STRUCTURABLE TORI

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Dedicated to the memory of Professor Issai Kantor

Abstract. The classification of structurable tori with nontrivial involution, which was begun by Allison and Yoshii, is completed. New examples of structurable tori are obtained using a construction of structurable algebras from a semilinear version of cubic forms satisfying the adjoint identity. The classification uses techniques borrowed from quadratic forms over $\mathbb{Z}_2$ and from the geometry of generalized quadrangles. Since structurable tori are the coordinate algebras for the centreless cores of extended affine Lie algebras of type BC_1, the results of this paper provide a classification and new examples for this class of Lie algebras.

1. Introduction

The purpose of is to complete the classification of structurable tori with nontrivial involution begun in [AY]. Structurable algebras are (in general nonassociative) algebras with involution that are defined by an identity that is needed for their use in the construction of 5-graded Lie algebras [K2, A2]. If Λ is a finitely generated abelian group, a *structurable* Λ*-torus* is a Λ-graded structurable algebra with the property that any nontrivial homogeneous component of $\mathcal{A}$ is spanned by an invertible element (as well as the convenient assumption that the support of $\mathcal{A}$ generates the group Λ).

Structurable tori with nontrivial involution arise as coordinate algebras of centreless cores of extended affine Lie algebras of type BC_1. The class of extended affine Lie algebras (EALAs) is an axiomatically defined class of Lie algebras over a field of characteristic 0 that contains finite dimensional split simple Lie algebras (the nullity 0 EALAs) and affine Kac-Moody Lie algebras (the nullity 1 EALAs) as motivating examples. Any EALA $\mathcal{E}$ has two compatible gradings, one by a finitely generated abelian group Λ (whose rank is called the *nullity* of $\mathcal{E}$) and the other by a finite root system (whose type is called the *type* of $\mathcal{E}$). Also, the structure of an EALA $\mathcal{E}$ is determined by the structure of its centreless core $\mathcal{E}_{\mathrm{cc}}$, in the sense that there is a general construction that produces all EALAs with a given centreless core [N]. For a EALA $\mathcal{E}$ of type BC_1, it is shown in [AY] that $\mathcal{E}_{\mathrm{cc}}$ can be constructed from a structurable Λ-torus with nontrivial involution using a Lie algebra construction due to Issai Kantor [K2]. (The result in [AY] is stated over a field $\mathbb{C}$ of complex numbers, but the same proof works for any field of characteristic 0.) Hence, our classification of structurable tori with nontrivial involution gives a corresponding

2000 *Mathematics Subject Classification.* Primary: 17A30, 17B60; Secondary: 17A75, 17C40, 17B67.

Bruce Allison gratefully acknowledge the support of the Natural Sciences and Engineering Research Council of Canada.

classification of the centreless cores of EALAs of type BC_1. Previously, the classification of the centreless cores over $\mathbb{C}$ had been completed for all types except BC_r [BGK, BGKN, Y1, AG]. So this paper, combined with the work in [F2] on type BC_2 and the work in [ABG] on type BC_r ($r \geq 3$), completes the classification of the centreless cores of EALAs over $\mathbb{C}$. (We note that the terminology for EALAs has not fully stabilized. We are using the term EALA in the sense of [N] where the base field is an arbitrary field of characteristic 0. Over $\mathbb{C}$ these form a slightly more general class of algebras than the class of tame EALAs studied earlier in [AABGP], but the centreless cores obtained from algebras in the two classes coincide. Also, EALAs in the sense of [N] coincide with tame EALAs of finite null rank studied recently in [MY].)

We now outline the contents of this paper in more detail. Since composition algebras will appear frequently in our constructions, we begin in §2 by recalling some basic definitions and facts about composition algebras. Then, in §3, we recall the definition and give some examples of structurable algebras and structurable tori. If the involution is trivial, a structurable torus is a Jordan torus and these were classified by Yoshii [Y1]. The remaining tori divide naturally into three classes: I, II, and III, and we recall that trichotomy from [AY] in §4.

In §5, we classify structurable tori of class I. These are the tori generated by the skew elements. We introduce maps ε measuring skewness, β measuring commutativity, and α measuring associativity. Indeed, β and α are multiplicative versions of the commutator and the associator. We show that ε is roughly (a multiplicative version of) a quadratic form on a $\mathbb{Z}_2$-vector space with polarization β. In the associative case, this is precisely correct and allows a classification of tori directly from the classification of quadratic forms over $\mathbb{Z}_2$. In the nonassociative case, although ε is not a quadratic form, we are able to use similar methods to obtain the classification. The main result of the section, which was announced without proof in [AFY], states roughly that a torus of class I is the tensor product of composition algebras over the algebra of Laurent polynomials.

In §6, we briefly recall from [AY] the classification of structurable tori of class II. These are all constructed from a diagonal graded hermitian form over a class I associative torus $\mathcal{B}$ with involution. We include this result since we now better understand the possibilities for $\mathcal{B}$ (in view of our work in §5) and since we need the construction using hermitian forms in any case to obtain some of the tori of class III (the class III(a) tori).

To prepare for the classification of tori of class III, we introduce in §7 a semilinear version of cubic forms satisfying the adjoint identity. We show that they can be used to construct certain structurable algebras. We connect this construction with previous constructions of structurable algebras known as the Cayley-Dickson process and the 2×2 matrix algebra construction. This section is written so that it can be read independently of the rest of the paper, since we believe that the material is of interest by itself in the theory of structurable algebras.

In §8, we associative a geometry $\mathcal{I}$ to a class III torus $\mathcal{A}$. The incidence structure in $\mathcal{I}$ is defined to encode the multiplication of homogeneous components in $\mathcal{A}$. Properties of the geometry $\mathcal{I}$ give a natural subdivision of class III into III(a), III(b), and III(c). In class III(b), $\mathcal{I}$ turns out to be a generalized quadrangle of order $(2, t)$. Although in the end we made no use of that fact, it did guide our thinking in developing geometric properties.

In §9, we use a recognition theorem from [AY] to show that all tori of class III(a) are constructed from a diagonal hermitian form over the quaternion algebra with nonstandard involution over the algebra of Laurent polynomials.

In §10, we classify tori of class III(b). We see that any such torus can be constructed using the Cayley-Dickson doubling process from a Jordan torus of degree 4 over the algebra of Laurent polynomials.

In the final three sections we turn our attention to class III(c) which we show contains precisely two examples. These examples are obtained using the construction in §7 of a structurable algebra from a cubic form. The cubic form is defined on the space $\mathcal{H}(\mathcal{C}_3)$ of 3×3-hermitian matrices over $\mathcal{C}$, where $\mathcal{C}$ is either a quaternion torus or an octonion torus. The first of the three sections contains two preparatory lemmas on $\mathcal{H}(\mathcal{C}_3)$, and the last two contain the construction and classification of class III(c) tori.

For ease of future reference, the classification theorems in this paper, Theorem 5.19 for class I, Theorem 6.2 for class II, Theorem 9.1 for class III(a), Theorem 10.6 for class III(b), and Theorem 13.3 for class III(c), are presented to stand alone as much as possible. We note also that the tori of class III(a) that are not associative, as well as the tori of class III(b) and III(c), are new in this work. As discussed above, each of these new structurable tori in turn determines a new family of EALAs of type BC_1.

Throughout the paper, *we assume that F is a field of characteristic $\neq 2$ or 3.* (Including the positive characteristic case requires no additional work.)

2. Composition algebras

At several points in this work, we will make use of composition algebras to construct examples. In this section, we recall some of the basic definitions and facts that we will need about composition algebras. We first fix some terminology and notation for algebras.

By an *algebra*, we mean a (not necessarily associative) unital algebra $\mathcal{A}$ over F. If $\mathcal{A}$ is an algebra and $x, y, z \in \mathcal{A}$, we use the notation $[x, y] = xy - yx$ and $(x, y, z) = (xy)z - x(yz)$ for the commutator and associator. If $x \in \mathcal{A}$, we define endomorphisms L_x and R_x of $\mathcal{A}$ by $L_x y = xy$ and $R_x y = yx$.

An *algebra with involution* is a pair $(\mathcal{A}, *)$ consisting of an algebra $\mathcal{A}$ with an anti-automorphism $*$ of period 2. When no confusion will result, we usually denote an algebra with involution $(\mathcal{A}, *)$ simply by $\mathcal{A}$. For $\sigma = \pm$ we let

$$\mathcal{A}_\sigma = \{x \in \mathcal{A} : x^* = \sigma x\},$$

and we call elements of $\mathcal{A}_-$ (resp. $\mathcal{A}_+$) *skew* (resp. *hermitian*). We define the *centre* of the algebra with involution $\mathcal{A}$ to be

$$\mathcal{Z}(\mathcal{A}) := \{z \in \mathcal{A}_+ : [z, \mathcal{A}] = (z, \mathcal{A}, \mathcal{A}) = (\mathcal{A}, z, \mathcal{A}) = (\mathcal{A}, \mathcal{A}, z) = 0\}. \tag{1}$$

If $\mathcal{A}$ is a commutative algebra, then the identity map is an involution on $\mathcal{A}$ called the *trivial involution.*

For the rest of this section, assume that K *is a commutative associative algebra over F.*

If $\mathcal{U}$ is a K-module, then a map $q : \mathcal{U} \to K$ is called a *quadratic form* over K if $q(ra) = r^2 q(a)$ for $r \in K$, $a \in \mathcal{U}$, and if $f(a, b) := q(a + b) - q(a) - q(b)$ defines a K-bilinear map $f : \mathcal{U} \times \mathcal{U} \to K$ called the *linearization* of q. In that case, q is said to be *nondegenerate* if f is nondegenerate ($f(a,\mathcal{U}) = 0 \implies a = 0$ for $a \in \mathcal{U}$). It is usual to abuse notation and denote the linearization of a quadratic form q by q.

Recall [Mc, p.156] that a *composition algebra* over K is an algebra $\mathcal{C}$ over K with a nondegenerate quadratic form $n : \mathcal{C} \to K$, called the *norm*, which permits *composition*:

$$n(1) = 1 \quad \text{and} \quad n(ab) = n(a)n(b), \text{ for } a, b \in \mathcal{C}.$$

We then define the *trace* $t : \mathcal{C} \to K$ by $t(a) = n(a, 1)$. It is known [Mc, p.156] that the map $^- : \mathcal{C} \to \mathcal{C}$ defined by

$$\bar{a} = t(a)1 - a$$

is an involution over K called the *canonical involution* on $\mathcal{C}$. Unless mentioned to the contrary, *we will always regard a composition algebra as an algebra with involution using the canonical involution.* It also known [ibid] that $\mathcal{C}$ is an alternative algebra and that $n(a)1 = a\bar{a} = \bar{a}a$. Therefore, $a \in \mathcal{C}$ is invertible if and only if $n(a)$ is invertible in K, in which case $a^{-1} = n(a)^{-1}\bar{a}$. Also, we have $n(\bar{a}) = n(a)$, $t(\bar{a}) = t(a)$ and

$$n(a, b) = t(a\bar{b}).$$

Further,

$$t((ab)c) = t(a(bc)),$$

so we can write $t(abc) = t((ab)c)$. Also, alternativity shows that $(ab)c = 1$ if and only if $a(bc) = 1$, which we can thus write as $abc = 1$. Finally, if $abc = 1$, then $bca = cab = 1$.

To construct examples of composition algebras we recall the classical *Cayley-Dickson process.* (See for example [BGKN, §1] or [Mc, pp. 160-163]). If $\mathcal{D}$ is an algebra with involution $^-$, and μ is a unit in the centre of $\mathcal{D}$, we let

$$\mathrm{CD}(\mathcal{D}, \mu) = \mathcal{D} \oplus u\mathcal{D}$$

be the algebra with involution whose product and involution are defined by

$$(a + ub)(c + ud) = (ac + \mu d\bar{b}) + u(\bar{a}d + cb) \quad \text{and} \quad \overline{a + ub} = \bar{b} - ub. \tag{2}$$

If μ_1, μ_2, μ_3 are units in K, we can iterate the CD-process starting at K with the trivial involution $^-$. We successively construct algebras with involution:

$$\begin{aligned} &K, \\ \mathrm{CD}(K, \mu_1) &= K \oplus x_1 K, \\ \mathrm{CD}(K, \mu_1, \mu_2) &= \mathrm{CD}(\mathrm{CD}(K, \mu_1), \mu_2) = \mathrm{CD}(K, \mu_1) \oplus x_2 \,\mathrm{CD}(K, \mu_1), \\ \mathrm{CD}(K, \mu_1, \mu_2, \mu_3) &= \mathrm{CD}(\mathrm{CD}(K, \mu_1, \mu_2), \mu_3) \\ &= \mathrm{CD}(K, \mu_1, \mu_2) \oplus x_3 \,\mathrm{CD}(K, \mu_1, \mu_2). \end{aligned} \tag{3}$$

It is well known (see for example the argument in [S, p. 58]) that these algebras are composition algebras over K with norm defined by $n(a) = a\bar{a}$ and that their involution (defined in (2)) is the canonical involution. The first two of these algebras are commutative and associative, the second is associative and the last is alternative. They have K-bases $\{1\}$, $\{1, x_1\}$, $\{1, x_1, x_2, x_2x_1\}$ and $\{1, x_1, x_2, x_2x_1, x_3, x_3x_1, x_3x_2, x_3(x_2x_1)\}$ respectively; and they have generating sets $\{1\}$, $\{x_1\}$, $\{x_1, x_2\}$ and $\{x_1, x_2, x_3\}$ respectively as algebras over K. The elements in these generating sets are called *canonical generators* and they satisfy $x_i^2 = \mu_i$, $x_ix_j = -x_jx_i$ for $i \neq j$ and $(x_ix_j)x_k = -x_i(x_jx_i)$ for $i, j, k \neq$. The algebras with involution

$\mathrm{CD}(K, \mu_1, \mu_2)$ and $\mathrm{CD}(K, \mu_1, \mu_2, \mu_3)$ are called *quaternion* and *octonion* algebras over K respectively.

The theorem of Hurwitz [S, Theorem 3.25] tells us that if K is a field, every composition algebra over K is obtained as in the previous paragraph. In particular, the *2-dimensional composition algebras* over F are the algebras E over F that have a basis $1, e$ satisfying $e^2 \in F^\times$. In that case the canonical involution of E is the automorphism σ_E satisfying $\sigma_E(e) = -e$. There are only two possibilities for a 2-dimensional composition algebra E: either E/F is a quadratic field extension and σ_E generates the Galois group of E/F, or E is isomorphic to $E = F \oplus F$ with the exchange involution. In the second case we say that E is *split*.

3. Structurable algebras and tori

In this section, we recall the definition and give some examples of structurable tori. We first recall some terminology and notation for graded algebras.

Assume that Λ *is an arbitrary abelian group.* A vector space $\mathcal{V}$ over F is *graded* by Λ if $\mathcal{V} = \bigoplus_{\lambda\in\Lambda} \mathcal{V}^\lambda$, where $\mathcal{V}^\lambda$ is a subspace of $\mathcal{V}$ for $\lambda \in \Lambda$. If M is a subgroup of Λ, let

$$\mathcal{V}^M := \bigoplus_{\lambda\in M} \mathcal{V}^\lambda,$$

an M-graded space. The *support of* $\mathcal{V}$ is

$$\mathrm{supp}(\mathcal{V}) := \{\lambda \in \Lambda : \mathcal{V}^\lambda \neq 0\}.$$

$\mathcal{V}$ is said to be *finely graded* if $\dim(\mathcal{V}^\lambda) = 1$ for all $\lambda \in \mathrm{supp}(\mathcal{V})$.

Two gradings of $\mathcal{V}$ by abelian groups Λ and Λ' are *isomorphic gradings* if there is a group isomorphism $\theta : \Lambda \to \Lambda'$ with $\mathcal{V}^\lambda = \mathcal{V}^{\theta(\lambda)}$. More generally, an *isograded isomorphism* is a linear isomorphism $\varphi : \mathcal{V} \to \mathcal{W}$ with $\varphi(\mathcal{V}^\lambda) = \mathcal{W}^{\theta(\lambda)}$, where $\mathcal{V}$ is Λ-graded, $\mathcal{W}$ is Λ'-graded, and $\theta : \Lambda \to \Lambda'$ is a group isomorphism. In this case, we write $\mathcal{V} \simeq_{ig} \mathcal{W}$. If $\Lambda = \Lambda'$ and $\theta = \mathrm{id}$, this reduces to the usual notion of a *graded isomorphism*, and we write $\mathcal{V} \simeq_\Lambda \mathcal{W}$.

A *graded algebra* is an algebra $\mathcal{A}$ which is a graded space with $\mathcal{A}^\lambda\mathcal{A}^\mu \subset \mathcal{A}^{\lambda+\mu}$. For a *graded algebra with involution*, we also require the involution to be a graded vector space isomorphism; i.e. $(\mathcal{A}^\lambda)^* = \mathcal{A}^\lambda$. In that case, the subspaces $\mathcal{A}_+$ and $\mathcal{A}_-$ are graded subspaces of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is a finely graded algebra with involution then $x^* = \pm x$ for each homogeneous element x of $\mathcal{A}$.

We extend the notions of isograded isomorphism and graded isomorphism to isomorphisms of graded algebras and isomorphisms of graded algebras with involution in the obvious fashion. If $\mathcal{A}$ and $\mathcal{A}'$ are graded algebras with involution, we write $\mathcal{A} \simeq_{ig} \mathcal{A}'$ to mean that $\mathcal{A}$ and $\mathcal{A}'$ are isograded isomorphic as algebras with involution. (If we mean that $\mathcal{A}$ and $\mathcal{A}'$ are isograded isomorphic as algebras without involution, we will say this specifically.)

Next we recall [A1, (3) and Corollary 5][1] that a *structurable algebra* is a (unital) algebra $\mathcal{A}$ with an involution $*$ satisfying

$$\{xy\{zwq\}\} - \{zw\{xyq\}\} = \{\{xyz\}wq\} - \{z\{yxw\}q\}$$

where

$$\{xyz\} := (xy^*)z + (zy^*)x - (zx^*)y.$$

[1]In [A1] there is an overriding assumption that algebras are finite dimensional. However that assumption is not used in the sections (1–5 and 8(iii)) of [A1] that we use in the infinite dimensional setting.

Examples of structurable algebras include associative algebras with involution, alternative algebras with involution and Jordan algebras with the trivial involution [A1, Theorem 13]. In fact, unless stated to the contrary, we will regard Jordan algebras as algebras with the trivial involution, and hence as structurable algebras.

As the term suggests, a Λ-*graded structurable algebra* is a Λ-graded algebra with involution $\mathcal{A}$ such that $\mathcal{A}$ is structurable.

Suppose that $\mathcal{A}$ is a finely Λ-graded structurable algebra. A homogeneous element x of $\mathcal{A}$ is said to be *invertible* if there exists $x^{-1} \in \mathcal{A}$ so that

$$xx^{-1} = x^{-1}x = 1 \quad \text{and} \quad [L_x, L_{x^{-1}}] = [R_x, R_{x^{-1}}] = 0.$$

In that case, x^{-1} is unique and if $x \in \mathcal{A}_\sigma^\lambda$, where $\lambda \in \Lambda$, $\sigma = \pm$, we have $x^{-1} \in \mathcal{A}_\sigma^{-\lambda}$. [AY, Prop. 3.1] . We call x^{-1} the *inverse* of x.

Remark 3.1. Suppose that $\mathcal{A}$ is a finely Λ-graded structurable algebra, and let x be a homogeneous element of $\mathcal{A}$.

(a) To check that x is invertible with inverse y, it is sufficient to check that $xy = 1$ and that $[L_x, L_y] = 0$. Indeed, in that case, we have $yx = L_yL_x1 = L_xL_y1 = xy = 1$. Also, conjugating the equality $[L_x, L_y] = 0$ by the involution, we get $[R_x, R_y] = 0$.

(b) x is invertible if and only if there exists an element $\hat{x}$ in $\mathcal{A}$ such that $\{x, \hat{x}, z\} = z$ for all $z \in \mathcal{A}$. (Elements x with the latter property are said to be *conjugate invertible.*) In that case, $\hat{x}$ is unique and $\hat{x} = \varepsilon x^{-1}$, where $x^* = \varepsilon x$ with $\varepsilon = \pm 1$ [AY, Prop. 3.1].

(c) If $x = s$ is skew, then s is invertible if and only if L_s (or equivalently R_s) is invertible, in which case $L_s^{-1} = L_{s^{-1}}$ and $R_s^{-1} = R_{s^{-1}}$ [AY, Lemma 2.9]. Moreover, in that case sw is invertible for any invertible homogeneous element w of $\mathcal{A}$ (by (b) and [AH, Prop. 8.2 and Theorem 11.4]).

(d) If $\mathcal{A}$ is associative, alternative or Jordan, then the notion of invertibility used here coincides with the usual notion.

Definition 3.2. A *structurable* Λ-*torus* is a Λ-graded structurable algebra $\mathcal{A}$ satisfying

(ST1) $\mathcal{A}$ is a finely graded.
(ST2) Each nonzero homogeneous element of $\mathcal{A}$ is invertible.
(ST3) Λ is generated as a group by $\mathrm{supp}(\mathcal{A})$.

Remark 3.3. If $\mathcal{A}$ is a structurable Λ-torus and M is a subgroup of Λ, then $\mathcal{A}^M$ is clearly a subalgebra of $\mathcal{A}$ satisfying (ST1) and (ST2). Thus, if M is generated by $\mathrm{supp}(\mathcal{A}^M)$, then $\mathcal{A}^M$ is a structurable M-torus. In particular, if $\mathrm{supp}(\mathcal{A}) = \Lambda$, then $\mathcal{A}^M$ is a structurable torus for any subgroup M of Λ.

Remark 3.4. If $\mathcal{A}$ is a structurable Λ-torus and x is a nonzero homogeneous element of $\mathcal{A}$, then the (unital) subalgebra of $\mathcal{A}$ that is generated by x and x^{-1} is commutative and associative [AY, Corollary 7.8]. Hence, the powers x^k for $k \in \mathbb{Z}$ are all nonzero.

Examples of structurable Λ-tori include Λ-graded associative algebras with involution satisfying (ST1)–(ST3), Λ-graded alternative algebras with involution satisfying (ST1)–(ST3), and Λ-graded Jordan algebras (with the trivial involution) satisfying (ST1)–(ST3) [AY, Examples 4.2 and 4.3]. These are called respectively *associative* Λ-*tori with involution, alternative* Λ-*tori with involution and Jordan* Λ-*tori*.

More specifically we have the following basic examples:

Example 3.5. Let
$$\mathcal{P}(n) := F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$
be the algebra of Laurent polynomials over F, where $n \geq 0$. We call the elements $t_1, \dots, t_n$ *canonical (Laurent) generators* for $\mathcal{P}(n)$. The algebra $\mathcal{P}(n)$ has a unique $\mathbb{Z}^n$-grading so that $\deg(t_i) = \varepsilon_i$ for each i, where $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the standard basis for $\mathbb{Z}^n$. Then $\mathcal{P}(n)$ is a commutative associative $\mathbb{Z}^n$-torus with trivial involution, and hence a structurable $\mathbb{Z}^n$-torus with trivial involution. (If $n = 0$, then $\mathbb{Z}^0 = 0$ and $\mathcal{P}(0) = F$.)

Example 3.6. ([BGKN, §1]) Suppose that $n = 1, 2, 3$, and let
$$\mathcal{C}(n) := \mathrm{CD}(\mathcal{P}(n), t_1, \dots t_n).$$
We give the algebra $\mathcal{C}(n)$ the *canonical* $\mathbb{Z}^n$-grading by assigning the degrees in the standard basis $\varepsilon_1, \dots, \varepsilon_n$ to the canonical generators $x_1, \dots, x_n$ of $\mathcal{C}(n)$ (in order). We use the notation $\mathcal{C}(n)$ for $(\mathcal{C}(n), \natural)$, where $\natural$ is the canonical involution (satisfying $x_i^\natural = -x_i$). We also set
$$\mathcal{C}(0) = F$$
(where, as in Example 3.5, we regard F as an algebra with trivial involution graded by the trivial group $\mathbb{Z}^0 = 0$). Then, $\mathcal{C}(n)$ is an alternative $\mathbb{Z}^n$-torus with involution and hence it is a structurable $\mathbb{Z}^n$-torus for $n = 0, 1, 2, 3$.

Example 3.7. The graded algebra $\mathcal{C}(2)$ described in Example 3.6 also has the involution $*_m$, called the *main involution*, that fixes the canonical generators. We use the notation $\mathcal{C}_*(2)$ for $(\mathcal{C}(2), *_m)$. Then $\mathcal{C}_*(2)$ is a structurable $\mathbb{Z}^2$-torus.

Remark 3.8. Unless indicated to the contrary, we will regard $\mathcal{P}(n)$, $\mathcal{C}(n)$ and (if $n = 2$) $\mathcal{C}_*(2)$ as $\mathbb{Z}^n$-graded algebras with involution as above. However it is occasionally convenient to grade these algebras with involution by a free abelian group Λ with basis $\lambda_1, \dots, \lambda_n$ by assigning the degree λ_i to the i^{th} canonical generator. In that case, we say that $\mathcal{P}(n)$, $\mathcal{C}(n)$ or $\mathcal{C}_*(2)$ has the Λ-grading *determined by the basis* $\lambda_1, \dots, \lambda_n$.

Remark 3.9. We sometimes refer to *associative* Λ*-tori* (without involution). By definition these are Λ-graded associative algebras (without involution) satisfying (ST1)–(ST3).

4. Classes I, II and III

For the remainder of this paper, *we assume that* Λ *is a free abelian group of finite rank.*

In [AY], structurable tori with nontrivial involution were placed into 3 classes I, II and III. To recall that trichotomy, we introduce some notation.

