TRIPLE DERIVATIONS ON VON NEUMANN ALGEBRAS

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ABSTRACT. It is well known that every derivation of a von Neumann algebra into itself is an inner derivation and that every derivation of a von Neumann algebra into its predual is inner. It is less well known that every triple derivation (defined below) of a von Neumann algebra into itself is an inner triple derivation.

We examine to what extent all triple derivations of a von Neumann algebra into its predual are inner. This rarely happens but it comes close. We prove a (triple) cohomological characterization of finite factors and a zero-one law for factors.

1. INTRODUCTION

Building on earlier work of Kadison [11], Sakai [15] proved that every derivation of a von Neumann algebra into itself is inner. Building on earlier work of Bunce and Paschke [2], Haagerup, on his way to proving that every C*-algebra is weakly amenable, showed in [5] that every derivation of a von Neumann algebra into its predual is inner. Thus the first Hochschild cohomology groups $H^1(M, M)$ and $H^1(M, M_*)$ vanish for any von Neumann algebra M.

It is also known that every triple derivation of a von Neumann algebra into itself is an inner triple derivation ([9, Theorem 2]), and every Jordan derivation of a von Neumann algebra into itself is an inner Jordan derivation [18, Theorem 3.10]. In [14], a study was made of automatic continuity of triple derivations from a Banach Jordan triple system into a Banach Jordan triple module, and in [10], the study of ternary weak amenability in operator algebras and triples was initiated. Triple derivations and inner triple derivations into a triple module are defined and discussed in [10] and in [14].[1] However, this paper is essentially independent of [10] and [14] as we shall only be concerned with these concepts for von Neumann algebras, with their associated Jordan structure.

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[1]In this paper, we use the terms 'triple' and 'ternary' interchangeably, while mindful that in some quarters 'triple' means 'Jordan triple' and 'ternary' refers to an associative triple setting, such as a TRO (ternary ring of operators).

Among other things, it was shown in [10] that every commutative (real or complex) C*-algebra A is **ternary weakly amenable**, that is, every triple derivation of A into its dual A^* is an inner triple derivation, but the C*-algebras $K(H)$ of all compact operators and $B(H)$ of all bounded operators on an infinite dimensional Hilbert space H do not have this property.

Two consequences of [10] are that finite dimensional von Neumann algebras and abelian von Neumann algebras have the property that every triple derivation into the predual is an inner triple derivation, analogous to the Haagerup result. We call this property **normal ternary weak amenability**, and we show that it rarely holds in a general von Neumann algebra, but that it comes close.

Our main results are Theorems 1 and 2. If the set of inner triple derivations from a von Neumann algebra M into its predual is norm dense in the real vector space of all triple derivations, then M must be finite (Theorem 1(a)). The converse holds if M acts on a separable Hilbert space, or is a factor (Theorem 1(b)). For all infinite factors, the norm closure of the set of all inner triple derivations into the predual has codimension 1 in the space of all triple derivations (Theorem 2), thus providing a cohomological characterization of finite factors and a zero-one law for factors.

Inner triple derivations on a von Neumann algebra M into its predual M_* are closely related to the span of commutators of normal functionals with elements of M, denoted by $[M_*, M]$. As noted above, for a finite factor of type I_n, all triple derivations into the predual are inner triple derivations. A consequence of Proposition 4.1 and the work of Dykema and Kalton [3] is that no factor of type II_1 is normally ternary weakly amenable (Corollary 4.3), even though each triple derivation into the predual is a norm limit of inner triple derivations by Theorem 1.

We also show that a finite countably decomposable von Neumann algebra M of type I_n is normally ternary weakly amenable if and only if every element of M_* of central trace zero belongs to $[M_*, M]$. In particular, for such an algebra M with a faithful normal finite trace, if every element of M_* of trace zero belongs to $[M_*, M]$ then M is normally ternary weakly amenable.

2. TRIPLE MODULES AND TRIPLE DERIVATIONS

2.1. **Varieties of modules.** Let A be an associative algebra. Let us recall that an A-**bimodule** is a vector space X, equipped with two bilinear products $(a, x) \mapsto ax$ and $(a, x) \mapsto xa$ from $A \times X$ to X satisfying the following

axioms:

$$a(bx) = (ab)x, \quad a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$. In the case where A is a Banach algebra and X is a Banach space which is also an A-bimodule, X is called a **Banach A-bimodule** if there is a constant C such that $\|ax\|, \|xa\| \leq C\|a\|\|x\|$ for every $a \in A$ and $x \in X$.

Let A be a Jordan algebra, that is, a commutative algebra with product denoted by $\circ$ satisfying $a^2 \circ (a \circ b) = a \circ (a^2 \circ b)$. A **Jordan A-module** is a vector space X, equipped with two bilinear products $(a, x) \mapsto a \circ x$ and $(a, x) \mapsto x \circ a$ from $A \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in A$ and $x \in X$. In the case where A is a Jordan Banach algebra and X is a Banach space which is also a Jordan A-module, X is called a **Jordan Banach A-bimodule** if there is a constant C such that $\|a \circ x\| \leq C\|a\|\|x\|$ for every $a \in A$ and $x \in X$.

