-1}b] = N_C(t)^{-1}g(t)[bg^{-1}(\bar{t})^{-1}]\,, \tag{5.2.3}$$

$$g^{[t]}(x) = g(t)[g(x)t^{-1}]\,. \qquad (x \in C)$$

c) *The multiplier of* $\mathcal{A}^{[t]}$ *is*

$$u^{[t]} = N_C(t)^{-1}u\,. \tag{5.2.4}$$

Proof. We confine ourselves to proving only those parts of this result that will be used later on, and these are (5.2.1) – (5.2.4). Observing (5.1.1), (5.1.3), we put

$$\mathcal{A}^0 = \mathcal{A}^{[t]0} = (E, \rho, M, Q, *) \tag{5.2.5}$$

and systematically make use of the abbreviations

$$t^- = u^{-1}\,\bar{t}\,,\ t^+ = \rho^2\big(Q(t^-)\big)^{-1}(t^- * t^-)\,,\ c^{[t]} = t^- * t^-\,, \tag{5.2.6}$$

$t^+, c^{[t]}$ being modelled after (3.6.1), (3.7.1), respectively. We now claim

$$Q(t^-) = u^{-1}N_C(t)\,, \tag{5.2.7}$$

$$c^{[t]} = \rho^2(u)^{-1}u^{-1}g(t)[g^2(t)b]\,, \tag{5.2.8}$$

Here (5.2.7) follows immediately from (3.2.1), (5.2.6). To prove (5.2.8), we compute

$$\begin{aligned} c^{[t]} &= \rho(u)^{-1}\rho^2(u)^{-1}(\bar{t} * \bar{t}) && \text{(by Lemma 1.3 a), (5.2.6))} \\ &= \rho(u)^{-1}\rho^2(u)^{-1}u_0 g(t)[g^2(t)b] && \text{(by (3.2.2))} \\ &= \rho^2(u)^{-1}u^{-1}g(t)[g^2(t)b]\,, && \text{(by (2.5.1))} \end{aligned}$$

and the proof of (5.2.8) is complete.
Now (5.2.4) follows immediately from (5.2.7) combined with (3.2.1), (5.1.3), (5.2.5) and Theorem 3.6. Combining (5.2.4) with (3.2.4), (3.2.1), we also obtain (5.2.1), (5.2.2). Finally, (5.2.3) derives from the following computation:

$$\begin{aligned} b^{[t]} &= \rho^2\big(Q(t^-)\big)^{-1}Q(t^-)^{-1}c^{[t]} && \text{(by (3.6.2), (5.2.6))} \\ &= \rho^2(u)\rho^2\big(N_C(t)\big)^{-1}uN_C(t)^{-1}\rho^2(u)^{-1}u^{-1}g(t)[g^2(t)b] && \text{(by (5.2.7), (5.2.8))} \\ &= N_C(t)^{-1}g(t)[g^2(\bar{t})^{-1}b] && \text{(by 2.2 CT4, Lemma 2.3 e))} \\ &= N_C(t)^{-1}g(t)[g^3g^{-1}(\bar{t}^{\,-1})b] && \\ &= N_C(t)^{-1}g(t)[bg^{-1}(\bar{t}\,)^{-1}]\,. && \text{(by (2.2.2))} \end{aligned}$$

□

5.3 Weak homomorphisms. Given two cyclic trisotopies

$$\mathcal{A} = (E, \rho, C, g, b), \mathcal{A}' = (E', \rho', C', g', b')$$

over k, a *weak homomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ is a homomorphism in the sense of 2.2 from $\mathcal{A}$ to $\mathcal{A}'^{[t']}$ for some $t' \in C'^{\times}$, necessarily unique by (5.2.1). We say that $\mathcal{A}, \mathcal{A}'$ are *weakly isomorphic* and write $\mathcal{A} \sim \mathcal{A}'$ if there exists a *weak isomorphism*, i.e., a bijective weak homomorphism, from $\mathcal{A}$ to $\mathcal{A}'$. By our previous results, considering cyclic trisotopies under weak homomorphisms amounts to considering cyclic compositions with (unspecified) base points under ordinary homomorphisms. More precisely, we obtain the following theorem.

5.4 Theorem. *The assignments* $(\mathcal{S}, a) \mapsto \mathrm{Alb}(\mathcal{S}, a)$ *and* $\mathcal{A} \mapsto \mathrm{Spr}(\mathcal{A})$ *defined in Theorems* 3.2, 3.6 *above canonically induce an equivalence of categories between free cyclic compositions on the one hand and cyclic trisotopies with weak homomorphisms on the other.* □

5.5 Cyclic compositions and trisotopies over rings. O. Loos has made the interesting observation that, aside from Proposition 1.7 and Theorem 1.8 above, *all of our previous results continue to hold if the base field k is replaced by an arbitrary commutative associative ring of scalars*, provided the pertinent modules are assumed throughout to be finitely generated projective. Indeed, the only proof demanding special care over rings is the one of Proposition 2.5, where we make use of Hilbert's Theorem 90, which by the standard proof known from field theory [6, pp. 288-9] is easily seen to hold locally, though it fails to hold globally. Therefore, given a cyclic trisotopy $\mathcal{A} = (E, \rho, C, g, b)$ over any commutative ring k (assuming, in particular, that E be finitely generated projective as a k-module), elements u satisfying the conditions of Proposition 2.5 exist locally and are unique, hence glue to give a unique global element of the desired kind.

5.6 Invariants of cyclic compositions. The preceding results enable us to find invariants of cyclic compositions. For simplicity working with a fixed cyclic cubic étale k-algebra (E, ρ) throughout, let $\mathcal{S}$ be a free cyclic composition of rank $2^n, 0 \le n \le 3$, over k and $\mathcal{A} = (E, \rho, C, g, b)$ a cyclic trisotopy with multiplier $u \in E^{\times}$ corresponding to $\mathcal{S}$ via Theorem 5.4. Since $\mathcal{A}$ is uniquely determined by $\mathcal{S}$ up to weak isomorphism, it follows from (5.2.2) that the n-fold Pfister form N_C over E up to isometry only depends on $\mathcal{S}$, as does the composition algebra C over E up to isomorphism; for $n = 3$, these invariants may both be identified with the base change from k to E of the 3-invariant mod 2 attached to the Albert algebra $J(\mathcal{S})$ ([5, § 40] or [13, § 4]). Similarly, it follows from (5.2.4) that the image of u in $E^{\times}/N_C(C^{\times})$ only depends on $\mathcal{S}$; it is called the *multiplier* of $\mathcal{S}$.

6. Cyclic trisotopies of low rank

6.1 In this section, we enumerate cyclic trisotopies of rank at most 2; rather than doing so directly, which would have been quite easy, we prefer to derive a somewhat more general result that turns out to be useful later on. As an application, we draw connections to the étale Tits process and to simple associative algebras of degree 3 with involution.

6.2 Theorem. *Let* $\mathcal{A} = (E, \rho, C, g, b)$ *be a cyclic trisotopy over* k. *Then the following statements are equivalent.*

(i) $g : C \to C$ *is a* (ρ*-semi-linear*) *automorphism of* C *having order* 3.

(ii) *There exist a composition algebra* C_0 *over* k *as well as invertible elements* $d_0 \in C_0, w_0 \in E$ *such that*

$$\mathcal{A}' := \big(E, \rho, C_0 \otimes E, \mathbf{1}_{C_0} \otimes \rho, d_0 \otimes \rho(w_0)^{-1}\big)$$

is a cyclic trisotopy isomorphic to $\mathcal{A}$.

(iii) $\mathcal{A}$ *has rank at most* 2 *or* $b \in E1_C$.

In this case, $u = N_{C_0}(d_0)N_E(w_0)^{-1}w_0$ *is the multiplier of* $\mathcal{A}'$ *and in the terminology of Proposition* 2.5 *we have*

$$u_0 = \rho(w_0)w_0^{-1}, \quad b_0 = N_{C_0}(d_0)N_E(w_0)^{-1}d_0 \in C_0^\times.$$

Proof. (i) $\Longrightarrow$ (ii). By hypothesis, $C_0 := \{x \in C \mid g(x) = x\} \subseteq C$ is a unital k-subalgebra, and N_C restricts to a quadratic form $N_{C_0} : C_0 \to k$. It therefore suffices to show that the inclusion $C_0 \hookrightarrow C$ induces an isomorphism $C_0 \otimes E \xrightarrow{\sim} C$ of E-modules. Here we are done if E is a field ([3, X, Lemma 2]). On the other hand, if (E, ρ) splits, the assertion follows easily by direct computation.
(ii) $\Longrightarrow$ (iii). By (ii), we may assume that there are a composition algebra C_0 over k and invertible elements $d_0 \in C_0, w_0 \in E$ satisfying $C = C_0 \otimes E, g = \mathbf{1}_{C_0} \otimes \rho, b = d_0 \otimes \rho(w_0)^{-1}$. In particular, g is a ρ-semi-linear automorphism of C having order 3, which, by (2.2.1), (2.2.2), implies $C^{(b,b^{-1})} = C$ and $bxb^{-1} = x$ for all $x \in C$. The former condition, in turn, is equivalent to b belonging to the nucleus of C [8, Proposition 7]. Since quaternion (resp. octonion) algebras over fields are known to be central (resp. to have trivial nucleus), hence enjoy these properties over E as well, even if E is not a field, (iii) holds.
(iii) $\Longrightarrow$ (i). If $\mathcal{A}$ has rank at most 2, C is commutative associative. Therefore, if (iii) holds, (2.2.1), (2.2.2) tell us that $g : C \to C$ is a ρ-semi-linear automorphism of order 3.
The remaining assertions of the theorem now follow immediately from Proposition 2.5. □

The following corollary, enumerating cyclic trisotopies of rank ≤ 2, is an immediate consequence of the preceding theorem.

6.3 Corollary. *Up to isomorphism, the cyclic trisotopies of rank* $r \leq 2$ *are precisely of the form*

$$\mathcal{A} = \big(E, \rho, L_0 \otimes E, \mathbf{1}_{L_0} \otimes \rho, d_0 \otimes \rho(w_0)^{-1}\big),$$

where (E, ρ) *is a cyclic cubic étale* k*-algebra,* L_0 *is a composition algebra of dimension* r *over* k *and* $d_0 \in L_0^\times, w_0 \in E^\times$. □

6.4 Proof of Theorem 1.8 a). Every free cyclic composition of rank $r \leq 2$ by Theorem 5.4 corresponds to a cyclic trisotopy as in Corollary 6.3. Setting $d := d_0, v := w_0^{-1}$ and applying Theorems 3.2, 6.2, the assertion follows. □

6.5 Algebras of degree 3 with involution. Let $\mathcal{A}$ be a cyclic trisotopy of rank $r \leq 2$ over k. Up to isomorphism, Corollary 6.3 allows us to assume

$$\mathcal{A} = (E, \rho, L, \sigma, b), \tag{6.5.1}$$

where (E, ρ) is a cyclic cubic étale k-algebra and

$$L = L_0 \otimes E, \quad \sigma = \mathbf{1}_{L_0} \otimes \rho, \quad b = d_0 \otimes \rho(w_0)^{-1} \tag{6.5.2}$$

for some composition algebra L_0 of dimension r over k, $d_0 \in L_0^\times, w_0 \in E^\times$. By Theorem 6.2, $u = N_{L_0}(d_0)N_E(w_0)^{-1}w_0$ is the multiplier of $\mathcal{A}$ and

$$u_0 = \rho(w_0)w_0^{-1}, \quad b_0 = N_{L_0}(d_0)N_E(w_0)^{-1}d_0 \in L_0^\times. \tag{6.5.3}$$

We now put

$$\lambda = \overline{b}\, b^{-1} = \overline{d_0}\, d_0^{-1} = \overline{b_0}\, b_0^{-1} \in L_0^\times\,, \tag{6.5.4}$$

forcing

$$\lambda \overline{\lambda} = 1_{L_0}\,. \tag{6.5.5}$$

Combining (2.5.2), (2.5.1) with (6.5.3), we also have

$$N_E(u) = N_{L_0}(b_0)\,. \tag{6.5.6}$$

Furthermore, we regard (L, σ) as a cyclic cubic étale L_0-algebra. Then $B := (L, \sigma, \lambda)$ is a cyclic Azumaya algebra of degree 3 over L_0. We may write $B = L \oplus LY \oplus LY^{-1}$ as a free L-module with basis $(1_L, Y, Y^{-1})$ and have the relations

$$Y^3 = \lambda 1_L\,,\ Ya = \sigma(a)Y\,. \qquad (a \in L) \tag{6.5.7}$$

In particular, Y is invertible in B with inverse $Y^{-1} = \overline{\lambda}\, Y^2$. Now put

$$u_0' = \rho(u_0)^{-1} = \rho^2(u)^{-1}\rho(u) \qquad \text{(by (2.5.1))} \tag{6.5.8}$$

and observe that $u_0u_0' \in E$ has norm 1. Thus we find a unique L_0/k-involution τ of B which extends the conjugation of L over E and satisfies

$$\tau(Y) = u_0u_0'Y^{-1}\,. \tag{6.5.9}$$

Writing $H(B, \tau) = \{x \in B|\, \tau(x) = x\}$ as usual for the Jordan algebra over k of τ-symmetric elements in B, the following proposition is an immediate consequence of the definitions.

6.6 Proposition. *Notations being as in* 6.5, *the involution* τ *satisfies the relations*

$$\tau(a_0 + u_0'a_1Y + u_0'a_{-1}Y^{-1}) = \overline{a_0} + u_0'\sigma(\overline{a_{-1}})Y + u_0'\sigma^{-1}(\overline{a_1})Y^{-1} \qquad (a_0, a_{\pm 1} \in L) \tag{6.6.1}$$

and

$$H(B, \tau) = \{v + u_0'\sigma^{-1}(a)Y + u_0'\sigma(\overline{a})Y^{-1} \,|\, v \in E, a \in L\}\,. \tag{6.6.2}$$

□

From now on, the generic norm of an algebra will always be regarded as a polynomial law in the sense of Roby [17], acting under the same notation on every base change of the algebra in the natural way. In what follows, we need explicit formulae for the generic norms and traces of B and $H(B, \tau)$. These may be derived along the approach suggested by Springer-Veldkamp [19, Lemma 4.7.4] or Engelberger [2, Lemma 2.1.10]. We omit the details.