Suppose that $\mathcal{A}$ is a structurable Λ-torus. Let
$$S = S(\mathcal{A}) := \mathrm{supp}(\mathcal{A})$$
and
$$S_\sigma = S_\sigma(\mathcal{A}) := \mathrm{supp}(\mathcal{A}_\sigma) = \{\lambda \in S : \mathcal{A}^\lambda \subset \mathcal{A}_\sigma\}.$$
We also set
$$\Lambda_- = \Lambda_-(\mathcal{A}) := \langle S_- \rangle\,,$$

the subgroup generated by S_-. Next let

$$\mathcal{Z} = \mathcal{Z}(\mathcal{A})$$

be the centre of the algebra with involution $\mathcal{A}$ (as defined in (1)). Then $\mathcal{Z}$ is a graded subalgebra of $\mathcal{A}$ and we let

$$\Gamma = \Gamma(\mathcal{A}) := \operatorname{supp}(\mathcal{Z}) = \{\lambda \in S : \mathcal{A}^\lambda \subset \mathcal{Z}\}.$$

By [AY, Prop. 6.7], Γ is a subgroup of Λ called the *central grading group.*

If $S_- = \emptyset$ (that is the involution is trivial), then $\mathcal{A}$ is a Jordan torus [AY, Example 4.3]. The classification of Jordan tori was done by Yoshii in [Y1], and consequently we are interested in this work in the case when $S_- \neq \emptyset$.

If $S_- \neq \emptyset$, let

$$\mathcal{E} := \mathcal{A}^{\Lambda_-} \quad \text{and} \quad \mathcal{W} := \mathcal{A}^{\Lambda \setminus \Lambda_-},$$

so

$$\mathcal{A} = \mathcal{E} \oplus \mathcal{W}$$

with $\mathcal{E}\mathcal{E} \subset \mathcal{E}$ and $\mathcal{E}\mathcal{W} + \mathcal{W}\mathcal{E} \subset \mathcal{W}$. Then, $\mathcal{E}$ is the subalgebra of $\mathcal{A}$ that is generated by $\mathcal{A}_-$ [AY, Prop. 8.1].

Definition 4.1. If $\mathcal{A}$ is a structurable Λ-torus, we say that $\mathcal{A}$ has *class* I, II, or III, if $S_- \neq \emptyset$ and the corresponding condition below holds:

I. $\mathcal{E} = \mathcal{A}$,
II. $\mathcal{E} \neq \mathcal{A}$ and $\mathcal{W}\mathcal{W} \subset \mathcal{E}$,
III. $\mathcal{E} \neq \mathcal{A}$ and $\mathcal{W}\mathcal{W} \not\subset \mathcal{E}$.

Allison and Yoshii [AY, Theorem 9.5] classified structurable tori of class II and obtained basic results which we shall use to obtain the classification in the remaining two classes.

5. Structurable tori of class I

In this section we obtain the construction and classification of structurable tori of class I. We will construct these tori as tensor products of the basic examples described in §3. Thus we begin with a few simple observations about tensor products of graded algebras.

Definition 5.1. If $\mathcal{A}'$ and $\mathcal{A}''$ are algebras with involutions (denoted by $*'$ and $*''$ respectively) then the tensor product algebra $\mathcal{A}' \otimes \mathcal{A}''$ is an algebra with involution $*' \otimes *''$. (Here and subsequently an unadorned symbol $\otimes$ means $\otimes_F$.) Also, if $\mathcal{A}'$ is a Λ'-graded algebra with involution and $\mathcal{A}''$ is a Λ''-graded algebra with involution, then the $\mathcal{A}' \otimes \mathcal{A}''$ is a $(\Lambda' \oplus \Lambda'')$-graded algebra with involution, where

$$(\mathcal{A}' \otimes \mathcal{A}'')^{(\lambda', \lambda'')} = \mathcal{A}'^{\lambda'} \otimes \mathcal{A}''^{\lambda''}.$$

We call this grading on $\mathcal{A}' \otimes \mathcal{A}''$ the *tensor product* grading. These definitions extend in the obvious fashion to tensor products of several graded algebras with involution. Of course all of these definitions are made in the same way for graded algebras without involution.

Lemma 5.2. *Suppose that $\mathcal{A}$ is a finely Λ-graded algebra with involution such that* $\operatorname{supp}(\mathcal{A}) = \Lambda$ *and the product of any two nonzero homogeneous elements of $\mathcal{A}$ is not zero. Suppose Λ' and Λ'' are subgroups of Λ such that $\Lambda = \Lambda' \oplus \Lambda''$,*

$$[\mathcal{A}^{\Lambda'}, \mathcal{A}^{\Lambda''}] = 0 \quad \textit{and} \quad (\mathcal{A}^{\Lambda'}, \mathcal{A}^{\Lambda''}, \mathcal{A}) = (\mathcal{A}, \mathcal{A}^{\Lambda'}, \mathcal{A}^{\Lambda''}) = 0.$$

Then $\mathcal{A} \simeq_\Lambda \mathcal{A}^{\Lambda'} \otimes \mathcal{A}^{\Lambda''}$.

Proof. Clearly, $\eta : \mathcal{A}^{\Lambda'} \otimes \mathcal{A}^{\Lambda''} \to \mathcal{A}$ with $\eta : x \otimes y \to xy$ is a homomorphism of Λ-graded vector spaces. Now, if $x \in \mathcal{A}^{\Lambda'}$ and $y \in \mathcal{A}^{\Lambda''}$, we have

$$(\eta(x \otimes y))^* = (xy)^* = y^* x^* = x^* y^* = \eta(x^* \otimes y^*) = \eta((x \otimes y)^*).$$

Also, if $x_i \in \mathcal{A}^{\Lambda'}$ and $y_i \in \mathcal{A}^{\Lambda''}$, we have

$$\begin{aligned}(x_1y_1)(x_2y_2) &= x_1(y_1(x_2y_2)) = x_1((y_1x_2)y_2) = x_1((x_2y_1)y_2) = x_1(x_2(y_1y_2)) \\ &= (x_1x_2)(y_1y_2)\end{aligned}$$

It remains then to show that η is a bijection. Since η is a homomorphism of finely graded vector spaces, it is enough to show that the restriction of η to $\mathcal{A}^{\lambda'} \otimes \mathcal{A}^{\lambda''}$ is nonzero for $\lambda' \in \Lambda'$ and $\lambda'' \in \Lambda''$. This follows from our assumptions that $\mathrm{supp}(\mathcal{A}) = \Lambda$ and the product of any two nonzero homogeneous elements of $\mathcal{A}$ is not zero. □

We also need the following lemma about base ring extension for tensor products.

Lemma 5.3. *Suppose we are given an F-algebra homomorphism $K_1 \otimes K_2 \to K$, where K_1, K_2 and K are associative and commutative F-algebras. If $\mathcal{A}_i$ is a K_i-algebra for $i = 1, 2$, then*

$$(K \otimes_{K_1} \mathcal{A}_1) \otimes_K (K \otimes_{K_2} \mathcal{A}_2) \simeq K \otimes_{K_1 \otimes K_2} (\mathcal{A}_1 \otimes \mathcal{A}_2). \tag{4}$$

as K-algebras. Moreover, if $\mathcal{A}_i$ is a K_i-algebra with involution for $i = 1, 2$, then (4) *is an isomorphism of K-algebras with involution.*

Proof. Note that, for $i = 1, 2$, we have the F-algebra homomorphism $K_i \to K_1 \otimes K_2 \to K$, so the base ring extensions $K \otimes_{K_i} \mathcal{A}_i$ on the left hand side of (4) make sense. Also, $A_1 \otimes A_2$ is naturally a $K_1 \otimes K_2$-algebra, so the base ring extension on the right hand side also makes sense.

It follows from the universal properties of the tensor product that there exists a unique additive map φ from the left hand side to the right hand side under which

$$(\alpha_1 \otimes_{K_1} a_1) \otimes_K (\alpha_2 \otimes a_2) \mapsto \alpha_1\alpha_2 \otimes_{K_1 \otimes K_2} (a_1 \otimes a_2)$$

for $\alpha_1, \alpha_2 \in K$, $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. Similarly there exists a unique additive map ψ from the right hand side to the left hand side under which

$$\alpha \otimes_{K_1 \otimes K_2} (a_1 \otimes a_2) \mapsto (\alpha \otimes_{K_1} a_1) \otimes_K (1 \otimes a_2)$$

for $\alpha \in K$, $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. One immediately sees that φ and ψ are homomorphisms of K-algebras with $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$. Furthermore, in the involutory case, these maps preserve the involutions. □

Remark 5.4. Lemma 5.3 has an obvious extension (with the same proof) allowing n tensor factors $\mathcal{A}_1, \dots, \mathcal{A}_n$, where $n \geq 1$.

Remark 5.5. (a) If $\mathcal{A}_i$ is a structurable Λ_i-torus for $i = 1, \dots, k$ and if $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_k$ is a structurable algebra, then it is clear that $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_k$ satisfies (ST1)-(ST3) and hence $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_k$ is a structurable $\Lambda_1 \oplus \cdots \oplus \Lambda_k$-torus.

(b) If $\mathcal{A}$ is a structurable Λ-torus, then it follows from (a) that $\mathcal{A} \otimes \mathcal{P}(r)$ is a structurable $\Lambda \oplus \mathbb{Z}^r$-torus for $r \geq 0$.

Finite dimensional simple structurable algebras can be constructed by taking the tensor product of two composition algebras over the base field [K1], [A1, §8]. (In [K1], Kantor worked with *conservative algebras of second degree with left unit* which were later seen to be structurable algebras with a modified product [AH].) We now present an infinite dimensional adaptation of this for class I structurable tori.

Proposition 5.6. *If $0 \le k \le 3$ and $r \ge 0$, then*

$$\mathcal{C}(3) \otimes \mathcal{C}(k) \otimes \mathcal{P}(r)$$

is a structurable $\mathbb{Z}^{3+k+r}$-torus of class I .

Proof. Let $\mathcal{A} = \mathcal{C}(3) \otimes \mathcal{C}(k) \otimes \mathcal{P}(r)$. Then $\mathcal{A}_-$ is spanned by elements of the form $s_1 \otimes h_2 \otimes f$ or $h_1 \otimes s_2 \otimes f$, where s_i is skew and h_i is hermitian. It is clear from this that $\mathcal{A}_-$ generates $\mathcal{A}$ as an algebra. Thus, it suffices to show that $\mathcal{A}$ is a structurable torus. By Remarks 3.3 and 5.5(b), we can assume that $\mathcal{A} = \mathcal{C}(3) \otimes \mathcal{C}(3)$. Furthermore, by Remark 5.5(a), it suffices to show that $\mathcal{A}$ is a structurable algebra. For this, we let $K_1 = K_2 = \mathcal{P}(3)$ and we let K be the quotient field of the integral domain $K_1 \otimes K_2$. Then, by Lemma 5.3, we have

$$(K \otimes_{K_1} \mathcal{C}(3)) \otimes_K (K \otimes_{K_2} \mathcal{C}(3)) \simeq K \otimes_{K_1 \otimes K_2} \mathcal{A}$$

as K-algebras. Now the left hand side is the tensor product of two octonion algebras over the field K and hence it is structurable [A1, §8(iv)]. Thus, since the F-algebra $\mathcal{A}$ embeds in the right hand side, it is structurable. □

We will show in the main theorem of this section (Theorem 5.19) that any class I structurable torus that is not associative is (up to isograded isomorphism) one of the tori described in the preceding proposition. In the same theorem we will show that any class I structurable torus that is associative arises as a tensor product of the tori $\mathcal{C}(1)$, $\mathcal{C}(2)$, $\mathcal{C}_*(2)$ and $\mathcal{P}(r)$.

With this goal in mind, we next discuss some properties of skew-elements in structurable algebras. If $\mathcal{A}$ is a structurable algebra then $\mathcal{A}$ is *skew-alternative*, which means that

$$(s, x, y) = -(x, s, y) = (x, y, s)$$

for $x, y \in \mathcal{A}$ and $s \in \mathcal{A}_-$ [A1, Prop. 1]. It follows that if $s \in \mathcal{A}_-$ and $x \in \mathcal{A}$, we have

$$s(sx) = s^2x, \quad (sx)s = s(xs) \quad \text{and} \quad (xs)s = xs^2.$$

Using these facts we can prove the following two lemmas:

Lemma 5.7. *The following hold in a structurable algebra $\mathcal{A}$ with $r, s, t \in \mathcal{A}_-$, $x, y \in \mathcal{A}$:*

(a) $r(s, r, x) + (r, sr, x) = 0$,

(b) $(r^2, x, y) = (r, rx, y) + r(r, x, y)$,

(c) $(r^2, s, x) = 2r(r, s, x) + (r, [r, s], x)$.

Proof. By expanding associators, (a) becomes $-r(s(rx)) + (r(sr))x = 0$, which is the left Moufang identity [A1, (43)]. Also, after expanding associators, (b) follows from $r(rx) = r^2x$. Letting $x = s$ and $y = x$ in (b), we have by (a)

$$\begin{aligned}(r^2, s, x) &= (r, rs, x) + r(r, s, x)\\ &= (r, rs, x) + r(r, s, x) - r(s, r, x) - (r, sr, x)\\ &= 2r(r, s, x) + (r, [r, s], x)\end{aligned}$$

by skew alternativity. ☐

In the following lemma, we use the notation $x \circ y = xy + yx$.

Lemma 5.8. *Let $\mathcal{A}$ be a structurable algebra that is generated as an algebra by $\mathcal{A}_-$. Then:*

(a) $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_- \circ \mathcal{A}_-$.

(b) If $(\mathcal{A}_-, \mathcal{A}_-, \mathcal{A}_-) = 0$ *then $\mathcal{A}$ is associative.*

(c) If $x \in \mathcal{A}$ and $[x, \mathcal{A}] = 0$, then $(x, \mathcal{A}, \mathcal{A}) = (\mathcal{A}, x, \mathcal{A}) = (\mathcal{A}, \mathcal{A}, x) = 0$.

Proof. (a) has been shown in [A1, Lemma 14]. In particular, $\mathcal{A}_+$ is spanned by r^2 with $r \in \mathcal{A}_-$. For (b), assume that $(\mathcal{A}_-, \mathcal{A}_-, \mathcal{A}_-) = 0$. Since $[r, s] \in \mathcal{A}_-$, letting $x = t$ in Lemma 5.7(c) gives $(r^2, s, t) = 0$. Hence, $(\mathcal{A}, \mathcal{A}_-, \mathcal{A}_-) = 0$, so also $(\mathcal{A}_-, \mathcal{A}, \mathcal{A}_-) = (\mathcal{A}_-, \mathcal{A}_-, \mathcal{A}) = 0$, by skew alternativity. Now Lemma 5.7(c) gives $(r^2, s, x) = 0$. Thus $(\mathcal{A}, \mathcal{A}_-, \mathcal{A}) = 0$, so also $(\mathcal{A}_-, \mathcal{A}, \mathcal{A}) = (\mathcal{A}, \mathcal{A}, \mathcal{A}_-) = 0$. Lemma 5.7(b) now gives $(r^2, x, y) = 0$, so $(\mathcal{A}, \mathcal{A}, \mathcal{A}) = 0$. Finally, (c) is proved in [A1, Lemma 24]. ☐

For the rest of this section, *we assume that $\mathcal{A}$ is a class I structurable Λ-torus. We use the notation S, S_σ, Λ_-, $\mathcal{Z}$, Γ, ... from §4.*

We will make frequent use of the following facts from [AY, Prop. 8.1 and 8.11] for class I tori:

- $\mathcal{A}$ is generated as an algebra by $\mathcal{A}_-$.
- $S = \Lambda$.
- If $0 \neq s \in \mathcal{A}_-^\lambda$, then $s^2 \in \mathcal{Z}$.

Lemma 5.9. *Let $\mathcal{A}$ be a structurable Λ-torus of class I and let r, h, x, y be homogeneous in $\mathcal{A}$ with $r \in \mathcal{A}_-$ and $h \in \mathcal{A}_+$. Then:*

(a) If $[r, x] = 0$, then r, x and y associate in all orders.

(b) $h = st$ for some commuting homogenous elements s, t of $\mathcal{A}_-$.

(c) If $x \neq 0 \neq y$, then $xy \neq 0$.

Proof. For (a), we can assume that $r \neq 0$. Suppose first that $x = s \in \mathcal{A}_-$ and $[r, s] = 0$. Since $r^2 \in \mathcal{Z}$, Lemma 5.7(c) shows $(r, s, y) = 0$. Suppose next that $x = h \in \mathcal{A}_+$ and $[r, h] = 0$. Then $r(r^{-1}h - hr^{-1})r = r(r^{-1}h)r - r(hr^{-1})r = hr - rh = 0$, so r^{-1} commutes with h. Set $s = hr^{-1} = r^{-1}h \in \mathcal{A}_-$ in which case $[r, s] = 0$. Now the first case and Lemma 5.7(a) give $(r, h, y) = (r, sr, y) = -r(s, r, y) = 0$. Thus, $(r, x, y) = 0$ in all cases. Applying the involution gives $(y, x, r) = 0$ and (a) holds by skew alternativity.

For (b), we know from Lemma 5.8(a) that we can write $h = st$ with s, t homogeneous in $\mathcal{A}_-$. Also, $h = h^* = ts$, so $[s, t] = 0$.

Clearly (c) holds if $x \in \mathcal{A}_-$. If $x = h \in \mathcal{A}_+$, we can write $h = st$ as in (b). Now $hy = (st)y = s(ty) \neq 0$ by (a). ☐

We next define functions

$$\varepsilon : \Lambda \to \{\pm 1\}, \quad \beta : \Lambda \times \Lambda \to F, \quad \alpha : \Lambda \times \Lambda \times \Lambda \to F, \quad \mu : \Lambda \times \Lambda \times \Lambda \to \{\pm 1\}.$$

We will see in the next lemma that β and α also take values in $\{\pm 1\}$.

Since the support of $\mathcal{A}$ is Λ, if $\lambda \in \Lambda$ we can choose a unique $\varepsilon(\lambda) = \pm 1$ so that

$$x^* = \varepsilon(\lambda)x$$

for $x \in \mathcal{A}^\lambda$. Also, if $\lambda_i \in \Lambda$, we can define $\beta(\lambda_1,\lambda_2) \in F^\times$ and $\alpha(\lambda_1,\lambda_2,\lambda_3) \in F^\times$ by

$$\begin{aligned} x_2x_1 &= \beta(\lambda_1,\lambda_2)x_1x_2, \\ x_1(x_2x_3) &= \alpha(\lambda_1,\lambda_2,\lambda_3)(x_1x_2)x_3, \end{aligned}$$

for $x_i \in \mathcal{A}^{\lambda_i}$. The scalars $\beta(\lambda_1,\lambda_2)$ and $\alpha(\lambda_1,\lambda_2,\lambda_3)$ are uniquely determined by Lemma 5.9(c). We can view β as a *multiplicative commutator* and α as a *multiplicative associator*. We also define

$$\mu(\lambda_1,\lambda_2,\lambda_3) = \varepsilon(\lambda_1)\varepsilon(\lambda_2)\varepsilon(\lambda_3)\varepsilon(\lambda_1+\lambda_2)\varepsilon(\lambda_1+\lambda_3)\varepsilon(\lambda_2+\lambda_3)\varepsilon(\lambda_1+\lambda_2+\lambda_3).$$

We set

$$\bar{\Lambda} = \Lambda/2\Lambda$$

and denote the canonical projection from Λ to $\bar{\Lambda}$ by $\lambda \mapsto \bar{\lambda}$. We regard $\bar{\Lambda}$ as a vector space over the field $\mathbb{Z}_2 = \{0,1\}$. Since $s^2 \in \mathcal{Z}$ for homogeneous $s \in \mathcal{A}^-$ and since $\mathcal{A}^-$ generates $\mathcal{A}$, we have $2\Lambda \subset \Gamma$, the support of $\mathcal{Z}$. Thus, ε, α, β, and μ are unchanged by adding an element $\gamma \in 2\Lambda$ to any one of their arguments. So ε, α β and μ induce maps $\varepsilon : \bar{\Lambda} \to \{\pm 1\}$, $\beta : \bar{\Lambda}\times\bar{\Lambda} \to F$, $\alpha : \bar{\Lambda}\times\bar{\Lambda}\times\bar{\Lambda} \to F$ and $\mu : \bar{\Lambda}\times\bar{\Lambda}\times\bar{\Lambda} \to \{\pm 1\}$ with arguments from the $\mathbb{Z}_2$-vector space $\bar{\Lambda}$.

Lemma 5.10. *Let $\mathcal{A}$ be a structurable Λ-torus of class I and let $\lambda_i \in \Lambda$ for $1 \le i \le 4$. Then:*

(a) The commutator β takes values in $\{\pm 1\}$, and

$$\beta(\lambda_2,\lambda_1) = \beta(\lambda_1,\lambda_2) = \varepsilon(\lambda_1)\varepsilon(\lambda_2)\varepsilon(\lambda_1+\lambda_2).$$

(b) $\mu(\lambda_1,\lambda_2,\lambda_3) = \beta(\lambda_1+\lambda_2,\lambda_3)\beta(\lambda_1,\lambda_3)\beta(\lambda_2,\lambda_3)$.
(c) $\mu(\lambda_1,\lambda_2,\lambda_3) = \alpha(\lambda_1,\lambda_2,\lambda_3)\alpha(\lambda_3,\lambda_1,\lambda_2)\alpha(\lambda_2,\lambda_3,\lambda_1)$.
(d) $\alpha(\lambda_2,\lambda_3,\lambda_4)\alpha(\lambda_1+\lambda_2,\lambda_3,\lambda_4)^{-1}\alpha(\lambda_1,\lambda_2+\lambda_3,\lambda_4)$
$$\alpha(\lambda_1,\lambda_2,\lambda_3+\lambda_4)^{-1}\alpha(\lambda_1,\lambda_2,\lambda_3) = 1.$$
(e) If $\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3$ are dependent then $\mu(\lambda_1,\lambda_2,\lambda_3) = 1$.
(f) If $\varepsilon(\lambda_i) = -1$ and $\beta(\lambda_i,\lambda_j) = 1$ for some $i \ne j$ then $\alpha(\lambda_1,\lambda_2,\lambda_3) = 1$
(g) If some $\varepsilon(\lambda_i) = -1$ then $\alpha(\lambda_1,\lambda_2,\lambda_3) = \mu(\lambda_1,\lambda_2,\lambda_3)$.
(h) The associator α take values in $\{\pm 1\}$ and $\alpha(\lambda_3,\lambda_2,\lambda_1) = \alpha(\lambda_1,\lambda_2,\lambda_3)$.

Proof. To simplify notation, write $\varepsilon_1 = \varepsilon(\lambda_1)$, $\varepsilon_{12} = \varepsilon(\lambda_1+\lambda_2)$, $\beta_{1,2} = \beta(\lambda_1,\lambda_2)$, $\beta_{12,3} = \beta(\lambda_1+\lambda_2,\lambda_3)$, etc.

For (a), if $x_i \in \mathcal{A}^{\lambda_i}$, then $x_2x_1 = (x_1^*x_2^*)^* = \varepsilon_1\varepsilon_2\varepsilon_{12}x_1x_2$. Thus, $\beta_{1,2} = \varepsilon_1\varepsilon_2\varepsilon_{12} \in \{\pm 1\}$, so $\beta_{2,1} = \beta_{1,2}$.

(b) follows by writing β in terms of ε using (a).

For (c), we have $(x_2x_3)x_1 = \beta_{1,23}x_1(x_2x_3) = \beta_{1,23}\alpha_{1,2,3}(x_1x_2)x_3$, so

$$\gamma_{1,2,3}\gamma_{3,1,2}\gamma_{2,3,1} = 1$$

for $\gamma_{1,2,3} = \beta_{1,23}\alpha_{1,2,3}$. But using (a), we have $\beta_{1,23}\beta_{3,12}\beta_{2,31} = \mu_{1,2,3} = \mu_{1,2,3}^{-1}$, so (c) follows.

For (d), we have

$$\begin{aligned} x_1(x_2(x_3x_4)) &= \alpha_{2,3,4}x_1((x_2x_3)x_4) \\ &= \alpha_{2,3,4}\alpha_{1,23,4}(x_1(x_2x_3))x_4 \\ &= \alpha_{2,3,4}\alpha_{1,23,4}\alpha_{1,2,3}((x_1x_2)x_3)x_4 \\ &= \alpha_{2,3,4}\alpha_{1,23,4}\alpha_{1,2,3}\alpha_{12,3,4}^{-1}(x_1x_2)(x_3x_4) \\ &= \alpha_{2,3,4}\alpha_{1,23,4}\alpha_{1,2,3}\alpha_{12,3,4}^{-1}\alpha_{1,2,34}^{-1}x_1(x_2(x_3x_4)). \end{aligned}$$

For (e), let $U = \mathbb{Z}_2^3$ have basis u_i and let φ be the linear map with $\varphi(u_i) = \bar{\lambda}_i$. Since $\varepsilon(0) = 1$, we see that

$$\mu_{1,2,3} = \prod_{u\in U} \varepsilon(\varphi(u)).$$

If $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$ are dependent, then each $\varphi(u)$ occurs $|\ker(\varphi)| = 2, 4$, or 8 times, so $\mu_{1,2,3} = 1$.

(f) is a restatement of Lemma 5.9(a).

Before continuing, we observe that applying the involution to the defining equation for α yields

$$\alpha_{1,2,3} = \alpha_{3,2,1}^{-1}. \tag{5}$$

Now for (g) suppose that $\varepsilon_i = -1$. If $\beta_{i,j} = 1$ for some $j \neq i$, then each factor on the right side of (c) is 1, so $\mu_{1,2,3} = 1 = \alpha_{1,2,3}$. Thus, we may assume that $\beta_{i,j} = \beta_{i,k} = -1$ for $\{i, j, k\} = \{1, 2, 3\}$. Note that

$$(x_i, x_j, x_k) = (1 - \alpha_{i,j,k})(x_ix_j)x_k,$$

whereas

$$(x_i, x_j, x_k) = -(x_j, x_i, x_k) = -(1 - \alpha_{j,i,k})(x_jx_i)x_k = (1 - \alpha_{j,i,k})(x_ix_j)x_k,$$

so $\alpha_{i,j,k} = \alpha_{j,i,k} = \alpha_{k,i,j}^{-1}$. Replacing, $1, 2, 3$ in (c) by i, j, k gives $\alpha_{j,k,i} = \mu_{i,j,k} = \mu_{1,2,3}$. Interchanging the roles of j and k, we also have $\alpha_{i,k,j} = \alpha_{k,i,j}$ and $\alpha_{k,j,i} = \mu_{1,2,3}$. Since $\alpha_{p,q,r} = \alpha_{r,q,p}^{-1}$, we see $\alpha_{p,q,r} = \mu_{1,2,3}$ for all $\{p, q, r\} = \{i, j, k\}$. In particular, $\alpha_{1,2,3} = \mu_{1,2,3}$.