In particular, any von Neumann algebra M is a Jordan algebra under the product $a \circ b = (ab + ba)/2$ and its predual M_* is an M-bimodule under the actions

$$a\varphi(x) = \varphi(xa) \ , \ \varphi a(x) = \varphi(ax), \text{ for } \varphi \in M_*, a, x \in M.$$

M_* is also a Jordan M-bimodule under the actions

$$a \circ \varphi(x) = \varphi(x \circ a) = \varphi \circ a(x), \text{ for } \varphi \in M_*, a, x \in M.$$

A complex (resp., real) **Jordan triple** is a complex (resp., real) vector space E equipped with a triple product

$$E \times E \times E \to E \qquad (xyz) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one and satisfying the so-called *"Jordan Identity"*:

$$D(a, b)D(x, y) - D(x, y)D(a, b) = D(D(a, b)x, y) - D(x, D(b, a)y),$$

for all a, b, x, y in E, where $D(x, y)z := \{x, y, z\}$.

A von Neumann algebra M is a complex Jordan triple under the triple product $\{abc\} = (ab^*c + cb^*a)/2$ and its predual M_* is a Jordan triple M-module, according to the general definition[2] in [14] and [10], if we define module actions as follows:

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\} \tag{2.1}$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}}, \tag{2.2}$$

for every $x, a, b \in M, \varphi \in M_*$. In this case, it is a tedious exercise to show that the identity

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

which is equivalent to the Jordan identity above, holds whenever exactly one of the elements belongs to M_*.

2.2. **Varieties of derivations.** Let X be a Banach A-bimodule over an (associative) Banach algebra A. A linear mapping $D : A \to X$ is said to be a **derivation** if $D(ab) = D(a)b + aD(b)$, for every a, b in A. For emphasis we call this a **binary (or associative) derivation**. We denote the set of all continuous binary derivations from A to X by $\mathcal{D}_b(A, X)$.

When X is a Jordan Banach module over a Jordan Banach algebra A, a linear mapping $D : A \to X$ is said to be a **derivation** if $D(a \circ b) = D(a) \circ b + a \circ D(b)$, for every a, b in A. For emphasis we call this a **Jordan derivation**. We denote the set of continuous Jordan derivations from A to X by $\mathcal{D}_J(A, X)$.

In the setting of Jordan Banach triples, a **triple** or **ternary derivation** from a (real or complex) Jordan Banach triple, E, into a Banach triple E-module, X, is a *conjugate* linear mapping $\delta : E \to X$ satisfying

$$\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}, \tag{2.3}$$

for every a, b, c in E. We denote the set of all continuous ternary derivations from E to X by $\mathcal{D}_t(E, X)$.

The conjugate linearity (as opposed to linearity) of ternary derivations from a complex Jordan triple into a Jordan triple module is a reflection of the fact that a complex Jordan triple is not necessarily a Jordan triple

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[2]We do not need the general definition since we are only interested in the special case considered here of a von Neumann algebra and its Jordan structure. It is worth noting however that the triple module actions are linear in some variables and conjugate-linear in other variables. For example, (2.2) is conjugate linear in a, b and φ.

module over itself (as explained in [14] and [10]). This anomaly has no effect on our results.

Let X be a Banach A-bimodule over an associative Banach algebra A. Given x_0 in X, the mapping $D_{x_0} : A \to X$, $D_{x_0}(a) = x_0 a - a x_0$ is a bounded (associative or binary) derivation. Derivations of this form are called **inner**. We shall use the customary notation $\mathrm{ad}\, x_0$ for these inner derivations. The set of all inner derivations from A to X will be denoted by $\mathcal{I}nn_b(A, X)$.

When x_0 is an element in a Jordan Banach A-module, X, over a Jordan Banach algebra, A, for each $b \in A$, the mapping

$$\delta_{x_0,b} = L(x_0)L(b) - L(b)L(x_0) : A \to X,$$

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0, \ \ (a \in A),$$

is a bounded derivation. Here $L(x_0)$ (resp. $L(b)$) denotes the module action $a \mapsto x_0 \circ a$ (resp. multiplication $a \mapsto b \circ a$). Finite sums of derivations of this form are called **inner**. The set of all inner Jordan derivations from A to X is denoted by $\mathcal{I}nn_J(A, X)$.

Let E be a complex (resp., real) Jordan triple and let X be a triple E-module. For each $b \in E$ and each $x_0 \in X$, it is known that the mapping

$$\delta = \delta(b, x_0) = L(b, x_0) - L(x_0, b) : E \to X,$$

defined by

(2.4) $\qquad \delta(a) = \delta(b, x_0)(a) := \{b, x_0, a\} - \{x_0, b, a\} \ \ (a \in E),$

is a ternary derivation from E into X. Finite sums of derivations of the form $\delta(b, x_0)$ are called **inner triple derivations**. The set of all inner ternary derivations from E to X is denoted by $\mathcal{I}nn_t(E, X)$. We shall only need this definition in the case that E is a von Neumann algebra M and X is M_*, with module action defined by (2.1) and (2.2).

The reader who is interested in more details concerning the more general definitions of the concepts in this section can consult [10, 14].

3. Normal ternary weak amenability for factors

We shall use the facts that if M is a finite von Neumann algebra, then every element of M of central trace zero is a finite sum of commutators ([4, Theoreme 3.2], [13, Theorem 1]), and if M is properly infinite (no finite central projections), then every element of M is a finite sum of commutators ([8], [6, Theorem 1], [7, Corollary to Theorem 8], [1, Lemma 3.1]). Thus for any von Neumann algebra, we have $M = Z(M) + [M, M]$, where $Z(M)$ is the center of M and $[M, M]$ is the set of finite sums of commutators in M.