6.7 Proposition. *Notations being as in* 6.5, *the generic norm and trace of* B *are given by the formulae*

$$N_B(x) = N_E(a_0) + \lambda N_E(a_1) + \lambda^{-1} N_E(a_{-1}) - T_E\big(a_0\sigma(a_1)\sigma^{-1}(a_{-1})\big)\,, \tag{6.7.1}$$

$$T_B(x, x') = T_E(a_0, a_0') + T_E\big(a_1, \sigma(a_{-1}')\big) + T_E\big(\sigma(a_{-1}), a_1'\big) \tag{6.7.2}$$

for

$$x = a_0 + a_1 Y + a_{-1} Y^{-1}\,,\; x' = a_0' + a_1' Y + a_{-1}' Y^{-1} \in B. \qquad (a_0, a_0', a_{\pm 1}, a_{\pm 1}' \in L)$$

□

6.8 Corollary. *Observing* (6.6.2), *the relations*

$$N_B(x_0) = N_E(v) + \lambda N_E(a) + \overline{\lambda N_E(a)} - T_E(u_0'^{-1} a v \overline{a})\,, \tag{6.8.1}$$

$$T_B(x_0, x_0') = T_E(v, v') + T_E(u_0'^{-1} a, \overline{a'}) + \overline{T_E(u_0'^{-1} a, \overline{a'})} \tag{6.8.2}$$

hold for all

$$x_0 = v + u_0'\sigma^{-1}(a)Y + u_0'\sigma(\overline{a})Y^{-1}\,,\; x_0' = v' + u_0'\sigma^{-1}(a')Y + u_0'\sigma(\overline{a'})Y^{-1} \in H(B, \tau)\,,$$

with arbitrary elements $v, v' \in E, a, a' \in L$. □

Our final aim in this section will be to realize the Jordan algebra $H(B, \tau)$ above by a slight generalization of the étale Tits process. Dispensing ourselves from the situation just discussed, here are the relevant facts to understand the details.

6.9 The Tits process. Let K be a composition algebra of dimension $r \leq 2$ over k, B a separable associative algebra of degree 3 over K (with the obvious meaning if $K \cong k \oplus k$ is split) and τ a K/k-involution of B. Assume that we are given an *admissible pair* for (B, τ) in the sense of [5, § 39, p. 525], i.e., a pair of invertible elements $u \in H(B, \tau)$ and $\mu \in K$ satisfying $N_B(u) = N_K(\mu)$. Then we may extend $N_B, \sharp$ (the adjoint of B^+ as a Jordan algebra of degree 3), 1_B as given on B and $H(B, \tau)$ to the vector space $W = H(B, \tau) \oplus B$ over k according to the rules

$$\begin{aligned} N(x_0 \oplus x) &= N_B(x_0) + \mu N_B(x) + \overline{\mu N_B(x)} - T_B\big(x_0, x u \tau(x)\big)\,, \\ (x_0 \oplus x)^\sharp &= \big(x_0^\sharp - x u \tau(x)\big) \oplus \big(\overline{\mu} \tau(x)^\sharp u^{-1} - x_0 x\big)\,, \\ 1 &= 1_B \oplus 0 \end{aligned} \tag{6.9.1}$$

for $x_0 \in H(B, \tau), x \in B$ to obtain a cubic norm structure whose corresponding Jordan algebra (cf. 4.1) will be written as $J = J(K, B, \tau, u, \mu)$. The associated trace form is given by

$$T(x_0 \oplus x, y_0 \oplus y) = T_B(x_0, y_0) + T_B\big(xu, \tau(y)\big) + T_B\big(yu, \tau(x)\big)$$

for $x_0, y_0 \in H(B, \tau), x, y \in B$. Furthermore, $H(B, \tau)$ identifies canonically with a subalgebra of J through the first factor.

6.10 The étale Tits process. In the remainder of this section, we will be interested in a specialization of the Tits process that is originally due to Petersson-Racine [14] and was taken up later by Petersson-Thakur [16]. Let L (resp. E) be a quadratic (resp. cubic) étale k-algebra and write ι for the non-trivial k-automorphism of L. Then we may specialize the Tits process to $K = L, B = E \otimes L, \tau = \mathbf{1}_E \otimes \iota$. Hence, given an admissible pair (u, μ) for (B, τ) , the Tits process leads to the Jordan algebra

$$J(E, L, u, \mu) = J(L, E \otimes L, \mathbf{1}_E \otimes \iota, u, \mu) = E \oplus (E \otimes L)\,,$$

which is said to arise from the parameters E, L, u, μ by means of the *étale Tits process.*

6.11 Theorem. *Keeping the notations of* 6.5, *the Jordan algebra of degree* 3 *attached to* $\mathcal{A}$ *via* 4.6 *may be realized by means of the étale Tits process as*

$$J(\mathcal{A}) = J(E, L_0, u, \overline{b_0})\,. \tag{6.11.1}$$

Furthermore, the map $\theta : H(B,\tau) \xrightarrow{\sim} J(\mathcal{A})$ *defined by* (cf. (6.6.2))

$$\theta\big(v + u_0'\sigma^{-1}(a)Y + u_0'\sigma(\overline{a})Y^{-1}\big) := v \oplus \rho^{-1}(u)\,\overline{b_0}^{\,-1}\,\overline{a} \qquad (v \in E, a \in L) \tag{6.11.2}$$

is an isomorphism of Jordan algebras.

Proof. We first note that the right-hand side of (6.11.1) makes sense by (6.5.6). Furthermore, thanks to 6.10, 4.2, (4.6.1), (6.5.2) both $J(\mathcal{A})$ and $J = J(E, L_0, u, \overline{b_0})$ live on the same vector space over k, namely $E \oplus L$. To complete the proof of (6.11.1), it therefore remains to show that they have the same unit elements and the same norms. Since the unit elements both agree with $1_E \oplus 0$, we only need consider the norms. To do so, write $\mathcal{A}^0 = (E, \rho, M, Q, *)$ for the cyclic composition attached to $\mathcal{A}$ via Theorem 3.2 and let $v \in E, a \in L$. Then

$$\begin{aligned}
N_{J(\mathcal{A})}(v \oplus a) &= N_{J(\mathcal{A}^0)}(v \oplus a) && \text{(by (4.6.1))}\\
&= N_E(v) - T_E\big(v, Q(a)\big) + N_{\mathcal{A}^0}(a) && \text{(by (4.2.1))}\\
&= N_E(v) - T_E\big(v, uN_{L_0}(a)\big) + N_{\mathcal{A}}(a) && \text{(by (3.2.1), (4.6.1))}\\
&= N_E(v) - T_E\big(v, uN_{L_0}(a)\big) + N_{L_0}\big(\sigma^2(a)\sigma(a)a, b_0\big) && \text{(by (4.6.2), (6.5.1))}\\
&= N_E(v) + N_{L_0}\big(b_0, N_E(a)\big) - T_E(v, au\overline{a})\\
&= N_E(v) + T_{L_0}\big(\overline{b_0}, N_E(a)\big) - T_E(v, au\overline{a})\\
&= N_E(v) + \overline{b_0}\,N_E(a) + b_0\,\overline{N_E(a)} - T_E(v, au\overline{a})\\
&= N_J(v \oplus a)\,, && \text{(by (6.9.1))}
\end{aligned}$$

and we have established (6.11.1). Since θ is a k-linear bijection preserving units, the proof of the theorem will be complete once we have shown that θ preserves norms as well. To do so, we compute, for $v \in E, a \in L$,

$$\begin{aligned}
N_{J(\mathcal{A})} \circ \theta\big(v{+}u_0'\sigma^{-1}(a)Y + u_0'\sigma(\overline{a})Y^{-1}\big)&\\
={}&N_{J(E,L_0,u,\overline{b_0})}\big(v \oplus \rho^{-1}(u)\,\overline{b_0}^{-1}\,\overline{a}\,\big) && \text{(by (6.11.1), (6.11.2))}\\
={}&N_E(v) + \overline{b_0}\,N_E\big(\rho^{-1}(u)\,\overline{b_0}^{-1}\,\overline{a}\,\big) + b_0\,\overline{N_E\big(\rho^{-1}(u)\,\overline{b_0}^{-1}\,\overline{a}\,\big)}\\
&- T_E\big(v, \rho^{-1}(u)\,\overline{b_0}^{-1}\,\overline{a}\,u\rho^{-1}(u)b_0^{-1}a\big)\,. && \text{(by (6.9.1), 6.10)}
\end{aligned}$$

The terms in the very last expression will now be treated separately as follows.

$$\begin{aligned}
\overline{b_0}\,N_E\big(\rho^{-1}(u)\overline{b_0}^{-1}\overline{a}\big) &= \overline{b_0}\,\overline{b_0}^{-3}N_E(u)N_E(\overline{a}) = \overline{b_0}^{-2}N_{L_0}(b_0)N_E(\overline{a}) && \text{(by (6.5.6))}\\
&= b_0\overline{b_0}^{-1}N_E(\overline{a}) = \overline{\lambda N_E(a)}\,, && \text{(by (6.5.4))}\\
\rho^{-1}(u)\,\overline{b_0}^{-1}u\rho^{-1}(u)b_0^{-1} &= \rho^{-1}(u)^2uN_{L_0}(b_0)^{-1} = \rho^{-1}(u)^2uN_E(u)^{-1} && \text{(by (6.5.6))}\\
&= \rho^{-1}(u)\rho(u)^{-1} = u_0'^{-1}\,. && \text{(by (6.5.8))}
\end{aligned}$$

Thus

$$\begin{aligned}
N_{J(\mathcal{A})} \circ \theta\big(v + u_0'\sigma^{-1}(a)Y + u_0'\sigma(\overline{a})Y^{-1}\big) &= N_E(v) + \lambda N_E(a) + \overline{\lambda N_E(a)}\\
&\quad - T_E(u_0'^{-1}av\overline{a})\\
&= N_B\big(v + u_0'\sigma^{-1}(a)Y + u_0'\sigma(\overline{a})Y^{-1}\big)\,, && \text{(by (6.8.1))}
\end{aligned}$$

as claimed. □

7. The core of a cyclic trisotopy

Our main concern in this section will be to introduce a number of technical concepts that will play a significant role in the structure theory of cyclic trisotopies later on. In particular, these concepts will permit the construction of sub-compositions and sub-trisotopies in the following sense.

7.1 Cyclic sub-compositions and sub-trisotopies. Let $\mathcal{S} = (E, \rho, M, Q, *)$ be a cyclic composition over k and $M' \subseteq M$ an E-submodule which is non-singular relative to Q and, at the same time, a k-subalgebra of $(M, *)$. Writing $Q' = Q|_{M'}$ for the restriction of Q to M' and $*'$ for the k-algebra structure induced by $*$ on M', then, clearly, $\mathcal{S}|_{M'} = (E, \rho, M', Q', *')$ is again a cyclic composition over k, called the *restriction* of $\mathcal{S}$ to M'. Cyclic compositions of this kind are called *sub-compositions* of $\mathcal{S}$; we write $\mathcal{S}|_{M'} \subseteq \mathcal{S}$. An element $a' \in M'$ is a *base point for* $\mathcal{S}|_{M'}$ if and only if it is one for $\mathcal{S}$, in which case we call $(\mathcal{S}, a')|_{M'} := (\mathcal{S}|_{M'}, a')$ the *restriction* of the pointed cyclic composition $(\mathcal{S}, a')$ to M'. Similar conventions apply to cyclic trisotopies over k. We omit the details. It is clear that passing to cyclic sub-compositions and -trisotopies is compatible with all the major constructions we have encountered so far.

7.2 The core; unital and étale cyclic trisotopies. Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy over k. Then $\mathrm{Core}(\mathcal{A}) := E[b]$, the unital E-subalgebra of C generated by b, which by (1.4.1) agrees with the E-submodule of C spanned by 1_C and b, is called the *core* of $\mathcal{A}$. By Lemma 2.3 a) and (2.5.3), the core of $\mathcal{A}$ is stabilized by g. We say that $\mathcal{A}$ is *unital* in case $b \in E1_C$, which happens if and only if $\mathrm{Core}(\mathcal{A}) = E1_C \cong E$ is a free E-module of rank 1. At the other extreme, $\mathcal{A}$ is said to be *étale* (resp. *split étale*) if $\mathrm{Core}(\mathcal{A})$ is a quadratic étale E-algebra (resp. $\mathrm{Core}(\mathcal{A}) \cong E \oplus E$ as E-algebras). If $\mathcal{A}$ is étale, it can be restricted to its core, and

$$\mathcal{A}|_{\mathrm{Core}(\mathcal{A})} = (E, \rho, \mathrm{Core}(\mathcal{A}), g|_{\mathrm{Core}(\mathcal{A})}, b) \subseteq \mathcal{A}$$

is a cyclic sub-trisotopy of rank 2 over k.

7.3 Convention. Until further notice, we fix a cyclic trisotopy $\mathcal{A} = (E, \rho, C, g, b)$ with multiplier $u \in E^\times$ over k.

7.4 Proposition. *Notations being as in* 7.3, *put* $L := \mathrm{Core}(\mathcal{A})$ *and* $b_0 := \rho(u)b$ *as in* (2.5.1). *Then* $\mathrm{Core}_0(\mathcal{A}) := L_0 := k[b_0]$, *the unital k-subalgebra of L generated by b_0, has dimension at most* 2, *and the inclusion $L_0 \hookrightarrow L$ induces identifications*

$$L = L_0 \otimes E\,,\; g = \mathbf{1}_{L_0} \otimes \rho \text{ (on } L)\,,\; b = b_0 \otimes \rho(u)^{-1}\,. \tag{7.4.1}$$

Proof. We clearly have $L = E[b_0]$, and Proposition 2.5 yields $g(b_0) = b_0$. Combining this with Lemma 2.3 b), we conclude $N_C(b_0), T_C(b_0) \in k$. Hence, X being an indeterminate, (1.4.1) yields isomorphisms

$$L \cong E[X]/\big(X^2 - T_C(b_0)X + N_C(b_0)\big)\,,\; L_0 \cong k[X]/\big(X^2 - T_C(b_0)X + N_C(b_0)\big)$$

that are compatible with the inclusion $L_0 \hookrightarrow L$. This gives the first and third identification of (7.4.1), while the second one follows from the fact that g is ρ-semi-linear and induces the identity on L_0. ☐

7.5 Corollary. *If $\mathcal{A}$ is not unital,* $\mathrm{Core}(\mathcal{A})$ *is a free E-module of rank* 2 *with basis* $1_C, b$. ☐

The following statement is an immediate consequence of Proposition 4.7.