For (h) it follows from (g) that $\alpha_{1,2,3} = \pm 1$ if $\varepsilon_1 = -1$. If $\varepsilon(\lambda) = 1$, by Lemma 5.9(b) we can write $\lambda = \lambda_1 + \lambda_2$ with $\varepsilon_1 = \varepsilon_2 = -1$. Now (d) shows $\alpha(\lambda, \lambda_3, \lambda_4) = \pm 1$. Thus by (5) we have (h). □

Notation 5.11. We now introduce some notation and terminology that we will apply both to the pair (Λ, ε) and the pair $(\bar{\Lambda}, \varepsilon)$.[2] To do this efficiently, suppose that (M, ε) is a pair consisting of an abelian group M and a map $\varepsilon : M \to \{\pm 1\}$ so that

$$\varepsilon(0) = 1 \quad \text{and} \quad \varepsilon \text{ is constant on cosets of } 2M. \tag{6}$$

This last condition means that $\varepsilon(u) = \varepsilon(v)$ whenever $u - v \in 2M$. Let $\beta : M \times M \to \{\pm 1\}$ be defined by

$$\beta(u, v) = \varepsilon(u + v)\varepsilon(u)\varepsilon(v).$$

If (M', ε) is another pair satisfying (6), we say (M, ε) is *isomorphic to* (M', ε'), written $(M, \varepsilon) \simeq (M', \varepsilon')$, if there is a group isomorphism $\varphi : M \to M'$ with $\varepsilon'(\varphi(u)) = \varepsilon(u)$ for $u \in M$. If $u \in M$, we say that u is *anisotropic* if $\varepsilon(u) = -1$.

[2]The motivation for the terminology comes from the case when $(M, \varepsilon) = (\bar{\Lambda}, \varepsilon)$ and $\mathcal{A}$ is associative. In that case, ε is a multiplicative version of a quadratic form (see Lemma 5.14) and the terminology is standard.

If N is a subgroup of M, we say that N is anisotropic if all vectors $u \in N$ that are not in $2N$ are anisotropic. Furthermore we say that the pair (M, ε) is anisotropic if M an anisotropic subgroup of itself. If $u, v \in M$, we say u is *orthogonal* to v, written $u \perp v$, if $\beta(u, v) = 1$.

Finally suppose that $M_1, \dots, M_k$ are subgroups of M. If $M = M_1 \oplus \cdots \oplus M_k$ and $\beta(M_i, \sum_{j \neq i} M_j) = 1$ for all i, we write $M = M_1 \perp \cdots \perp M_k$. If $M = M_1 \oplus \cdots \oplus M_k$ it is easy to check that

$$\begin{aligned} M = M_1 \perp \cdots \perp M_k &\iff \varepsilon(\sum_i u_i) = \prod_i \varepsilon(u_i) \text{ for } u_i \in M_i \\ &\iff \beta(M_i, \sum_{j > i} M_j) = 1 \text{ for all } i < k. \end{aligned}$$

Lemma 5.12. *If $V_1, \dots, V_k$ are subgroups of $\bar{\Lambda}$ with $\bar{\Lambda} = V_1 \perp \cdots \perp V_k$, then there exist subgroups $\Lambda_1, \dots, \Lambda_k$ of Λ so that $\Lambda = \Lambda_1 \perp \cdots \perp \Lambda_k$ and $\bar{\Lambda}_i = V_i$ for $i = 1, \dots, k$.*

Proof. We can assume without loss of generality that $\Lambda = \mathbb{Z}^n$ and identify $\bar{\Lambda} = \mathbb{Z}_2^n$. We first note that $\mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}_2)$ is surjective. Indeed, $\mathrm{GL}(n, \mathbb{Z}_2) = \mathrm{SL}(n, \mathbb{Z}_2) = \mathrm{E}(n, \mathbb{Z}_2)$, the elementary group, and $\mathrm{E}(n, \mathbb{Z}) \to \mathrm{E}(n, \mathbb{Z}_2)$ is surjective.

For $1 \leq i \leq k$, let $d_i = \dim(V_i)$ and choose a basis $\bar{B}_i$ for the subspace V_i of $\bar{\Lambda}$. Let $\bar{A}$ be the element of $\mathrm{GL}(n, \mathbb{Z}_2)$ whose rows are the elements of $\bar{B}_1, \dots, \bar{B}_k$ in order. Choose $A \in \mathrm{GL}(n, \mathbb{Z})$ with $A \to \bar{A}$. Let B_1 be the set of the first d_1 rows of A, let B_2 be the set of the next d_2 rows of A, and so on. Let Λ_i be the subgroup of Λ generated by B_i for $i = 1, \dots, k$. Clearly, $\Lambda_1, \dots, \Lambda_k$ have the properties claimed. □

Lemma 5.13. *Let $\mathcal{A}$ be a structurable Λ-torus of class I. Then the dimension of an anisotropic subspace of $\bar{\Lambda}$ is at most 3. Moreover, for $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$, we have $\alpha(\lambda_1, \lambda_2, \lambda_3) = \varepsilon(\lambda_1) = \varepsilon(\lambda_2) = \varepsilon(\lambda_3) = -1$ if and only if $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$ is a basis for an anisotropic subspace of $\bar{\Lambda}$.*

Proof. If $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$ is a basis for an anisotropic subspace, then $\alpha(\lambda_1, \lambda_2, \lambda_3) = \mu(\lambda_1, \lambda_2, \lambda_3) = -1$ by Lemma 5.10(g) and the definition of μ. Thus, $\bar{\Lambda}$ does not contain an anisotropic subspace of dimension 4, since otherwise, using Lemma 5.10(d), we would have $(-1)^5 = 1$.

It remains to show that if $\alpha(\lambda_1, \lambda_2, \lambda_3) = \varepsilon(\lambda_i) = -1$, then $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$ is a basis for an anisotropic subspace of $\bar{\Lambda}$. Indeed Lemma 5.10(f) shows $\beta(\lambda_i, \lambda_j) = -1$ for $i \neq j$, so $\varepsilon(\lambda_i + \lambda_j) = -1$. Also, by Lemma 5.10(g), $\mu(\lambda_1, \lambda_2, \lambda_3) = -1$, so $\varepsilon(\lambda_1 + \lambda_2 + \lambda_3) = -1$ by the definition of μ. Thus, $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$ span an anisotropic subspace V. Moreover, $\dim(V) = 3$ by Lemma 5.10(e). □

Lemma 5.14. *Let $\mathcal{A}$ be a structurable Λ-torus of class I and suppose that $\mathcal{A}$ is associative. Let $\hat{\varepsilon} : \bar{\Lambda} \to \mathbb{Z}_2$ and $\hat{\beta} : \bar{\Lambda} \times \bar{\Lambda} \to \mathbb{Z}_2$ be the additive versions of the maps ε and β on $\bar{\Lambda}$; that is, define $\hat{\varepsilon} : \bar{\Lambda} \to \mathbb{Z}_2$ and $\hat{\beta} : \bar{\Lambda} \times \bar{\Lambda} \to \mathbb{Z}_2$ by*

$$\varepsilon(v) = (-1)^{\hat{\varepsilon}(v)} \quad \textit{and} \quad \beta(v_1, v_2) = (-1)^{\hat{\beta}(v_1, v_2)}$$

for $v, v_1, v_2 \in \bar{\Lambda}$. Then $\hat{\varepsilon}$ is a quadratic form over $\mathbb{Z}_2$ with linearization $\hat{\beta}$. (A quadratic form over $\mathbb{Z}_2$ is defined exactly as in §2 even though $\mathbb{Z}_2$ has characteristic 2.)

Proof. By Lemma 5.10(a), we have $\hat{\beta}(v_1, v_2) = \hat{\varepsilon}(v_1 + v_2) + \hat{\varepsilon}(v_1) + \hat{\varepsilon}(v_2)$. In other words $\hat{\beta}$ is the linearization of $\hat{\varepsilon}$. Also by Lemma 5.10(b) and (c), we have $\beta(v_1 +$

$v_2, v_3) = \beta(v_1, v_3)\beta(v_2, v_3)$, so $\hat{\beta}(v_1 + v_2, v_3) = \hat{\beta}(v_1, v_3) + \hat{\beta}(v_2, v_3)$. Hence $\hat{\varepsilon}$ is a quadratic form. $\square$

Notation 5.15. In the next results, we will make use of two particular pairs (M, ε) that satisfy the conditions (6) above. If $r \geq 0$ we let $(\mathbb{Z}^r, \varepsilon_-)$ be the anisotropic pair with underlying group $\mathbb{Z}^r$. In other words

$$\varepsilon_-(\lambda) = \begin{cases} 1 & \text{if } \lambda \in 2\mathbb{Z}^r \\ -1 & \text{otherwise.} \end{cases}$$

(If $r = 0$, $\mathbb{Z}^r = 0$ and $\varepsilon_- = 1$.) We let $(\mathbb{Z}^2, \varepsilon_0)$ be the pair with

$$\varepsilon_0(\lambda_1) = \varepsilon_0(\lambda_2) = 1 \quad \text{and} \quad \varepsilon_0(\lambda_1 + \lambda_2) = -1,$$

where λ_1, λ_2 is the standard basis for $\mathbb{Z}^2$.

Proposition 5.16. *Let $\mathcal{A}$ be a class I structurable Λ-torus. If $\mathcal{A}$ is associative, then there exist subgroups $\Lambda'_1, \ldots, \Lambda'_k, \Lambda'', \Lambda'''$ of Λ so that*

$$\Lambda = \Lambda'_1 \perp \ldots \perp \Lambda'_k \perp \Lambda'' \perp \Lambda''',$$

where $k \geq 0$,

$$\begin{aligned} &(\Lambda'_i, \varepsilon) \simeq (\mathbb{Z}^2, \varepsilon_-) \textit{ for } 1 \leq i \leq k, \\ &(\Lambda'', \varepsilon) \simeq 0, (\mathbb{Z}, \varepsilon_-) \textit{ or } (\mathbb{Z}^2, \varepsilon_0) \textit{ with } (\Lambda'', \varepsilon) \simeq (\mathbb{Z}, \varepsilon_-) \textit{ if } k = 0, \\ &\Lambda''' \subset \Gamma. \end{aligned}$$

Proof. Let $\hat{\varepsilon} : \bar{\Lambda} \to \mathbb{Z}_2$ and $\hat{\beta} : \bar{\Lambda} \times \bar{\Lambda} \to \mathbb{Z}_2$ be defined as in Lemma 5.14. The classification of quadratic forms over $\mathbb{Z}_2$ (see [D, Chapitre I, §16] or Remark 5.17 below) shows that

$$\bar{\Lambda} = V'_1 \perp \ldots \perp V'_k \perp V'' \perp V'''$$

where $k \geq 0$, each V'_i is anisotropic of dimension 2, V'' is either 0, anisotropic of dimension 1, or has a basis v_1, v_2 with $\hat{\varepsilon}(v_1) = \hat{\varepsilon}(v_2) = 0$ and $\hat{\beta}(v_1, v_2) = 1$, and $\hat{\varepsilon}(V''') = 0$. Moreover, since $\mathcal{A}$ has class I, $\bar{\Lambda}$ is spanned by anisotropic vectors. Thus, if $k = 0$, V'' must be anisotropic. The result now follows from Lemma 5.12. $\square$

Remark 5.17. The classification of quadratic forms over $\mathbb{Z}_2$ can be done as follows. Suppose that $\hat{\varepsilon}$ is an arbitrary quadratic form on a finite dimensional space V over $\mathbb{Z}_2$, and let $\hat{\beta}$ be the linearization of $\hat{\varepsilon}$. Since $\hat{\beta}$ is an alternating bilinear form, V is the orthogonal direct sum, relative to $\hat{\beta}$, of copies of F with basis x and copies of the hyperbolic plane H with basis x, y and $\hat{\beta}(x, y) = 1$. The quadratic form $\hat{\varepsilon}$ is determined by specifying $(F, \hat{\varepsilon}(x))$ or $(H, \hat{\varepsilon}(x), \hat{\varepsilon}(y))$ for each copy. We have

(a) $(F, 1) \perp (F, 1) \simeq (F, 1) \perp (F, 0)$,
(b) $(H, 1, 0) \simeq (H, 0, 0)$,
(c) $(H, 0, 0) \perp (H, 0, 0) \simeq (H, 1, 1) \perp (H, 1, 1)$,
(d) $(H, 0, 0) \perp (F, 1) \simeq (H, 1, 1) \perp (F, 1)$.

Indeed, for (a) we can start with a basis x_1, x_2 for the left hand side, where x_1 is a basis for the first summand and x_2 is a basis for the second. The new basis $x_1, x_1 + x_2$ then gives a decomposition as on the right hand side. Similarly (with obvious notation): for (b) use $x + y, y$, for (c) use $x_1 + y_1, x_1 + x_2 + y_2, x_2 + y_2, x_1 + y_1 + x_2$, and for (d) use x_1+x_2, y_1+x_2, x_2. It follows from (a), (b), (c) and (d) that we can arrange to have no copies of $(H,1,0)$ and at most one copy of $(F,1)$ or $(H,0,0)$, but not both.

Proposition 5.18. *Let $\mathcal{A}$ be a class I structurable Λ-torus. If $\mathcal{A}$ is not associative, then there exist subgroups Λ', Λ'' and Λ''' of Λ so that*

$$\Lambda = \Lambda' \perp \Lambda'' \perp \Lambda'''$$

where $(\Lambda',\varepsilon) \simeq (\mathbb{Z}^3, \varepsilon_-)$, $(\Lambda'',\varepsilon) \simeq (\mathbb{Z}^k, \varepsilon_-)$ with $0 \le k \le 3$, and $\Lambda''' \subset \Gamma$.

Proof. In view of Lemmas 5.12 and 5.13, it suffices to show that there are subspaces V', V'' and V''' of $\bar{\Lambda}$ such that V and V' are anisotropic of dimensions 3 and k respectively, $V''' \subset \Gamma/2\Lambda$ and

$$\bar{\Lambda} = V' \perp V'' \perp V'''.$$

Since $\mathcal{A}$ is not associative, there are $s_i \in \mathcal{A}^-$ with $(s_1,s_2,s_3) \neq 0$ by Lemma 5.8(b). We can assume s_i is homogeneous with $s_i \in \mathcal{A}^{\sigma_i}$. Thus, $\varepsilon(\sigma_i) = -1$ and $\alpha(\sigma_1,\sigma_2,\sigma_3) = -1$. Let V' be the anisotropic subspace with basis $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3$ as in Lemma 5.13.

We next observe that if $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ and some $\varepsilon(\lambda_i) = -1$ then

$$\lambda_1 \perp \lambda_3,\ \lambda_2 \perp \lambda_3 \implies (\lambda_1+\lambda_2) \perp \lambda_3 \tag{7}$$

Indeed we have $\mu(\lambda_1,\lambda_2,\lambda_3) = \alpha(\lambda_1,\lambda_2,\lambda_3) = 1$ by Lemma 5.10(g) and (f). Thus, $\beta(\lambda_1+\lambda_2,\lambda_3) = \beta(\lambda_1,\lambda_3)\beta(\lambda_2,\lambda_3) = 1$ by Lemma 5.10(b), so $(\lambda_1+\lambda_2) \perp \lambda_3$.

Now let

$$W = V'^{\perp} = \{\bar{\lambda} \in \bar{\Lambda} : \beta(\bar{\lambda}, V') = 1\}.$$

Then, it follows from (7) that W is a subspace of $\bar{\Lambda}$. We will show that

$$\bar{\Lambda} = V' \perp W. \tag{8}$$

Indeed, $V' \cap W = \{0\}$ since $\beta(\lambda_1,\lambda_2) = -1$ for any two distinct nonzero elements $\bar{\lambda}_1$, $\bar{\lambda}_2$ in V'. Moreover, $\beta(V',W) = 1$ by definition of W. So to prove (8) it remains to show that $\bar{\Lambda} = V' + W$. Since S_- spans Λ, we can do this by showing that $\bar{\sigma} \in V' + W$ for all $\bar{\sigma} \notin V' \cup W$ with $\varepsilon(\sigma) = -1$. Since $\bar{\sigma} \notin W$, we have $\beta(\sigma,\lambda_3) = -1$ and hence $\varepsilon(\lambda_3+\sigma) = -1$ for some $\bar{\lambda}_3 \in V'$. Let $\bar{\mu}_1$, $\bar{\mu}_2$, $\bar{\mu}_3$ be a basis for V'. Since $\alpha(\mu_1,\mu_2,\mu_3) = -1$, Lemma 5.10(d) gives

$$\alpha(\mu_2,\mu_3,\sigma)\alpha(\mu_1+\mu_2,\mu_3,\sigma)\alpha(\mu_1,\mu_2+\mu_3,\sigma)\alpha(\mu_1,\mu_2,\mu_3+\sigma) = -1.$$

At least one of the factors on the left hand side of this equality must equal -1, so we can replace σ by $\mu_3+\sigma$ or rechoose the basis $\bar{\mu}_1$, $\bar{\mu}_2$, $\bar{\mu}_3$ of V' to get $\alpha(\mu_1,\mu_2,\sigma) = -1$. Thus, by Lemma 5.13, $X + \mathbb{Z}_2\bar{\sigma}$ is anisotropic where $X = \mathbb{Z}_2\bar{\mu}_1 + \mathbb{Z}_2\bar{\mu}_2$. Since $V' + \mathbb{Z}_2\bar{\sigma}$ is not anisotropic by Lemma 5.13, there is some $\bar{\tau} \in V' \setminus X$ with $\varepsilon(\tau+\sigma) = 1$. Then Lemma 5.10(a) shows

$$\beta(\tau+\sigma,\tau) = 1.$$

Next, if $i = 1,2$, the space $\mathbb{Z}_2\bar{\tau} + \mathbb{Z}_2\bar{\sigma} + \mathbb{Z}_2\bar{\mu}_i$ is not anisotropic since $\varepsilon(\tau+\sigma) = 1$. Therefore, by Lemma 5.13, we have $\alpha(\tau,\sigma,\mu_i) = 1$. So, by Lemma 5.10(g), $\mu(\tau,\sigma,\mu_i) = 1$, and hence, by Lemma 5.10(b), $\beta(\tau+\sigma,\mu_i) = \beta(\tau,\mu_i)\beta(\sigma,\mu_i)$. But, since V' and $X + \mathbb{Z}_2\bar{\sigma}$ are anisotropic, we have $\beta(\tau,\mu_i) = -1$ and $\beta(\sigma,\mu_i) = -1$. So

$$\beta(\tau+\sigma,\mu_i) = 1$$

for $i = 1, 2$. Since $\bar\tau$, $\bar\mu_1$, $\bar\mu_2$ span V', we see that $\bar\tau + \bar\sigma \in W$ by (7). Thus, $\bar\sigma = \bar\tau + (\bar\tau + \bar\sigma) \in V' + W$, proving (8).

Next since $\beta(V', W) = 1$, we have $\varepsilon(\lambda + \tau) = \varepsilon(\lambda)\varepsilon(\tau)$ for $\bar\lambda \in W$ and $\bar\tau \in V'$. Thus, for $\bar\lambda_i \in W$ and $\bar\tau_i \in V'$, we have

$$\mu(\lambda_1 + \tau_1, \lambda_2 + \tau_2, \lambda_3 + \tau_3) = \mu(\lambda_1, \lambda_2, \lambda_3)\mu(\tau_1, \tau_2, \tau_3) \tag{9}$$

and

$$\beta(\lambda_1 + \tau_1, \lambda_2 + \tau_2) = \beta(\lambda_1, \lambda_2)\beta(\tau_1, \tau_2). \tag{10}$$

Now let

$$V''' = \{\bar\lambda \in W : \varepsilon(\lambda) = 1\}.$$

We first argue that

$$\beta(V''', V) = 1. \tag{11}$$

Indeed if $\bar\lambda_1 \in V'''$ and $\bar\lambda_2 \in W$, we have

$$\mu(\lambda_1 + \sigma_1, \lambda_2 + \sigma_2, \sigma_3) = \mu(\lambda_1, \lambda_2, 0)\mu(\sigma_1, \sigma_2, \sigma_3) = (1)(-1) = -1$$

by (9), and

$$\varepsilon(\lambda_1 + \sigma_1) = \varepsilon(\lambda_1)\varepsilon(\sigma_1)\beta(\lambda_1, \sigma_1) = (1)(-1)(1) = -1.$$

(Here $\bar\sigma_1, \bar\sigma_2, \bar\sigma_3$ is the basis for V' chosen at the beginning of this proof.) So by Lemma 5.10(f) and (g), we have $\beta(\lambda_1 + \sigma_1, \lambda_2 + \sigma_2) = -1$. Thus, by (10), $\beta(\lambda_1, \lambda_2)\beta(\sigma_1, \sigma_2) = -1$, so

$$\beta(\lambda_1, \lambda_2) = 1.$$

Then, if $\bar\lambda_1 \in V'''$, $\bar\lambda_2 \in W$ and $\bar\tau \in V'$, we have using (10) that $\beta(\lambda_1, \lambda_2 + \tau) = \beta(\lambda_1, \lambda_2)\beta(0, \tau) = (1)(1) = 1$ which shows (11).

Next if $\bar\lambda_1, \bar\lambda_2 \in V'''$ then $\beta(\lambda_1, \lambda_2) = 1$ by (11), so $\varepsilon(\lambda_1 + \lambda_2) = \varepsilon(\lambda_1)\varepsilon(\lambda_2) = 1$. This shows that V''' is a subspace of W.

We next show that

$$V''' \subset \Gamma/2\Lambda \tag{12}$$

For this suppose that $\bar\lambda_1 \in V'''$. Then, by (11), $\beta(\lambda_1, \Lambda) = 1$. Hence, if $x \in \mathcal{A}^{\lambda_1}$, we have $[x, \mathcal{A}] = 0$. So, by Lemma 5.8(c), x associates with any two elements of $\mathcal{A}$. Therefore, since $\varepsilon(\bar\lambda_1) = 1$, we have $\lambda_1 \in \Gamma$, proving (12).

Now let V'' be a maximal anisotropic subspace of W. It remains to show that $W = V'' \perp V'''$. Certainly $V'' \cap V''' = 0$ and, by (11), $\beta(V'', V''') = 1$. So we only have to show that $W = V'' + V'''$. To do this let $\bar\lambda \in W$ with $\bar\lambda \notin V'' \cup V'''$. Then $\varepsilon(\lambda) = -1$. Also, $\mathbb{Z}_2\bar\lambda + V''$ is not anisotropic, so $\varepsilon(\lambda + \sigma) = 1$ for some $\bar\sigma \in V''$. Thus, $\bar\lambda = -\bar\sigma + (\bar\lambda + \bar\sigma) \in V'' + V'''$. □

The following is the main result of this section:

Theorem 5.19. *Let $\mathcal{A}$ be a structurable Λ-torus of class I. Then:*

(a) If $\mathcal{A}$ is associative, then $\mathcal{A}$ is isograded isomorphic to

$$\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_k \otimes \mathcal{A}_{k+1} \otimes \mathcal{P}(r)$$

where $k \geq 0, r \geq 0$, and

$$\mathcal{A}_i = \mathcal{C}(2) \text{ for } 1 \leq i \leq k, \tag{13}$$

$$\mathcal{A}_{k+1} = F,\ \mathcal{C}(1),\ \text{or } \mathcal{C}_*(2) \text{ with } \mathcal{A}_{k+1} = \mathcal{C}(1) \text{ if } k = 0. \tag{14}$$

(b) If $\mathcal{A}$ is not associative, then $\mathcal{A}$ is isograded isomorphic to

$$\mathcal{C}(3) \otimes \mathcal{C}(k) \otimes \mathcal{P}(r)$$

where $0 \le k \le 3$ *and* $r \ge 0$.

Conversely, each of the graded algebras with involution in (a) and (b) are class I structurable tori.

Proof. We prove (a) and (b) together. First, we have orthogonal decompositions of Λ as in Theorems 5.16 and 5.18. By Remark 3.3, we obtain the M-tori $\mathcal{A}^M$ with $M = \Lambda_1', \dots, \Lambda_k', \Lambda'', \Lambda'''$, if $\mathcal{A}$ is associative, and $M = \Lambda', \Lambda'', \Lambda'''$, if $\mathcal{A}$ is not associative. Repeated use of Lemma 5.2 now yields the isomorphisms

$$\mathcal{A} \simeq_\Lambda \mathcal{A}^{\Lambda_1'} \otimes \dots \otimes \mathcal{A}^{\Lambda_k'} \otimes \mathcal{A}^{\Lambda''} \otimes \mathcal{A}^{\Lambda'''} \tag{15}$$

$$\mathcal{A} \simeq_\Lambda \mathcal{A}^{\Lambda'} \otimes \mathcal{A}^{\Lambda''} \otimes \mathcal{A}^{\Lambda'''}, \tag{16}$$

in the two cases. Indeed, if $\mathcal{A}$ is associative, the argument for this is clear. If $\mathcal{A}$ is not associative, we first get $\mathcal{A} \simeq_\Lambda \mathcal{A}^{\Lambda' \oplus \Lambda''} \otimes \mathcal{A}^{\Lambda'''}$ by Lemma 5.2, since $\Lambda''' \subset \Gamma$. Next $[\mathcal{A}^{\Lambda'}, \mathcal{A}^{\Lambda''}] = 0$ since $\beta(\Lambda', \Lambda'') = 1$. Also since Λ' is spanned by anisotropic vectors and $\beta(\Lambda', \Lambda'') = 1$, we have $\alpha(\Lambda', \Lambda'', \Lambda) = \alpha(\Lambda, \Lambda', \Lambda'') = 1$ by Lemma 5.10(f). So we have $(\mathcal{A}^{\Lambda'}, \mathcal{A}^{\Lambda''}, \mathcal{A}) = (\mathcal{A}, \mathcal{A}^{\Lambda'}, \mathcal{A}^{\Lambda''}) = 0$ and we can apply Lemma 5.2 to get $\mathcal{A}^{\Lambda' \oplus \Lambda''} \simeq_{\Lambda' \oplus \Lambda''} \mathcal{A}^{\Lambda'} \otimes \mathcal{A}^{\Lambda''}$ which gives the isomorphism (16).