3.1. **Cohomological characterization of finiteness.** Let M be a von Neumann algebra and consider the submodule $M_* \subset M^*$. Then

(3.1) $$\mathcal{D}_t(M, M_*) = \mathcal{I}nn_b^*(M, M_*) \circ * + \mathcal{I}nn_t(M, M_*),$$

where for a linear operator D, $D \circ *(x) = D(x^*)$. This was stated and proved for M semifinite in [10, Corollary 3.10] but the same proof holds by using [5, Theorem 4.1] in place of [2, Theorem 3.2].

Let M be any von Neumann algebra and let ϕ_0 be any fixed normal state. Then

(3.2) $$M_* = \{1\}_\perp + \mathbb{C}\phi_0,$$

where

$$\{1\}_\perp = \{\psi \in M_* : \psi(1) = 0\}.$$

Lemma 3.1. *If M is a von Neumann algebra, then $\overline{[M_*, M]} = Z(M)_\perp$ (norm closure), where $Z(M)$ is the center of M. In particular, if M is a factor, then $\overline{[M_*, M]} = \{1\}_\perp$.*

Proof. It is clear that $[M_*, M] \subset Z(M)_\perp$. If $x \in M$ satisfies $\langle x, [\varphi, b] \rangle = 0$ for all $\varphi \in M_*$ and $b \in M$, then $\varphi(bx - xb) = 0$, and so x belongs to the center of M and $\langle x, Z(M)_\perp \rangle = 0$, proving the lemma. $\qquad\square$

Lemma 3.2. *If M is a properly infinite von Neumann algebra, if $\psi \in M_*$ and if $D_\psi \circ *$ belongs to the norm closure of $\mathcal{I}nn_t(M, M_*)$, where $D_\psi = ad\,\psi$, then $\psi(Z(M)) = 0$, and in particular, $\psi(1) = 0$.*

Proof. For $\epsilon > 0$, there exist $\varphi_j \in M_*$ and $b_j \in M$ such that

$$\left\| D_\psi \circ * - \sum_{j=1}^n (L(\varphi_j, b_j) - L(b_j, \varphi_j)) \right\| < \epsilon.$$

For $x, a \in M$, direct calculations yield

$$\left| \psi(a^*x - xa^*) - \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^*\varphi_j^*)(a^*x) - \frac{1}{2}\sum_{j=1}^n (b_j\varphi_j - \varphi_j^*b_j^*)(xa^*) \right| < \epsilon\|a\|\|x\|,$$

where for a linear functional φ, $\varphi^*(x) = \overline{\varphi(x^*)}$.

We set $x = 1$ to get

$$\left| \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^*\varphi_j^*)(a^*) + \frac{1}{2}\sum_{j=1}^n (b_j\varphi_j - \varphi_j^*b_j^*)(a^*) \right| < \epsilon\|a\|,$$

and therefore

$$\left| \psi(a^*x - xa^*) - \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^*\varphi_j^*)(a^*x - xa^*) \right| < 2\epsilon\|a\|\|x\|,$$

for every $a, x \in M$, that is,

(3.3) $$\left| \psi([a,x]) - \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^* \varphi_j^*)([a,x]) \right| < 2\epsilon \|a\|\|x\|,$$

and therefore

(3.4) $$\left| \psi([a,x]) - \frac{1}{2}\sum_{j=1}^{n}(\varphi_j^* b_j^* - b_j \varphi_j)([a,x]) \right| < 3\epsilon \|a\|\|x\|.$$

Let us now write

$$\sum_j (\varphi_j b_j - b_j^* \varphi_j^*) = \sum_{j=1}^{n}(\varphi_j b_j - b_j \varphi_j + b_j \varphi_j - \varphi_j^* b_j^* + \varphi_j^* b_j^* - b_j^* \varphi_j^*)$$

so that

$$2\psi - \sum_j (\varphi_j b_j - b_j^* \varphi_j^*) =$$

$$2\psi - \sum_j [\varphi_j, b_j] - \sum_j (b_j \varphi_j - \varphi_j^* b_j^*) + 2\psi - 2\psi - \sum_j [\varphi_j^*, b_j^*]$$

and

$$4\psi - \sum_j [\varphi_j, b_j] - \sum_j [\varphi_j^*, b_j^*] =$$

$$2\psi - \sum_j (\varphi_j b_j - b_j^* \varphi_j^*) + 2\psi + \sum_j (b_j \varphi_j - \varphi_j^* b_j^*).$$

Thus, by (3.3) and (3.4),

$$\left| 4\psi([a,x]) - \sum_j [\varphi_j, b_j]([a,x]) - \sum_j [\varphi_j^*, b_j^*]([a,x]) \right| \le$$

$$\left| 2\psi([a,x]) - \sum_j (\varphi_j b_j - b_j^* \varphi_j^*)([a,x]) \right|$$

$$+ \left| 2\psi([a,x]) + \sum_j (b_j \varphi_j - \varphi_j^* b_j^*)([a,x]) \right| < 10\epsilon \|a\|\|x\|.$$

Since $[M, M] = M$, for any $z \in Z(M)$, if $z = \sum_k [a_k, x_k]$ we have

$$|4\psi(z)| \le 10\epsilon \sum_k \|x_k\|\|a_k\|,$$

proving that $\psi(z) = 0$. $\qquad\square$

Variants of the argument used in the preceding proof will be used in Proposition 4.1(b) and in Proposition 4.5.