7.6 Proposition. *Notations being as in* 7.3, $\mathrm{Core}(\mathcal{A}^{(w)}) = \mathrm{Core}(\mathcal{A})$ *for all* $w \in E^\times$. *In particular,* $\mathcal{A}^{(w)}$ *is unital (resp. [split] étale) if and only if* $\mathcal{A}$ *is.* □

7.7 Proposition. *Notations being as in* 7.3, *let* $t \in C^\times$. *Then the* t*-hybrid* $\mathcal{A}^{[t]}$ *is unital if and only if* $(tg(t))g^2(t) \in E\bar{b}$.

Proof. By Lemma 2.3, g commutes with the inversion and the canonical involution of C; also, $x^{-1} \in E\,\bar{x}$ for all $x \in C^\times$. Hence we obtain the following chain of equivalent statements.

$$\begin{aligned}
\mathcal{A}^{[t]} \text{ is unital} &\iff b^{[t]} \in E1_{C^{[t]}} && \text{(by 7.2)} \\
&\iff g(t)[g^2(\bar{t})^{-1}b] \in Et^{-1} && \text{(by (5.2.1), (5.2.3))} \\
&\iff b \in Eg^2(\bar{t})[g(\bar{t})\bar{t}] = E\overline{\big(tg(t)\big)g^2(t)} \\
&\iff \big(tg(t)\big)g^2(t) \in E\bar{b}\,.
\end{aligned}$$

□

7.8 Corollary. *If* $T_C(b) = 0$, *then* $\mathcal{A}^{[b]}$ *is unital.*

Proof. Setting $t = b$ in Proposition 7.7 and observing 2.2 CT 5 as well as $\bar{b} = -b$, we obtain $(bg(b))g^2(b) \in Eb^3 = Eb = E\bar{b}$. □

7.9 Proposition. *Notations being as in* 7.3, *the following statements are equivalent.*

(i) $\mathcal{A}$ *is neither unital nor étale.*

(ii) *Precisely one of the following holds.*
a) k *has characteristic* 2, *and* $b \notin E1_C$ *satisfies* $T_C(b) = 0$.
b) k *has characteristic not* 2, *and there are* $w \in E^\times, y \in C$ *satisfying*

$$b^{(w)} = 1_C + y\,,\ y \neq 0 = y^2\,,\ g(b^{(w)}) = b^{(w)}\,. \tag{7.9.1}$$

Proof. (ii) $\Longrightarrow$ (i). Obvious by Proposition 7.6.
(i) $\Longrightarrow$ (ii). Replacing $\mathcal{A}$ by its u-isotope and observing Corollary 4.8, we may assume $g(b) = b$, which by Lemma 2.3 b) implies $T_C(b), N_C(b) \in k$. By hypothesis and Corollary 7.5, $E[b]$ is a free E-module of rank 2 with basis $1_C, b$ but not étale as an E-algebra, forcing the discriminant $4N_C(b) - T_C(b)^2 \in k$ to be a non-unit in E. This implies $T_C(b)^2 = 4N_C(b)$. Hence, for $\mathrm{char}\,k = 2$, we are in case (ii) a). On the other hand, setting $\alpha = \frac{1}{2}T_C(b), y' = b - \alpha 1_C$ for $\mathrm{char}\,k \neq 2$, we obtain $b = \alpha 1_C + y', y' \neq 0$ since $\mathcal{A}$ is not unital, and $y'^2 = 0$. Hence, setting $w = \alpha^{-1}1_E, y = \alpha^{-1}y'$ and observing (4.7.1), we end up with (7.9.1). □

7.10 Proposition. *If* $\mathcal{A}$ *is not étale or* $N_E : E \otimes \mathrm{Core}_0(\mathcal{A}) \to \mathrm{Core}_0(\mathcal{A})$ *is surjective* (*in particular, if* E *is not a field or* k *is finite*), *then some hybrid of* $\mathcal{A}$ *is unital.*

Proof. Assuming as we may that $\mathcal{A}$ is not unital, suppose first $\mathcal{A}$ is not étale. Then either a) or b) of Proposition 7.9 (ii) holds. If a) holds, the assertion follows from Corollary 7.8. If b) holds, we choose $v \in E$ such that $T_E(v) = 1$ and put $t = 1_C - vy$. Since g, fixing 1_C by Lemma 2.3 a) and $b^{(w)}$ by (7.9.1), fixes y as well, we conclude

$$\begin{aligned}
\big(tg(t)\big)g^2(t) &= \big(1_C - vy\big)\big(1_C - \rho(v)y\big)\big(1_C - \rho^2(v)y\big) \\
&= 1_C - \big(v + \rho(v) + \rho^2(v)\big)y && \text{(by (7.9.1))} \\
&= 1_C - T_E(v)y = 1_C - y = \overline{b^{(w)}} \in E\bar{b}\,, && \text{(by (4.7.1))}
\end{aligned}$$

so by Proposition 7.7, $\mathcal{A}^{[t]}$ is unital. Adopting the notations of Proposition 7.4, we are left with the case that $N_E : L \to L_0$ is surjective. In particular, there exists an element $t \in L$ such that $\big(tg(t)\big)g^2(t) = N_E(t)$ (by (7.4.1)) $= \overline{b_0} \in E\overline{b}$, forcing $\mathcal{A}^{[t]}$ to be unital by Proposition 7.7. □

8. Reduced, isotropic, hyperbolic cyclic compositions and trisotopies

The concepts introduced in the previous section will now be used to characterize those cyclic compositions and trisotopies whose associated cubic Jordan algebras are reduced. In particular, we will present a self-contained proof for the fact that this holds always true if the rank is 1 or 4. We then proceed to investigate isotropic and hyperbolic compositions, the latter being an elaborate version of the former. Finally, we will show that every cyclic trisotopy $\mathcal{A}$ of rank > 1 up to weak isomorphism contains a cyclic sub-trisotopy of rank 2 which one may choose to be isotropic if $\mathcal{A}$ was isotropic to begin with.

Since the property of a cubic Jordan algebra to be reduced is equivalent to saying that its associated norm form represents zero non-trivially, we are lead to the following natural definitions.

8.1 Reduced cyclic compositions and trisotopies. A cyclic composition $\mathcal{S} = (E, \rho, M, Q, *)$ over k is said to be *reduced* if the cubic form $N_{\mathcal{S}}$ of Lemma 1.3 c) is isotropic in the sense that some non-zero element $x \in M$ satisfies $N_{\mathcal{S}}(x) = 0$. Notice that this concept does not formally agree with the one introduced under the same name by Springer-Veldkamp [19, 4.1.8]. However, thanks to Theorem 8.2 below, which should be regarded as an elaborate version of [19, Theorem 4.1.10], the two are always equivalent unless the rank is 1.

A cyclic trisotopy $\mathcal{A}$ over k is said to be *reduced* if the cyclic composition $\mathcal{A}^0$ attached to $\mathcal{A}$ via (3.2.3) is reduced.

8.2 Theorem. *Given a cyclic trisotopy $\mathcal{A} = (E, \rho, C, g, b)$ over k and writing $\mathcal{A}^0 = (E, \rho, M, Q, *)$ for its associated cyclic composition in the sense of Theorem* 3.2, *consider the following conditions on $\mathcal{A}$ and $\mathcal{A}^0$.*

(i) *$\mathcal{A}^0$ is reduced.*

(ii) *$\mathcal{A}$ is reduced.*

(iii) *There exists a non-zero element $x \in M$ satisfying $x * x \in Ex$.*

(iv) *There exists a non-zero element $v \oplus x \in E \oplus M$ satisfying $v^\sharp = Q(x)$ and $x * x = vx$.*

(v) *$J(\mathcal{A})$ is reduced.*

(vi) *If E is a field, there exists a non-zero element $x \in M$ satisfying $x * x \in Ex$.*

(vii) *There exists a vector $x \in M$ which is strongly anisotropic relative to Q and satisfies $x*x \in Ex$.*

(viii) *Some hybrid of $\mathcal{A}$ is unital.*

(ix) *There exist a composition algebra C_0 over k and an invertible element $w_0 \in E$ satisfying*

$$\mathcal{A} \sim \mathcal{A}' := \big(E, \rho, C_0 \otimes E, \mathbf{1}_{C_0} \otimes \rho, 1_{C_0} \otimes \rho(w_0)^{-1}\big)\,, \tag{8.2.1}$$

i.e., $\mathcal{A}$ is weakly isomorphic to the unital cyclic trisotopy $\mathcal{A}'$.

Then the following implications hold.

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longleftrightarrow \text{(iv)} \Longleftrightarrow \text{(v)} \Longleftrightarrow \text{(vi)} \Longleftrightarrow \text{(vii)} \Longleftrightarrow \text{(viii)} \Longleftrightarrow \text{(ix)}\,.$$

Furthermore, if $\mathcal{A}$ has rank > 1, then all statements (i) - (ix) *are equivalent.*

Proof. We proceed in several steps.

a) The first step consists in establishing the following set of implications.

$$\text{(ii)} \Longleftrightarrow \text{(i)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(vi)} \Longrightarrow \text{(v)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(vi)} \Longrightarrow \text{(vii)} \Longrightarrow \text{(viii)} \Longrightarrow \text{(ix)} \Longrightarrow \text{(viii)} \Longrightarrow \text{(vi)} \tag{8.2.2}$$

(ii) $\Longleftrightarrow$ (i). Obvious by 8.1.

(i) $\Longrightarrow$ (iii). Cf. [19, proof of Theorem 4.1.10, (ii) $\Rightarrow$ (iii), p. 74]

(iii) $\Longrightarrow$ (vi). Trivial.

(vi) $\Longrightarrow$ (v). Assuming (vi), we must show that $J = J(\mathcal{A})$ is reduced. Since J contains E^+ as a subalgebra, we are done if E is not a field. Otherwise, (vi) produces a non-zero element $x \in M$ satisfying $y := x * x = vx$ for some $v \in E$. For $y = 0$, (4.2.1), (1.3.3) yield the relation $N_J(0 \oplus x)1_E = N_{\mathcal{A}^0}(x)1_E = Q(x, x * x) = Q(x, y) = 0$, which implies that J is reduced, so we may assume $y \neq 0$. Setting $v' = Q(x)$, the element $v' \oplus y \in J$ is non-zero as well and satisfies

$$(v' \oplus y)^\sharp = \big(v'^\sharp - Q(y)\big) \oplus (-v'y + y * y)\,, \qquad \text{(by (4.2.2))}$$

where

$$\begin{aligned}
v'^\sharp - Q(y) &= \rho\big(Q(x)\big)\rho^2\big(Q(x)\big) - Q(x * x) = 0\,, && \text{(by (1.2.1))}\\
y * y - v'y &= (x * x) * (x * x) - Q(x)(x * x) \\
&= N_{\mathcal{A}^0}(x)x - 2Q(x)(x * x) && \text{(by (1.3.4))}\\
&= Q(x, vx)x - 2Q(x)vx && \text{(by (1.3.3) and since } x * x = vx)\\
&= 2vQ(x)x - 2vQ(x)x = 0\,.
\end{aligned}$$

Summing up, we have thus obtained $(v' \oplus y)^\sharp = 0 \neq v' \oplus y$, whence J is reduced, as claimed.

(v) $\Longrightarrow$ (iv). Since $J(\mathcal{A})$ is reduced, some non-zero element $v \oplus x \in J(\mathcal{A})$ satisfies $(v \oplus x)^\sharp = 0$. By (4.2.2), this is (iv).

(iv) $\Longrightarrow$ (vi). For $v \oplus x$ as in (iv), the assumption $x = 0$ would imply $v^\sharp = Q(x) = 0$, hence $v = 0$ since E as in (vi) is a field. This contradiction shows $x \neq 0$, and we have (vi).

(vi) $\Longrightarrow$ (vii). If E is a field, the argument given in [19, pp. 74-6] yields (vii). On the other hand, if E is not a field, Proposition 7.10 implies that some hybrid of $\mathcal{A}$ is unital. But since (vii) only depends on $\mathcal{A}^0$, which remains unaffected by passing to a hybrid, we may assume that $\mathcal{A}$ itself is unital to begin with. Then $x = 1_C$ is strongly anisotropic relative to Q and satisfies, using Lemma 2.3 a), (3.2.2) as well as the unitality of $\mathcal{A}$, $x * x = u_0 b \in E1_C = Ex$.