To obtain our conclusions in (a) and (b), it now suffices to show that if M is a subgroup of Λ, then $\mathcal{A}^M$ is isograded isomorphic to

(i) $\mathcal{C}(k)$, if $(M, \varepsilon) \simeq (\mathbb{Z}^k, \varepsilon_-)$ with $1 \le k \le 3$,
(ii) $\mathcal{C}_*(2)$, if $(M, \varepsilon) \simeq (\mathbb{Z}^2, \varepsilon_0)$ and $\mathcal{A}^M$ is associative,
(iii) $\mathcal{P}(r)$, if $\varepsilon(M) = 1$ and $M \subset \Gamma$.

In cases (ii) and (iii) $\mathcal{A}^M$ is associative and the conclusions are well known and easy to verify (using an argument similar to the one below for (i)).

So we consider only the case (i) and suppose that $(M, \varepsilon) \simeq (\mathbb{Z}^k, \varepsilon_-)$ with $1 \le k \le 3$. Choose a basis $B = \{\lambda_1, \dots, \lambda_k\}$ for M and $0 \ne x_i \in \mathcal{A}^{\lambda_i}$. We get a basis $\{x^\lambda\}_{\lambda \in M}$ for $\mathcal{A}^M$ by choosing for each $\lambda \in M$ an element $x^\lambda \in \mathcal{A}^\lambda$ as follows: Write λ in a fixed way as

$$\lambda = \lambda_{i_1} + \dots + \lambda_{i_p} - \lambda_{j_1} - \dots - \lambda_{j_q}, \tag{17}$$

and choose x^λ to be a product of $x_{i_1}, \dots, x_{i_p}, x_{j_1}^{-1}, \dots, x_{j_q}^{-1}$ in some fixed order and association. A change in the expression (17) for λ or in the order and association of the product changes x^λ by a sign which is determined by α and β on M. Thus, the multiplication table for $\mathcal{A}^M$ is determined by α and β while the involution table is determined by ε, α and β. But β and μ are determined by ε, and, by Lemma 5.10(e) and (g), $\alpha = \mu$ on $\mathcal{A}^M$. Thus, the multiplication and involution tables for $\mathcal{A}^M$ are determined by ε. This argument shows that if $\mathcal{B}$ is any other class I structurable N-torus with $(M, \varepsilon) \simeq (N, \varepsilon)$, then $\mathcal{A}^M \simeq_{ig} \mathcal{B}$. In particular, $\mathcal{A}^M \simeq_{ig} \mathcal{C}(k)$.

Conversely, suppose that $\mathcal{A}$ is a Λ-graded algebra with involution $\mathcal{A}$ satisfying the conclusions in (a) or (b). In case (b), $\mathcal{A}$ is a structurable torus of class I by Proposition 5.6. In case (a), $\mathcal{A}$ is associative, hence structurable, and hence a structurable torus by Remark 5.5(a). Moreover, one easily checks using (13) and (14) that $\mathcal{A}$ is generated by its skew-elements. □

Remark 5.20. If $\mathcal{A}$ is an associative Λ-torus with involution (not necessarily of class I), then the arguments in this section with almost no change show that $\mathcal{A}$ is isograded isomorphic to

$$\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_k \otimes \mathcal{A}_{k+1} \otimes \mathcal{P}(r),$$

where $k \geq 0$, $r \geq 0$, $\mathcal{A}_i = \mathcal{C}(2)$ for $1 \leq i \leq k$, and $\mathcal{A}_{k+1} = \mathcal{C}(0)$, $\mathcal{C}(1)$ or $\mathcal{C}_*(2)$. (In other words we drop the restriction in Theorem 5.19 on $\mathcal{A}_{k+1}$ when $k = 0$.) This result has previously been obtained by Yoshii in [Y2] using different methods. It can also be deduced from K.-H. Neeb's recent classification of rational associative tori (without involution) [Neeb] along with (a)-(d) in Remark 5.17.

6. Structurable tori of class II

In this section we briefly recall from [AY] the construction and classification of structurable tori of class II.

We first recall a construction of structurable algebras from hermitian forms. Suppose that $\mathcal{B}$ is an associative algebra with involution $*$, and let $\mathcal{X}$ be a left $\mathcal{B}$-module. Suppose that $k : \mathcal{X} \times \mathcal{X} \to \mathcal{B}$ is a hermitian form over $\mathcal{B}$; that is

$$k(au, v) = ak(u, v), \quad k(u, av) = k(u, v)a^*$$

and

$$k(v, u) = k(u, v)^*$$

for $u, v \in \mathcal{X}$ and $a \in \mathcal{B}$. Let

$$\mathcal{A}(k) := \mathcal{B} \oplus \mathcal{X}$$

with product and involution defined respectively by

$$(a, u)(b, v) = (a \cdot_{\text{op}} b + k(u, v), av + b^*u) \quad \text{and} \quad (a, u)^* = (a^*, u)$$

for $a, b \in \mathcal{B}$ and $u, v \in \mathcal{X}$, where $a \cdot_{\text{op}} b := ba$ is the product in the opposite algebra $\mathcal{B}^{\text{op}}$ of $\mathcal{B}$.[3] The algebra with involution $\mathcal{A}(k)$ is a structurable algebra called the *structurable algebra associated with* k [A1, §8].

Example 6.1. Let M be a subgroup of Λ such that $2\Lambda \subset M$, and let $\mathcal{B}$ be an associative M-torus with involution $*$. We regard $\mathcal{B}$ as a Λ-graded algebra with involution by setting $\mathcal{B}^\lambda = 0$ for $\lambda \in \Lambda \setminus M$. Let $m \geq 1$; let $\rho_1, \ldots, \rho_m$ be elements of $\Lambda \setminus M$ such that

$$\Lambda = \langle M, \rho_1, \ldots, \rho_m \rangle, \quad \rho_i - \rho_j \notin M \text{ for } i \neq j, \quad \text{and} \quad 2\rho_i \in S_+(\mathcal{B}) \text{ for all } i;$$

and let $b_1, \ldots, b_m$ be elements of $\mathcal{B}$ such that

$$0 \neq b_i \in \mathcal{B}^{2\rho_i} \text{ for all } i.$$

We construct a graded hermitian form from this data as follows. Let $\mathcal{X}$ be the left $\mathcal{B}$-module that is free of rank m with basis $v_1, \ldots, v_m$, and assign $\mathcal{X}$ the Λ-grading such that bx_i has degree $\sigma + \rho_i$ if $b \in \mathcal{B}^\sigma$, $\sigma \in M$ and $1 \leq i \leq m$. Then *the module* $\mathcal{X}$ *is* Λ-*graded*; that is $\mathcal{B}^\lambda \mathcal{X}^{\lambda'} \subset \mathcal{X}^{\lambda+\lambda'}$ for $\lambda, \lambda' \in \Lambda$. Let $k : \mathcal{X} \times \mathcal{X} \to \mathcal{B}$ be the hermitian form such that

$$k(v_i, v_j) = \delta_{i,j} b_i$$

for $1 \leq i, j \leq m$. Then *the hermitian form* k *is* Λ-*graded*; that is, $k(\mathcal{X}^\lambda, \mathcal{X}^{\lambda'}) \subset \mathcal{B}^{\lambda+\lambda'}$ for $\lambda, \lambda' \in \Lambda$. We call a graded hermitian form k constructed in this way (using some choice of $\rho_1, \ldots, \rho_m$ and $b_1, \ldots, b_m$) a *diagonal* Λ-*graded hermitian form over* $\mathcal{B}$. Now $\mathcal{A}(k) = \mathcal{B} \oplus \mathcal{X}$ is a structurable Λ-torus with Λ-grading extending the

[3] The product on $\mathcal{A}(k)$ defined in [A1] and [AY] is the opposite of the product defined here. However, the involution on $\mathcal{A}(k)$ is an isomorphism of $\mathcal{A}(k)$ with $\mathcal{A}(k)^{\text{op}}$ as algebras with involution.

Λ-grading on $\mathcal{B}$ and $\mathcal{X}$ [AY, Example 4.6].[4] We call $\mathcal{A}(k)$ the *structurable Λ-torus associated with k.*

In the next theorem, proved in [AY], we restrict our attention to coordinate algebras $\mathcal{B}$ that are class I associative tori with involution. These are precisely the associative tori with involution that were classified in Theorem 5.19(a).

Theorem 6.2. *If M is a subgroup of Λ such that $2\Lambda \subset M$, $\mathcal{B}$ is an class I associative M-torus with involution, and k is a diagonal Λ-graded hermitian form over $\mathcal{B}$, then the structurable Λ-torus $\mathcal{A}(k)$ associated with k (as in Example 6.1) is of class II. Moreover, any structurable Λ-torus of class II is graded-isomorphic to a structurable Λ-torus $\mathcal{A}(k)$ obtained in this way.*

Proof. For the first statement, since $\mathcal{B}$ is of class I, we have $\Lambda_- = M$, $\mathcal{E} = \mathcal{B}^{\mathrm{op}}$ and $\mathcal{W} = \mathcal{X}$ (using the Notation from §4 for $\mathcal{A} = \mathcal{A}(k)$). So, since $\mathcal{X} \neq 0$, it is clear that $\mathcal{A}(k)$ is of class II. The second statement is Theorem 9.5 of [AY]. □

7. Cubic forms and structurable algebras

Before beginning our study of class III structurable tori, we pause to introduce a new construction of structurable algebras. As background for this construction, we mention that there are three constructions of finite dimensional simple structurable algebras $\mathcal{A}$ with $\mathcal{A}_-$ one dimensional: (i) a construction of a 2×2-matrix algebra with coefficients from a cubic Jordan algebra [K1], [A1, §8]; (ii) a Cayley Dickson process for doubling a degree 4 Jordan algebra [AF]; (iii) a construction using a self-adjoint norm semisimilarity of a cubic Jordan algebra over a quadratic extension of the base field [A2, p.1861], [Se, Chapter 8]. In this section we adapt the third of these constructions to the infinite dimensional setting. However, rather than initially using a norm similarity, we base our construction on a pair (h, N) consisting of a hermitian form and a cubic form satisfying the adjoint identity. We then see that the first two constructions (adapted to the infinite dimensional setting) can be viewed as special cases of this construction.

Since F is a field of characteristic $\neq 2$ or 3, we may define homogeneous polynomial maps of degree n for $1 \le n \le 4$ in the following simple fashion. Suppose K is a commutative, associative algebra over F, and suppose $\mathcal{U}$ and $\mathcal{V}$ are left K-modules. If $1 \le n \le 4$, a map $g : \mathcal{U} \to \mathcal{V}$ is called a *homogeneous polynomial map of degree n* over K if there is a symmetric K-multilinear map $f : \mathcal{U}^n \to \mathcal{V}$ with

$$g(x) = \frac{1}{n} f(x, \ldots, x)$$

for $x \in \mathcal{U}$. In that case, given $u, v \in \mathcal{U}$, there exist unique elements $c_{i,u,v} \in \mathcal{V}$ for $1 \le i \le n$ such that

$$g(v + au) = g(v) + \sum_{i=1}^{n} a^i c_{i,u,v},$$

for $a \in F$. We define $\partial_u g|_v = c_{1,u,v}$, in which case

$$\partial_u g|_v = \frac{1}{(n-1)} f(u, v, \ldots, v).$$

[4] In [AY], the M-torus $\mathcal{B}$ is realized as a quantum torus with involution (see [AY, Prop. 4.5]).

Thus, for $u \in \mathcal{U}$, we have the map $\partial_u g : \mathcal{U} \to \mathcal{V}$. If $n \geq 2$, $\partial_u g$ is a homogeneous polynomial map of degree $n-1$; whereas, if $n = 1$, $\partial_u g$ is the constant map $x \mapsto g(u)$. Finally, we have

$$\partial_{u_1} \dots \partial_{u_{n-1}} g|_{u_n} = f(u_1, \dots, u_n)$$

for $u_1, \dots, u_n \in \mathcal{U}$, which shows that we can recover f from g. We call f the *full linearization* of g.

If K is a commutative, associative algebra over F and if $\mathcal{U}$ is a left K-module, a *cubic* (resp. *quadratic*) form on $\mathcal{U}$ over K is a homogeneous polynomial map $g : \mathcal{U} \to K$ of degree 3 (resp. 2) over K. (This notion of quadratic form is clearly equivalent to the one described in §2.)

Suppose now that $\mathcal{E}$ is a commutative, associative algebra with involution $*$ over F and $\mathcal{W}$ is a left $\mathcal{E}$-module. Let $N : \mathcal{W} \to \mathcal{E}$ be a cubic form over $\mathcal{E}$ and let $h : \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ be a hermitian form (see §6) that is nondegenerate (i.e. $h(\mathcal{W}, u) = 0 \Longrightarrow u = 0$ for $u \in \mathcal{W}$). We say that (h, N) has an *adjoint* if for each $v \in \mathcal{W}$ there is an element $v^\natural \in \mathcal{W}$ with

$$\partial_u N|_v = h(u, v^\natural) \text{ for all } u \in \mathcal{W}.$$

Clearly, the nondegeneracy of h shows that the adjoint map $v \mapsto v^\natural$ is unique if it exists.[5]

If (h, N) has an adjoint, we let

$$u \diamond v = (u + v)^\natural - u^\natural - v^\natural,$$

in which case

$$h(u, v \diamond w) = \partial_v \partial_u N|_w$$

is a symmetric trilinear form (the full linearization of N). Moreover,

$$v^\natural = \frac{1}{2} v \diamond v \quad \text{and} \quad N(u) = \frac{1}{6} h(u, u \diamond u).$$

Since $h(u, v \diamond w)$ is symmetric in its arguments, $\diamond$ is a *bi-semilinear product* on $\mathcal{W}$ (that is $\diamond$ is biadditive and $(au) \diamond v = u \diamond (av) = a^*(u \diamond v)$ for $u, v \in \mathcal{W}$, $a \in \mathcal{E}$). Hence the map $\natural : \mathcal{W} \to \mathcal{W}$ is *semiquadratic* (that is $v^\natural = \frac{1}{2} v \diamond v$ where $\diamond$ is symmetric and bi-semilinear).

Conversely, if h is a nondegenerate hermitian form on $\mathcal{W}$ and $\diamond$ is a bi-semilinear product on $\mathcal{W}$ with $h(u, v \diamond w)$ symmetric, then $N(u) = \frac{1}{6} h(u, u \diamond u)$ defines a cubic form and (h, N) has adjoint $u^\natural = \frac{1}{2}(u \diamond u)$.

Remark 7.1. In the special case when the involution $*$ on $\mathcal{E}$ is trivial ($=$ id), pairs (h, N) with an adjoint play an important role in the study of cubic Jordan algebras [Mc, Chapter II.4]. In that case, we usually write T for h, $\#$ for $\natural$, and $\times$ for $\diamond$ (thereby following the usual notational conventions).

Given (h, N) with an adjoint, we define

$$\mathcal{A}(h, N) = \mathcal{E} \oplus \mathcal{W}$$

with product and involution given respectively by

$$(a, v)(b, w) = (ab + h(v, w), aw + b^* v + v \diamond w) \quad \text{and} \quad (a, v)^* = (a^*, v). \tag{18}$$

We will write $a + v$ for (a, v) in $\mathcal{A}(h, N)$.

---

[5] Nondegeneracy of h is equivalent to the map $\alpha : u \to h(\ , u)$ being an injection of $\mathcal{W}$ into its dual space for $\mathcal{E}$. If α is a bijection, then (h, N) automatically has an adjoint.

We note that if h is a nondegenerate hermitian form and $N = 0$ then $(h, 0)$ has an adjoint (the zero map) and, since $\mathcal{E}$ is commutative, $\mathcal{A}(h, N) = \mathcal{A}(h)$ (see §6).

We now wish to determine when the algebra with involution $\mathcal{A}(h, N)$ is structurable.

We say that (h, N) satisfies the *adjoint identity* if (h, N) has an adjoint and

(ADJ) $(v^\natural)^\natural = N(v)v$ for all $v \in \mathcal{W}$.

If (h, N) satisfies the *adjoint identity* then, since $|F| \geq 5$, (h, N) also satisfies the polarizations

(ADJ1) $(w \diamond v) \diamond v^\natural = N(v)w + h(w, v^\natural)v$,
(ADJ2) $(w \diamond u) \diamond v^\natural + (w \diamond v) \diamond (u \diamond v) = h(u, v^\natural)w + h(w, u \diamond v)v + h(w, v^\natural)u$,
(ADJ3) $(w \diamond u) \diamond (x \diamond v) + (w \diamond x) \diamond (u \diamond v) + (w \diamond v) \diamond (u \diamond x)$
$= h(u, x \diamond v)w + h(w, u \diamond x)v + h(w, u \diamond v)x + h(w, x \diamond v)u$,
(ADJ4) $(v \diamond w)^\natural + v^\natural \diamond w^\natural = h(w, v^\natural)w + h(v, w^\natural)v$.

Also, $h(v \diamond w, u) = h(u, v \diamond w)^* = h(v, u \diamond w)^*$. Hence, applying $h(\ , u)$ to (ADJ1) gives $h(w, (u \diamond v^\natural) \diamond v) = h(w, N(v)^* u) + h(w, h(u, v)v^\natural)$, so

(ADJ5) $(u \diamond v^\natural) \diamond v = N(v)^* u + h(u, v)v^\natural$

Theorem 7.2. *Suppose that $\mathcal{E}$ is a commutative associative algebra with involution, $N : \mathcal{W} \to \mathcal{E}$ is a cubic form, and $h : \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ is a nondegenerate hermitian form such that (h, N) has an adjoint. If (h, N) satisfies the adjoint identity, then $\mathcal{A}(h, N)$ is a structurable algebra. If $\mathcal{E}$ contains a skew element that is invertible in $\mathcal{E}$, then the converse holds.*

Proof. We recall from [A1, Theorem 13] that $\mathcal{A}$ is structurable if and only if

(i) $\mathcal{A}$ is skew-alternative,
(ii) $(a, b, c) - (c, a, b) = (b, a, c) - (c, b, a)$ for $a, b, c \in \mathcal{A}_+$,
(iii) $D_{a^2, a}(b) = 0$ for $a, b \in \mathcal{A}_+$,

where

$$D_{a,b}(c) = \frac{2}{3}[[a, b], c] + (c, b, a) - (c, a, b)$$

for $a, b, c \in \mathcal{A}_+$.

We shall first show that (i) and (ii) hold automatically for $\mathcal{A}(h, N)$ without assuming the adjoint identity. We note that $\mathcal{W}$ is a bimodule for $\mathcal{E}$ using the product in $\mathcal{A}(h, N)$, so

$$(\mathcal{E}, \mathcal{E}, \mathcal{W}) = (\mathcal{E}, \mathcal{W}, \mathcal{E}) = (\mathcal{W}, \mathcal{E}, \mathcal{E}) = 0.$$

Skew-alternativity reduces to

$$\begin{aligned}(s, u, v) + (u, s, v) = h(su, v) &+ (su) \diamond v - sh(u, v) - s(u \diamond v) \\ &+ h(us, v) + (us) \diamond v - h(u, sv) - u \diamond (sv) = 0\end{aligned}$$

for $s \in \mathcal{A}(h, N)_- = \mathcal{E}_-$ and $u, v \in \mathcal{W}$, which holds since $us = -su$, h is sesquilinear and $\diamond$ is bi-semilinear. Since $(a, b, c)^* = -(c, b, a)$ for $a, b, c \in \mathcal{A}_+$, (ii) is equivalent to $(a, b, c) - (c, a, b) \in \mathcal{A}_+$ for $a, b, c \in \mathcal{A}_+$. We note that $\mathcal{E}_+$ is in the centre of the algebra with involution $\mathcal{A}(h, N)$ and that $\mathcal{A}(h, N)_+ = \mathcal{E}_+ \oplus \mathcal{W}$, so it suffices to check the $\mathcal{E}$-component of $(u, v, w) - (w, u, v)$ for $u, v, w \in \mathcal{W}$. Now, the $\mathcal{E}$-component of $u(vw)$ is $h(u, v \diamond w)$ which is symmetric in u, v, w, as are the $\mathcal{E}$-components of $(uv)w = (w(vu))^*$ and (u, v, w). Thus, the $\mathcal{E}$-component of $(u, v, w) - (w, u, v)$ is 0.

We next reduce condition (iii) without using the adjoint identity. If $a \in \mathcal{E}_+ \subset \mathcal{Z}(\mathcal{A}(h,N))$ and $x, y \in \mathcal{A}(h,N)_+$, then

$$D_{x,y}(a) = 0, \quad D_{ax,x} = aD_{x,x} = 0, \quad D_{a,x} = D_{x,a} = 0.$$

Let $x = (a,v) = a + v \in \mathcal{A}(h,N)_+$ so $x^2 = a^2 + h(v,v) + 2av + v \diamond v$. We see that

$$D_{x^2,x} = D_{2av+v\diamond v,v} = D_{v\diamond v,v} = 2D_{v^\natural,v}.$$

Thus, (iii) reduces to

$$D_{v^\natural,v}(w) = 0$$

for $v, w \in \mathcal{W}$.

Since $\diamond$ is symmetric, we have $[v^\natural, v] = h(v^\natural, v) - h(v, v^\natural) = -3(N(v) - N(v)^*)$, so

$$\frac{2}{3}[[v^\natural, v], w] = -4(N(v) - N(v)^*)w.$$

Also, since $(wu)v = h(w \diamond u, v) + h(w,u)v + (w \diamond u) \diamond v$ and since $h(w \diamond u, v)$ is symmetric in u, v, we see that

$$\begin{aligned}(w, v, v^\natural) - (w, v^\natural, v) &= (wv)v^\natural - (wv^\natural)v + w[v^\natural, v] \\ &= h(w,v)v^\natural + (w \diamond v) \diamond v^\natural - h(w, v^\natural)v - (w \diamond v^\natural) \diamond v + 3(N(v) - N(v)^*)w.\end{aligned}$$

Thus,

$$D_{v^\natural,v}(w) = -(N(v) - N(v)^*)w + h(w,v)v^\natural + (w \diamond v) \diamond v^\natural - h(w, v^\natural)v - (w \diamond v^\natural) \diamond v$$

and (iii) is equivalent to

(iii$'$) $N(v)w + h(w, v^\natural)v - (w \diamond v) \diamond v^\natural = N(v)^*w + h(w,v)v^\natural - (w \diamond v^\natural) \diamond v.$

If the adjoint identity holds, both sides of (iii$'$) are 0 by (ADJ1) and (ADJ5) and $\mathcal{A}(h,N)$ is a structurable algebra. Conversely, if $\mathcal{A}(h,N)$ is a structurable algebra and $s \in \mathcal{E}_-$ is invertible in $\mathcal{E}$, $w = v$ in (iii$'$) gives

(iii$''$) $4(N(v)v - (v^\natural)^\natural) = N(v)^*v + h(v,v)v^\natural - (v \diamond v^\natural) \diamond v.$

Replacing v by sv in (iii$''$) multiplies the left side by s^4 and the right side by $-s^4$. Thus, both sides are 0 and the adjoint identity holds. □

We will also need the following simple fact:

Lemma 7.3. *Suppose that $\mathcal{E}$ is a commutative associative algebra with involution, $N : \mathcal{W} \to \mathcal{E}$ is a cubic form, and $h : \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ is a nondegenerate hermitian form such that (h,N) has an adjoint. If $\mathcal{E}$ contains a skew element that is invertible in $\mathcal{E}$, then $\mathcal{Z}(\mathcal{A}(h,N)) = \mathcal{E}_+$.*

Proof. Suppose that $s \in \mathcal{E}_-$ is invertible in $\mathcal{E}$. If $x = a + v \in \mathcal{Z}(\mathcal{A}(h,N))$, then $a^* = a$ and $0 = [x,s] = (a+v)s - s(a+v) = -2sv$, which forces $v = 0$. Conversely, it is easy to check that $\mathcal{E}_+ \subset \mathcal{Z}(\mathcal{A}(h,N))$. □

We shall now relate $\mathcal{A}(h,N)$ to two other constructions of structurable algebras.

We first consider the Cayley-Dickson process for structurable algebras that was studied in [AF]. Let $\mathcal{J}$ be a commutative algebra with 1 over a commutative associative F-algebra K. In practice, $\mathcal{J}$ will be Jordan, but we do not require that immediately. Let $x \to x^\theta$ be a period two K-linear map on $\mathcal{J}$ and let $\mu \in K^\times$. The *Cayley-Dickson* process (see [AF, p. 200]) gives an algebra with involution

$$\mathrm{CD}(\mathcal{J}, \theta, \mu) = \mathcal{J} \oplus \mathcal{J},$$

with (a,b) written as $a+s_0b$,

$$(a+s_0b)(c+s_0d)=(ac+\mu(bd^\theta)^\theta)+s_0(a^\theta d+(b^\theta c^\theta)^\theta), \tag{19}$$

and

$$(a+s_0b)^*=a-s_0b^\theta. \tag{20}$$

Note that $\mathcal{E}=K[s_0]$ with $s_0^2=\mu$ is a subalgebra of $\mathrm{CD}(\mathcal{J},\theta,\mu)$.

Lemma 7.4. *Let K be a commutative and associative algebra which we regard as an algebra with involution with the trivial involution. Let $N_\mathcal{V}:\mathcal{V}\to K$ be a cubic form on a K-module $\mathcal{V}$ and let $T:\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ be a nondegenerate symmetric K-bilinear form such that $(T,N_\mathcal{V})$ satisfies the adjoint identity. Denote the adjoint of $(T,N_\mathcal{V})$ by $\#$ and the linearization of $\#$ by $\times$ (see Remark 7.1). Form the commutative algebra*

$$\mathcal{J}=\mathcal{A}(T,N_\mathcal{V})=K\oplus\mathcal{V}$$

(with trivial involution) and define $\theta:\mathcal{J}\to\mathcal{J}$ by $(\alpha+v)^\theta=\alpha-v$ for $\alpha\in K,v\in\mathcal{V}$. Also form $\mathcal{E}=K[s_0]$ with $s_0^2=\mu\in K^\times$ and K-linear involution $*$ with $s_0^*=-s_0$, and form the left $\mathcal{E}$-module $\mathcal{W}=\mathcal{E}\otimes_K\mathcal{V}$ with $\mathcal{V}$ identified with $1\otimes\mathcal{V}$. Extend T to a hermitian form $h:\mathcal{W}\times\mathcal{W}\to\mathcal{E}$, extend $\times$ to a bi-semilinear map $\diamond:\mathcal{W}\times\mathcal{W}\to\mathcal{W}$, and set $w^\natural=\frac{1}{2}w\diamond w$ and $N(w)=\frac{1}{6}h(w,w\diamond w)$ for $w\in\mathcal{W}$. Then:

(a) h is nondegenerate, $N:\mathcal{W}\to\mathcal{E}$ is the unique cubic form extending $N_\mathcal{V}$, and (h,N) satisfies the adjoint identity with adjoint $\natural$.