Lemma 3.3. *If M is a von Neumann algebra, and $\psi \in M_*$ satisfies $\psi^* = -\psi$, then D_ψ is a self-adjoint mapping. Conversely, if M is properly infinite and D_ψ is self-adjoint, then $\psi^* = -\psi$.*

Proof. For $a, b \in M$, by direct calculation $D_\psi(a^*)(b) = \psi(a^*b - ba^*)$ and $D_\psi(a)^*(b) = \psi^*(ba^* - a^*b)$ (cf. [10, Lemma 3.2]). This proves the first statement. The second statement follows from this since $M = [M, M]$. $\qquad\square$

Theorem 1. *Let M be a von Neumann algebra.*

(a) *If every triple derivation of M into M_* is approximated in norm by inner triple derivations, then M is finite.*

(b) *If M is a finite von Neumann algebra acting on a separable Hilbert space or if M is a finite factor, then every triple derivation of M into M_* is approximated in norm by inner triple derivations.*

Proof. (a) Assume that every triple derivation of M into M_* is a norm limit of inner triple derivations and also assume for the moment that M is properly infinite. If $\psi \in M_*$ satisfies $\psi^* = -\psi$, then by Lemma 3.3, and (3.1), $D_\psi \circ * \in \mathcal{D}_t(M, M_*)$. Then by Lemma 3.2, $\psi(1) = 0$. This is a contradiction if we take $\psi = i\phi_0$ where ϕ_0 is any normal state of M. This proves that M cannot be properly infinite.

If M is arbitrary, write $M = pM + (1-p)M$ for some central projection p, where pM is finite and $(1-p)M$ is properly infinite. It is easy to see that if $\delta \in \mathcal{D}_t(M, M_*)$, then $p\delta \in \mathcal{D}_t(pM, (pM)_*)$ and similarly for $(1-p)\delta$ and that if $\mathcal{I}nn_t(M, M_*)$ is norm dense in $\mathcal{D}_t(M, M_*)$, then $\mathcal{I}nn_t(pM, (pM)_*)$ is norm dense in $\mathcal{D}_t(pM, (pM)_*)$, and $\mathcal{I}nn_t((1-p)M, ((1-p)M)_*)$ is norm dense in $\mathcal{D}_t((1-p)M, ((1-p)M)_*)$. By the preceding paragraph, $1-p = 0$, so that M is finite.

(b) Suppose first that M is a finite factor. Let $\psi \in M_*$ be such that the inner derivation $D_\psi : x \mapsto \psi \cdot x - x \cdot \psi$, is self adjoint, that is, $D_\psi \in \mathcal{I}nn_b^*(M, M_*)$. By the proof of Lemma 3.3, $\psi^* = -\psi$ on $[M, M]$. Let us assume temporarily that $\psi(1) \in i\mathbb{R}$, so that $\psi^* = -\psi$ on $M = \mathbb{C}1 + [M, M]$. We also assume, temporarily, that $\psi = \hat{x}_\psi$ for some $x_\psi \in M$, that is, $\psi(y) = \text{tr}\,(yx_\psi)$ for $y \in M$, where tr is a faithful normal finite trace on M.

We then have

(3.5) $$ x_\psi = \text{tr}\,(x_\psi)1 + \sum_j [a_j + ib_j, c_j + id_j] $$

where a_j, b_j, c_j, d_j are self adjoint elements of M. Expanding the right side of (3.5) and using the fact that $x_\psi^* = -x_\psi$, we have

$$ x_\psi = \text{tr}\,(x_\psi)1 + \sum_j ([a_j, c_j] - [b_j, d_j]) $$

so that

$$ \hat{x}_\psi = \text{tr}\,(x_\psi)\text{tr}\,(\cdot) + \sum_j ([a_j, c_j]\hat{} - [b_j, d_j]\hat{}). $$

It is easy to check that for $a = a^*, b = b^*, x, y \in M$,

$$[a, b]\hat{\ }([x^*, y]) = \{\hat{a}, 2b, x\}(y) - \{2b, \hat{a}, x\}(y).$$

Thus

$$D_\psi(x^*)(y) = \psi(x^*y - yx^*) = \operatorname{tr}\left(\sum_j([a_j, c_j] - [b_j, d_j])[x^*, y]\right)$$

so that

$$(3.6) \qquad D_\psi \circ * = \sum_j \Big(L(\hat{a}_j, 2c_j) - L(2c_j, \hat{a}_j) - L(\hat{b}_j, 2d_j) + L(2d_j, \hat{b}_j) \Big)$$

belongs to $\mathcal{I}nn_t(M, M_*)$.

By replacing ψ by $\psi' = \psi - \Re\psi(1)\operatorname{tr}(\cdot)$, so that $\psi'(1) \in i\mathbb{R}$ and $D_\psi = D_{\psi'}$, we now have that if $\psi = \hat{x}_\psi$ for some $x_\psi \in M$, then $D_\psi \circ * \in \mathcal{I}nn_t(M, M_*)$. Since elements of the form $\hat{x}$ are dense in M_* (cf. [16, Theorem 2.18, p.321] or [17, Lemma 2.12, p.175]) and $\|D_\psi\| \le 2\|\psi\|$, it follows that for every $\psi \in M_*$, $D_\psi \circ *$ belongs to the norm closure of $\mathcal{I}nn_t(M, M_*)$. From (3.1), $\mathcal{I}nn_t(M, M_*)$ is norm dense in $\mathcal{D}_t(A, A^*)$.

Now suppose that M is a finite von Neumann algebra acting on a separable Hilbert space, so that it admits a faithful normal finite trace tr. Let $\psi \in M_*$ be such that the inner derivation $D_\psi : x \mapsto \psi \cdot x - x \cdot \psi$, is self adjoint, that is, $D_\psi \in \mathcal{I}nn_b^*(M, M_*)$. As above, $\psi^* = -\psi$ on $[M, M]$. Assuming temporarily that $\psi^* = -\psi$ on $Z(M)$, the above proof in the factor case shows that $\mathcal{I}nn_t(M, M_*)$ is norm dense in $\mathcal{D}_t(A, A^*)$.