(vii) $\Longrightarrow$ (viii). By (3.2.1), we have $x \in C^\times$, allowing us to put $t = x^{-1}$. Since g, by Lemma 2.3, commutes with the canonical involution of C as well as with taking inverses, we conclude from (5.2.3) that

$$\begin{aligned}
b^{[t]} &= N_C(t)^{-1}\rho^2\big(N_C(t)\big)^{-1}g(t)[g^2(t)b] \\
&= u_0^{-1}N_C(t)^{-1}\rho^2\big(N_C(t)\big)^{-1}(\bar t * \bar t) && \text{(by (3.2.2))}\\
&= u_0^{-1}N_C(t)^{-1}\rho\big(N_C(t)\big)(t^{-1} * t^{-1}) && \text{(by Lemma 1.3 a))}\\
&\in Et^{-1} && \text{(by (vii), since } x = t^{-1})\\
&= E1_{C^{[t]}}\,, && \text{(by (5.2.1))}
\end{aligned}$$

so the t-hybrid $\mathcal{A}^{[t]}$ of $\mathcal{A}$ is unital.
(viii) $\Longrightarrow$ (ix). Let $\mathcal{A}^{[t]}, t \in C^\times$, be a hybrid of $\mathcal{A}$ which is unital. Then Theorem 6.2 implies that $\mathcal{A}^{[t]}$ is isomorphic, hence $\mathcal{A}$ is weakly isomorphic, to $\mathcal{A}'$ for some cyclic trisotopy $\mathcal{A}'$ as indicated in (ix).
(ix) $\Longrightarrow$ (viii). Obvious.
(viii) $\Longrightarrow$ (vi). Since $\mathcal{A}^0$ does not change when passing to a hybrid, we may assume that $\mathcal{A}$ is unital. Then (3.2.2) and Lemma 2.3 a) yield $1_C * 1_C = u_0 b \in E1_C$, and $x = 1_C$ satisfies (vi).
b) In the second step we wish to prove that, if $\mathcal{A}$ has rank > 1, all nine conditions are equivalent. By (8.2.2), it suffices to establish the implication (ix) $\Rightarrow$ (i). Again there is no harm in assuming $\mathcal{A} = \mathcal{A}'$. Then one simply follows the argument of [19, Theorem 4.1.10, (i) $\Rightarrow$ (ii), p. 74] to complete the proof.
c) In the final step, returning to the case of arbitrary rank and observing (8.2.2), it remains to prove (iv) $\Rightarrow$ (iii). Since (iii) holds trivially if $\mathcal{A}$ has rank 1, we may assume $\operatorname{rank} \mathcal{A} > 1$. But then, as we have seen in b), conditions (i) - (ix) are all equivalent, so (iii) must hold again. □

8.3 Corollary. *Every non-reduced cyclic trisotopy of rank > 1 is étale.*

Proof. Since condition (viii) of Theorem 8.2 fails to hold, the assertion follows from Proposition 7.10. □

8.4 Corollary. *Let $\mathcal{A}$ be a cyclic trisotopy of rank 1 over k. Then $J(\mathcal{A})$ is reduced.*

Proof. Since condition (iii) of Theorem 8.2 holds, so does (v). □

8.5 Example. Let (E, ρ) be a cyclic cubic field extension of k, $w_0 \in E^\times$ and assume char $k \neq 2$. By Corollary 6.3, $\mathcal{A} = (E, \rho, E, \rho, \rho(w_0)^{-1})$ is the most general cyclic trisotopy of rank 1 over k that can be built up from (E, ρ). By Corollary 8.4, $J(\mathcal{A})$ is reduced. On the other hand, one checks easily that $N_{\mathcal{A}} = \langle 2N_E(w_0)^{-1}\rangle \,.\, N_E$. Therefore, since E is a field, $\mathcal{A}$ cannot be reduced, and we have shown in Theorem 8.2 that (iii) does not imply (ii) if $\mathcal{A}$ has rank 1. On the other hand, comparing [19, (4.9)] with (8.2.1) , we see that condition (ix) of Theorem 8.2 is equivalent to the cyclic composition $\mathcal{A}^0$ being reduced in the sense of [19, Definition 4.1.8]. In particular, thanks to our example, [19, Theorem 4.1.10] fails to hold for cyclic compositions of rank 1, as does [5, Theorem (36.24)].

8.6 Theorem. *Every cyclic trisotopy of rank 4 over k is reduced.*

Proof. Let $\mathcal{A} = (E, \rho, C, g, b)$ be a cyclic trisotopy of rank 4 over k. By Theorem 8.2, it suffices to show that some hybrid of $\mathcal{A}$ is unital. Hence Proposition 7.10 allows us to assume that $\mathcal{A}$ is étale and E is a field. Moreover, since passing to hybrids and isotopes are commuting operations, we may combine Corollary 4.8 with Proposition 7.6 to reduce to the case $g(b) = b$. In the terminology of Proposition 2.5, this implies $u_0 = 1$, and the multiplier, γ, of $\mathcal{A}$ belongs to k. We now put $L := \operatorname{Core}(\mathcal{A}), L_0 := \operatorname{Core}_0(\mathcal{A})$ as in 7.2, Proposition 7.4, and obtain the identifications

$$L = L_0 \otimes E\,,\; g = \mathbf{1}_{L_0} \otimes \rho \text{ (on } L)\,,\; b = \gamma^{-1} b_0 \in L_0 \tag{8.6.1}$$

from (7.4.1). By Springer's Lemma [19, Lemma 4.2.11], anisotropic cubic forms remain anisotropic under quadratic field extensions. Hence, extending scalars if necessary, we may assume that L_0 splits over k. But then L and C split over E; in fact, we may assume that L, L_0 sit diagonally in $C = \operatorname{Mat}_2(E), C_0 = \operatorname{Mat}_2(k)$, respectively:

$$L = \operatorname{Diag}_2(E) \subseteq C = \operatorname{Mat}_2(E)\,,\; L_0 = \operatorname{Diag}_2(k) \subseteq C_0 = \operatorname{Mat}_2(k)\,. \tag{8.6.2}$$

Now, letting ρ act componentwise on C, we obtain a ρ-semi-linear automorphism, also denoted by ρ, which, in view of (8.6.1),(8.6.2) , agrees with g on L. On the other hand, since C is associative, 2.2 CT4 implies that g is a ρ-semi-linear automorphism of C, forcing $g \circ \rho^2 : C \to C$ to be an E-linear automorphism. Thus the Skolem-Noether theorem yields an invertible element $a \in C$ satisfying

$$g(x) = a\rho(x)a^{-1}\,. \qquad (x \in C) \tag{8.6.3}$$

Since $g = \rho$ on L, a centralizes L, and we conclude $a \in L^\times$. On the other hand, a threefold application of g combined with (2.2.2), (8.6.3) gives $bxb^{-1} = g^3(x) = N_E(a)xN_E(a)^{-1}$ for all $x \in C$, forcing $N_E(a) = vb$ for some $v \in E^\times$. Setting $t = \overline{a} \in L^\times$, we conclude $\big(tg(t)\big)g^2(t) = t\rho(t)\rho^2(t) = N_E(\overline{a}) = v\overline{b} \in E\overline{b}$ and $\mathcal{A}^{[t]}$ is unital by Proposition 7.7. □

Remark. The advantage of the above proof derives from the fact that it neither depends on the structure theory for finite-dimensional simple Jordan algebras nor on fundamental facts about (associative) algebras with involution. Dito for the proof of Corollary 8.4 above.

8.7 Proof of Theorem 1.8 b). The cyclic trisotopy corresponding via Theorem 5.4 to a free cyclic composition of rank 4 over k is reduced by Theorem 8.6, hence up to weak isomorphism has the form (8.2.1) for some quaternion algebra C_0 over k. Applying Theorems 3.2, 6.2 and setting $v = w_0^{-1}$, the assertion follows. □

8.8 Isotropic compositions. A cyclic composition $\mathcal{S} = (E, \rho, M, Q, *)$ over k is said to be *isotropic* if the quadratic form Q has this property, so $Q(x) = 0$ for some non-zero element $x \in M$. Similarly, a cyclic trisotopy $\mathcal{A} = (E, \rho, C, g, b)$ over k is said to be *isotropic* if N_C, the norm of C, has this property. Using standard facts about composition algebras over fields, this holds true if and only if the quadratic space (C, N_C) over E is hyperbolic. Also, $\mathcal{A}$ is isotropic if and only if $\mathcal{A}^0$ is.

In order to deal with isotropic cyclic compositions and trisotopies effectively, more elaborate versions of these notions need to be investigated.

8.9 Hyperbolic cyclic compositions. By a *hyperbolic* cyclic composition we mean a pair $(\mathcal{S}, e)$ consisting of a cyclic composition $\mathcal{S} = (E, \rho, M, Q, *)$ over k and an element $e \in M$ satisfying $Q(e) = 0, N_{\mathcal{S}}(e) \neq 0$, where $N_{\mathcal{S}}$ is the cubic form attached to $\mathcal{S}$ in the sense of Lemma 1.3 c). A *homomorphism* $(\varphi, \phi) : (\mathcal{S}, e) \longrightarrow (\mathcal{S}', e')$ of hyperbolic cyclic compositions is a homomorphism $(\varphi, \phi) : \mathcal{S} \to \mathcal{S}'$ of cyclic compositions satisfying $\phi(e) = e'$. The following observation may be found in Springer-Veldkamp [19, 4.5, p. 95].

8.10 Proposition. *Let $(\mathcal{S}, e)$ be a hyperbolic cyclic composition over k and write*

$$\mathcal{S} = (E, \rho, M, Q, *)$$

as usual. Then, putting

$$\gamma = N_{\mathcal{S}}(e)\,,\; e^* = \gamma^{-1}(e * e)\,,$$

(e, e^*) is a hyperbolic pair for the quadratic space (M, Q) such that

$$e * e = \gamma e^*\,,\; e^* * e^* = \gamma^{-1}e\,,\; e * e^* = e^* * e = 0\,.$$

Furthermore,

(8.10.1) $$a = e + e^*$$

satisfies the relations

$$Q(a) = 1\,,\ a * e = \gamma e^*\,,\ a * e^* = \gamma^{-1} e\,,\ a * a = \gamma^{-1} e + \gamma e^*\,.$$

In particular, $(\mathcal{S}, a)$ is a pointed cyclic composition. □

8.11 Hyperbolic cyclic trisotopies. By Proposition 8.10, every hyperbolic cyclic composition may be viewed as a pointed cyclic composition in a natural way, hence connects with a cyclic trisotopy via Theorem 3.2. To make this connection more explicit, we introduce the following terminology. A *hyperbolic* cyclic trisotopy over k is a pair $(\mathcal{A}, e)$ consisting of a cyclic trisotopy $\mathcal{A} = (E, \rho, C, g, b)$ having multiplier 1 and an idempotent $e \in C$, $e \neq 0, 1_C$ such that

(8.11.1) $$g(e) = e\,,\ b = \gamma^{-1} e + \gamma \bar{e}$$

for some $\gamma \in k^\times$. A *homomorphism* $(\varphi, \phi) : (\mathcal{A}, e) \to (\mathcal{A}', e')$ of hyperbolic cyclic trisotopies is a homomorphism $(\varphi, \phi) : \mathcal{A} \to \mathcal{A}'$ of cyclic trisotopies satisfying $\phi(e) = e'$.

The following proposition can now be checked by a straightforward computation.

8.12 Proposition. *Let $(\mathcal{S}, e)$ be a hyperbolic cyclic composition over k and define a as in* (8.10.1). *Then* $\mathrm{Alb}(\mathcal{S}, e) = (\mathrm{Alb}(\mathcal{S}, a), e)$ *is a hyperbolic cyclic trisotopy. Conversely, let $(\mathcal{A}, e)$ be a hyperbolic cyclic trisotopy over k. Then* $\mathrm{Spr}(\mathcal{A}, e) = (\mathcal{A}^0, e)$*, $\mathcal{A}^0$ being defined as in* (3.2.3)*, is a hyperbolic cyclic composition. The assignments* $(\mathcal{S}, e) \mapsto \mathrm{Alb}(\mathcal{S}, e), (\mathcal{A}, e) \mapsto \mathrm{Spr}(\mathcal{A}, e)$ *define inverse isomorphisms between the category of hyperbolic cyclic compositions and the category of hyperbolic cyclic trisotopies.* □

8.13 Theorem. *Every cyclic trisotopy $\mathcal{A}$ of rank > 1 over k contains up to weak isomorphism a cyclic sub-trisotopy of rank* 2*. Moreover, if $\mathcal{A}$ is isotropic, this sub-trisotopy may be so chosen as to be isotropic as well.*

Proof. First suppose $\mathcal{A}$ is reduced. Then Theorem 8.2 implies that up to weak isomorphism we may assume $\mathcal{A} = (E, \rho, C_0 \otimes E, \mathbf{1}_{C_0} \otimes \rho, 1_{C_0} \otimes \rho(w_0)^{-1})$ where (E, ρ) is a cyclic cubic étale k-algebra, C_0 is a composition algebra over k of dimension at least 2 and $w_0 \in E^\times$. Let $L_0 \subseteq C_0$ be a composition subalgebra of dimension 2. Then the restriction $\mathcal{A}|_{L_0 \otimes E} \subseteq \mathcal{A}$ in the sense of 7.1 is a cyclic sub-trisotopy of rank 2. Furthermore, if $\mathcal{A}$ is isotropic, C_0 must be split, which follows from Springer's theorem if E is a field and by direct computation otherwise. Hence one may choose L_0 to be split as well, forcing $\mathcal{A}|_{L_0 \otimes E}$ to be isotropic. We are left with the case that $\mathcal{A} = (E, \rho, C, g, b)$ is not reduced, forcing it to be étale by Corollary 8.3. But then the restriction of $\mathcal{A}$ to its core yields a cyclic sub-composition of rank 2. Finally, suppose $\mathcal{A}$ is isotropic. Then so is $\mathcal{A}^0 = (E, \rho, M, Q, *)$, its associated cyclic composition, and we must show that some hybrid of $\mathcal{A}$ contains an isotropic cyclic sub-trisotopy of rank 2. Following 8.8, we pick a non-zero element $e \in M$ satisfying $Q(e) = 0$. Since $\mathcal{A}$ is not reduced, we conclude $N_{\mathcal{A}^0}(e) = N_{\mathcal{A}}(e) \neq 0$. Thus $(\mathcal{A}^0, e)$ is a hyperbolic cyclic composition, so Proposition 8.12 yields a hyperbolic cyclic trisotopy $\mathrm{Alb}(\mathcal{A}^0, e) = (\mathcal{A}', e), \mathcal{A}' = \mathrm{Alb}(\mathcal{A}^0, a)$ for some invertible element $a \in C$. In particular, by (5.1.2), $\mathcal{A}'$ is a hybrid of $\mathcal{A}$. Inspecting (8.11.1) for $\mathcal{A}'$ in place of $\mathcal{A}$, we see that $L = Ee + E\,\bar{e} \subseteq C$ is a split étale subalgebra containing b and stabilized by g. We may therefore restrict $\mathcal{A}'$ to L and obtain an isotropic cyclic rank 2 sub-trisotopy of $\mathcal{A}'$. □

8.14 Corollary. *Every free cyclic composition $\mathcal{S}$ of rank > 1 over k contains a free cyclic sub-composition of rank 2 which one may choose to be isotropic if $\mathcal{S}$ is isotropic.* $\square$

Remark. The preceding corollary is closely related to [2, Proposition 1.2.5] and [19, Lemma 4.2.12].