(b) $\mathrm{CD}(\mathcal{J},\theta,\mu)=\mathcal{A}(h,N)$ as algebras with involution.

(c) $\mathcal{Z}(\mathrm{CD}(\mathcal{J},\theta,\mu))=K$ and $\mathrm{CD}(\mathcal{J},\theta,\mu)$ is a structurable algebra.

Proof. For (a) it is clear that h is nondegenerate and that N is a cubic form extending $N_\mathcal{V}$. Also, if $1\le n\le 4$, any degree n homogeneous polynomial map from $\mathcal{W}$ to $\mathcal{E}$ that vanishes on $\mathcal{V}$ is 0. This tells us that the extension of $N_\mathcal{V}$ is unique. Furthermore, since $\partial_xN|_w=h(x,w^\natural)$ and $(w^\natural)^\natural=N(w)w$ hold for $w\in\mathcal{V}$ they hold for $w\in\mathcal{W}$. For (b), observe first that as K-modules

$$\mathrm{CD}(\mathcal{J},\theta,\mu)=\mathcal{J}\oplus s_0\mathcal{J}=K\oplus\mathcal{V}\oplus s_0K\oplus s_0\mathcal{V}=\mathcal{E}\oplus\mathcal{W}=\mathcal{A}(h,N).$$

It is also clear that the involutions on $\mathrm{CD}(\mathcal{J},\theta,\mu)$ and $\mathcal{A}(h,N)$ are identical. Furthermore, using the multiplication in $\mathrm{CD}(\mathcal{J},\theta,\mu)$, we have $s_0(s_0v)=\mu v=s_0^2v$, so $\mathcal{V}\oplus s_0\mathcal{V}=\mathcal{W}$ as left $\mathcal{E}$-modules. Since $*$ is an involution, we also have $wb=(b^*w)^*=b^*w$ in $\mathrm{CD}(\mathcal{J},\theta,\mu)$ for $b\in\mathcal{E}$ and $w\in\mathcal{W}$. Also, if $u,v,x,y\in\mathcal{V}$, we can compute in $\mathrm{CD}(\mathcal{J},\theta,\mu)$:

$$\begin{aligned}(u+s_0v)(x+s_0y)&=ux-\mu(vy)^\theta-s_0(uy)+s_0(vx)^\theta\\&=T(u,x)-\mu T(v,y)-s_0T(u,y)+s_0T(v,x)\\&\qquad+u\times x+\mu(v\times y)-s_0(u\times y)-s_0(v\times x)\\&=h(u+s_0v,x+s_0y)+(u+s_0v)\diamond(x+s_0y).\end{aligned}$$

So $\mathrm{CD}(\mathcal{J},\theta,\mu)=\mathcal{A}(h,N)$ as algebras with involution. Finally, (c) follows from Lemma 7.3 and Theorem 7.2. □

Suppose next that K is a commutative associative algebra, $\mathcal{J}$ is a Jordan algebra over K, and $t:\mathcal{J}\to K$ is a K-linear form. Following [BZ, § 0.10], we say that t is a *trace form* on $\mathcal{J}$ if $t(x(yz))=t((xy)z)$ for $x,y,z\in\mathcal{J}$. Also, we say that t is *nondegenerate* if the K-bilinear form $(x, y) \mapsto t(xy)$ is nondegenerate. If $t(1) = 4$, we say that t satisfies the *Cayley-Hamilton trace identity of degree 4* if

$$\mathrm{ch}_4(x) := x^4 + q_3(x)x^3 + q_2(x)x^2 + q_1(x)x + q_0(x)1 = 0$$

for $x \in \mathcal{J}$, where the coefficients $q_i(x) \in K$ are defined by

$$q_3(x) = -t(x),\ q_2(x) = \frac{1}{2}(t(x)^2 - t(x^2)),\ q_1(x) = \frac{1}{6}(3t(x)t(x^2) - 2t(x^3) - t(x)^3),$$
$$q_0(x) = \frac{1}{24}(3t(x^2)^2 + 8t(x)t(x^3) - 6t(x^4) - 6t(x)^2t(x^2) + t(x)^4).$$

(These expressions are of course familiar from Newton's identities for the coefficients of the characteristic polynomial of a 4×4-matrix.)

An important classical example of the preceding occurs when K is a field, $\mathcal{J}$ is a finite dimensional separable degree 4 Jordan algebra over K and t is the generic trace on $\mathcal{J}$ [J2, §VI.3 and VI.6] . In that case, it is well known that $t(1) = 4$ and t is a nondegenerate trace form that satisfies $\mathrm{ch}_4(x) = 0$ (see for example [AF, Prop. 5.1]).

Corollary 7.5. *Suppose $\mathcal{J}$ is a Jordan algebra, K is a subalgebra of the centre of $\mathcal{J}$, and $t : \mathcal{J} \to K$ is a nondegenerate trace form such that $t(1) = 4$. Let $\mathcal{V} = \{v \in \mathcal{J} : t(v) = 0\}$, and define a symmetric K-bilinear form $T : \mathcal{V} \times \mathcal{V} \to K$ and a cubic form $N_{\mathcal{V}} : \mathcal{V} \to K$ by*

$$T(u, v) = \frac{1}{4}t(uv) \quad \textit{and} \quad N_{\mathcal{V}}(v) = \frac{1}{24}t(v^3).$$

Then:

(a) T is nondegenerate, $(T, N_{\mathcal{V}})$ has an adjoint, and $\mathcal{J} = \mathcal{A}(T, N_{\mathcal{V}})$.

(b) $(T, N_{\mathcal{V}})$ satisfies the adjoint identity if and only if t satisfies the Cayley-Hamilton identity of degree 4.

(c) If t satisfies the Cayley-Hamilton identity of degree 4, if $\mu \in K^\times$, and if we define $\theta : \mathcal{J} \to \mathcal{J}$ by $\theta = \frac{1}{2}t - \mathrm{id}$, then $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ is a structurable algebra with centre K.

Proof. (a): Note that we have $\mathcal{J} = K \oplus \mathcal{V}$ as K-modules. Hence, since t is nondegenerate, T is nondegenerate. Also, since t is a trace form, we have $\partial_v N_{\mathcal{V}}|_u = \frac{1}{8}t(vu^2)$ for $u, v \in \mathcal{V}$. Thus, $\partial_v N_{\mathcal{V}}|_u = T(v, \frac{1}{2}u^2) = T(v, \frac{1}{2}u^2 - \frac{1}{8}t(u^2))$, so $(T, N_{\mathcal{V}})$ has an adjoint given by

$$v^\# = \frac{1}{2}(v^2 - \frac{1}{4}t(v^2)). \tag{21}$$

Next, it follows from (21) that $uv = T(u, v) + u \times v$, so $\mathcal{J} = \mathcal{A}(T, N_{\mathcal{V}})$ as algebras (with trivial involution).

(b): Now $q_i : \mathcal{J} \to K$ is a homogeneous polynomial map of degree $4 - i$ over K for $0 \le i \le 3$, and it is easy to check that

$$\partial_1 q_i|_x = -(i + 1)q_{i+1}(x) \tag{22}$$

for $x \in \mathcal{J}$ and $1 \le i \le 3$, where $q_4(x) := 1$. Also, $\mathrm{ch}_4 : \mathcal{J} \to \mathcal{J}$ is a homogeneous polynomial map of degree 4 over K, and it follows from the definition of ch_4 and (22) that $\partial_1 \mathrm{ch}_4|_x = 0$. Let f be the full linearization of ch_4. Then, $\partial_1 \mathrm{ch}_4|_x = \frac{1}{6}f(1, x, x, x)$, so $f(1, x, x, x) = 0$ for $x \in \mathcal{J}$. Linearization gives $f(1, x_1, x_2, x_3) = 0$ for $x_1, x_2, x_3 \in \mathcal{J}$. So, if $a \in K$ and $v \in \mathcal{V}$, we have

$$\mathrm{ch}_4(a + v) = \frac{1}{24}f(a + v, a + v, a + v, a + v) = \frac{1}{24}f(v, v, v, v) = \mathrm{ch}_4(v).$$

Another easy calculation shows that $8((v^\#)^\# - N_{\mathcal{V}}(v)v) = \mathrm{ch}_4(v)$ for $v \in \mathcal{V}$. Thus, $\mathrm{ch}_4(a+v) = 8((v^\#)^\# - N_{\mathcal{V}}(v)v)$ for $a \in K$ and $v \in \mathcal{V}$.

(c): We have $\alpha^\theta = \alpha$ and $v^\theta = -v$ for $\alpha \in K$, $v \in \mathcal{V}$. Thus (c) follows from Lemma 7.4(c). □

Corollary 7.5(c) generalizes [AF, Theorem 6.6] which dealt with the case when K is a field and $\mathcal{J}$ is finite dimensional over K. Corollary 7.5(c) can be thought of as a degree 4 analog of the generalized Tits' construction of Lie algebras described in [BZ, Prop. 24], since the structurable algebra $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ can in turn be used to construct a $\mathbb{Z}$-graded Lie algebra using Kantor's construction [K2, A2].

Although we will not use the matrix construction of structurable algebras (see [K2, p. 257], [A1, § 8(v)] in the finite dimensional case), we now briefly consider it as a special case of our construction using cubic forms. For this purpose, we require a pair formulation of the adjoint identity. Suppose that K is a commutative and associative F-algebra, that $(\mathcal{V}_+, \mathcal{V}_-)$ is a pair of K-modules, that $N_\sigma : \mathcal{V}_\sigma \to K$ is a pair of cubic forms, and that $T : \mathcal{V}_+ \times \mathcal{V}_- \to K$ is a nondegenerate K-bilinear pairing, i.e., for $v_\sigma \in \mathcal{V}_\sigma$ we have $T(v_+, \mathcal{V}_-) = 0$ implies $v_+ = 0$ and $T(\mathcal{V}_+, v_-) = 0$ implies $v_- = 0$. To allow symmetry of notation, we set $T(v_-, v_+) = T(v_+, v_-)$. We say (T, N_+, N_-) has an *adjoint* if for $\sigma = \pm$ and for each $u \in \mathcal{V}_\sigma$ there is $u^\# \in \mathcal{V}_{-\sigma}$ with $\partial_v N_\sigma|_u = T(v, u^\#)$ for all $v \in \mathcal{V}_\sigma$. If (T, N_+, N_-) has an adjoint, the *adjoint map* $u \mapsto u^\#$ from $\mathcal{V}_\sigma$ to $\mathcal{V}_{-\sigma}$ is unique, and we define $u \times v = (u \times v)^\# - u^\# - v^\#$ for $u, v \in \mathcal{V}_\sigma$. We say (T, N_+, N_-) satisfies the *adjoint identity* if it has an adjoint and $(v^\#)^\# = N_\sigma(v)v$ for all $v \in \mathcal{V}_\sigma$, $\sigma = \pm$.

If (T, N_+, N_-) has an adjoint, following [A1, p. 148], we form the algebra with involution
$$\mathcal{M}(T, N_+, N_-) = \left\{ \begin{bmatrix} \alpha_+ & v_+ \\ v_- & \alpha_- \end{bmatrix} : \alpha_\sigma \in K, v_\sigma \in \mathcal{V}_\sigma \right\}$$
with
$$\begin{bmatrix} \alpha_+ & v_+ \\ v_- & \alpha_- \end{bmatrix} \begin{bmatrix} \beta_+ & u_+ \\ u_- & \beta_- \end{bmatrix} = \begin{bmatrix} \alpha_+\beta_+ + T(v_+, u_-) & \alpha_+ u_+ + \beta_- v_+ + v_- \times u_- \\ \beta_+ v_- + \alpha_- u_- + v_+ \times u_+ & \alpha_-\beta_- + T(v_-, u_+) \end{bmatrix}$$
and
$$\begin{bmatrix} \alpha_+ & v_+ \\ v_- & \alpha_- \end{bmatrix}^* = \begin{bmatrix} \alpha_- & v_+ \\ v_- & \alpha_+ \end{bmatrix}.$$

Part (c) of the next corollary generalizes [A1, §8, Example (v)] which dealt with the case when K is a field and the pair $(\mathcal{V}_+, \mathcal{V}_-)$ is finite dimensional.

Corollary 7.6. *Suppose that K is a commutative associative algebra over F and that the triple (T, N_+, N_-), defined on a pair of K-modules $(\mathcal{V}_+, \mathcal{V}_-)$, has an adjoint. Let $\mathcal{E} = K \oplus K$ with the exchange involution $*$ and let $\mathcal{W} = \mathcal{V}_+ \oplus \mathcal{V}_-$. For $\alpha = (\alpha_+, \alpha_-) \in \mathcal{E}$ and $v = (v_+, v_-), u = (u_+, u_-) \in \mathcal{W}$, we set $\alpha v = (\alpha_+ v_+, \alpha_- v_-)$,*
$$N(v) = (N_+(v_+), N_-(v_-)) \quad \textit{and} \quad h(v, u) = (T(v_+, u_-), T(v_-, u_+)).$$
Then:

(a) $\mathcal{W}$ is a left $\mathcal{E}$-module, N is a cubic form, h is a nondegenerate hermitian form, (h, N) has an adjoint, and $\mathcal{M}(T, N_+, N_-) \simeq \mathcal{A}(h, N)$ as algebras with involutions.

(b) If (T, N_+, N_-) satisfies the adjoint identity, then so does (h, N).

(c) If (T, N_+, N_-) satisfies the adjoint identity then $\mathcal{M}(T, N_+, N_-)$ is a structurable algebra with centre $K1$.

Proof. For (a), it is clear that $\mathcal{W}$ is a left $\mathcal{E}$-module, N is a cubic form, and h is a nondegenerate hermitian form. Also, (h, N) has an adjoint given by $(v_+, v_-)^\natural = (v_-^\#, v_+^\#)$. Identifying $(\alpha_+, \alpha_-)+(v_+, v_-)$ in $\mathcal{A}(h, N)$ with $\left[\begin{smallmatrix}\alpha_+ & v_+\\ v_- & \alpha_-\end{smallmatrix}\right]$ in $\mathcal{M}(T, N_+, N_-)$, we see that $\mathcal{M}(T, N_+, N_-) = \mathcal{A}(h, N)$ as algebras with involution. (b) is clear, and (c) follows from (a), (b), Theorem 7.2 and Lemma 7.3. $\square$

8. The geometry of class III tori

With the general structurable algebra constructions from §7 in hand, we now begin our classification of class III structurable tori. In this section we associate a finite incidence geometry to any class III torus. Our analysis of this geometry will limit the possibilities for class III tori.

Throughout the section we assume that $\mathcal{A}$ is a class III structurable torus, and we use the notation S, S_σ, Λ_-, $\mathcal{Z}$, Γ, $\mathcal{E}$ and $\mathcal{W}$ from §4.

Recall from §4 that $\mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ with $\mathcal{E}\mathcal{E} \subset \mathcal{E}$ and $\mathcal{E}\mathcal{W} + \mathcal{W}\mathcal{E} \subset \mathcal{W}$. Moreover, we have

$$we = e^* w$$

for $e \in \mathcal{E}$ and $w \in \mathcal{W}$ [AY, Prop. 8.3(a)]. Also, we recall the following from [AY, Prop. 8.13].

Proposition 8.1. *If $\mathcal{A}$ is a structurable Λ-torus of class III, $\sigma_0 \in S_-$, and $0 \neq s_0 \in \mathcal{A}^{\sigma_0}$, then*

(a) $\mathcal{E}$ is a commutative, associative Λ_--torus with involution.

(b) $\mathcal{Z} = \mathcal{Z}(\mathcal{E}) = \mathcal{E}_+$. Consequently, if x is a homogeneous element of $\mathcal{A}$, then $x^2 \in \mathcal{E} \iff x^2 \in \mathcal{Z}$.

(c) $\mathcal{E}_- = s_0\mathcal{Z}$, $s_0^2 \in \mathcal{Z}$, and $\mathcal{E} = \mathcal{Z} \oplus s_0\mathcal{Z}$.

(d) $S_- = \sigma_0 + \Gamma$, $2\sigma_0 \in \Gamma$, and $\Lambda_- = \Gamma \cup (\sigma_0 + \Gamma)$.

(e) $(\Lambda_- : \Gamma) = 2$.

(f) $4\Lambda \subset \Gamma \subset \Lambda_- \subset \Lambda$.

We now let $\mathcal{E}$ act on $\mathcal{W}$ by means of the left multiplication action $(a, w) \mapsto aw$ in $\mathcal{A}$. We define maps $h : \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ and $\diamond : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ by the equality

$$w_1 w_2 = h(w_1, w_2) + w_1 \diamond w_2$$

for $w_1, w_2 \in \mathcal{W}$. We also define a map $N : \mathcal{W} \to \mathcal{E}$ by

$$N(w) = \frac{1}{6} h(w, w \diamond w)$$

for $w \in \mathcal{W}$.

Proposition 8.2. *Suppose that $\mathcal{A}$ is a structurable torus of class III. Then*

(a) $\mathcal{W}$ is a graded left $\mathcal{E}$-module relative to the action $(a, w) \mapsto aw$.

(b) h is a nondegenerate graded hermitian form on $\mathcal{W}$ over $\mathcal{E}$ and $\diamond$ is an $\mathcal{E}$-bi-semilinear symmetric graded map.

(c) N is a graded cubic form on $\mathcal{W}$ over $\mathcal{E}$; that is, N is cubic form and its full linearization is graded.

(d) The pair (h, N) satisfies the adjoint identity with $w^\natural = \frac{1}{2} w \diamond w$ for $w \in \mathcal{W}$.

(e) $\mathcal{A} = \mathcal{A}(h, N)$ as Λ-graded algebras, where $\mathcal{A}(h, N)$ has the Λ-grading extending the Λ-grading on $\mathcal{E}$ and $\mathcal{W}$.

Proof. (a): For general structurable tori, $\mathcal{W}$ is a left $\mathcal{E}$-module relative to $a \circ w = a^* w$ [AY, Prop. 8.4(a)] . However, since $\mathcal{A}$ is of class III, $\mathcal{E}$ is commutative, so $\mathcal{W}$ is an $\mathcal{E}$-module relative to aw. The action aw is clearly graded.

(b): h and $\diamond$ are clearly graded. Also, in the notation of [AY, §8], we have $h(w_1, w_2) = \chi(w_2, w_1)$ and $w_1 \diamond w_2 = \xi(w_2, w_1)$. (b) then follows from the corresponding statements about χ and ξ [AY, Prop. 8.4 (b)–(e)].

(c): We have $h(w_1, w_2 \diamond w_3) = \overline{\chi(w_1, \xi(w_3, w_2)}$ and the right hand side is symmetric in its arguments by [AY, p. 128]. Hence, $h(w_1, w_2 \diamond w_3)$ is symmetric in its arguments, so N is a cubic form over $\mathcal{E}$ with full linearization $h(w_1, w_2 \diamond w_3)$.

(d) and (e): Since $\partial_u N \mid_v = \frac{1}{2} h(u, v \diamond v)$, the pair (h, N) has an adjoint $v^\natural = \frac{1}{2} v \diamond v$. Also, we have

$$(a + v)(b + w) = ab + h(v, w) + aw + b^* v + v \diamond w \quad \text{and} \quad (a + v)^* = a^* + v$$

for $a, b \in \mathcal{E}$ and $v, w \in \mathcal{W}$. Therefore $\mathcal{A} = \mathcal{A}(h, N)$ as graded algebras with involution. Hence (h, N) satisfies the adjoint identity by Theorem 7.2. $\square$

Let

$$\bar{\Lambda} = \Lambda / \Lambda_- \quad \text{and} \quad \bar{\alpha} = \alpha + \Lambda_- \in \bar{\Lambda},$$

for $\alpha \in \Lambda$. So

$$4\bar{\alpha} = 0$$

for $\alpha \in \Lambda$ by Proposition 8.1(f).

If $\beta \in \bar{\alpha}$ for $0 \neq \bar{\alpha} \in \bar{S}$, then $\mathcal{A}^\beta = e\mathcal{A}^\alpha$ for any $0 \neq e \in \mathcal{E}^{\beta - \alpha}$. Since $h(w_1, w_2 \diamond w_3)$ is symmetric and trilinear, we see that for $0 \neq \bar{\alpha}_i \in \bar{S}$, the condition $h(\mathcal{A}^{\alpha_1}, \mathcal{A}^{\alpha_2} \diamond \mathcal{A}^{\alpha_3}) \neq 0$ does not depend on the order of the α_i or on the choice of representatives for $\bar{\alpha}_i$.

We can now define an incidence geometry $\mathcal{I}$ associated with $\mathcal{A}$. The *points* of $\mathcal{I}$ are the elements $\bar{\alpha} \in \bar{\Lambda}$ for $\alpha \in S \setminus \Lambda_-$. The *lines* of $\mathcal{I}$ are the unordered 3-tuples $[\bar{\alpha}, \bar{\beta}, \bar{\gamma}]$ of points $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ with

$$h(\mathcal{A}^\alpha, \mathcal{A}^\beta \diamond \mathcal{A}^\gamma) \neq 0.$$

Note, in this case, that $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$. We also note that since $\mathcal{A}$ is of class III, $\mathcal{I}$ has at least one point and one line.

If $[\bar{\alpha}, \bar{\beta}, \bar{\gamma}]$ is a line, we say that $\bar{\alpha}$ is *incident* to $[\bar{\alpha}, \bar{\beta}, \bar{\gamma}]$ and we say that $\bar{\alpha}$ and $\bar{\beta}$ are *collinear.* Since h is nondegenerate, we see for points $\bar{\alpha}, \bar{\beta}$ that $h(\mathcal{A}^\alpha, \mathcal{A}^\beta) \neq 0$ if and only if $\bar{\alpha} + \bar{\beta} = 0$. Thus, for points $\bar{\alpha}, \bar{\beta}$,

$$\bar{\alpha} \text{ and } \bar{\beta} \text{ are collinear } \iff \mathcal{A}^\alpha \diamond \mathcal{A}^\beta \neq 0.$$

We write $|\bar{\alpha}|$ for the order of $\bar{\alpha} \in \bar{\Lambda}$, in which case $|\bar{\alpha}| = 1$, 2 or 4. Moreover, if $\bar{\alpha}$ is a point, $|\bar{\alpha}| = 2$ or 4.

It is important to note that our definition of collinear allows the possibility of a point $\bar{\alpha}$ being collinear with itself. Indeed, by definition, this holds if and only if $[\bar{\alpha}, \bar{\alpha}, \bar{\gamma}]$ is a line for some point $\bar{\gamma}$. Moreover, we see in part (a) of the next lemma that this is the case if and only if $|\bar{\alpha}| = 4$. We also note that if $\bar{\alpha}$ is a point, then so is $-\bar{\alpha}$ (by (ST2)), but $\bar{\alpha}$ is clearly never collinear with $-\bar{\alpha}$.

Lemma 8.3. *Suppose that $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ are points in the geometry $\mathcal{I}$ for a class III structurable Λ-torus $\mathcal{A}$. Then:*

(a) If $|\bar{\alpha}| = 4$, then $[\bar{\alpha}, \bar{\alpha}, 2\bar{\alpha}]$ is a line. These are the only lines not having three distinct points and the only lines with points of both orders, 2 *and* 4.

(b) If $[\bar{\alpha}, \bar{\beta}, \bar{\gamma}]$ is a line, then $\bar{\delta}$ is collinear with at least one of $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$.

(c) If $\left|\bar\delta\right| = 2$ and $\bar\delta$ is collinear with $\bar\alpha$, then $\bar\delta$ is collinear with $-\bar\alpha$.

(d) If $[\bar\alpha, \bar\beta, \bar\gamma]$ is a line, if $\bar\delta$ is collinear with both $\bar\alpha$ and $\bar\beta$, and if $\left|\bar\delta\right| = 2$, then $\bar\delta = \bar\gamma$.

Proof. We let $0 \neq a \in \mathcal{A}^\alpha$, $0 \neq b \in \mathcal{A}^\beta$, $0 \neq c \in \mathcal{A}^\gamma$, and $0 \neq d \in \mathcal{A}^\delta$.

(a): If $|\bar\alpha| = 4$, then $h(\mathcal{A}^\alpha, \mathcal{A}^\alpha) = 0$, so $a \diamond a = a^2 \neq 0$ by Remark 3.4. Hence, $2\bar\alpha = -2\bar\alpha$ is a point and $[\bar\alpha, \bar\alpha, 2\bar\alpha]$ is a line. Moreover, if $[\bar\alpha, \bar\alpha, \bar\gamma]$ is a line, then $\bar\gamma = -\bar\alpha - \bar\alpha = 2\bar\alpha$, so $|\bar\alpha| = 4$. Also, if $[\bar\alpha, \bar\beta, \bar\gamma]$ is a line with $|\bar\alpha| = 4$ and $|\bar\gamma| = 2$, then $|\bar\alpha + \bar\gamma| = 4$. Thus, $c^\natural = \frac{1}{2} c \diamond c = 0$ and $(a \diamond c)^\natural \neq 0$. Using (ADJ4), we see that $h(c, a^\natural)c = (a \diamond c)^\natural \neq 0$ so $\bar\gamma + 2\bar\alpha = 0$. Therefore, $\bar\beta = -\bar\alpha - \bar\gamma = -\bar\alpha + 2\bar\alpha = \bar\alpha$

(b): Suppose $\bar\delta$ is not collinear with $\bar\alpha, \bar\beta, \bar\gamma$. Since $d \diamond a = d \diamond b = d \diamond c = 0$, (ADJ3) gives $h(a, b \diamond c)d = 0$, a contradiction.