To reduce to the case that $\psi^* = -\psi$ on $Z(M)$, we use direct integrals. Write M as a direct integral of factors: $M = \int_\Omega^\oplus M(\omega)\, d\mu(\omega)$, where $M(\omega)$ is a finite factor with canonical trace tr_ω, which is an element of $M(\omega)_*$. Since $M_* = \int_\Omega^\oplus M(\omega)_*\, d\mu(\omega)$ ([16, p.285]) we have $\psi = \int_\Omega^\oplus \psi(\omega)\, d\mu(\omega)$ for suitable $\psi(\omega) \in M(\omega)_*$. It follows that if $x \in Z(M)$ then $\psi^*(x) = -\psi(x)$ if and only if $\operatorname{tr}_\omega(\psi(\omega))$ is purely imaginary for almost every ω. Here we are using the same notation for the extension of tr_ω to a linear functional on $M(\omega)_*$ ([17, p.174]). By replacing ψ by ψ', where $\psi'(\omega) = \psi(\omega) - \Re\operatorname{tr}_\omega(\psi(\omega))\operatorname{tr}_\omega$, so that $D_\psi = D_{\psi'}$, we now have the desired result. $\square$

Variants of the argument used in the preceding proof of (b) will be used in Theorem 2 and Propositions 4.1(a) and 4.5.

Corollary 3.4. *If M acts on a separable Hilbert space, or if M is a factor, then M is finite if and only if every triple derivation of M into M_* is approximated in norm by inner triple derivations.*

3.2. **Zero-One law for factors.**

Theorem 2. *If M is a properly infinite factor, then the real vector space of triple derivations of M into M_*, modulo the norm closure of the inner triple derivations, has dimension 1.*

Proof. Let ϕ_0 be a fixed normal state and let $D_\psi \in \mathcal{I}nn_b^*(M, M_*)$ so that again by the proof of Lemma 3.3, since $M = [M, M]$, we have $\psi^* = -\psi$ and so $\psi(1) = i\lambda$ for some $\lambda \in \mathbb{R}$. Write, by (3.2),

$$(3.7) \qquad\qquad \psi = \varphi + i\lambda\phi_0$$

with $\varphi(1) = 0$. By Lemma 3.1, for every $\epsilon > 0$, there exist $\varphi_j \in M_*$ and $b_j \in M$, such that with $\varphi_\epsilon = \sum_j [\varphi_j, b_j]$, we have $\|\varphi - \varphi_\epsilon\| < \epsilon$. Since $\varphi^* = -\varphi$ we may assume $\varphi_\epsilon^* = -\varphi_\epsilon$.

If we write $\varphi_j = \xi_j + i\eta_j$ and $b_j = c_j + id_j$ where $\xi,, \eta_j, c_j, d_j$ are selfadjoint, then it follows from $\varphi_\epsilon^* = -\varphi_\epsilon$ that

$$\varphi_\epsilon = \sum_j ([\xi_j, c_j] - [\eta_j, d_j]).$$

Further calculation shows that for all $x \in M$,

$$D_{\varphi_\epsilon}(x^*) = \sum_j (\{\xi_j, 2c_j, x\} - \{2c_j, \xi_j, x\} - \{\eta_j, 2d_j, x\} + \{2d_j, \eta_j, x\}).$$

This shows that $D_{\varphi_\epsilon} \circ * \in \mathcal{I}nn_t(M, M_*)$ so that $D_\varphi \circ *$ belongs to the norm closure of $\mathcal{I}nn_t(M, M_*)$.

According to (3.1), every $\delta \in \mathcal{D}_t(M, M_*)$ has the form $\delta = \delta_0 + \delta_1$, where $\delta_0 = D_\psi \circ *$ is selfadjoint, and $\delta_1 \in \mathcal{I}nn_t(M, M_*)$ is the inner triple derivation $\frac{1}{2}L(\delta(1), 1) - \frac{1}{2}L(1, \delta(1))$. Lemma 3.2 shows now that the map

$$\delta + \overline{\mathcal{I}nn_t(M, M_*)} \mapsto \lambda$$

is an isomorphism

$$\mathcal{D}_t(M, M_*)/\overline{\mathcal{I}nn_t(M, M_*)} \sim \mathbb{R},$$

where $\lambda = \Im m(\psi(1))$ is defined by (3.7).

Explicitly, we define a map $\Phi : \mathcal{D}_t(M, M_*)/\overline{\mathcal{I}nn_t(M, M_*)} \to \mathbb{R}$ as follows. If $\delta \in \mathcal{D}_t(M, M_*)$, say $\delta = D_\psi \circ * + \delta_1$ as above, and $[\delta] = \delta + \overline{\mathcal{I}nn_t(M, M_*)}$, let $\Phi([\delta]) = -i\psi(1) \in \mathbb{R}$. It follows from Lemmas 3.3 and 3.2 that Φ is well defined, and it is easily seen to be linear, onto and one to one.

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$$

Corollary 3.5. *If M is a factor, the linear space of triple derivations into the predual, modulo the norm closure of the inner triple derivations, has*

dimension 0 or 1: It is zero if the factor is finite; and it is 1 if the factor is infinite.

4. SUMS OF COMMUTATORS IN THE PREDUAL

For any von Neumann algebra M, we shall write $(M_*)_0$ for the set of elements $\psi \in M_*$ such that $\psi(1) = 0$ (in (3.1) we called this space $\{1\}_\perp$). If M is finite and admits a faithful normal finite trace tr, which therefore extends to a trace on M_* ([17, p.174]), then $(M_*)_0 = \mathrm{tr}^{-1}(0)$.