9. Quadrupling composition algebras

The technique of quadrupling composition algebras was devised by Thakur [20] to construct octonions over arbitrary commutative rings containing $\frac{1}{2}$. It will be recast here, mostly without proofs, in a slightly more general setting.

9.1 The general set-up. Throughout this section, we let R be a commutative associative ring of scalars and fix a non-zero étale R-algebra L of (constant) rank $r \leq 2$. We write

$$\iota := \iota_L : L \longrightarrow L\,,\; a \longmapsto \overline{a} := \iota(a)\,, \tag{9.1.1}$$

for the conjugation of L (which we agree to be the identity for $r = 1$), and $N_L : L \to R$ for the quadratic form given by the formula

$$a\,\overline{a} = N_L(a)1_L\,. \qquad (a \in L) \tag{9.1.2}$$

If we assume $\frac{1}{2} \in R$ or $r = 2$, L is a composition algebra over R in the sense of [11, 1.4] with norm N_L.

9.2 Hermitian spaces. By a *hermitian space* over L we mean a pair (V, h) such that V is a finitely generated projective *right* L-module of constant rank and $h : V \times V \to L$ is a hermitian form, linear in the second variable, anti-linear in the first, which is *non-singular* in the sense that the assignment $x \mapsto h(x, -)$ defines an L-linear bijection from V onto its twisted dual V^*; here V^* is the ordinary L-module $\mathrm{Hom}_L(V, L)$ with scalar multiplication twisted by ι . In this case, $\mathrm{rank}\,(V, h) = \mathrm{rank}\,V$ is called the *rank* of (V, h). Given a hermitian space (V, h) of rank n over L, $\det(V, h) = \bigwedge^n(V, h)$ is a hermitian space of rank one over L, called the *determinant* of (V, h).

If $T \in \mathrm{GL}_n(L)$ is an invertible hermitian matrix of size n,

$$\langle T\rangle_{\mathrm{sesq}} : L^n \times L^n \longrightarrow L\,,\; (x, y) \longmapsto \langle T\rangle_{\mathrm{sesq}}(x, y) = \overline{x}^{\,t}Ty\,,$$

defines a hermitian space $(L^n, \langle T\rangle_{\mathrm{sesq}})$ of rank n over L and, up to isometry, all hermitian spaces whose underlying L-module is free of rank n have this form.

Now let (V, h) be a hermitian space of rank n and suppose $\Delta : \bigwedge^n V \xrightarrow{\sim} L$ is an (L-linear) isomorphism. By the above, there exists a unique element $\det_\Delta h \in R^\times$, called the Δ-*determinant* of (V, h), such that $\Delta : \bigwedge^n(V, h) \xrightarrow{\sim} (L, \langle \det_\Delta h\rangle_{\mathrm{sesq}})$ is a bijective isometry of hermitian spaces over L. Moreover,

$$\det{}_{a\Delta}\, h = N_L(a)^{-1}\det{}_\Delta\, h \tag{9.2.1}$$

for all $a \in L^\times$.

9.3 The hermitian vector product. Let (V, h) be a hermitian space over L which is *ternary* in the sense that it has rank 3 and suppose $\Delta : \bigwedge^3 V \xrightarrow{\sim} L$ is an isomorphism. By the non-singularity of h, there exists a unique map $V \times V \to V, (x, y) \mapsto x \times_{h,\Delta} y$, such that

$$h(x \times_{h,\Delta} y, z) = \Delta(x \wedge y \wedge z)\,. \qquad (x, y, z \in V)$$

We call $\times_{h,\Delta}$ the *hermitian vector product* induced by h and Δ. It is obviously alternating and anti-linear in both arguments. Moreover, the expression $h(x \times_{h,\Delta} y, z)$ remains unaffected by a cyclic change of variables and vanishes if two of them coincide.

9.4 Example. Consider the free L-module L^3 with standard basis e_1, e_2, e_3, and let $T \in \mathrm{GL}_3(L)$ be any hermitian matrix. Then $(L^3, \langle T\rangle_{\mathrm{sesq}})$ is a ternary hermitian space.

Writing $\Delta_0 : \bigwedge^3 L \xrightarrow{\sim} L$ for the isomorphism determined by the condition

$$\Delta_0(e_1 \wedge e_2 \wedge e_3) = 1\,, \tag{9.4.1}$$

any other isomorphism $\bigwedge^3 L \xrightarrow{\sim} L$ has the from $\Delta = a\Delta_0$ $(a \in L^\times)$. It is then easily checked that the hermitian vector product relative to $\langle T\rangle_{\mathrm{sesq}}$ and Δ may be expressed in terms of the ordinary vector product in 3-space by the formulae

$$x \times_{\langle T\rangle_{\mathrm{sesq}},\Delta} y = \overline{a}\, T^{-1}(\,\overline{x} \times \overline{y}\,) = (\det T)^{-1}\, \overline{a}\,(\,\overline{Tx} \times \overline{Ty})\,. \tag{9.4.2}$$

9.5 Proposition. *Let (V, h) be a ternary hermitian space over L and suppose $\Delta : \bigwedge^3 V \xrightarrow{\sim} L$ is an isomorphism. Then the hermitian vector product induced by h and Δ satisfies the* hermitian Grassmann identity

$$(x \times_{h,\Delta} y) \times_{h,\Delta} z = (\det_\Delta h)^{-1}\big(h(z, x)y - h(z, y)x\big)\,. \qquad (x, y, z \in V) \tag{9.5.1}$$

Proof. The question is local on L, so we may assume that $V = L^3$ is a free L-module of rank 3, $\Delta = a\Delta_0$ for some $a \in L^\times$ and $h = \langle T\rangle_{\mathrm{sesq}}$ for some hermitian matrix $T \in \mathrm{GL}_3(L)$ as in Example 9.4. Then (9.5.1) follows easily by invoking (9.4.2) and the Grassmann identity for the ordinary vector product. □

The next two theorems are due to Thakur [20] for $\frac{1}{2} \in R$ and $r = 2$. In the generality required here, they follow easily from the preceding discussion. We omit the details.

9.6 Theorem. *Under the hypotheses of* 9.1, *assume $\frac{1}{2} \in R$ or $r = 2$, let (V, h) be a ternary hermitian space over L and suppose $\Delta : \bigwedge^3 V \xrightarrow{\sim} L$ is an isomorphism such that $\det_\Delta h = 1$. Then the R-module $L \oplus V$ becomes a composition algebra of constant rank $4r$ over R under the multiplication*

$$(a \oplus x)(b \oplus y) := \big(ab - h(x, y)\big) \oplus (y\,\overline{a} + xb + x \times_{h,\Delta} y)\,. \qquad (a, b \in L, x, y \in V) \tag{9.6.1}$$

Identifying $R \subseteq L$ canonically, this composition algebra, written as $C = \mathrm{Quad}(L; V, h, \Delta)$, has unit element, norm, polarized norm, trace, conjugation given by

$$1_C = 1_L \oplus 0\,,$$

$$N_C(a \oplus x) = N_L(a) + h(x, x)\,, \tag{9.6.2}$$

$$N_C(a \oplus x, b \oplus y) = N_L(a, b) + T_L\big(h(x, y)\big)\,, \tag{9.6.3}$$

$$T_C(a \oplus x) = T_L(a)\,,$$

$$\overline{a \oplus x} = \overline{a} \oplus (-x) \tag{9.6.4}$$

for all $a, b \in L, x, y \in V$. Finally, the map $L \to C, a \mapsto a \oplus 0$, is an embedding of composition algebras, allowing us to view L as a composition subalgebra of C. □

9.7 Theorem. *Let C be a composition algebra of constant rank $4r$ ($r \leq 2$) over R and suppose $L \subseteq C$ is a composition subalgebra of rank r. Then there exist a ternary hermitian space (V, h) over L and an isomorphism $\Delta : \bigwedge^3 V \xrightarrow{\sim} L$ satisfying $\det_\Delta h = 1$ such that the inclusion $L \hookrightarrow C$ extends to an isomorphism $\mathrm{Quad}(L; V, h, \Delta) \xrightarrow{\sim} C$.* □

The construction presented in Theorem 9.6 is clearly functorial in the parameters involved. More precisely, we will present without proof the following elaborate version of a result of Thakur [20, Theorem 2.2].

9.8 Proposition. *Under the hypotheses of 9.1, assume $\frac{1}{2} \in R$ or $r = 2$, let (V_i, h_i) $(i = 1, 2)$ be ternary hermitian spaces over L and suppose $\Delta_i : \bigwedge^3 V_i \xrightarrow{\sim} L$ are isomorphisms such that $\det_{\Delta_i} h_i = 1$. Given any map $\chi : V_1 \to V_2$ and setting*

$$C_i := \mathrm{Quad}(L; V_i, h_i, \Delta_i) = L \oplus V_i\,, \qquad (i = 1, 2)$$

the following statements are equivalent.

(i) $\chi : (V_1, h_1, \Delta_1) \xrightarrow{\sim} (V_2, h_2, \Delta_2)$ *is an* isomorphism , *i.e.,* $\chi : (V_1, h_1) \xrightarrow{\sim} (V_2, h_2)$ *is a bijective isometry satisfying* $\Delta_2 \circ (\bigwedge^3 \chi) = \Delta_1$.

(ii) $\chi : (V_1, h_1) \xrightarrow{\sim} (V_2, h_2)$ *is a bijective isometry satisfying*

$$\chi(x \times_{h_1, \Delta_1} y) = \chi(x) \times_{h_2, \Delta_2} \chi(y)\,. \qquad (x, y \in V)$$

(iii) $\mathbf{1}_L \oplus \chi : C_1 \xrightarrow{\sim} C_2$ *is an isomorphism of composition algebras over R.*

□

10. Quadrupling cyclic trisotopies

We now extend the quadrupling procedure for composition algebras described in the preceding section to the setting of cyclic trisotopies. We show that all cyclic trisotopies of rank 4 and 8 may be obtained in this way and match the result with the Tits process. Before doing so, however, we will explain

10.1 Why Cayley-Dickson doubling doesn't work for cyclic trisotopies. It is a standard fact that all composition algebras over a field can be obtained from composition subalgebras of rank ≤ 2 (which always exist) by a repeated application of the Cayley-Dickson doubling process. We claim that *this fact does not carry over to cyclic trisotopies.* Indeed, let $\mathcal{A} = (E, \rho, C, g, b)$ be a non-reduced cyclic trisotopy of rank 8. (For example, one could start from a first Tits construction Albert division algebra J, pick a cyclic cubic subfield $E \subseteq J$ and a generator ρ of its Galois group, apply [5, (36.12), (38.6)] to find a cyclic composition $\mathcal{S} = (E, \rho, M, Q, *)$ satisfying $J(\mathcal{S}) \cong J$, pick a base point $a \in M$ for $\mathcal{S}$, and put $\mathcal{A} = \mathrm{Alb}(\mathcal{S}, a)$.) By Corollary 8.3, $\mathcal{A}$ is étale, so restricting it to its core $L = \mathrm{Core}(\mathcal{A})$ yields a cyclic sub-trisotopy $\mathcal{A}_0 = \mathcal{A}|_L \subseteq \mathcal{A}$ of rank 2. However, there does not exist a quaternion algebra $L \subseteq B \subseteq C$ stabilized by g and thus giving rise to a cyclic sub-trisotopy $\mathcal{A}_0 \subseteq \mathcal{A}|_B \subseteq \mathcal{A}$ because, thanks to Theorem 8.6, $\mathcal{A}|_B$, being of rank 4, would be reduced and hence so would $\mathcal{A}$, a contradiction.

10.2 Cyclic trisotopies of rank at least 2 revisited. We know from Theorem 8.13 that every cyclic trisotopy of rank > 1 up to weak isomorphism contains a cyclic sub-trisotopy of rank 2. Conversely, let us consider a cyclic trisotopy $\mathcal{A}$ of rank $r \leq 2$ over k, the case $r = 1$ by 3.3 being permitted only for char $k \neq 2$. Up to isomorphism, Corollary 6.3 allows us to assume

$$\mathcal{A} = \big(E, \rho, L_0 \otimes E, \mathbf{1}_{L_0} \otimes \rho, d_0 \otimes \rho(w_0)^{-1}\big)\,,$$

where (E, ρ) is a cyclic cubic étale k-algebra, L_0 is a composition algebra of dimension r over k and $d_0 \in L_0, w_0 \in E$ are invertible elements. We then have the notational and conceptual framework of 6.5 – 6.11 at our disposal. It will be taken for granted from now on. We begin by presenting a number of technical tools, and eventually a criterion that will facilitate the task to construct cyclic trisotopies of rank > 2 later on.

10.3 Determinants of semi-linear maps. Let R be any commutative associative ring of scalars, M an R-module, and suppose $\Delta : \bigwedge^n M \xrightarrow{\sim} R$ is an R-linear bijection. If $\rho \in \mathrm{Aut}(R)$ and $\varphi : M \to M$ is a ρ-semi-linear map, we find a unique element $\det_\Delta \varphi \in R$ such that

$$\Delta \circ \big(\bigwedge^n \varphi\big) = (\det_\Delta \varphi)(\rho \circ \Delta)\,. \tag{10.3.1}$$

We call $\det_\Delta \varphi$ the Δ-*determinant* of φ which, contrary to what one is used to from ordinary determinants, does depend on the choice of Δ. Indeed, for $\varepsilon \in R^\times$,

$$\det_{\varepsilon\Delta} \varphi = \varepsilon\rho(\varepsilon)^{-1}(\det_\Delta \varphi)\,. \tag{10.3.2}$$

Also, for $\sigma \in \mathrm{Aut}(R)$ and a σ-semi-linear map $\psi : M \to M$, the map $\varphi \circ \psi : M \to M$ is $\rho \circ \sigma$-semi-linear, and $\det_\Delta(\varphi \circ \psi) = (\det_\Delta \varphi)\rho(\det_\Delta \psi)$.