(c): We can assume that $|\bar\alpha| = 4$. Since $\bar\delta$ is collinear with $\bar\alpha$, we have $\bar\delta = 2\bar\alpha$ by (a). Therefore, $\bar\delta = 2(-\bar\alpha)$, so $\bar\delta$ is collinear with $-\bar\alpha$ by (a).

(d): We know that $\bar\delta$ is collinear with $-\bar\alpha$ and $-\bar\beta$ (by (c)). So $[\bar\delta, -\bar\alpha, \bar\alpha - \bar\delta]$ and $[\bar\delta, -\bar\beta, \bar\beta - \bar\delta]$ are lines. Let $0 \neq u \in \mathcal{A}^{\alpha-\delta}$ and $0 \neq v \in \mathcal{A}^{\beta-\delta}$. Thus, $0 \neq (d \diamond u) \diamond (d \diamond v) \in \mathcal{A}^\alpha \diamond \mathcal{A}^\beta$. Since $d^\natural = 0$, (ADJ2) shows that $h(d, u \diamond v)d \neq 0$. Thus $h(d, u \diamond v) \neq 0$, so $\bar\delta = -(\bar\alpha - \bar\delta) - (\bar\beta - \bar\delta) = -\bar\alpha - \bar\beta = \bar\gamma$. □

If a point $\bar\alpha$ is collinear with all points $\bar\beta \neq \bar\alpha$ and $\bar\alpha$ is incident to each line, we say that $\mathcal{I}$ is a *star* with *centre* $\bar\alpha$.

Corollary 8.4. *Let $\mathcal{A}$ be a structurable Λ-torus of class III.*

(a) If $\mathcal{I}$ is a star, then any centre $\bar\alpha$ of $\mathcal{I}$ has order 2.

(b) If $\mathcal{I}$ is a star, the centre of $\mathcal{I}$ is unique unless $\mathcal{I}$ consists of 3 distinct points $\bar\alpha$, $\bar\beta$, $\bar\gamma$ of order 2 and one line $[\bar\alpha, \bar\beta, \bar\gamma]$.

(c) If some point $\bar\alpha$ is collinear with all points $\bar\beta \neq \pm\bar\alpha$, then $\mathcal{I}$ is a star.

(d) $\mathcal{I}$ is a star if and only if there exists a homogeneous element $x \in \mathcal{W}$ such that L_x is invertible.

Proof. (a): Suppose that $\mathcal{I}$ is a star with centre $\bar\alpha$. Since $\bar\alpha$ is not collinear with $-\bar\alpha$, we have $-\bar\alpha = \bar\alpha$, so $|\bar\alpha| = 2$.

(b): Suppose that $\bar\alpha \neq \bar\beta$ are centres of $\mathcal{I}$. Then there is a line $[\bar\alpha, \bar\beta, \bar\gamma]$ and $|\bar\alpha| = \left|\bar\beta\right| = 2$, so $|\bar\gamma| = \left|-\bar\alpha - \bar\beta\right| = 2$. Let $\bar\delta$ be a point not equal to $\bar\alpha$ or $\bar\beta$. If $\left|\bar\delta\right| = 4$, then Lemma 8.3(a) tells us that $\bar\alpha = 2\bar\delta = \bar\beta$, a contradiction. So $\left|\bar\delta\right| = 2$, and thus $\bar\delta = \bar\gamma$ by Lemma 8.3(d). Therefore, $\bar\alpha$, $\bar\beta$ and $\bar\gamma$ are the only points and $[\bar\alpha, \bar\beta, \bar\gamma]$ is the only line.

(c): If $|\bar\alpha| = 2$, then $\bar\alpha$ is incident to each line by Lemma 8.3(d), so $\mathcal{I}$ is a star with centre $\bar\alpha$. Suppose then that $|\bar\alpha| = 4$. If $\bar\beta$ is any point of order 2, then $\bar\beta$ is collinear with $\bar\alpha$, so $\bar\beta = 2\bar\alpha$ by Lemma 8.3(a). Thus, if $\bar\beta$ is any point not equal to $2\bar\alpha$, we have $\left|\bar\beta\right| = 4$ and hence $\left|2\bar\beta\right| = 2$. But then, as we've just seen, $2\bar\beta = 2\bar\alpha$, so $2\bar\alpha$ is collinear with $\bar\beta$ by Lemma 8.3(a). Thus, $2\bar\alpha$ is collinear with all points $\bar\beta \neq 2\bar\alpha$, and the first case tells us that $\mathcal{I}$ is a star with centre $2\bar\alpha$.

(d): Suppose first that $x \in \mathcal{W}^\alpha$ with L_x invertible. Then, $\bar\alpha$ is a point, and $\mathcal{A}^\alpha \diamond \mathcal{A}^\beta = L_x\mathcal{A}^\beta \neq 0$ for all points $\bar\beta \neq -\bar\alpha$. So $\mathcal{I}$ is a star by (c). Conversely, suppose that $\mathcal{I}$ is a star with centre $\bar\alpha$. We choose $0 \neq x \in \mathcal{W}^\alpha$ and show that L_x is invertible. Since $\mathcal{A}$ is finely graded, it is sufficient to show that $L_x\mathcal{A}^\beta \neq 0$ for all $\beta \in S$. But if $\beta \in \Lambda_-$, this is clear from the multiplication in $\mathcal{A} = \mathcal{A}(h, N)$. So we can assume that $\bar\beta$ is a point. If $\bar\beta \neq \bar\alpha$, then $\bar\beta$ and $\bar\alpha$ are collinear, so $L_x\mathcal{A}^\beta \neq 0$. Finally, if $\bar\beta = \bar\alpha$, then $\mathcal{A}^\beta = \mathcal{E}^{\beta-\alpha}x$, so $L_x\mathcal{A}^\beta = \mathcal{E}^{\beta-\alpha}x^2 \neq 0$ by Remark 3.4. □

It is natural and convenient now to subdivide class III structurable tori into three subclasses:

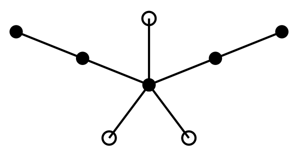
Definition 8.5. If $\mathcal{A}$ is a class III structurable Λ-torus, we say that $\mathcal{A}$ has *class* III(a), III(b), or III(c), if the corresponding condition below holds:

III(a): $\mathcal{I}$ is a star.
III(b): $\mathcal{I}$ is not a star and all points have order 2.
III(c): $\mathcal{I}$ is not a star and there is a point of order 4.

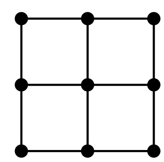
In view of Lemma 8.3(a), condition III(b) can be equivalently stated as: $\mathcal{I}$ is not a star and each line of $\mathcal{I}$ has 3 distinct points. Similarly, condition III(c) can be stated as: $\mathcal{I}$ is not a star and there is a line of $\mathcal{I}$ with a repeated point. So, the trichotomy in Definition 8.5 is purely based on the properties of the incidence geometry $\mathcal{I}$.

Example 8.6. To provide some intuition, we now give examples of incidence geometries that are associated with structurable tori of class III(a), III(b) and III(c). In the diagrams, solid circles represent points of order 2 and open circles represent points of order 4.

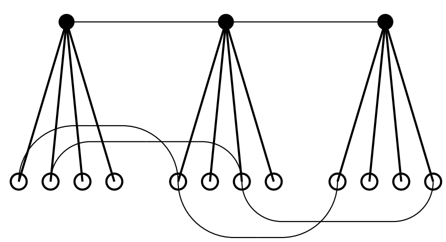
(a)



(b)



(c)



In example (c), we have drawn only 2 of the 16 lines that are made up of points from the 3 clusters at the bottom of the picture. Indeed, if we label the points in each of these clusters by the elements of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, there are 16 lines of the form $[\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3]$, where the points $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\alpha}_3$ are chosen one from each cluster with labels summing to 0. (See Lemma 13.2 below in the case when all $k_i = 1$.)

Remark 8.7. If $\mathcal{A}$ is of class III(b), then, using Lemma 8.3, $\mathcal{I}$ can be seen to be a generalized quadrangle $\mathcal{Q}$ of order $(2, t)$ (see [PT, §1.1]) imbedded in the projective space $\mathcal{P}$ of the $\mathbb{Z}_2$-vector space Λ/Λ_-. Furthermore, one can classify such pairs $(\mathcal{Q}, \mathcal{P})$. Methods similar to those in [F1] should then classify the pairs (h, N) leading to class III(b) tori. Although we have been guided in our research by these facts, in the end we adopted a different approach to the classification of class III(b) tori (see §10).

9. Tori of class III(a)

In this section we obtain the classification of structurable tori of class III(a).

Recall that in Example 6.1, we constructed the structurable torus $\mathcal{A}(k)$ associated with a diagonal graded hermitian form k over an associative torus with involution $\mathcal{B}$. We saw in Theorem 6.2 that if we take $\mathcal{B}$ of class I in this construction we obtain all class II structurable tori. On the other hand, there are, up to isograded-isomorphism, precisely two associative tori with involution that are not of class I: $\mathcal{P}(r)$ and $\mathcal{C}_*(2) \otimes \mathcal{P}(r)$ (see Remark 5.20). If we use the first of these for $\mathcal{B}$ in the construction, we obtain Jordan tori [Y1, § 5]. If we use the second, we now see that we obtain all class III(a) structurable tori.

Theorem 9.1. *If M is a subgroup of Λ such that $2\Lambda \subset M$, if $\mathcal{B}$ is an associative M-torus with involution that is isograded-isomorphic to $\mathcal{C}_*(2)\otimes\mathcal{P}(r)$ for some $r \geq 0$, and if k is a diagonal Λ-graded hermitian form over $\mathcal{B}$, then the structurable Λ-torus $\mathcal{A}(k)$ associated with k (as in Example 6.1) is of class III(a). Moreover, any structurable Λ-torus of class III(a) is graded-isomorphic to a structurable Λ-torus $\mathcal{A}(k)$ obtained in this way.*

Proof. For the first statement, assume that $\mathcal{A} = \mathcal{A}(k) = \mathcal{B} \oplus \mathcal{X}$, where $M, \mathcal{B}$, $\mathcal{X}$ and k are as in Example 6.1, and $\mathcal{B}$ is isograded isomorphic to $\mathcal{C}_*(2) \otimes \mathcal{P}(r)$. We use the notation S, S_σ, Λ_-, $\mathcal{Z}$, Γ, $\mathcal{E}$ and $\mathcal{W}$ from §4 for $\mathcal{A}$. Now the centre $\mathcal{Z}$ of the algebra with involution $\mathcal{A}$ is also the centre of the algebra with involution $\mathcal{B}$, so we have $\Gamma = \Gamma(\mathcal{B})$. Thus, since $\mathcal{B}$ is isograded-isomorphic to $\mathcal{C}_*(2) \otimes \mathcal{P}(r)$, we may choose a basis $\{\mu_1, \mu_2, \dots, \mu_{r+2}\}$ of M and nonzero elements $a_1 \in \mathcal{B}^{\mu_1}$, $a_2 \in \mathcal{B}^{\mu_2}$ such that $\Gamma = \langle 2\mu_1, 2\mu_2, \mu_3, \dots, \mu_{r+2}\rangle$,

$$\mathcal{B} = \mathcal{Z} \oplus \mathcal{Z}a_1 \oplus \mathcal{Z}a_2 \oplus \mathcal{Z}a_1a_2,$$

$a_i^2 \in \mathcal{Z}$, $a_2a_1 = -a_1a_2$, $a_1^* = a_1$ and $a_2^* = a_2$. Then $S_- = \Gamma + \mu_1 + \mu_2$, so $\Lambda_- = \langle \Gamma, \mu_1 + \mu_2\rangle$,

$$\mathcal{E} = \mathcal{Z} \oplus \mathcal{Z}a_1a_2 \quad \text{and} \quad \mathcal{W} = \mathcal{Z}a_1 \oplus \mathcal{Z}a_2 \oplus \mathcal{X}.$$

Now $\mathcal{W}\mathcal{W} \supset \mathcal{X}\mathcal{X} = k(\mathcal{X},\mathcal{X}) = \mathcal{B}$ and hence $\mathcal{W}\mathcal{W} \not\subset \mathcal{E}$. Thus, $\mathcal{A}$ is of class III. Moreover, L_{a_1} is invertible, so, by Corollary 8.4(d), $\mathcal{I}$ is a star. (This can also easily be seen directly.) So, $\mathcal{A}$ is of class III(a).

For the second statement, suppose that $\mathcal{A}$ is a structurable torus of class III(a), and we use the notation from §4 and §8 for $\mathcal{A}$. We use the recognition theorem [AY, Prop. 9.8]. Indeed, according to that result, it is sufficient to show that there exists a subgroup M of Λ satisfying the following conditions:

(a) $\Lambda_- \subset M \subset S$ and $(M : \Lambda_-) = 2$
(b) $\mathcal{A}^{S\setminus M}\mathcal{A}^{S\setminus M} \subset \mathcal{A}^M$
(c) For each $\sigma \in S \setminus M$ there exists $\tau \in S \setminus M$ so that $\mathcal{A}^\sigma\mathcal{A}^\tau \not\subset \mathcal{E}$.

To do this, let $\bar\delta$ be a centre of the geometry $\mathcal{I}$ associated with $\mathcal{A}$. Then, by Corollary 8.4(a), $|\bar\delta| = 2$. Let $M = \langle \Lambda_-, \delta\rangle = \Lambda_- \cup (\delta + \Lambda_-)$, in which case we have condition (a). To show (b), it suffices to check that $\mathcal{A}^\alpha\mathcal{A}^\beta \subset \mathcal{A}^M$ for points $\bar\alpha, \bar\beta$ distinct from $\bar\delta$. If $\bar\alpha + \bar\beta = 0$, then $\mathcal{A}^\alpha\mathcal{A}^\beta = h(\mathcal{A}^\alpha, \mathcal{A}^\beta) \subset \mathcal{E}$. If $\bar\alpha + \bar\beta \neq 0$ and $\bar\alpha$ and $\bar\beta$ are not collinear, then $\mathcal{A}^\alpha\mathcal{A}^\beta = \mathcal{A}^\alpha \diamond \mathcal{A}^\beta = 0$. If $[\bar\alpha, \bar\beta, \bar\gamma]$ is a line, then $\bar\delta = \bar\gamma$ (since $\bar\delta$ is a centre of $\mathcal{I}$) and $\mathcal{A}^\alpha\mathcal{A}^\beta = \mathcal{A}^\alpha \diamond \mathcal{A}^\beta \subset \mathcal{A}^M$ (since $\bar\alpha + \bar\beta = -\bar\delta$). Thus, we have (b). Finally, if $\sigma \in S \setminus M$, then $[\bar\sigma, \bar\delta, \bar\tau]$ is a line for $\tau = -\sigma - \delta \notin M$, so $\mathcal{A}^\sigma\mathcal{A}^\tau = \mathcal{A}^\sigma \diamond \mathcal{A}^\tau = \mathcal{A}^{-\delta} \not\subset \mathcal{E}$. $\square$

10. Tori of class III(b)

In this section we show that each structurable torus of class III(b) can be constructed by doubling a Jordan torus of degree 4 using the Cayley-Dickson process (see §7).

We will need the following simple fact from Jordan theory:

Lemma 10.1. *Suppose that $\mathcal{J}$ is a finite dimensional central simple Jordan algebra over F. If x is an element of $\mathcal{J}$ such that $x^2 \in F$ and L_x is invertible, then $x \in F$.*

Proof. By extending the base field and replacing x by a scalar multiple of x, we can assume that $x^2 = 1$. Then $e := \frac{1}{2}(1-x)$ is an idempotent and we have the Peirce decomposition $\mathcal{J} = \mathcal{J}_0 \oplus \mathcal{J}_1 \oplus \mathcal{J}_{\frac{1}{2}}$ of $\mathcal{J}$ relative to e [J2, §III.1] Since $x = 1 - 2e$, we have $L_x \mathcal{J}_{\frac{1}{2}} = 0$, so $\mathcal{J}_{\frac{1}{2}} = 0$ and $\mathcal{J} = \mathcal{J}_0 \oplus \mathcal{J}_1$ where $\mathcal{J}_0$ and $\mathcal{J}_1$ are ideals of $\mathcal{J}$. Thus, by simplicity, $e = 0$ or $e = 1$, so $x = \pm 1$. $\square$

Suppose that $\mathcal{J}$ is a Jordan Λ-torus with centre $\mathcal{Z}$ and central grading group Γ. Then $\mathcal{J}$ is a free $\mathcal{Z}$-module [Y1, Lemma 3.9(ii)]. Indeed the rank (possibly infinite) of this module is easily seen to be the cardinality $|S(\mathcal{J})/\Gamma|$ of the set $S(\mathcal{J})/\Gamma$, where $S(\mathcal{J})$ is the support of $\mathcal{J}$ and

$$S(\mathcal{J})/\Gamma = \{\alpha + \Gamma \;:\; \alpha \in S(\mathcal{J})\}.$$

Also, $\mathcal{Z}$ is an integral domain [Y1, Lemma 3.6(i)], so we can form the field $\tilde{\mathcal{Z}}$ of fractions of $\mathcal{Z}$. The Jordan algebra

$$\tilde{\mathcal{J}} = \tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{J}$$

over $\tilde{\mathcal{Z}}$ is called the *central closure* of $\mathcal{J}$. Since $\mathcal{J}$ is a free $\mathcal{Z}$-module, it follows that the $\mathcal{Z}$-algebra $\mathcal{J}$ embeds in $\tilde{\mathcal{J}}$ as $1 \otimes_{\mathcal{Z}} \mathcal{J}$. Moreover, each element of $\tilde{\mathcal{J}}$ can be expressed in the form $z^{-1} \otimes_{\mathcal{Z}} x$, where $0 \neq z \in \mathcal{Z}$ and $x \in \mathcal{J}$. Again since $\mathcal{J}$ is a free $\mathcal{Z}$-module, $\mathcal{J}$ is finitely generated as $\mathcal{Z}$-module if and only if $\tilde{\mathcal{J}}$ is finite dimensional over $\tilde{\mathcal{Z}}$. In fact if $\mathcal{J}$ is finitely generated as a $\mathcal{Z}$-module, its central closure $\tilde{\mathcal{J}}$ is a finite dimensional central division algebra over $\tilde{\mathcal{Z}}$ [Y1, 3.6, 2.6 and 2.10], and thus the degree of $\tilde{\mathcal{J}}$ over $\tilde{\mathcal{Z}}$ is defined as the degree of the generic minimum polynomial of each element of $\tilde{\mathcal{J}}$ [J2, §VI.3]. Following [Y1, Definition 6.4], we say that a Jordan Λ-torus $\mathcal{J}$ has *degree* n (or central degree n) if $\mathcal{J}$ is finitely generated as a $\mathcal{Z}$-module and the degree of $\tilde{\mathcal{J}}$ (in the above sense) over $\tilde{\mathcal{Z}}$ is n.

Proposition 10.2. *(a) Let M be a subgroup of Λ such that $(\Lambda : M) = 2$; let $\mathcal{J}$ be a degree 4 Jordan M-torus with centre $\mathcal{Z}$ and central grading group Γ satisfying $2M \subset \Gamma$; let σ_0 be an element of Λ such that*

$$\Lambda = \langle M, \sigma_0 \rangle \quad \textit{and} \qquad 2\sigma_0 \in \Gamma;$$

and let $0 \neq \mu \in \mathcal{Z}^{2\sigma_0}$. Define $\theta : \mathcal{J} \to \mathcal{J}$ by

$$\theta(x) = \begin{cases} x, & \textit{if } x \in \mathcal{J}^{\Gamma} \\ -x, & \textit{if } x \in \mathcal{J}^{M \setminus \Gamma} \end{cases}, \tag{23}$$

and give the algebra with involution $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ the Λ-grading defined by

$$\mathrm{CD}(\mathcal{J}, \theta, \mu)^{\tau} = \mathcal{J}^{\tau} \quad \textit{and} \quad \mathrm{CD}(\mathcal{J}, \theta, \mu)^{\sigma_0 + \tau} = s_0 \mathcal{J}^{\tau}. \tag{24}$$

for $\tau \in M$. Then, $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ is a class III(b) structurable Λ-torus with centre $\mathcal{Z}$.

(b) Any class III(b) structurable Λ-torus is graded isomorphic to a structurable torus $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ constructed from some M, $\mathcal{J}$, σ_0 and μ as in (a).

Proof. (a): Let $\tilde{\mathcal{Z}}$ be the field of fractions of $\mathcal{Z}$, let $\tilde{\mathcal{J}} = \tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{J}$, and let $\tilde{t} : \tilde{\mathcal{J}} \to \tilde{\mathcal{Z}}$ be the generic trace on $\tilde{\mathcal{J}}$.

We first claim that the restriction of $\tilde{t}$ to $\mathcal{J}^{M\setminus\Gamma}$ is zero.[6] Indeed, by [NY, Prop. 4.9], we have $\mathcal{J} = \mathcal{Z} \oplus (\mathcal{J},\mathcal{J},\mathcal{J})$, where $(\mathcal{J},\mathcal{J},\mathcal{J})$ is the F-span of the associators (x, y, z), $x, y, z \in \mathcal{J}$. Thus, since $(\mathcal{J},\mathcal{J},\mathcal{J})$ is a graded subspace of $\mathcal{J}$, it follows that $\mathcal{J}^{M\setminus\Gamma} = (\mathcal{J},\mathcal{J},\mathcal{J})$. Then the claim follows from the fact that the generic trace $\tilde{t}$ is a trace form on $\tilde{\mathcal{J}}$.

Now, by assumption, $\tilde{\mathcal{J}}$ is a finite dimensional central division algebra of degree 4 over $\tilde{\mathcal{J}}$ and hence it is a finite dimensional separable algebra of degree 4 over $\tilde{\mathcal{J}}$. So, by Corollary 7.5(c) (and the paragraph preceding Corollary 7.5), $\mathrm{CD}(\tilde{\mathcal{J}}, \tilde{\theta}, \mu)$ is a structurable algebra, where $\tilde{\theta} : \tilde{\mathcal{J}} \to \tilde{\mathcal{J}}$ is defined by $\tilde{\theta} = \frac{1}{2}\tilde{t} - \mathrm{id}$. But since $\tilde{t}$ is $\tilde{\mathcal{Z}}$-linear, $\tilde{t}(1) = 4$, and $\tilde{t}$ is zero on $\mathcal{J}^{M\setminus\Gamma}$, it follows that $\tilde{\theta}$ restricts to θ on $\mathcal{J}$. Hence, setting $\mathcal{A} = \mathrm{CD}(\mathcal{J}, \theta, \mu)$, we see that $\mathcal{A}$ is a subalgebra with involution of $\mathrm{CD}(\tilde{\mathcal{J}}, \tilde{\theta}, \mu)$, so $\mathcal{A}$ is structurable.

It is clear that $\mathcal{A}$ is a finely Λ-graded structurable algebra using the grading given by (24). Moreover, $\mathrm{supp}(\mathcal{A}) = \mathrm{supp}(\mathcal{J}) \cup (\sigma_0 + \mathrm{supp}(\mathcal{J}))$ generates Λ. If $0 \neq x \in \mathcal{J}^{\tau}$, where $\tau \in M$, then x has a Jordan inverse $y \in \mathcal{J}^{-\tau}$, in which case $xy = yx = 1$ and $[L_x, L_y]\mathcal{J} = 0$. Then, since $\theta(x) = -x$ if and only if $\theta(y) = -y$, we have

$$[L_x, L_y](s_0\mathcal{J}) = s_0([L_{\theta(x)}, L_{\theta(y)}]\mathcal{J}) = s_0([L_x, L_y]\mathcal{J}) = 0,$$

so y is the inverse of x in $\mathcal{A}$. Also, since L_{s_0} is invertible, it follows using Remark 3.1(c) that s_0x is invertible. Therefore $\mathcal{A}$ is a structurable torus. Also, by Corollary 7.5(c), $\mathcal{Z}(\mathcal{A}) = \mathcal{Z}$ and hence $\Gamma(\mathcal{A}) = \Gamma$.

Next, using the notation from §4 and §8 for $\mathcal{A}$, we have $S_- = \sigma_0 + \Gamma$, so $\Lambda_- = \langle \Gamma, \sigma_0 \rangle$, $\mathcal{E} = \mathcal{Z} \oplus s_0\mathcal{Z}$ and $\mathcal{W} = \mathcal{J}^{M\setminus\Gamma} \oplus s_0\mathcal{J}^{M\setminus\Gamma}$. But, since $\tilde{\mathcal{J}}$ is of degree 4, we have $\mathcal{J}^{M\setminus\Gamma}\mathcal{J}^{M\setminus\Gamma} \not\subset \mathcal{Z}$, so $\mathcal{W}\mathcal{W} \not\subset \mathcal{E}$ and $\mathcal{A}$ is of class III.

It remains to show that $\mathcal{A}$ is of class III(b). Since $2M \subset \Gamma$, we have $2\Lambda \subset \Lambda_-$, so every point in the geometry $\mathfrak{I}$ associated with $\mathcal{A}$ has order 2. Suppose finally for contradiction that $\mathfrak{I}$ is a star. So, by Corollary 8.4(d), $L_x : \mathcal{A} \to \mathcal{A}$ is invertible for some homogeneous $x \in \mathcal{W}$. It follows then from the multiplication and grading in $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ that $L_x : \mathcal{J} \to \mathcal{J}$ is invertible for some homogeneous $x \in \mathcal{J}^{M\setminus\Gamma}$. Moreover, since $2M \subset \Gamma$, we have $x^2 \in \mathcal{Z}$. This contradicts Lemma 10.1 applied to the finite dimensional central simple algebra $\tilde{\mathcal{J}}$ over $\tilde{\mathcal{Z}}$.

(b): Suppose that $\mathcal{A}$ is a structurable Λ-torus of class III(b). We use the notation of §4 and §8 for $\mathcal{A}$. As in Proposition 8.1, we fix a choice of $\sigma_0 \in S_-$, in which case $\sigma_0 \notin \Gamma$ and $2\sigma_0 \in \Gamma$. We also fix a choice of $0 \neq s_0 \in \mathcal{A}^{\sigma_0}$ and we let

$$\mu = s_0^2 \in \mathcal{Z}^{2\sigma_0}.$$

Now, since every point of $\mathfrak{I}$ has order 2, we have $2(S \setminus \Lambda_-) \subset \Lambda_-$. So $2S \subset \Lambda_-$. But $2S \subset S_+$ [AY, Prop. 7.1], so $2S \subset \Lambda_- \cap S_+ = \Gamma$. Therefore $2\Lambda \subset \Gamma$.