4.1. **Factors of type II_1.** Recall that a finite factor M of type I is both normally ternary weakly amenable and satisfies $(M_*)_0 = [M_*, M]$. For a finite factor of type II, the corresponding statements with $[M_*, M]$ and $\mathcal{I}nn_t(M, M_*)$ replaced by their norm closures are also both true by Theorem 1 and Lemma 3.1. More is true in the type II case.

Proposition 4.1. *Let M be a finite von Neumann algebra.*

(a) *If M acts on a separable Hilbert space or is a factor (hence admits a faithful normal finite trace), and if $(M_*)_0 = [M_*, M]$, then M is normally ternary weakly amenable.*

(b) *If M is a factor and M is normally ternary weakly amenable, then $(M_*)_0 = [M_*, M]$.*

Proof. (a) Assume first that M is a factor. Suppose that $(M_*)_0 = [M_*, M]$. Let $\psi \in M_*$ be such that the inner derivation $D_\psi : x \mapsto \psi \cdot x - x \cdot \psi$, is self adjoint, that is, $D_\psi \in \mathcal{I}nn_b^*(M, M_*)$. By the proof of Lemma 3.3, $\psi^* = -\psi$ on $[M, M]$. Let us assume temporarily that $\psi(1) \in i\mathbb{R}$, so that $\psi^* = -\psi$ on $M = \mathbb{C}1 + [M, M]$.

By our assumption, we then have (here tr is the extended trace, [17, p.174])

(4.1) $$\psi = \mathrm{tr}\,(\psi)\mathrm{tr} + \sum_j [\xi_j + i\eta_j, c_j + id_j]$$

where ξ_j, η_j are self adjoint elements of M_* and c_j, d_j are self adjoint elements of M. Expanding the right side of (4.1) and using the fact that $\psi^* = -\psi$, we have

$$\psi = \mathrm{tr}\,(\psi)\mathrm{tr} + \sum_j ([\xi_j, c_j] - [\eta_j, d_j]).$$

It is easy to check that for $\xi = \xi^* \in M_*$ and $c = c^*, x, y \in M$,

$$[\xi, c]([x^*, y]) = \{\xi, 2c, x\}\,(y) - \{2c, \xi, x\}\,(y).$$

Thus

(4.2) $D_\psi \circ * = \sum_j \left(L(\xi_j, 2c_j) - L(2c_j, \xi) - L(\eta_j, 2d_j) + L(2d_j, \eta_j) \right),$

which belongs to $\mathcal{I}nn_t(M, M_*)$.

By replacing ψ by $\psi' = \psi - \Re\psi(1)\operatorname{tr}(\cdot)$, so that $D_\psi = D_{\psi'}$, we now have that for every ψ, $D_\psi \circ * \in \mathcal{I}nn_t(M, M_*)$. From (3.1), $\mathcal{D}_t(A, A^*) = \mathcal{I}nn_t(M, M_*)$ proving that M is normally ternary weakly amenable.

The case where M is not necessarily a factor, but acts on a separable Hilbert space is proved using direct integrals as in the proof of Theorem 1(b).

(b) Suppose that M is a finite factor and that M is normally ternary weakly amenable. Let $\psi \in M_*$ with $\operatorname{tr}(\psi) = \psi(1) = 0$. Suppose first that $\psi^* = -\psi$ so that D_ψ is self adjoint and therefore $D_\psi \circ *$ belongs to $\mathcal{D}_t(M, M)$. By our assumption, there exist $\varphi_j \in M_*$ and $b_j \in M$ such that $D_\psi \circ * = \sum_{j=1}^n (L(\varphi_j, b_j) - L(b_j, \varphi_j))$ on M.

For $x, a \in M$, direct calculations yield

$$\psi(a^*x - xa^*) = \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*)(a^*x) + \frac{1}{2}\sum_{j=1}^n (b_j \varphi_j - \varphi_j^* b_j^*)(xa^*).$$

We set $x = 1$ to get

(4.3) $0 = \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*)(a^*) + \frac{1}{2}\sum_{j=1}^n (b_j \varphi_j - \varphi_j^* b_j^*)(a^*),$

and therefore

(4.4) $\psi(a^*x - xa^*) = \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*)(a^*x - xa^*),$

for every $a, x \in M$.

Since $M = \mathbb{C}1 + [M, M]$ and $\psi(1) = 0$ it follows from (4.3) (with $a = 1$) that

$$\psi = \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*) = \frac{1}{2}\sum_{j=1}^n (\varphi_j^* b_j^* - b_j \varphi_j).$$

Hence

$$\begin{aligned} 2\psi &= \sum_{j=1}^n (\varphi_j b_j - b_j \varphi_j + b_j \varphi_j - \varphi_j^* b_j^* + \varphi_j^* b_j^* - b_j^* \varphi_j^*) \\ &= \sum_{j=1}^n [\varphi_j, b_j] - 2\psi + \sum_{j=1}^n [\varphi_j^*, b_j^*], \end{aligned}$$

which shows that $\psi \in [M_*, M]$.