The following supplement to Proposition 9.8 may be established by a straightforward computation.

10.4 Proposition. *Let (V, h) be a ternary hermitian space over L, $\Delta : \bigwedge^3 V \xrightarrow{\sim} L$ an L-linear bijection and $\varphi : (V, h) \xrightarrow{\sim} (V, h)$ a bijective σ-semi-linear isometry. Then*

$$\varphi(x \times_{h,\Delta} y) = \overline{\det_\Delta \varphi}^{-1}\big(\varphi(x) \times_{h,\Delta} \varphi(y)\big)\,. \qquad (x, y \in V)$$

□

10.5 Proposition. *Setting $y = \mathrm{diag}\big(u, \rho(u), \rho^2(u)\big) \in H_3(L)^\times$ and viewing L canonically as a quadratic étale E-algebra in the set-up described by* 10.2*, the map*

$$\omega : H(B, \tau) \longrightarrow H_3(L)^{(y)}$$

defined by

$$\omega(x) := \begin{pmatrix} u^{-1}v & u^{-1}\sigma^2(\overline{a}) & \lambda\rho^2(u)^{-1}\sigma(a) \\ u^{-1}\sigma^2(a) & \rho(u)^{-1}\rho(v) & \rho(u)^{-1}\overline{a} \\ \overline{\lambda}\rho^2(u)^{-1}\sigma(\overline{a}) & \rho(u)^{-1}a & \rho^2(u)^{-1}\rho^2(v) \end{pmatrix} \tag{10.5.1}$$

for

$$x = v + u_0'\sigma^{-1}(a)Y + u_0'\sigma(\overline{a})Y^{-1} \in H(B, \tau)\,, \tag{10.5.2}$$

where $v \in E, a \in L$ (cf. (6.6.2)), is a unital embedding of Jordan algebras over k. In particular,

(10.5.3) $$\det \omega(x) = N_E(u)^{-1} N_B(x)\,.$$

Proof. Since ω is k-linear and preserves units, we only have to prove that it preserves norms as well. Since $\det y = N_E(u)$, this will follow once we have shown (10.5.3). To do so, one computes the determinant of the matrix on the right of (10.5.1) by brute force and applies (6.5.8), (6.8.1). □

10.6 Proposition. *Notations and assumptions being as in 10.2, let (V, h) be a ternary hermitian space over L and $\Delta : \bigwedge^3 V \xrightarrow{\sim} L$ an isomorphism that satisfies*

(10.6.1) $$\det_\Delta h = 1\,.$$

Furthermore, put

(10.6.2) $$C = \mathrm{Quad}(L; V, h, \Delta) = L \oplus V, \quad \text{(as right } L\text{-modules)}$$

let $\varphi : V \to V$ be a k-linear bijection and consider the map

(10.6.3) $$g = \sigma \oplus \varphi : C \longrightarrow C\,.$$

Then the following statements are equivalent.

(i) *$\mathcal{A}' = (E, \rho, C, g, b)$ is a cyclic trisotopy over k.*

(ii) *The map g satisfies the relations*

(10.6.4) $$g(xy) = [g(x)b][b^{-1}g(y)]\,, \qquad (x, y \in C)$$

(10.6.5) $$g^3(x) = bxb^{-1}\,. \qquad (x \in C)$$

(iii) *$\varphi : (V, h) \xrightarrow{\sim} (V, h)$ is a σ-semi-linear isometry satisfying*

(10.6.6) $$\det_\Delta \varphi = \lambda\,,\ \varphi^3 = \lambda \mathbf{1}_V\,.$$

In this case, $\mathcal{A}'$ is a cyclic trisotopy of rank $4r$ over k and contains $\mathcal{A}$ as a cyclic sub-trisotopy. Conversely, every cyclic trisotopy of rank $4r$ over k containing $\mathcal{A}$ as a cyclic sub-trisotopy has this form.

Proof. We begin by establishing the following auxiliary assertions.
a) *g satisfies the relation*

(10.6.7) $$g(ra) = [g(r)b][b^{-1}g(a)] \qquad (r \in V, a \in L)$$

if and only if $\varphi : V \to V$ is σ-semi-linear. This follows immediately from (10.6.3) and the property of V being a right L-module.
b) *If* (10.6.7) *holds, so does*

(10.6.8) $$g(as) = [g(a)b][b^{-1}g(s)]\,. \qquad (a \in L, s \in V)$$

This follows immediately from the fact that, by (9.6.4) and (10.6.3), g commutes with the conjugation of C.
c) *g satisfies the relation*

(10.6.9) $$g(rs) = [g(r)b][b^{-1}g(s)] \qquad (r, s \in V)$$

if and only if $\varphi : (V,h) \to (V,h)$ *is a* σ*-semi-linear isometry such that*

$$\det_\Delta \varphi = \lambda\,. \tag{10.6.10}$$

To see this, one simply expands both sides of (10.6.9) by using (9.6.1), 9.3 and Proposition 10.4.
d) g *satisfies the relation*

$$g^3(r) = brb^{-1} \qquad (r \in V) \tag{10.6.11}$$

if and only if $\varphi^3 = \lambda\mathbf{1}_V$. For $r \in V$, we apply (9.6.1) to obtain $brb^{-1} = r\,\bar{b}\,b^{-1} = r\lambda$ (by (6.5.4)), and the assertion follows.

We are now prepared to enter into the proof proper of the proposition.

(i) $\Longrightarrow$ (ii). Obvious.

(ii) $\Longrightarrow$ (iii). (10.6.4) (resp. (10.6.5)) implies (10.6.7), (10.6.9) (resp. (10.6.11)) as special cases. Hence (iii) follows from a), c), d).

(iii) $\Longrightarrow$ (i). $\mathcal{A}'$ trivially satisfies the conditions 2.2 CT1 – 3, so we only have to worry about CT4, CT5. Since φ is σ-semi-linear by (iii), g is ρ-semi-linear by (10.6.3). Since σ stabilizes the line through b, so does g, again by (10.6.3). It therefore remains to prove (2.2.1), (2.2.2). While (2.2.1), being trivially fulfilled for $x = a \in L, y = a' \in L$, is equivalent to (10.6.7) – (10.6.9) by bilinearity, (2.2.2), being trivially fulfilled for $x = a \in L$, is equivalent to (10.6.11) for the same reason. Hence both (2.2.1) and (2.2.2) follow from (iii) and a) – d).

Clearly, if (i) – (iii) hold, $\mathcal{A}'$ has rank $4r$ and contains $\mathcal{A}$ as a cyclic sub-trisotopy. Conversely, let $\mathcal{A}' = (E, \rho, C, g, b)$ be a cyclic trisotopy with these properties. Since $\mathcal{A}$ is a cyclic sub-trisotopy of $\mathcal{A}'$, L is a composition subalgebra of C stabilized by g, and g agrees with σ on L. Then Theorem 9.7 yields a ternary hermitian space (V,h) over L and an isomorphism $\Delta : \bigwedge^3 V \xrightarrow{\sim} L$ satisfying $\det_\Delta h = 1$ such that C identifies with (10.6.2). Furthermore, g, being a ρ-semi-linear isometry relative to N_C by Lemma 2.3 b), stabilizes $V = L^\perp$ and hence has the form indicated in (10.6.3). This completes the proof. □

10.7 The hermitian form of a symmetric element. Notations and assumptions being as in 10.2, we put

$$S := \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{10.7.1}$$

and let $z \in H(B,\tau)$ be a symmetric element of B relative to τ. Writing

$$z = w + u_0'\sigma^{-1}(\bar{c})Y + u_0'\sigma(c)Y^{-1} \tag{10.7.2}$$

for some $w \in E, c \in L$ according to (6.6.2), and e_1, e_2, e_3 for the unit vectors in (column) 3-space over L, on which σ acts componentwise under the same notation, there is a unique hermitian form $h_z : L^3 \times L^3 \to L$ satisfying

$$h_z(e_1, e_1) = u^{-1}w\,,\; h_z(e_2, e_3) = \rho(u)^{-1}c$$

and making $S \circ \sigma : (L^3, h_z) \xrightarrow{\sim} (L^3, h_z)$ a σ-semi-linear isometry. The matrix of h_z relative to e_1, e_2, e_3 agrees with $\omega(z)$ as given by (10.5.1), which implies (in the notation of Example 9.4)

$$\det_{\Delta_0} h_z = N_E(u)^{-1}N_B(z) \tag{10.7.3}$$

by (10.5.3), so (L^3, h_z) is a ternary hermitian space if and only if z is invertible in B.

10.8 Quadrupling cyclic trisotopies by means of admissible pairs. In the set-up of 10.2, let (z, μ) be an admissible pair for (B, τ), so $z \in H(B, \tau), \mu \in L_0$ are invertible elements satisfying $N_B(z) = N_{L_0}(\mu)$. In the notation of Example 9.4 we put

$$\Delta_\mu := b_0^{-1}\mu\Delta_0 : \bigwedge^3 L^3 \xrightarrow{\sim} L \tag{10.8.1}$$

and conclude from (6.5.6), (9.2.1), (10.7.3) that $\det_{\Delta_\mu} h_z = 1$. Applying Theorem 9.6, we now put

$$C_{z,\mu} := \mathrm{Quad}(L; L^3, h_z, \Delta_\mu) = L \oplus L^3 \quad \text{(as right } L\text{-modules)}\,,$$

define $g_S : C \to C$ by

$$g_S = \sigma \oplus \varphi\,, \varphi = S \circ \sigma\,, \tag{10.8.2}$$

and claim *that*

$$\mathcal{A}' := \mathrm{Quad}(\mathcal{A}; z, \mu) := (E, \rho, C_{z,\mu}, g, b)$$

is a cyclic trisotopy of rank $4r$ *over* k *containing* $\mathcal{A}$ *as a cyclic sub-trisotopy.* To see this, it suffices to verify the conditions of Proposition 10.6 (iii). By 10.7 we know that $\varphi : (L^3, h_z) \xrightarrow{\sim} (L^3, h_z)$ is a σ-semi-linear isometry, so it remains to prove $\det_{\Delta_\mu} \varphi = \lambda$, $\varphi^3 = \lambda \mathbf{1}_V$. While the second part is immediate since S commutes with σ on L^3, the first part follows from

$$\begin{aligned}
\det_{\Delta_\mu} \varphi &= \det_{b_0^{-1}\mu\Delta_0}(S \circ \sigma) && \text{(by (10.8.1), (10.8.2))}\\
&= b_0^{-1}\mu\sigma(b_0^{-1}\mu)^{-1}\det_{\Delta_0}(S \circ \sigma) && \text{(by (10.3.2))}\\
&= \det_{\Delta_0}(S \circ \sigma) && \text{(since } b_0, \mu \in L_0 \text{ by (6.5.3))}\\
&= \det S = \lambda\,. && \text{(by (10.7.1))}
\end{aligned}$$

In the remainder of this paper, we wish to prove that, conversely, every cyclic trisotopy of rank $4r$ over k containing $\mathcal{A}$ as a cyclic sub-trisotopy has the form $\mathcal{A}' = \mathrm{Quad}(\mathcal{A}; z, \mu)$ for some admissible pair (z, μ) relative to (B, τ). Moreover, we wish to match $J(\mathcal{A}')$, the associated Jordan algebra of degree 3, with the Tits process algebra $J(B, \tau, z, \mu)$. Up to a point, these objectives can be pursued simultaneously.

10.9 Technical preparations. Returning to the set-up described in 10.2, let $\mathcal{A}'$ be a cyclic trisotopy of rank $4r$ over k containing $\mathcal{A}$ as a cyclic sub-trisotopy. By Proposition 10.6, we may and will assume that $\mathcal{A}'$ has the form described therein.

a) Observing (6.5.8), we put

$$\varphi' := \rho(u_0')\varphi : V \longrightarrow V\,,\; r \longmapsto \varphi'(r) = \varphi(r)\rho(u_0')\,. \tag{10.9.1}$$

Since φ is σ-semi-linear, so is φ', and the relation $N_E(u_0') = 1$ implies $\varphi'^3 = \lambda \mathbf{1}_V$.

b) Now consider the homomorphism

$$\varepsilon_0 : L \longrightarrow \mathrm{End}_{L_0}(V)\,,\; a \longmapsto \varepsilon_0(a) : V \longrightarrow V\,, r \longmapsto \varepsilon_0(a)(r) := ra\,, \tag{10.9.2}$$

of unital L_0-algebras, which connects with φ' through the relation $\varphi' \circ \varepsilon_0(a) = \varepsilon_0\big(\sigma(a)\big) \circ \varphi'$ $(a \in L)$. Combining this with a), we therefore conclude that there exists a unique L_0-homomorphism $\varepsilon : B = (L, \sigma, \lambda) \to \mathrm{End}_{L_0}(V)$ satisfying

$$\varepsilon|_L = \varepsilon_0\,,\; \varepsilon(Y) = \varphi'\,. \tag{10.9.3}$$

c) By means of ε, V may thus be regarded as a left B-module, with the corresponding exterior operation

$$(x, r) \longmapsto x.r := \varepsilon(x)(r)\,. \qquad (x \in B, r \in V) \tag{10.9.4}$$

Since V, as a vector space over k, has dimension $12r - 3r = 9r$ by (10.6.2) and 3.3, and is free as an L_0-module, it is, in fact, a free L_0-module of rank $\frac{9r}{r} = 9$. On the other hand, B is an Azumaya algebra of degree 3 over L_0, forcing $V \cong {}_BB$ as left B-modules. Accordingly, let $\psi : {}_BB \xrightarrow{\sim} V$ be an isomorphism.

d) Setting $e := \psi(1_B) \in V$, it follows easily that

$$\psi(a_0 + a_1 Y + a_{-1} Y^{-1}) = e a_0 + \varphi(e)\rho(u_0')a_1 + \varphi^2(e) u_0'^{-1}\,\overline{\lambda}\, a_{-1}\,, \qquad (a_0, a_{\pm 1} \in L) \tag{10.9.5}$$

which immediately implies that $e \in V$ *is a cyclic vector relative to* φ, i.e.,

$$\big(e, \varphi(e), \varphi^2(e)\big) \text{ is a basis of } V \text{ over } L\,. \tag{10.9.6}$$

e) Next we prove

$$\mu := b_0 \Delta\big(e \wedge \varphi(e) \wedge \varphi^2(e)\big) \in L_0^\times\,. \tag{10.9.7}$$