So Λ/Γ is a $\mathbb{Z}_2$-vector space and $\sigma_0 \notin \Gamma$, and therefore we can find a subgroup M of Λ such that

$$\Gamma \subset M, \quad (\Lambda : M) = 2 \quad \text{and} \quad \sigma_0 \notin M.$$

We fix a choice of M with these properties. Then certainly $\Lambda = \langle M, \sigma_0 \rangle$.

[6]It is not difficult to give a direct argument for this without using [NY, Prop. 4.9], a result that uses the classification of Jordan tori.

Let

$$\mathcal{J} = \mathcal{A}^M.$$

Since $M \cap S_- = \emptyset$, we see that $v^* = v$ for all $v \in \mathcal{J}$. Thus $\mathcal{J}$ is a structurable algebra with trivial involution, so $\mathcal{J}$ is a Jordan algebra [A1, §1]. Since $\Lambda = M \cup (\sigma_0 + M)$ and $\sigma_0 + S = S$, we have $S = (M \cap S) \cup (\sigma_0 + (M \cap S))$. Since $\Lambda = \langle S \rangle$, $\sigma_0 \notin M$ and $2\sigma_0 \in \Gamma \subset M \cap S$, we see that $M = \langle M \cap S \rangle$. Therefore $\mathcal{J}$ is a Jordan M-torus.

Now

$$\mathcal{A} = \mathcal{A}^M \oplus \mathcal{A}^{\sigma_0 + M} = \mathcal{J} \oplus s_0\mathcal{J} \quad \text{and} \quad \mathcal{J} = \mathcal{Z} \oplus \mathcal{V},$$

where

$$\mathcal{V} = \mathcal{J}^{M \setminus \Gamma}.$$

Thus, $\mathcal{W} = \mathcal{V} \oplus s_0\mathcal{V} \simeq \mathcal{E} \otimes \mathcal{V}$ as $\mathcal{E}$-modules. Let T and $N_{\mathcal{V}}$ be the restrictions of h and N respectively to $\mathcal{V}$ (where h and N are defined as in §8). Then T and $N_{\mathcal{V}}$ take values in $\mathcal{E}^M = \mathcal{Z}$. Now, by Proposition 8.2 (d) and (e), (h, N) satisfies the adjoint identity and $\mathcal{A} = \mathcal{A}(h, N)$. Thus, $(T, N_{\mathcal{V}})$ satisfies the adjoint identity and $\mathcal{J} = \mathcal{A}(T, N_{\mathcal{V}})$. So, by Lemma 7.4 (a) and (b) (with $K = \mathcal{E}$), we have

$$\mathcal{A} = \mathrm{CD}(\mathcal{J}, \theta, \mu)$$

as algebras with involution, where $\theta : \mathcal{J} \to \mathcal{J}$ is defined by (23). Also, if we give $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ a Λ-grading by (24), this is an equality of Λ-graded algebras with involution.

Next, the centre $\mathcal{Z}$ of $\mathcal{A}$ is clearly contained in $\mathcal{Z}(\mathcal{J})$. On the other hand, if x is a nonzero homogeneous element of $\mathcal{Z}(\mathcal{J})$, then $L_x : \mathcal{J} \to \mathcal{J}$ is invertible, so $L_x : \mathcal{A} \to \mathcal{A}$ is invertible. Since $\mathfrak{I}$ is not a star, this implies that $x \in \mathcal{Z}$ by Corollary 8.4(d). Thus, $\mathcal{Z}(\mathcal{J}) = \mathcal{Z}$ and therefore $\Gamma(\mathcal{J}) = \Gamma$. Also, since $2\Lambda \subset \Gamma$, we have $2M \subset \Gamma$.

Now $\mathcal{J}$ is a free $\mathcal{Z}$-module of rank $|S(\mathcal{J})/\Gamma|$, and $S(\mathcal{J})/\Gamma \subset M/\Gamma$ is finite. Hence, the degree n of the M-torus $\mathcal{J}$ is defined. But (h, N) satisfies the adjoint identity by Proposition 7.2, so $(T, N_{\mathcal{V}})$ satisfies the adjoint identity $(v^{\#})^{\#} = N_{\mathcal{V}}(v)v$ for $v \in \mathcal{V}$, where $\#$ is the restriction of $\natural$ to $\mathcal{V}$. Hence, since $v^{\#} = \frac{1}{2}(v^2 - T(v, v))$, we see that each $v \in \mathcal{V}$ is a root of a monic polynomial of degree 4 over $\mathcal{Z}$. Thus, the same is true for each element $x \in \mathcal{J}$. So each element of $\tilde{\mathcal{J}}$ is a root of a monic polynomial of degree 4 over $\tilde{\mathcal{Z}}$. Consequently, $n \le 4$ [J2, p. 224] (since $\tilde{\mathcal{Z}}$ is infinite). Now if $n = 3$, then, by [Y1, Prop. 6.7], $3M \subset \Gamma$, which contradicts $2M \subset \Gamma$ and $M \ne \Gamma$. Also, if $\mathcal{J}$ has degree 2, then $\tilde{\mathcal{J}}$ is the Jordan algebra of a symmetric bilinear form over $\tilde{\mathcal{Z}}$ [J2, p. 207]. Thus, $\mathcal{J}^{M\setminus\Gamma}\mathcal{J}^{M\setminus\Gamma} \subset \mathcal{Z}$ [Y1, p. 153], so $\mathcal{W}\mathcal{W} \subset \mathcal{E}$, contradicting the fact that $\mathcal{A}$ is of class III. Of course, $n \ne 1$ since $\mathcal{J} \ne F$. So $\mathcal{J}$ is a Jordan M-torus of degree 4. □

We call the structurable Λ-torus $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ constructed in Theorem 10.2(a) a *Cayley-Dickson torus obtained from* $\mathcal{J}$.

Remark 10.3. It is interesting to note that in the proof of Proposition 10.2(a) we did not use the assumption $2\Lambda \subset M$ to show that $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ is a class III structurable torus. That assumption was only used to show that $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ is of class III(b). In fact, if we drop the assumption that $2\Lambda \subset M$, we do obtain an extra torus that is of class III(c), but it is not possible to obtain all class III(c) tori in this way.

To explain the dependence of the Cayley-Dickson torus $\mathrm{CD}(\mathcal{J}, \theta, \mu)$ on the scalar μ, we prove the following:

Lemma 10.4. *Suppose that* $\mathrm{CD}(\mathcal{J},\theta,\mu)$ *and* $\mathrm{CD}(\mathcal{J}',\theta',\mu')$ *are Cayley-Dickson tori constructed as in Theorem 10.2(a) (with the assumptions therein). Suppose that there exists an isograded algebra isomorphism* $\varphi:\mathcal{J}\to\mathcal{J}'$ *such that*

$$\varphi(\mu)={z'}^2\mu' \tag{25}$$

for some nonzero homogeneous $z'\in\mathcal{Z}'(\mathcal{J}')$*. Then* $\mathrm{CD}(\mathcal{J},\theta,\mu)\simeq_{ig}\mathrm{CD}(\mathcal{J}',\theta',\mu')$*.*

Proof. Let $\mathcal{A}=\mathrm{CD}(\mathcal{J},\theta,\mu)$ and $\mathcal{A}'=\mathrm{CD}(\mathcal{J}',\theta',\mu')$. We use the notation Λ, M, $\mathcal{J}$, $\mathcal{Z}$, Γ, σ_0, μ and s_0 for $\mathcal{A}$ as in Theorem 10.2(a), and we use corresponding primed notation for $\mathcal{A}'$.

Now by assumption we have a group isomorphism $\varepsilon:M\to M'$ such that

$$\varphi(\mathcal{J}^\tau)={\mathcal{J}'}^{\varepsilon(\tau)}$$

for $\tau\in M$. Since φ is an algebra isomorphism it follows that $\varepsilon(\Gamma)=\Gamma'$, so $\varepsilon(M\setminus\Gamma)=M'\setminus\Gamma'$. Thus,

$$\varphi\theta=\theta'\varphi. \tag{26}$$

Next let $\gamma'\in\Gamma'$ be the degree of $z'\in\mathcal{Z}'$. Then, since $\mu\in\mathcal{Z}^{2\sigma_0}$ and $\mu'\in{\mathcal{Z}'}^{2\sigma_0'}$, it follows from (25) that

$$\varepsilon(2\sigma_0)=2\gamma'+2\sigma_0'. \tag{27}$$

We now define $\varepsilon_\Lambda:\Lambda\to\Lambda'$ by

$$\varepsilon_\Lambda(\tau)=\varepsilon(\tau)\quad\text{and}\quad\varepsilon_\Lambda(\sigma_0+\tau)=\sigma_0'+\varepsilon(\tau)+\gamma'$$

for $\tau\in M$. Then, one checks using (27) that ε_Λ is an isomorphism of groups.

Next we define $\psi:\mathcal{A}\to\mathcal{A}'$ by

$$\psi(a+s_0b)=\varphi(a)+s_0'(z'\varphi(b))$$

for $a,b\in\mathcal{J}$. Then, ψ is a linear bijection, and one checks directly using the definitions of ψ, ε_Λ and the gradings on $\mathcal{A}$ and $\mathcal{A}'$ (see (24)) that

$$\psi(\mathcal{A}^\lambda)={\mathcal{A}'}^{\varepsilon_\Lambda(\lambda)}.$$

for $\lambda\in\Lambda$. Also, since $z'\in\mathcal{Z}'$, we have

$$z'{a'}^{\theta'}=(z'a')^{\theta'}$$

for $a'\in\mathcal{J}'$. Using this fact and (26), as well as the definitions of the involutions and products, one checks directly that ψ preserves the products and involutions. $\square$

As mentioned previously, Yoshii has classified Jordan tori in [Y1]. In view of Theorem 10.2, it important for us to identity in his list the Jordan Λ-tori $\mathcal{J}$ of degree 4 that satisfy $2\Lambda\subset\Gamma(\mathcal{J})$. (Of course we will then apply this with Λ replaced by M.) To do this, we recall some standard terminology from Jordan theory.

If $\mathcal{A}$ is an associative algebra, the *plus algebra* of $\mathcal{A}$ is the algebra $\mathcal{A}^+$ with underlying vector space $\mathcal{A}$ and product $x\cdot y=\frac{1}{2}(xy+yx)$. In that case, $\mathcal{A}^+$ is a Jordan algebra. If $\mathcal{A}$ is an associative algebra with involution $*$ then

$$\mathcal{H}(\mathcal{A}):=\mathcal{A}_+=\{x\in\mathcal{A}\ :\ x^*=x\}$$

is a subalgebra of the Jordan algebra $\mathcal{A}^+$. If we wish to emphasize the role of the involution $*$, we write $\mathcal{H}(\mathcal{A})$ as $\mathcal{H}(\mathcal{A},*)$. If $\mathcal{A}$ is a Λ-graded algebra, then $\mathcal{A}^+$ is a Λ-graded algebra (with the same grading). Also, if $\mathcal{A}$ is a Λ-graded algebra with involution, then $\mathcal{H}(\mathcal{A})$ is a Λ-graded subalgebra of $\mathcal{A}^+$. Finally, if $\mathcal{A}$ is an associative Λ-torus (without involution), then $\mathcal{A}^+$ is a Jordan Λ-torus; and if $\mathcal{A}$

is an associative Λ-torus with involution such that $\langle S_+(\mathcal{A})\rangle = \Lambda$, then $\mathcal{H}(\mathcal{A})$ is a Jordan Λ-torus

If E is a split 2-dimensional composition algebra (with its canonical involution) over F and $\mathcal{A}$ is a Λ-graded algebra with involution $*$, then it is well known that

$$\mathcal{H}(\mathcal{A} \otimes E) \simeq_\Lambda \mathcal{A}^+. \tag{28}$$

(If we identify $E = F \oplus F$, an isomorphism from right to left in (28) is $a \mapsto a \otimes (1,0) + a^* \otimes (0,1)$.)

Proposition 10.5. *Suppose that $\mathcal{J}$ is a Λ-graded Jordan algebra. Then,*

$$\mathcal{J} \text{ is a Jordan } \Lambda\text{-torus of degree 4 satisfying } 2\Lambda \subset \Gamma(\mathcal{J}) \tag{29}$$

if and only if $\mathcal{J}$ is isograded isomorphic to

$$\mathcal{H}(\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes \mathcal{T} \otimes \mathcal{P}(r)),$$

where $r \geq 0$, $\mathcal{T} = \mathcal{C}(0)$, $\mathcal{C}(1)$, $\mathcal{C}(2)$ or E, and E is a 2-dimensional composition algebra over F.

Proof. "$\Leftarrow$" We can assume that $\mathcal{J} = \mathcal{H}(\mathcal{A})$, where

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3 \otimes \mathcal{A}_4,$$

with $\mathcal{A}_1 = \mathcal{C}(2)$, $\mathcal{A}_2 = \mathcal{C}(2)$,

$$\mathcal{A}_3 = \mathcal{C}(0),\ \mathcal{C}(1),\ \mathcal{C}(2) \quad \text{or} \quad E, \tag{30}$$

and $\mathcal{A}_4 = \mathcal{P}(r)$ with $r \geq 0$. Then $\mathcal{A}$ is a $\mathbb{Z}^n$-graded associative algebra with involution, where $n = 2 + 2 + k + r$, with $k = 0$, 1, 2 or 0 in the cases covered by (30) (in order). (Here we are making the obvious identification of $\mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^k \oplus \mathbb{Z}^r$ with $\mathbb{Z}^n$.) So $\mathcal{J}$ is a Jordan $\mathbb{Z}^n$-torus by Remark 5.5(a). (To see this in the case when $\mathcal{T} = E$, we can extend the base field, assume that E is split, and use (28).) Next let $\mathcal{Z}_i = \mathcal{Z}(\mathcal{A}_i)$ for $1 \leq i \leq 4$, and let $\mathcal{Z} = \mathcal{Z}(\mathcal{J})$. Using the fact that $\mathcal{Z}$ is a graded subspace of $\mathcal{J}$, it is easy to check that $\mathcal{Z} = \mathcal{Z}_1 \otimes \mathcal{Z}_2 \otimes \mathcal{Z}_3 \otimes \mathcal{Z}_4$. Thus $\Gamma(\mathcal{J}) = 2\mathbb{Z}^2 \oplus 2\mathbb{Z}^2 \oplus 2\mathbb{Z}^k \oplus \mathbb{Z}^r$, so $2\mathbb{Z}^n \subset \Gamma(\mathcal{J})$. Finally, let $\tilde{\mathcal{Z}}$ be the quotient field of $\mathcal{Z}$. Then, by Lemma 5.3 (extended to more than two factors), we have

$$(\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_1} \mathcal{A}_1) \otimes_{\tilde{\mathcal{Z}}} (\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_2} \mathcal{A}_2) \otimes_{\tilde{\mathcal{Z}}} (\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_3} \mathcal{A}_3) \otimes_{\tilde{\mathcal{Z}}} (\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_4} \mathcal{A}_4) \simeq \tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A}$$

as algebras with involution over $\tilde{\mathcal{Z}}$. So, since $\mathcal{A}_4 = \mathcal{Z}_4$, we have

$$\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{J} = \tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{H}(\mathcal{A}) = \mathcal{H}(\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A}) \simeq \mathcal{H}((\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_1} \mathcal{A}_1) \otimes_{\tilde{\mathcal{Z}}} (\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_2} \mathcal{A}_2) \otimes_{\tilde{\mathcal{Z}}} (\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_3} \mathcal{A}_3)).$$

as algebras over $\tilde{\mathcal{Z}}$. Finally, since $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_1} \mathcal{A}_1$ and $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_2} \mathcal{A}_2$ are quaternion algebras with canonical involution over $\tilde{\mathcal{Z}}$ and $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}_3} \mathcal{A}_3$ is a composition algebra with canonical involution of dimension 1, 2, 4 or 2 over $\tilde{\mathcal{Z}}$, it is well known that the righthand side of this isomorphism is a degree 4 Jordan algebra over $\tilde{\mathcal{Z}}$ [J2, §V.7, Theorem 11].

"$\Rightarrow$" Suppose that (29) holds, and let $\mathcal{Z} = \mathcal{Z}(\mathcal{J})$ and $\Gamma = \Gamma(\mathcal{J})$. Then the central closure $\tilde{\mathcal{J}} = \tilde{\mathcal{Z}} \otimes_{\mathcal{J}} \mathcal{J}$ of $\mathcal{J}$ is a finite dimensional central simple Jordan division algebra of degree 4 over $\tilde{\mathcal{J}}$. Thus, the dimension $\tilde{\mathcal{J}}$ over $\tilde{\mathcal{Z}}$ is 10, 16 or 28 [ibid]. So

$$d := |S(\mathcal{J})/\Gamma| = 10, 16 \text{ or } 28.$$

We now use the classification of Jordan tori of degree ≥ 4. Indeed, since $\tilde{\mathcal{J}}$ has degree ≥ 4, [Y1, Theorem 7.1] tells us that one of the following holds:

(i) $\mathcal{J} \simeq_\Lambda \mathcal{A}^+$, where $\mathcal{A}$ is an associative Λ-torus,
(ii) $\mathcal{J} \simeq_\Lambda \mathcal{H}(\mathcal{A})$, where $\mathcal{A}$ is an associative Λ-torus with involution,

(iii) $\mathcal{J} \simeq_\Lambda \mathcal{H}(\mathcal{B}, \sigma)$, where $\mathcal{B}$ is an associative Λ-torus over E, E/F is a quadratic field extension, and σ is a graded σ_E-semilinear involution of $\mathcal{B}$.

(In fact [Y1, Theorem 7.1] says more in each case, but this is all we need.) We consider the cases (i), (ii) and (iii) separately.

(i) Suppose that $\mathcal{J} = \mathcal{A}^+$, where $\mathcal{A}$ is an associative Λ-torus. Then $S(\mathcal{J}) = \Lambda$, so $d = |\Lambda/\Gamma|$ is a power of 2. Thus $d = 16$.

Since $2\Lambda \subset \Gamma$, we may choose a basis $\lambda_1, \dots, \lambda_r$ for Λ such that $n_1\lambda_1, \dots, n_r\lambda_r$ is a basis for Γ with $n_i = 1$ or 2. We then choose $0 \neq x_i \in \mathcal{A}^{\lambda_i}$, in which case the elements $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ generate the algebra $\mathcal{A}$ and satisfy $x_j x_i = q_{ij} x_j x_i$, where $\mathbf{q} = (q_{ij}) \in F_{n\times n}$ is a *quantum matrix* (that is $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$). In other words, $\mathcal{A}$ is the *quantum torus* $F_{\mathbf{q}}$ (see for example [Y1, p.129]). Now, since $\mathcal{J}$ is a Jordan torus, the centre $\mathcal{Z}$ of the Jordan algebra $\mathcal{J} = \mathcal{A}^+$ is also the centre of the associative algebra $\mathcal{A}$ (without involution) [Y1, Lemma 3.6(i) and 2.5]. So the elements $x_j^{n_j} \in \mathcal{Z}$ commute with the elements x_i of $\mathcal{A}$. But $x_j^{n_j} x_i = q_{ij}^{n_j} x_i x_j^{n_j}$, so $q_{ij}^{n_j} = 1$. Thus, $q_{ij} = \pm 1$; that is, $\mathbf{q}$ is an *elementary* quantum matrix. Therefore $\mathcal{A}$ has a graded involution $*$ such that $x_i^* = x_i$ [Y1, Example 4.3]. So, using the classification of associative tori *with* involution (see Remark 5.20), we conclude that we have

$$\mathcal{A} \simeq_{ig} \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_p \otimes \mathcal{P}(r),$$

as graded algebras *without* involution, where $p, r \geq 0$ and $\mathcal{A}_i = \mathcal{C}(2)$ for each i. (Here we have used the fact that $\mathcal{C}_*(2) \simeq_{ig} \mathcal{C}(2)$ and $\mathcal{C}(1) \simeq_{ig} \mathcal{P}(1)$ as graded algebras without involution.) But Γ is the support of the centre of the associative algebra $\mathcal{A}$ (without involution), and therefore the same is true for $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_p \otimes \mathcal{P}(r)$. Thus $d = |\Lambda/\Gamma| = 4^p$. So, $p = 2$, and hence $\mathcal{A} \simeq_{ig} \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{P}(r)$ as graded algebras without involution. Thus $\mathcal{J} \simeq_{ig} (\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes \mathcal{P}(r))^+$. So, by (28), $\mathcal{J} \simeq_{ig} \mathcal{H}(\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes E \otimes \mathcal{P}(r))$, where E is the split 2-dimensional composition algebra.

(ii) Suppose $\mathcal{J} = \mathcal{H}(\mathcal{A})$, where $\mathcal{A}$ is an associative Λ-torus with involution. By Remark 5.20, we may assume that $\Lambda = \mathbb{Z}^n$ and

$$\mathcal{A} = \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_l \otimes \mathcal{A}_{l+1} \otimes \mathcal{P}(r)$$

where $\ell, r \geq 0$, $\mathcal{A}_i = \mathcal{C}(2)$ for $1 \leq i \leq l$, and $\mathcal{A}_{l+1}$ is $\mathcal{C}(0)$, $\mathcal{C}(1)$, or $\mathcal{C}_*(2)$. From this it is easy to check that $\mathcal{Z} = \mathcal{Z}_1 \otimes \dots \otimes \mathcal{Z}_{\ell+1} \otimes \mathcal{P}(r)$, where $\mathcal{Z}_i$ is the centre of $\mathcal{A}_i$ for $1 \leq i \leq \ell + 1$. If $\mathcal{C}$ and $\mathcal{D}$ are algebras with involution, then

$$\mathcal{H}(\mathcal{C} \otimes \mathcal{D}) = (\mathcal{C} \otimes \mathcal{D})_+ = (\mathcal{C}_+ \otimes \mathcal{D}_+) \oplus (\mathcal{C}_- \otimes \mathcal{D}_-).$$

Thus, we can use induction on l to compute $d = |S(\mathcal{J})/\Gamma|$ getting

	$\mathcal{A}_{l+1} = \mathcal{C}(0)$	$\mathcal{C}(1)$	$\mathcal{C}_*(2)$
$l = 0$	1	1	3
1	1	4	6
2	10	16	36
3	28	64	
4	136		

Since d increases with increasing l and since $d = 10, 16$, or 28, we have only the following possibilities: $\ell = 2$ and $\mathcal{A}_{\ell+1} = F$; $\ell = 3$ and $\mathcal{A}_{\ell+1} = F$; or $\ell = 2$ and $\mathcal{A}_{\ell+1} = \mathcal{C}(1)$. Therefore, $\mathcal{J} \simeq_{ig} \mathcal{H}(\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes \mathcal{C}(k) \otimes \mathcal{P}(r))$, where $0 \leq k \leq 2$.

(iii) Suppose that $\mathcal{J} = \mathcal{H}(\mathcal{B}, \sigma)$, with $\mathcal{B}$, E/F and σ as above in (iii). Now $E \otimes \mathcal{J} \simeq \mathcal{B}^+$ as graded Jordan algebras [Y1, Example 4.3(3)], and hence $\mathcal{B}^+$ is a Jordan Λ-torus over E that satisfies the condition (29). Thus, by the argument in (i) (with F replaced by E and $\mathcal{A}$ replaced by $\mathcal{B}$), we may identify

$$\mathcal{B} = \mathcal{B}_1 \otimes_E \mathcal{B}_2 \otimes_E \mathcal{B}_3,$$

as Λ-graded algebras without involution over E, where $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$, $\mathcal{B}_i$ is a Λ_i-graded associative algebra without involution over E for $1 \le i \le 3$,

$$\mathcal{B}_1 \simeq_{ig} E \otimes \mathcal{C}(2), \quad \mathcal{B}_2 \simeq_{ig} E \otimes \mathcal{C}(2) \quad \text{and} \quad \mathcal{B}_3 \simeq_{ig} E \otimes \mathcal{P}(r)$$

as graded associative algebras without involution for some $r \ge 0$. Since σ is graded, it has the form $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$, where σ_i is a σ_E-semilinear involution of $\mathcal{B}_i$ for $1 \le i \le 3$. But for $\lambda_i \in \Lambda_i$ we have $\mathcal{B}_i^{\lambda_i} \ne 0$ and therefore (since $\sigma_i|_{\mathcal{B}_i^{\lambda_i}}$ is a graded σ_E-semilinear map of period 2) $\mathcal{B}_i$ contains an nonzero element fixed by σ_i and a nonzero element antifixed by σ_i. Hence, rechoosing canonical generators, we have $(\mathcal{B}_i, \sigma_i) \simeq_{ig} (E \otimes \mathcal{C}(2), \sigma_E \otimes \natural)$ for $i = 1, 2$, and $(\mathcal{B}_3, \sigma_3) \simeq_{ig} (E \otimes \mathcal{P}(r), \sigma_E \otimes 1)$ as graded algebras with involution over F. Thus, as graded algebras with involution over F, we have

$$\begin{aligned}(\mathcal{B}, \sigma) &\simeq_{ig} \big(E \otimes (\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes \mathcal{P}(r)), \sigma_E \otimes (\natural \otimes \natural \otimes 1)\big)\\ &\simeq_{ig} (\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes E \otimes \mathcal{P}(r), \natural \otimes \natural \otimes \sigma_E \otimes 1)\end{aligned}$$

So $\mathcal{J} \simeq_{ig} \mathcal{H}(\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes E \otimes \mathcal{P}(r))$. $\square$

We now put together our results to give a precise description of structurable tori of class III(b).