Now let $\psi \in (M_*)_0$ and write $\psi = \psi_1 + \psi_2$, where $\psi_1^* = \psi_1$ and $\psi_2^* = -\psi_2$. Since $0 = \operatorname{tr}(\psi) = \operatorname{tr}(\psi_1) + \operatorname{tr}(\psi_2)$ and $\operatorname{tr}(\psi_1) = \psi_1(1)$ is real and $\operatorname{tr}(\psi_2) =$

$\psi_2(1)$ is purely imaginary, $\text{tr}\,(\psi_1) = 0 = \text{tr}\,(\psi_2)$. By the previous paragraph, $i\psi_1, \psi_2 \in [M_*, M]$ and so $\psi = -i(i\psi_1) + \psi_2 \in [M_*, M]$, completing the proof. □

Corollary 4.2. *Let M be a factor of type II_1. Then M is normally ternary weakly amenable if and only if $(M_*)_0 = [M_*, M]$.*

After proving Proposition 4.1, we learned from Ken Dykema that, as a consequence of [3, Theorem 4.6], $(M_*)_0 \neq [M_*, M]$ for any II_1-factor M. He also kindly provided the following detailed explanation. Consider a self adjoint operator in $L^1(M)$ of the form $A = A_1 - A_2$ where A_1 and A_2 are the positive and negative parts of A and where A_i has singular numbers

$$\mu_t(A_1) = \begin{cases} 1/(t\log^2 t) & 0 < t < 1/2 \\ 0 & 1/2 < t < 1 \end{cases}, \quad \mu_t(A_2) = \begin{cases} 2\log 2 & 0 < t < 1/2 \\ 0 & 1/2 < t < 1. \end{cases}$$

Then A is in $L^1(M)$, has trace zero and A is not a sum of $(M, L^1(M))$-commutators. The latter statement follows from [3, Theorem 4.6] because when $0 < r < 1/2$

$$\text{tr}\,(AE_{|A|}([0, \mu_r(A)])) = -\log(2) + \int_r^{1/2} 1/(t\log^2 t)\, dt = 1/\log r$$

but the function $r \mapsto 1/r \log r$ is not integrable.

Hence we have

Corollary 4.3. *A factor of type II_1 is never normally ternary weakly amenable.*

No infinite factor can be approximately normally ternary weakly amenable by Theorem 1, much less normally ternary weakly amenable. As for the case of a factor $M = B(H)$ of type I_∞, we also have $(M_*)_0 \neq [M_*, M]$, due to the work of Weiss: [20, Main Theorem], [19, Theorem 10], [21, Theorem 2.1] (See Problem 4).

4.2. **Finite von Neumann algebras of type I.** Now let M be a finite von Neumann algebra of type I_n, with $n < \infty$. Then we can assume

$$M = L^\infty(\Omega, \mu, M_n(\mathbb{C})) = M_n(L^\infty(\Omega, \mu)),$$
$$M_* = L^1(\Omega, \mu, M_n(\mathbb{C})_*) = M_n(L^1(\Omega, \mu))$$

and

$$Z(M) = L^\infty(\Omega, \mu)1.$$

It is known that the center valued trace is given by

$$\text{TR}\,(x) = \frac{1}{n}\left(\sum_1^n x_{ii}\right)1 \quad, \qquad \text{for } x = [x_{ij}] \in M$$

(see [12, p.1409] or [4, p.65]). We thus define, for a finite von Neumann algebra of type I_n which has a faithful normal finite trace tr ,

$$\mathrm{TR}\,(\psi) = \frac{1}{n}\left(\sum_1^n \psi_{ii}\right)\mathrm{tr} \quad , \qquad \text{for } \psi = [\psi_{ij}] \in M_*.$$

By modifying slightly the proof of Proposition 4.1 we have the following proposition. First we note the following elementary remark.

Remark 4.4. Let M be a finite von Neumann algebra of type I_n admitting a faithful normal finite trace and let $\psi \in M_*$.

(a) If $\mathrm{TR}\,(\psi) = 0$, then ψ vanishes on the center $Z(M)$ of M.
(b) $\psi^* = -\psi$ on $Z(M)$ if and only if $\mathrm{tr}\,(\psi(\omega))$ is purely imaginary for almost every ω.

Proof. If $x \in Z(M)$, then $x = f \cdot 1$ with $f \in L^\infty$ and

$$
\begin{aligned}
\psi(x) &= \int_\Omega \langle \psi(\omega), x(\omega)\rangle\,d\mu(\omega) = \int_\Omega \mathrm{tr}\,(\psi(\omega)x(\omega))\,d\mu(\omega) \\
&= \int_\Omega \mathrm{tr}\,([\sum_k \psi_{ik}(\omega)x_{kj}(\omega)])\,d\mu(\omega) \\
&= \int_\Omega (\sum_i \sum_k \psi_{ik}(\omega)x_{ki}(\omega))\,d\mu(\omega) \\
&= \int_\Omega (\sum_k \psi_{kk}(\omega)x_{kk}(\omega))\,d\mu(\omega) \\
&= \int_\Omega (\sum_k \psi_{kk}(\omega))f(\omega)\,d\mu(\omega) \\
&= \int_\Omega (\mathrm{TR}\,\psi)(\omega)f(\omega)\,d\mu(\omega),
\end{aligned}
$$

proving (a). As for (b), use $\psi(f \cdot 1) = \int_\Omega f(\omega)\mathrm{tr}\,(\psi(\omega))\,d\mu(\omega)$. $\qquad \square$

Proposition 4.5. *Let M be a finite von Neumann algebra of type I_n with $n < \infty$, which admits a faithful normal finite trace tr (equivalently, M is countably decomposable, also called σ-finite). Then M is normally ternary weakly amenable if and only if*

$$TR^{-1}(0) = [M_*, M].$$

In particular, if $tr^{-1}(0) = [M_*, M]$, then M is normally ternary weakly amenable.