Since $\Delta\big(e \wedge \varphi(e) \wedge \varphi^2(e)\big)$ by (10.9.6) is a unit in L, we must only show that it remains fixed under σ, which follows from

$$\begin{aligned}
\sigma\Big(\Delta\big(e \wedge \varphi(e) \wedge \varphi^2(e)\big)\Big)\lambda &= (\det\nolimits_\Delta \varphi)(\sigma \circ \Delta)\big(e \wedge \varphi(e) \wedge \varphi^2(e)\big) && \text{(by (10.6.6))} \\
&= (\Delta \circ \bigwedge^3 \varphi)\big(e \wedge \varphi(e) \wedge \varphi^2(e)\big) && \text{(by (10.3.1))} \\
&= \Delta\big(\varphi(e) \wedge \varphi^2(e) \wedge \varphi^3(e)\big) = \lambda\Delta\big(\varphi(e) \wedge \varphi^2(e) \wedge e\big) && \text{(by (10.6.6))} \\
&= \Delta\big(e \wedge \varphi(e) \wedge \varphi^2(e)\big)\lambda\,.
\end{aligned}$$

f) We now put

$$w := uh(e, e) \in E\,,\ c := \rho(u)\sigma\big(h(e, \varphi(e)\big) = \rho(u)h\big(\varphi(e), \varphi^2(e)\big) \in L\,, \tag{10.9.8}$$

$$z := w + u_0'\sigma^{-1}(\overline{c})Y + u_0'\sigma(c)Y^{-1} \in H(B, \tau) \tag{10.9.9}$$

and use the property of $\varphi : (V, h) \xrightarrow{\sim} (V, h)$ being a σ-semi-linear isometry (Proposition 10.6 (iii)) to conclude from (10.9.6), 10.7 that the L-linear bijection $\chi : V \xrightarrow{\sim} L^3$ determined by

$$\chi\big(\varphi^i(e)\big) = e_{i+1}\,. \qquad (i = 0, 1, 2) \tag{10.9.10}$$

is, in fact, a bijective isometry $(V, h) \xrightarrow{\sim} (L^3, h_z)$.

g) On the other hand, writing $\Delta_e : \bigwedge^3 V \xrightarrow{\sim} L$ for the L-isomorphism satisfying

$$\Delta_e\big(e \wedge \varphi(e) \wedge \varphi^2(e)\big) = 1\,, \tag{10.9.11}$$

and observing f), (10.7.3), (6.5.6), we obtain

$$\det\nolimits_{\Delta_e} h = \det\nolimits_{\Delta_0} h_z = N_{L_0}(b_0)^{-1} N_B(z)\,. \tag{10.9.12}$$

h) Comparing (10.9.11) with (10.9.7), we conclude

$$\Delta = b_0^{-1}\mu\Delta_e\,, \tag{10.9.13}$$

which implies

$$\begin{aligned} N_{L_0}(b_0)^{-1}N_B(z) &= \det{}_{\Delta_e} h && \text{(by (10.9.12))} \\ &= \det{}_{(b_0^{-1}\mu)^{-1}\Delta} h = N_{L_0}(b_0^{-1}\mu)\det{}_\Delta h && \text{(by (9.2.1))} \\ &= N_{L_0}(b_0)^{-1}N_{L_0}(\mu)\,, && \text{(by (10.6.1))} \end{aligned}$$

hence $N_B(z) = N_{L_0}(\mu)$, so (z, μ) is an admissible pair for (B, τ).

10.10 Theorem. *Notations and assumptions being as in* 10.2, *let* $\mathcal{A}'$ *be a cyclic trisotopy of rank* $4r$ *over* k *containing* $\mathcal{A}$ *as a cyclic sub-trisotopy. Then there exists an admissible pair* (z, μ) *for* (B, τ) *such that the inclusion* $\mathcal{A} \hookrightarrow \mathrm{Quad}(\mathcal{A}; z, \mu)$ *extends to an isomorphism* $\mathcal{A}' \xrightarrow{\sim} \mathrm{Quad}(\mathcal{A}; z, \mu)$.

Proof. From (9.4.1), (10.9.10), (10.9.11), (10.9.13), (10.8.1) we conclude that $\chi : (V, h, \Delta) \xrightarrow{\sim} (L^3, h_z, \Delta_\mu)$ is an isomorphism in the sense of Proposition 9.8 (i), consequently giving rise to an isomorphism

(10.10.1) $$\phi := \mathbf{1}_L \oplus \chi : C = \mathrm{Quad}(L; V, h, \Delta) \xrightarrow{\sim} C_{z,\mu} = \mathrm{Quad}(L; L^3, h_z, \Delta_\mu)\,,$$

of composition algebras over E, which in turn is easily seen to induce an isomorphism

$$\Phi := (\mathbf{1}_E, \phi) : \mathcal{A}' \xrightarrow{\sim} \mathrm{Quad}(\mathcal{A}; z, \mu)$$

of cyclic trisotopies extending the identity of $\mathcal{A}$. □

10.11 Proof of Theorem 1.8 c). Let $\mathcal{S}$ be a free cyclic composition of rank 8 over k. By Theorems 8.13, 10.10, there is a cyclic trisotopy corresponding to $\mathcal{S}$ via Theorem 5.4 and having the form $\mathcal{A}' = \mathrm{Quad}(\mathcal{A}; z, \mu)$, where $\mathcal{A}$ satisfies 10.2 and (z, μ) is an admissible pair for (B, τ). Since $b_0^{-1}\mu$ belongs to L_0, 10.8 yields a basis (q_1, q_2, q_3) of L^3 over L that remains fixed under σ and satisfies $\Delta_\mu(q_1 \wedge q_2 \wedge q_3) = 1$. Converting S and h_z into matrices P and T, respectively, relative to this new basis, the assertion follows from Theorems 3.2, 9.6 and (9.4.2) by a straightforward computation. □

10.12 Some identifications. We return to the set-up described in 10.9 and deduce from 4.2, 4.6 that

(10.12.1) $$J(\mathcal{A}) = E \oplus L\,,$$

(10.12.2) $$J(\mathcal{A}') = E \oplus C = E \oplus L \oplus V = J(\mathcal{A}) \oplus V$$

as vector spaces over k. Using this, we obtain

(10.12.3) $$T_{J(\mathcal{A}')}(v \oplus a \oplus r, v' \oplus a' \oplus r') = T_E(v, v') + T_E\big(uN_{L_0}(a, a')\big) + T_E\Big(uT_{L_0}\big(h(r, r')\big)\Big)$$

for all $v, v' \in E, a, a' \in L, r, r' \in V$ from (4.2.3), (3.2.1), (9.6.3). This immediately implies

(10.12.4) $$V = J(\mathcal{A})^\perp$$

relative to the generic trace if $J(\mathcal{A})$ and V are canonically identified in $J(\mathcal{A}')$.

10.13 Theorem. *Notations and assumptions being as in* 10.9, 10.12, *the map*

$$\Theta := \theta \oplus \psi : J(B,\tau,z,\mu) \xrightarrow{\sim} J(\mathcal{A}')$$

given by

$$\Theta(x_0 \oplus x) = \theta(x_0) \oplus \psi(x) \qquad (x_0 \in H(B,\tau)\,,\, x \in B)$$

is an isomorphism of Jordan algebras extending θ.

Proof. Since Θ trivially extends θ and hence preserves units, it remains to show that it preserves norms as well. This will be accomplished in several steps.

1^0. We consider arbitrary elements

$$x = a_0 + a_1Y + a_{-1}Y^{-1}\,,\; x' = a_0' + a_1'Y + a_{-1}'Y^{-1} \in B \tag{10.13.1}$$

for $a_0, a_0', a_{\pm1}, a_{\pm1}' \in L$ and

$$x_0 = v + u_0'\sigma^{-1}(a)Y + u_0'\sigma(\overline{a})Y^{-1}\,,\; x_0' = v' + u_0'\sigma^{-1}(a')Y + u_0'\sigma(\overline{a'})Y^{-1} \in H(B,\tau) \tag{10.13.2}$$

for $v, v' \in E, a, a' \in L$. *Setting* $J := J(B,\tau,z,\mu)$, *we must show*

$$N_{J(\mathcal{A}')} \circ \Theta(x_0 \oplus x) = N_J(x_0 \oplus x)\,. \tag{10.13.3}$$

2^0. The right-hand side of (10.13.3) by (6.9.1) attains the form

$$N_J(x_0 \oplus x) = N_B(x_0) + \mu N_B(x) + \overline{\mu N_B(x)} - T_B\big(\tau(x)x_0x, z\big)\,. \tag{10.13.4}$$

On the other hand, expanding the left-hand side of (10.13.3), and observing (10.12.1), (10.12.2), we obtain

$$\begin{aligned} N_{J(\mathcal{A}')} \circ \Theta(x_0 \oplus x)\big) =& N_{J(\mathcal{A})}\big(\theta(x_0)\big) + T_{J(\mathcal{A}')}\Big(\big(\theta(x_0) \oplus 0\big)^\sharp, 0 \oplus 0 \oplus \psi(x)\Big) + \\ & T_{J(\mathcal{A}')}\Big(\theta(x_0) \oplus 0, \big(0 \oplus 0 \oplus \psi(x)\big)^\sharp\Big) + N_{J(\mathcal{A}')}\big(0 \oplus 0 \oplus \psi(x)\big)\,. \end{aligned}$$

Here the first summand agrees with $N_B(x_0)$ by Theorem 6.11, while (10.12.4) implies that the second summand vanishes since $J(\mathcal{A})$, being a subalgebra of $J(\mathcal{A}')$, is stabilized by the adjoint map. Thus

$$\begin{aligned} N_{J(\mathcal{A}')} \circ \Theta(x_0 \oplus x) =& N_B(x_0) + T_{J(\mathcal{A}')}\Big(\theta(x_0) \oplus 0, \big(0 \oplus 0 \oplus \psi(x)\big)^\sharp\Big) + \\ & N_{J(\mathcal{A}')}\big(0 \oplus 0 \oplus \psi(x)\big)\,. \end{aligned} \tag{10.13.5}$$

3^0. Our next aim will be to match the last summand of (10.13.5) with the two mid-terms of (10.13.4). Before doing so, we will have to prove

$$\varphi^2(r)[\varphi(r)r] \equiv \overline{\Delta\big(r \wedge \varphi(r) \wedge \varphi^2(r)\big)} \bmod V\,, \qquad (r \in V) \tag{10.13.6}$$

$$\Delta\big(\psi(x) \wedge \varphi\psi(x) \wedge \varphi^2\psi(x)\big) = b_0^{-1}\mu N_B(x)\,. \tag{10.13.7}$$

(10.13.6) follows easily from (9.6.1) and standard properties of the hermitian vector product (cf. 9.3); we omit the details. (10.13.7) is more troublesome. First we combine (10.9.5) with (10.13.1) and the σ-semi-linearity of φ to obtain

$$\begin{aligned} \psi(x) &= ea_0 + \varphi(e)\rho(u_0')a_1 + \varphi^2(e)u_0'^{-1}\overline{\lambda}\,a_{-1}\,, \\ \varphi\psi(x) &= e\rho(u_0')^{-1}\sigma(a_{-1}) + \varphi(e)\sigma(a_0) + \varphi^2(e)\rho^2(u_0')\sigma(a_1) \qquad \text{(by (6.5.5), (10.6.6))} \\ \varphi^2\psi(s) &= eu_0'\lambda\sigma^2(a_1) + \varphi(e)\rho^2(u_0')^{-1}\sigma^2(a_{-1}) + \varphi^2(e)\sigma^2(a_0)\,. \end{aligned}$$

Using (6.5.5) and $N_E(u_0') = 1$ (by (6.5.8)), this implies

$$\begin{aligned}
&\Delta\big(\psi(x) \wedge \varphi\psi(x) \wedge \varphi^2\psi(x)\big) = \\
&\qquad = \det \begin{pmatrix} a_0 & \rho(u_0')a_1 & \overline{\lambda}\, u_0'^{-1}a_{-1} \\ \rho(u_0')^{-1}\sigma(a_{-1}) & \sigma(a_0) & \rho^2(u_0')\sigma(a_1) \\ \lambda u_0'\sigma^2(a_1) & \rho^2(u_0')^{-1}\sigma^2(a_{-1}) & \sigma^2(a_0) \end{pmatrix} \Delta\big(e \wedge \varphi(e) \wedge \varphi^2(e)\big) \\
&\qquad = [N_E(a_0) + \lambda N_E(a_1) + \lambda^{-1}N_E(a_{-1}) - \\
&\qquad\quad T_E\big(a_0\sigma(a_1)\sigma^{-1}(a_{-1})\big)]\Delta\big(e \wedge \varphi(e) \wedge \varphi^2(e)\big) \\
&\qquad = N_B(x)b_0^{-1}\mu
\end{aligned}$$

by (6.7.1), (10.9.7), as claimed. We can now compute,

$$\begin{aligned}
N_{J(\mathcal{A}')}\big(0 \oplus 0 \oplus \psi(x)\big) &= N_{\mathcal{A}'}\big(0 \oplus \psi(x)\big) && \text{(by (4.2.1), (4.6.1))} \\
&= N_C\big(\varphi^2\psi(x)[\varphi\psi(x)\psi(x)], b_0\big) && \text{(by (4.6.2), (10.6.3))} \\
&= N_C\Big(\overline{\Delta\big(\psi(x) \wedge \varphi\psi(x) \wedge \varphi^2\psi(x)\big)}\,, b_0\Big) && \text{(by (10.13.6))} \\
&= T_C\Big(b_0\Delta\big(\psi(x) \wedge \varphi\psi(x) \wedge \varphi^2\psi(x)\big)\Big) && \text{(by (3.2.5))} \\
&= T_C\big(\mu N_B(x)\big) && \text{(by (10.13.7))} \\
&= \mu N_B(x) + \overline{\mu N_B(x)}\,.
\end{aligned}$$

Inserting this into (10.13.5) and comparing with (10.13.4), we see that (10.13.3) and hence *our theorem will follow once we have shown*

$$T_{J(\mathcal{A}')}\Big(\theta(x_0) \oplus 0, \big(0 \oplus 0 \oplus \psi(x)\big)^\sharp\Big) = -T_B\big(\tau(x)x_0x, z\big)\,. \tag{10.13.8}$$

4^0. We wish to make the left-hand side of (10.13.8) more explicit. To this end, we put

$$H_1(x, x') := h\big(\psi(x), \psi(x')\big)\,,\ H_2(x, x') := h\big(\psi(x), \varphi\psi(x')\big) \tag{10.13.9}$$

and write $\mathcal{A}'^0 = (E, \rho, M, Q, *)$ for the cyclic composition corresponding to $\mathcal{A}'$ to establish the relations

$$Q\big(0 \oplus \psi(x)\big) = uH_1(x, x)\,, \tag{10.13.10}$$

$$\big(0 \oplus \psi(x)\big) * \big(0 \oplus \psi(x)\big) \equiv -\Big(u^{-1}b_0\sigma\big(H_2(x, x)\big) \oplus 0\Big) \bmod V\,. \tag{10.13.11}$$

The first one of these follows immediately from (3.2.1), (9.6.2), (10.13.9). For the second one, we identify $V \subseteq C$ canonically and compute

$$\begin{aligned}
\big(0 \oplus \psi(x)\big) * \big(0 \oplus \psi(x)\big) &= \psi(x) * \psi(x) \\
&= u_0 g\big(\overline{\psi(x)}\big)[g^2\big(\overline{\psi(x)}\big)b] && \text{(by (3.2.2))} \\
&= u_0\varphi\big(\psi(x)\big)[\varphi^2\big(\psi(x)\big)b] && \text{(since } \overline{\psi(x)} = -\psi(x)) \\
&= \rho(u)^{-1}u_0\varphi\big(\psi(x)\big)[\varphi^2\big(\psi(x)\big)b_0] && \text{(by (2.5.1))} \\
&= u^{-1}\varphi\big(\psi(x)\big)\varphi^2\big(\psi(x)b_0\big) && \text{(by (2.5.1), } \sigma(b_0) = b_0) \\
&\equiv -u^{-1}h\Big(\varphi\big(\psi(x)\big), \varphi^2\big(\psi(x)b_0\big)\Big) \bmod V && \text{(by (9.6.1))} \\
&\equiv -u^{-1}\sigma\Big(h\big(\psi(x), \varphi(\psi(x)b_0)\big)\Big) \bmod V && \text{(by Proposition 10.6 (iii))} \\
&\equiv -u^{-1}b_0\sigma\Big(h\big(\psi(x), \varphi\psi(x)\big)\Big) \bmod V\,,
\end{aligned}$$

and the identifications made together with (10.13.9) yield (10.13.11). We can now treat the left-hand side of (10.13.8) by using (10.13.10), (10.13.11) in the following way:

$$
\begin{aligned}
&T_{J(\mathcal{A}')}\Big(\theta(x_0)\oplus 0, \big(0\oplus 0\oplus\psi(x)\big)^{\sharp}\Big)\\
&= T_{J(\mathcal{A}')}\Big(\theta(x_0)\oplus 0, -Q\big(0\oplus\psi(x)\big)\oplus[(0\oplus\psi(x))*(0\oplus\psi(x))]\Big) && \text{(by (4.2.2))}\\
&= -\,T_{J(\mathcal{A}')}\Big(v\oplus\rho^{-1}(u)\,\overline{b_0}^{\,-1}\,\overline{a}\,, uH_1(x,x)\oplus u^{-1}b_0\sigma(H_2(x,x))\Big) && \text{(by (6.11.2), (10.12.4))}\\
&= -\,T_E\big(v, uH_1(x,x)\big)-\\
&\quad -T_E\Big(Q\big(\rho^{-1}(u)\,\overline{b_0}^{\,-1}\overline{a}, u^{-1}b_0\sigma\big(H_2(x,x)\big)\big)\Big)\,, && \text{(by (4.2.3))}
\end{aligned}
$$

where the second summand agrees with

$$
\begin{aligned}
&-\,T_E\Big(\rho^{-1}(u)N_C\big(\overline{b_0^{-1}a}, b_0\sigma(H_2(x,x))\big)\Big) && \text{(by (3.2.1))}\\
&= -\,T_E\Big(\rho^{-1}(u)T_C\big(b_0^{-1}ab_0\sigma(H_2(x,x))\big)\Big) && \text{(by (3.2.5))}\\
&= -\,T_E\Big(\rho^{-1}(u)a\sigma\big(H_2(x,x)\big)\Big)-\overline{T_E\Big(\rho^{-1}(u)a\sigma\big(H_2(x,x)\big)\Big)}\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
T_{J(\mathcal{A}')}\Big(\theta(x_0)\oplus 0, \big(0\oplus 0\oplus\psi(x)\big)^{\sharp}\Big) = &-T_E\big(v, uH_1(x,x)\big)-T_E\Big(\rho^{-1}(u)a\sigma\big(H_2(x,x)\big)\Big)\\
&-\overline{T_E\Big(\rho^{-1}(u)a\sigma\big(H_2(x,x)\big)\Big)}\,,
\end{aligned}
$$

and, comparing with (10.13.8), we see that, *in order to complete the proof of the theorem, we are reduced to showing*

(10.13.12)
$$
\begin{aligned}
T_B\big(\tau(x)x_0x, z\big) = T_E\big(v, uH_1(x,x)\big)+&\overline{T_E\Big(\rho^{-1}(u)a\sigma\big(H_2(x,x)\big)\Big)}\\
+&\overline{T_E\Big(\rho^{-1}(u)a\sigma\big(H_2(x,x)\big)\Big)}\,.
\end{aligned}
$$

5^0. We claim *that* (10.13.12) *will follow once we have shown*

(10.13.13)
$$T_B\big(\tau(x)p\,x', z\big) = T_E\big(p, uH_1(x,x')\big)\,. \qquad (p\in L)$$

Suppose this has been done. Since $\psi:\ {}_BB\xrightarrow{\sim}V$ is an isomorphism of left B-modules by 10.9 c), we obtain, for $d\in L$,

$$
\begin{aligned}
\psi(dx) = d.\psi(x) &= \varepsilon(d)\big(\psi(x)\big) && \text{(by (10.9.4))}\\
&= \varepsilon_0(d)\big(\psi(x)\big)\,, && \text{(by (10.9.3))}
\end{aligned}
$$

and (10.9.2) yields

(10.13.14)
$$\psi(dx) = \psi(x)d\,. \qquad (d\in L)$$

Similarly, $\psi(Yx') = Y.\psi(x') = \varepsilon(Y)(\psi(x')) = \varphi'(\psi(x')) = \varphi(\psi(x'))\rho(u_0')$, which implies

$$\varphi\psi(x') = \psi(Yx')\rho(u_0')^{-1} = \psi(Yx')\rho^{-1}(u_0) \qquad \text{(by (6.5.8))}$$

and hence

$$\varphi\psi(x') = \psi\big(\rho^{-1}(u_0)Yx'\big) \tag{10.13.15}$$

by (10.13.14). Using the preceding relations, we now compute

$$\begin{aligned}
T_E\big(\rho(u)aH_2(x,x')\big) &= T_E\Big(\rho(u)ah\big(\psi(x),\varphi\psi(x')\big)\Big) && \text{(by (10.13.9))}\\
&= T_E\Big(uh\big(\psi(x)\rho(u)u^{-1}\,\overline{a}\,,\psi(\rho^{-1}(u_0)Yx')\big)\Big) && \text{(by (10.13.15))}\\
&= T_E\Big(uh\big(\psi(u_0\overline{a}x),\psi(\rho^{-1}(u_0)Yx')\big)\Big) && \text{(by (10.13.14), (2.5.1))}\\
&= T_E\Big(uH_1\big(u_0\overline{a}x,\rho^{-1}(u_0)Yx'\big)\Big) && \text{(by (10.13.9))}\\
&= T_B\big(\tau(u_0\overline{a}x)\rho^{-1}(u_0)Yx',z\big) && \text{(by (10.13.13) for } p=1)\\
&= T_B\big(\tau(x)au_0\rho^{-1}(u_0)Yx',z\big) &&\\
&= T_B\big(\tau(x)\rho(u_0)^{-1}aYx',z\big)\,, && \text{(since } N_E(u_0)=1)
\end{aligned}$$

and (6.5.8) implies

$$T_B\big(\tau(x)[u_0'aY]x',z\big) = T_E\big(\rho(u)aH_2(x,x')\big)\,. \tag{10.13.16}$$

Returning to (10.13.12), we now obtain

$$\begin{aligned}
T_B\big(\tau(x)x_0x,z\big) =\,& T_B\big(\tau(x)vx,z\big) + T_B\big(\tau(x)[u_0'\sigma^{-1}(a)Y]x,z\big)+\\
& T_B\big(\tau(x)\tau(u_0'\sigma^{-1}(a)Y)x,z\big) && \text{(by (10.13.2), (6.6.1))}\\
=\,& T_B\big(\tau(x)vx,z\big) + T_B\big(\tau(x)[u_0'\sigma^{-1}(a)Y]x,z\big)+\\
& \overline{T_B\big(\tau(x)[u_0'\sigma^{-1}(a)Y]x,z\big)} && \text{(since } \tau(z)=z)\\
=\,& T_E\big(v,uH_1(x,x)\big) + T_E\big(\rho(u)\sigma^{-1}(a)H_2(x,x)\big)+\\
& \overline{T_E\big(\rho(u)\sigma^{-1}(a)H_2(x,x)\big)} && \text{(by (10.13.13), (10.13.16))}\\
=\,& T_E\big(v,uH_1(x,x)\big) + T_E\Big(\rho^{-1}(u)a\sigma\big(H_2(x,x)\big)\Big)+\\
& \overline{T_E\Big(\rho^{-1}(u)a\sigma\big(H_2(x,x)\big)\Big)}\,,
\end{aligned}$$

which proves (10.13.12), hence our claim.

6^0. We now turn to the proof of (10.13.13). It is straightforward to check that, *if* (10.13.13) *holds for* (x,x') *and all* $p\in L$, *so it does for* (x',x) *as well as* $(dx,d'x')\,(d,d'\in L)$ *and all* $p\in L$. Therefore, all we have to do is check (10.13.13) in the six cases

$$(x,x') = (1,1),(1,Y),(1,Y^{-1}),(Y,Y),(Y,Y^{-1}),(Y^{-1},Y^{-1})\,.$$

We only treat the case $(x,x')=(Y,Y^{-1})$ since the others can be handled in a similar manner. On the one hand, we obtain

$$\begin{aligned}
T_E\big(p, uH_1(Y,Y^{-1})\big) &= T_E\Big(p, uh\big(\psi(Y), \psi(Y^{-1})\big)\Big) && \text{(by (10.13.9))}\\
&= T_E\Big(p, uh\big(\varphi(e)\rho(u_0'), \varphi^2(e){u_0'}^{-1}\,\overline{\lambda}\big)\Big) && \text{(by (10.9.5))}\\
&= T_E\Big(p, u\rho(u_0'){u_0'}^{-1}\,\overline{\lambda}\,h\big(\varphi(e), \varphi^2(e)\big)\Big) &&\\
&= T_E\big(p, uu^{-1}\rho^2(u)\rho^2(u)\rho(u)^{-1}\,\overline{\lambda}\,\rho(u)^{-1}c\big) && \text{(by (6.5.8), (10.9.8))}\\
&= T_E\Big(p, \big(\rho^2(u)\rho(u)^{-1}\big)^2\overline{\lambda}c\Big) &&\\
&= T_E\big(p, \rho(u_0)^2\overline{\lambda}c\big)\,, && \text{(by (2.5.1))}
\end{aligned}$$

while on the other

$$\begin{aligned}
T_B\big(\tau(Y)pY^{-1}, z\big) &= T_B(u_0u_0'Y^{-1}pY^{-1}, z) && \text{(by (6.5.9))}\\
&= T_B\big(u_0u_0'\sigma^2(p)Y^{-2}, z\big) = T_B\big(u_0u_0'\,\overline{\lambda}\,\sigma^2(p)Y, z\big) && \text{(by (6.5.7))}\\
&= T_E\big(u_0u_0'\overline{\lambda}\sigma^2(p), \rho(u_0')\sigma^2(c)\big) && \text{(by (6.7.2), (10.9.9))}\\
&= T_E\big(p, \rho(u_0)\rho(u_0')\rho^2(u_0')\overline{\lambda}c\big) = T_E\big(p, \rho(u_0){u_0'}^{-1}\overline{\lambda}c\big) && \text{(since } N_E(u_0') = 1)\\
&= T_E\big(p, \rho(u_0)^2\,\overline{\lambda}\,c\big)\,, && \text{(by (6.5.8))}
\end{aligned}$$

and comparing we end up with (10.13.13), which completes the proof of the theorem. □

Remark. For an analogue of Theorem 10.13 where B is split over L_0 and $J(\mathcal{A}')$ is replaced by an appropriate Jordan matrix algebra, see [10, Theorem 1.1]. The connection between first Tits constructions and isotropic cyclic compositions of rank 8 is worked out in [19, 8.5]. Finally, [2, Chapter 5] describes the connection between arbitrary twisted compositions of rank 8 and the Tits process but relies on descent and has to distinguish between the isotropic and the non-isotropic case. All the results mentioned exclude characteristics 2 and 3.

Acknowledgements

A preliminary version of this material was presented to the non-associative algebra seminar of the University of Ottawa in September-October 2003. The author is grateful to its participants, most notably to S. Krutelevič, O.Loos, E. Neher, M.L. Racine, for useful comments and stimulating discussions. The author is particularly grateful to O. Loos, who went through the paper with great care and made many helpfull suggestions; in particular, 5.5 is due to him. Thanks are also due to the Department of Mathematics and Statistics of the University of Ottawa for its hospitality. Finally, the author would like to thank U. Stenger for many enlightening conversations on the subject.

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