Proof. Suppose that M, a finite von Neumann algebra of type I_n admitting a faithful normal finite trace, is normally ternary weakly amenable. Let $\psi \in M_*$ with $\mathrm{TR}\,(\psi) = 0$. Suppose first that $\psi^* = -\psi$ so that D_ψ is self

adjoint and therefore $D_\psi \circ *$ belongs to $\mathcal{D}_t(M, M)$. By our assumption, there exist $\varphi_j \in M_*$ and $b_j \in M$ such that $D_\psi \circ * = \sum_{j=1}^n (L(\varphi_j, b_j) - L(b_j, \varphi_j))$ on M.

For $x, a \in M$, direct calculations yield

$$\psi(a^*x - xa^*) = \frac{1}{2} \sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*)(a^*x) + \frac{1}{2} \sum_{j=1}^n (b_j \varphi_j - \varphi_j^* b_j^*)(xa^*).$$

We set $x = 1$ to get

(4.5) $$0 = \frac{1}{2} \sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*)(a^*) + \frac{1}{2} \sum_{j=1}^n (b_j \varphi_j - \varphi_j^* b_j^*)(a^*),$$

and therefore

(4.6) $$\psi(a^*x - xa^*) = \frac{1}{2} \sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*)(a^*x - xa^*),$$

for every $a, x \in M$.

Since $M = Z(M) + [M, M]$ and $\psi(Z(M)) = 0$ (since $\mathrm{TR}\, \psi = 0$) and $\sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*)(Z(M)) = 0$ (by (4.5)) it follows that

$$\psi = \frac{1}{2} \sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*) = \frac{1}{2} \sum_{j=1}^n (\varphi_j^* b_j^* - b_j \varphi_j).$$

Hence

$$\begin{aligned} 2\psi &= \sum_{j=1}^n (\varphi_j b_j - b_j \varphi_j + b_j \varphi_j - \varphi_j^* b_j^* + \varphi_j^* b_j^* - b_j^* \varphi_j^*) \\ &= \sum_{j=1}^n [\varphi_j, b_j] - 2\psi + \sum_{j=1}^n [\varphi_j^*, b_j^*], \end{aligned}$$

which shows that $\psi \in [M_*, M]$.

Now let $\psi \in \mathrm{TR}^{-1}(0)$ and write $\psi = \psi_1 + \psi_2$, where $\psi_1^* = \psi_1$ and $\psi_2^* = -\psi_2$. Since $0 = \mathrm{TR}\,(\psi) = \mathrm{TR}\,(\psi_1) + \mathrm{TR}\,(\psi_2)$ and $\mathrm{TR}\,(\psi_1)$ is real valued and $\mathrm{TR}\,(\psi_2)$ is purely imaginary valued, $\mathrm{TR}\,(\psi_1) = 0 = \mathrm{TR}\,(\psi_2)$. By the previous paragraph, $i\psi_1, \psi_2 \in [M_*, M]$ and so $\psi = -i(i\psi_1) + \psi_2 \in [M_*, M]$, completing the proof of one direction.

To prove the other direction, suppose that $\mathrm{TR}^{-1}(0) = [M_*, M]$. Let $\psi \in M_*$ be such that the inner derivation $D_\psi : x \mapsto \psi \cdot x - x \cdot \psi$, is self adjoint, that is, $D_\psi \in \mathcal{I}nn_b^*(M, M_*)$. By Lemma 3.3, $\psi^* = -\psi$ on $[M, M]$. Let us assume temporarily that $\psi^* = -\psi$ on $Z(M)$ so that $\psi^* = -\psi$.

By our main assumption, we have

(4.7) $$\psi = \mathrm{TR}\,(\psi) + \sum_j [\varphi_j + i\xi_j, c_j + id_j]$$

where φ_j, ξ_j are self adjoint elements of M_* and c_j, d_j are self adjoint elements of M. Expanding the right side of (4.7) and using the fact that $\psi^* = -\psi$, we have

$$\psi = \mathrm{TR}\,(\psi) + \sum_j ([\varphi_j, c_j] - [\xi_j, d_j]).$$

It is easy to check that for $\varphi \in M_*$ and $c = c^*, x, y \in M$,

$$[\varphi, c]([x^*, y]) = \{\varphi, 2c, x\}\,(y) - \{2c, \varphi, x\}\,(y).$$

Thus

$$(4.8) \quad D_\psi \circ * = \sum_j \left(L(\varphi_j, 2c_j) - L(2c_j, \varphi_j) - L(\xi_j, 2d_j) + L(2d_j, \xi_j) \right),$$

which belongs to $\mathcal{I}nn_t(M, M_*)$.

By replacing ψ by ψ', where $\psi'(\omega) = \psi(\omega) - \Re\mathrm{tr}\,(\psi(\omega))1_n$, we have $D_\psi = D_{\psi'}$ and $(\psi')^* = -\psi'$ so that $D_\psi \circ * = D_{\psi'} \circ * \in \mathcal{I}nn_t(M, M_*)$. From (3.1), $\mathcal{D}_t(M, M_*) = \mathcal{I}nn_t(M, M_*)$ proving that M is normally ternary weakly amenable. $\qquad\square$

Problem 1. Does Theorem 1(b) hold for finite von Neumann algebras on non separable Hilbert spaces?

Problem 2. Does Theorem 2 hold for all properly infinite von Neumann algebras?

Problem 3. Is a finite countably decomposable von Neumann algebra of type I normally ternary weakly amenable?

Problem 4. For a factor M of type III, do we have $\{1\}_\perp = [M_*, M]$?

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