Theorem 10.6. *(a) Let*

$$\mathcal{J} = \mathcal{H}(\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes \mathcal{T} \otimes \mathcal{P}(r)),$$

where $r \ge 1$,

$$\mathcal{T} = \mathcal{C}(0),\ \mathcal{C}(1),\ \mathcal{C}(2)\ \textit{or}\ E, \tag{31}$$

and E is a 2-dimensional composition algebra over F. Set $k = 0$, 1, 2 or 0 in the cases covered by (31) in order; and set $q = 4 + k$ and $n = q + r$. Let Λ be a free abelian group with basis $\lambda_1, \dots, \lambda_n$, and let

$$M = \langle \lambda_1, \dots, \lambda_q, 2\lambda_{q+1}, \lambda_{q+2}, \dots, \lambda_n \rangle.$$

Decompose M as $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$, where

$$M_1 = \langle \lambda_1, \lambda_2 \rangle,\ M_2 = \langle \lambda_3, \lambda_4 \rangle,\ M_3 = \langle \lambda_5, \dots, \lambda_q \rangle,\ M_4 = \langle 2\lambda_{q+1}, \lambda_{q+1}, \dots, \lambda_n \rangle.$$

(So $M_3 = 0$ if $\mathcal{T} = \mathcal{C}(0)$ or E.) Give the associative algebra with involution $\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes \mathcal{T} \otimes \mathcal{P}(r)$ the tensor product M-grading, where the M_1, M_2, M_3 and M_4 gradings on the factors $\mathcal{C}(2)$, $\mathcal{C}(2)$, $\mathcal{T}$ and $\mathcal{P}(r)$ are determined respectively by the bases $\{\lambda_1, \lambda_2\}$, $\{\lambda_3, \lambda_4\}$, $\{\lambda_5, \dots, \lambda_q\}$ and $\{2\lambda_{q+1}, \lambda_{q+1}, \dots, \lambda_n\}$ (see Remark 3.8). Give the Jordan algebra $\mathcal{J}$ an M-grading as a graded subalgebra of $(\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes \mathcal{T} \otimes \mathcal{P}(r))^+$. Set

$$\sigma_0 = \lambda_{q+1} \in \Lambda \quad \textit{and} \quad \mu = 1 \otimes 1 \otimes 1 \otimes t_1 \in \mathcal{Z}(\mathcal{J})^{2\sigma_0}.$$

Define $\theta : \mathcal{J} \to \mathcal{J}$ by (23) and construct the Λ-graded algebra with involution

$$\mathrm{CD}(\mathcal{J}, \theta, \mu) = \mathcal{J} \oplus s_0 \mathcal{J},$$

with multiplication, involution and grading given by (19), (20) *and* (24). *Then,* $\mathrm{CD}(\mathcal{J},\theta,\mu)$ *is a structurable* Λ*-torus of class III(b) with centre* $\mathcal{Z}(\mathcal{J})$*.*

(b) Any structurable Λ*-torus of class III(b) is isograded isomorphic to a Cayley-Dickson torus* $\mathrm{CD}(\mathcal{J},\theta,\mu)$ *constructed as in (a).*

Proof. (a): By Proposition 10.5, $\mathcal{J}$ is an M-graded Jordan torus of degree 4 with $2M \subset \Gamma(\mathcal{J})$. Then the conclusion follows from Theorem 10.2(a).

(b): Suppose that $\mathcal{A}$ is a structurable Λ-torus of class III(b). We use the notation of §4 and §8 for $\mathcal{A}$. Choose $\sigma_0 \in S_-$ and, as in the proof of Theorem 10.2(b), choose a subgroup M of Λ such that $\Gamma \subset M$, $(\Lambda : M) = 2$ and $\sigma_0 \notin M$. Choose $0 \neq s_0 \in \mathcal{A}^{\sigma_0}$, let $\mu = s_0^2 \in \mathcal{Z}^{2\sigma_0}$, and let $\mathcal{J} = \mathcal{A}^M$. Then, by the proof of Theorem 10.2(b), $\mathcal{J}$ is a Jordan M-torus of degree 4 with $\mathcal{Z}(\mathcal{J}) = \mathcal{Z}$, $\Gamma(\mathcal{J}) = \Gamma$ and $2M \subset \Gamma$. Moreover, $\mathcal{A} = \mathrm{CD}(\mathcal{J},\theta,\mu)$ as Λ-graded algebras with involution, where $\theta : \mathcal{J} \to \mathcal{J}$ is defined by (23) and the Λ-grading on $\mathcal{A}$ is defined by (24).

Now, by Proposition 10.5, $\mathcal{J}$ is isograded isomorphic to $\mathcal{H}(\mathcal{C}(2)\otimes\mathcal{C}(2)\otimes\mathcal{T}\otimes\mathcal{P}(r))$, where $r \geq 0$,

$$\mathcal{T} = \mathcal{C}(0),\ \mathcal{C}(1),\ \mathcal{C}(2) \text{ or } E,$$

and E is a 2-dimensional composition algebra. Let $k = 0$, 1, 2 or 0 respectively; and let $q = 4 + k$ and $n = q + r$. Then, we may identify $M = \mathbb{Z}^{2+2+k+r} = \mathbb{Z}^n$ and

$$\mathcal{J} = \mathcal{H}(\mathcal{C}(2)\otimes\mathcal{C}(2)\otimes\mathcal{T}\otimes\mathcal{P}(r)).$$

Next, as we saw in the proof of Proposition 10.5(a), we have $\mathcal{Z} = \mathcal{P}(2)\otimes\mathcal{P}(2)\otimes\mathcal{Z}(\mathcal{T})\otimes\mathcal{P}(r)$, with $\mathcal{Z}(\mathcal{T}) = \mathcal{P}(k)$ if $\mathcal{T} = \mathcal{A}(k)$ and $\mathcal{Z}(\mathcal{T}) = F$ if $\mathcal{T} = E$. So, letting $\varepsilon_1,\dots,\varepsilon_n$ be the standard basis of $M = \mathbb{Z}^n$, we have

$$\Gamma = \sum_{i=1}^{q} 2\mathbb{Z}\varepsilon_i + \sum_{i=q+1}^{n} \mathbb{Z}\varepsilon_i.$$

Also $\sigma_0 \notin M$, so $2\sigma_0 \notin 2M$. Thus $2M \neq \Gamma$, so $r \geq 1$.

Now let $\lambda_1,\dots,\lambda_n$ (in $(\frac{1}{2}\mathbb{Z})^n$) be defined by $\lambda_i = \varepsilon_i$ if $i \neq q{+}1$ and $\lambda_{q+1} = \frac{1}{2}\varepsilon_{q+1}$. Let $\Lambda' = \langle\lambda_1,\dots,\lambda_n\rangle$, in which case $M = \langle\lambda_1,\dots,\lambda_q,2\lambda_{q+1},\lambda_{q+2},\dots,\lambda_n\rangle$ in Λ'. Let

$$\sigma_0' = \lambda_{q+1} \in \Lambda' \quad \text{and} \quad \mu' = 1\otimes 1\otimes 1\otimes t_1 \in \mathcal{Z}(\mathcal{J})^{2\sigma_0'}.$$

Construct the Cayley-Dickson Λ'-torus $\mathrm{CD}(\mathcal{J},\theta,\mu')$ as in (a) (using Λ', σ_0' and μ' rather that Λ, σ_0 and μ). To complete the proof of (b), we show that $\mathrm{CD}(\mathcal{J},\theta,\mu') \simeq_{ig} \mathrm{CD}(\mathcal{J},\theta,\mu)$. To do this it suffices, by Lemma 10.4, to show that there exists an isograded automorphism $\varphi : \mathcal{J} \to \mathcal{J}$ such that

$$\varphi(\mu') = z^2\mu$$

for some nonzero homogeneous $z \in \mathcal{Z}(\mathcal{J})$.

Now, since $2\sigma_0 \in \Gamma$, we can write

$$2\sigma_0 = \sum_{i=1}^{q} 2a_i\varepsilon_i + \sum_{i=q+1}^{n} a_i\varepsilon_i,$$

where $a_i \in \mathbb{Z}$ for all i. Moreover, since $2\sigma_0 \notin 2M$, a_{q+j} is odd for some $j \geq 1$. Replacing μ by $z^2\mu$ for some $z \in \mathcal{Z}$, we can assume that $a_{q+j} = 1$. Next, there exists an isograded automorphism ψ of $\mathcal{P}(r)$ that permutes the canonical generators and exchanges t_1 and t_j. Replacing μ by $(1\otimes 1\otimes 1\otimes\psi)(\mu)$, we can assume that $a_{q+1} = 1$. So we have

$$w := \mu(\mu')^{-1} \in \mathcal{Z}^{\sum_{i\neq q+1}\mathbb{Z}\varepsilon_i}. \tag{32}$$

Define $\varphi : \mathcal{J} \to \mathcal{J}$ by $\varphi(x) = w^m x$ for $x \in \mathcal{J}^{m\varepsilon_{q+1}+\sum_{i\neq q+1}\mathbb{Z}\varepsilon_i}$, $m \in \mathbb{Z}$. Then, one checks directly that φ is an algebra homomorphism, and (using (32)) that φ is invertible with inverse defined by $x \mapsto w^{-m}x$ for $x \in \mathcal{J}^{m\varepsilon_{q+1}+\sum_{i\neq q+1}\mathbb{Z}\varepsilon_i}$, $m \in \mathbb{Z}$. Finally, it is clear that $\varphi(\mu') = \mu$ and that φ is isograded. $\square$

Remark 10.7. In Theorem 10.6(a), suppose that $\mathcal{T} = E$. If E split (which holds automatically when F is algebraically closed) then instead of choosing $\mathcal{J} = \mathcal{H}(\mathcal{C}(2)\otimes \mathcal{C}(2) \otimes E \otimes \mathcal{P}(r))$ we may, by (28), choose

$$\mathcal{J} = (\mathcal{C}(2) \otimes \mathcal{C}(2) \otimes \mathcal{P}(r))^+$$

with the tensor product grading by $M = M_1 \oplus M_2 \oplus M_4$. In that case our choice of μ is $\mu = 1 \otimes 1 \otimes t_1$.

11. The space of hermitian matrices $\mathcal{H}(\mathcal{C}_3)$

In the next section we will construct structurable tori of class III(c) using a cubic form defined on the space of hermitian matrices $\mathcal{H}(\mathcal{C}_3)$ over a composition algebra $\mathcal{C}$. To prepare for that we prove two lemmas about $\mathcal{H}(\mathcal{C}_3)$ in this section, a coordinatization lemma and a lemma about gradings.

Throughout the section, we assume that K *is a commutative associative algebra over* F.

If $\mathcal{C}$ is a composition algebra over K with canonical involution $^-$, we let $\mathcal{H}(\mathcal{C}_3)$ denote the K-module of all 3×3-hermitian matrices

$$x = \begin{bmatrix} a_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & a_2 & x_1 \\ x_2 & \bar{x}_1 & a_3 \end{bmatrix}.$$

over $\mathcal{C}$, where $a_i \in K$, $x_i \in \mathcal{C}$. We can also write the above element $x \in \mathcal{H}(\mathcal{C}_3)$ as

$$x = \sum_i a_i[ii] + \sum_{(i,j,k)\circlearrowleft} x_i[jk],$$

where $(i,j,k) \circlearrowleft$ means that (i,j,k) is a cyclic permutation of $(1,2,3)$. For this x and for $y = \sum_i b_i[ii] + \sum_{(i,j,k)\circlearrowleft} y_i[jk]$, we set

$$T(x,y) = \sum_i a_i b_i + \sum_i n(x_i, y_i) \in K, \tag{33}$$

$$N(x) = a_1a_2a_3 - a_1n(x_1) - a_2n(x_2) - a_3n(x_3) + t(x_1x_2x_3) \in K, \tag{34}$$

where n and t denote the norm and trace on $\mathcal{C}$ respectively. Then, T is a nondegenerate K-bilinear form on $\mathcal{H}(\mathcal{C}_3)$ and N is a cubic form on $\mathcal{H}(\mathcal{C}_3)$ over K. We call T and N the *standard trace and norm* on $\mathcal{H}(\mathcal{C}_3)$. It is well known [Mc, p. 488] that the pair (T, N) satisfies the adjoint identity with adjoint

$$x^\# = \sum_{(i,j,k)\circlearrowleft} (a_ia_j - n(x_k))[kk] + \sum_{(i,j,k)\circlearrowleft} (\overline{x_ix_j} - a_kx_k)[ij].$$

Hence, for x and y as above in $\mathcal{H}(\mathcal{C}_3)$, we have

$$x\times y = \sum_{(i,j,k)\circlearrowleft} (a_ib_j+b_ia_j-n(x_k,y_k))[kk]+\sum_{(i,j,k)\circlearrowleft} (\overline{x_iy_j}+\overline{y_ix_j}-a_ky_k-b_kx_k)[ij]. \tag{35}$$

We now prove a coordinatization result for a pair $(\tilde{T}, \tilde{N})$ satisfying the adjoint identity.

Lemma 11.1. *For $i = 1, 2, 3$, let $\mathcal{W}_i$ be a K-module with a quadratic form $n_i : \mathcal{W}_i \to K$ and let $\tau : \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{W}_3 \to K$ be trilinear. On $\mathcal{W} = K^3 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, define*

$$\tilde{T}(x, y) = \sum_i a_i b_i + \sum_i n_i(x_i, y_i),$$
$$\tilde{N}(x) = a_1 a_2 a_3 - a_1 n_1(x_1) - a_2 n_2(x_2) - a_3 n_3(x_3) + \tau(x_1, x_2, x_3),$$

for $x = (a_1, a_2, a_3, x_1, x_2, x_3)$ and $y = (b_1, b_2, b_3, y_1, y_2, y_3)$. Suppose $u_i \in \mathcal{W}_i$ with $n_i(u_i) = 1$ for $i = 1, 2$, $\tilde{T}$ is nondegenerate, and the pair $(\tilde{T}, \tilde{N})$ satisfies the adjoint identity. Then there is a composition algebra $\mathcal{C}$ and K-linear isomorphisms $\eta_i : \mathcal{C} \to \mathcal{W}_i$ such that $\eta_i(1) = u_i$, where $u_3 = u_1 \times u_2$, and such that the K-linear isomorphism $\eta : \mathcal{H}(\mathcal{C}_3) \to \mathcal{W}$ given by

$$\eta\Big(\sum_i a_i[ii] + \sum_{(i,j,k)\circlearrowleft} x_i[jk]\Big) = (a_1, a_2, a_3, \eta_1(x_1), \eta_2(x_2), \eta_3(x_3)), \tag{36}$$

satisfies $\tilde{T}(\eta(x), \eta(y)) = T(x, y)$ and $\tilde{N}(\eta(x)) = N(x)$ for all $x, y \in \mathcal{H}(\mathcal{C}_3)$.

Proof. Since $\tilde{T}$ is nondegenerate, each n_i is nondegenerate. Using $\partial_y \tilde{N}|_x = \tilde{T}(y, x^\#)$ to compute $x^\#$ for $x = (a_1, a_2, a_3, x_1, x_2, x_3)$, we get

$$x^\# = (c_1, c_2, c_3, z_1, z_2, z_3), \quad \text{where}$$
$$c_k = a_i a_j - n_k(x_k) \in K, \quad z_k = x_i \times x_j - a_k x_k \in \mathcal{W}_k$$

for $(i, j, k) \circlearrowleft$. In particular, if e_1, e_2, e_3 is the standard basis of K^3 and $x_k \in \mathcal{W}_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then

$$e_i \times e_j = e_k, \quad x_k^\# = -n_k(x_k) e_k, \quad e_k \times x_k = -x_k,$$

so $\tilde{T}(x_i, x_j^\#) = \tilde{T}(x_i, -n_j(x_j) e_j) = 0$. Thus, (ADJ4) for $(\tilde{T}, \tilde{N})$ gives $(x_i \times x_j)^\# + x_i^\# \times x_j^\# = 0$; that is

$$n_k(x_i \times x_j) = n_i(x_i) n_j(x_j). \tag{37}$$

Since $n_i(u_i) = 1$ and $u_i^\# = -e_i$ for $i = 1, 2$, we can set $u_3 = u_1 \times u_2 \in \mathcal{W}_3$ to get $n_3(u_3) = 1$ and $u_3^\# = -e_3$.

If $x_j, y_j \in \mathcal{W}_j$ and $(i, j, k) \circlearrowleft$, then $n_k(u_i \times x_j, u_i \times y_j) = n_i(u_i) n_j(x_j, y_y) = n_j(x_j, y_j)$ by the polarization of (37). Since n_j is nondegenerate, we see that $y_j \to u_i \times y_j$ is a linear monomorphism of $\mathcal{W}_j$ into $\mathcal{W}_k$. Polarizing (ADJ5) for $(\tilde{T}, \tilde{N})$, we get

$$(u \times (v \times w)) \times v + (u \times v^\#) \times w = \tilde{T}(w, v^\#) u + \tilde{T}(u, w) v^\# + \tilde{T}(u, v)(v \times w).$$

Taking $u = v = u_i$ and $w = x_j \in \mathcal{W}_j$ with $i \neq j$, we get

$$\begin{aligned} u_i \times (u_i \times (u_i \times x_j))) &= -(u_i \times u_i^\#) \times x_j + \tilde{T}(x_j, u_i^\#) u_i \\ &\quad + \tilde{T}(u_i, x_j) u_i^\# + \tilde{T}(u_i, u_i)(u_i \times x_j) \\ &= -u_i \times x_j + 2(u_i \times x_j) = u_i \times x_j, \end{aligned}$$

since $\tilde{T}(x_j, u_i^\#) = \tilde{T}(u_i, x_j) = 0$, $-(u_i \times u_i^\#) = u_i \times e_i = -u_i$, and $\tilde{T}(u_i, u_i) = 2n_i(u_i) = 2$. Since $y_j \to u_i \times y_j$ is a linear monomorphism, we have

$$u_i \times (u_i \times x_j) = x_j. \tag{38}$$

Since $u_3 = u_1 \times u_2$ by definition, (38) shows that $u_i \times u_j = u_k$ for all $\{i,j,k\} = \{1,2,3\}$. Let $\mathcal{C} = \mathcal{W}_3$ and define a bilinear product on $\mathcal{C}$ by

$$xy = (x \times u_1) \times (u_2 \times y)$$

for $x, y \in \mathcal{C}$. Clearly, $1_{\mathcal{C}} := u_3$ is the identity element for $\mathcal{C}$ by (38). Moreover, $n := n_3$ has $n(1_{\mathcal{C}}) = 1$ and $n(xy) = n(x)n(y)$ by (37). Thus, $\mathcal{C}$ is a composition algebra. We denote the norm, trace and canonical involution on $\mathcal{C}$ by n, t and $^-$ respectively. Define $\eta_i : \mathcal{C} \to \mathcal{W}_i$ by $\eta_1(x) = u_2 \times \bar{x}$, $\eta_2(x) = u_1 \times \bar{x}$, and $\eta_3 = \mathrm{id}$. By (38), η_i is a K-linear isomorphism satisfying $\eta_i(1_{\mathcal{C}}) = u_i$. Now define η by (36). Then, if $x \in \mathcal{C}$, using (37) we have $n_1(\eta(x[23])) = n_1(\eta_1(x)) = n_1(u_2 \times \bar{x}) = n_2(u_2)n_3(\bar{x}) = n(\bar{x}) = n(x)$, and similarly $n_2(\eta(x[31])) = n(x)$. Thus

$$n_1(\eta(x[23]) = n_2(\eta(x[31])) = n_3(\eta(x[12])) = n(x) \tag{39}$$

for $x \in \mathcal{C}$. Next, if $x_i \in \mathcal{W}_i$, we have $\tau(x_1, x_2, x_3) = \partial_{x_1}\partial_{x_2}\tilde{N}|_{x_3} = \tilde{T}(x_1 \times x_2, x_3) = n_3(x_1 \times x_2, x_3)$. So for $x, y, z \in \mathcal{C}$,

$$\begin{aligned} \tau(\eta(x[23]), \eta(y[31]), \eta(z[12])) &= n_3(\eta(x[23]) \times \eta(y[31]), \eta(z[12])) \\ &= n_3((\bar{y} \times u_1) \times (u_2 \times \bar{x}), z) \\ &= n(\bar{y}\bar{x}, z) = n(\overline{xy}, z) = t(xyz). \end{aligned} \tag{40}$$

Thus for $x, y \in \mathcal{H}(\mathcal{C}_3)$, we see, using (39) and (40), that $\tilde{N}(\eta(x)) = N(x)$ and $\tilde{T}(\eta(x), \eta(y)) = T(x, y)$. $\square$

We next characterize certain gradings of $\mathcal{H}(\mathcal{C}_3)$.

Lemma 11.2. *Suppose that K is Λ-graded as an algebra, and suppose $\mathcal{C}$ is a composition algebra over K.*

(a) If $\mathcal{C}$ is Λ-graded as an algebra with involution and as a K-module and if $\lambda_i \in \Lambda$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$, then letting

$$a[ij] \in \mathcal{H}(\mathcal{C}_3) \quad \text{have degree} \quad \lambda_i + \lambda_j + \mu, \tag{41}$$

for $a \in \mathcal{C}^\mu$, $\mu \in \Lambda$, $(i,j,k) \circlearrowleft$, and for $a \in K^\mu$, $\mu \in \Lambda$, $i = j$, defines a Λ-grading of $\mathcal{H}(\mathcal{C}_3)$ as a K-module for which N is a graded cubic form.

(b) Conversely, suppose that we are given a Λ-grading of $\mathcal{H}(\mathcal{C}_3)$ as a K-module with the properties that N is a graded cubic form, $1[ij]$ is homogeneous for $(i,j,k) \circlearrowleft$ or $i = j$, and each $\mathcal{C}[ij]$ with $(i,j,k) \circlearrowleft$ is a graded submodule of $\mathcal{H}(\mathcal{C}_3)$. Then, there exists a Λ-grading on $\mathcal{C}$ as an algebra with involution and as a K-module so that the grading on $\mathcal{H}(\mathcal{C}_3)$ is given by (41) where $-\lambda_i$ is the degree of $1[jk]$ for $(i,j,k) \circlearrowleft$.

Proof. (a): Clearly, (41) gives a Λ-grading of $\mathcal{H}(\mathcal{C}_3)$ as an K-module. Also, N is a graded cubic form. Indeed, each term in the full linearization of N involves one factor from each row and one from each column, so the contribution to the degree from the matrix locations is $2(\lambda_1 + \lambda_2 + \lambda_3) = 0$.

(b): Given a Λ-grading of $\mathcal{H}(\mathcal{C}_3)$ as specified in the converse, let $-\lambda_i$ be the degree of $1[jk]$ for $(i,j,k) \circlearrowleft$ and let ρ_i be the degree of $1[ii]$. We can obtain three gradings of $\mathcal{C}$ as a K-module by translating the gradings of the submodules $\mathcal{C}[jk]$. Specifically, let $\mathcal{C}_{(i)} = \mathcal{C}$ with the grading given by $\mathcal{C}^\mu_{(i)}[jk] = (\mathcal{C}[jk])^{-\lambda_i + \mu}$ for $(i,j,k) \circlearrowleft$. Thus, $1 \in \mathcal{C}^0_{(i)}$. If $x_i \in \mathcal{C}^{\mu_i}_{(i)}$, we have

$$t(x_1x_2x_3) = T(x_1[23], x_2[31] \times x_3[12]) \in K^{-(\lambda_1+\lambda_2+\lambda_3)+(\mu_1+\mu_2+\mu_3)}.$$

Taking each $x_i = 1$ shows $\lambda_1+\lambda_2+\lambda_3 = 0$, so the trilinear map $\mathcal{C}_{(1)}\times\mathcal{C}_{(2)}\times\mathcal{C}_{(3)} \to K$ with $(x_1, x_2, x_3) \to t(x_1x_2x_3)$ is graded. Also, if $x \in \mathcal{C}^{\lambda}_{(i)}$ and $y \in \mathcal{C}^{\mu}_{(i)}$, then

$$n(x, y) = -T(1[ii], x[jk] \times y[jk]) \in K^{\rho_i-2\lambda_i+\lambda+\mu}.$$

Again $x = y = 1$ shows $\rho_i = 2\lambda_i$, so $n : \mathcal{C}_{(i)}\times\mathcal{C}_{(i)} \to K$ and $t : \mathcal{C}_{(i)} \to K$ are graded. Thus, $\bar{x} = t(x)1 - x \in \mathcal{C}^{\lambda}_{(i)}$, so $-$ is graded on $\mathcal{C}_{(i)}$. Since n is nondegenerate and graded on $\mathcal{C}_{(i)}$, it is easy to see that $\mathcal{C}^{\mu}_{(i)} = \{x \in \mathcal{C} \mid n(\mathcal{C}^{\lambda}_{(i)}, x) \in K^{\lambda+\mu}\}$. Now if $x \in \mathcal{C}^{\mu}_{(j)}\mathcal{C}^{\nu}_{(k)}$, then

$$n(\mathcal{C}^{\lambda}_{(i)}, x) = n(\overline{\mathcal{C}^{\lambda}_{(i)}}, x) \subset t(\mathcal{C}^{\lambda}_{(i)}\mathcal{C}^{\mu}_{(j)}\mathcal{C}^{\nu}_{(k)}) \subset K^{\lambda+\mu+\nu}$$

so $\mathcal{C}^{\mu}_{(j)}\mathcal{C}^{\nu}_{(k)} \subset \mathcal{C}^{\mu+\nu}_{(i)}$. In particular, $\mathcal{C}^{\lambda}_{(1)} = 1\mathcal{C}^{\lambda}_{(3)} = \mathcal{C}^{\lambda}_{(2)}1$, so the three gradings coincide. Now writing $\mathcal{C}^{\lambda} = \mathcal{C}^{\lambda}_{(i)}$, we see that $\mathcal{C}$ is Λ-graded as an algebra with involution. Since $K^{\mu}[ii] = K^{\mu}(1[ii]) \subset \mathcal{H}(\mathcal{C}_3)^{2\lambda_i+\mu}$ and $\mathcal{C}^{\mu}[jk] = (\mathcal{C}[jk])^{-\lambda_i+\mu} \subset \mathcal{H}(\mathcal{C}_3)^{-\lambda_j-\lambda_k+\mu}$, we have the desired grading of $\mathcal{H}(\mathcal{C}_3)$. □

12. Construction of tori of class III(c)

In this section we construct structurable tori of class III(c) using the construction from §7 of a structurable algebra $\mathcal{A}(h, N)$ from a pair (h, N) satisfying the adjoint identity. These tori will be used in the next section to classify all tori of class III(c).

We will use the following proposition which can be viewed as a converse to Proposition 8.2.

Proposition 12.1. *Suppose $\mathcal{E}$ is a Λ-graded associative commutative algebra with involution $*$ and centre $\mathcal{Z}$; suppose $\mathcal{W}$ is a Λ-graded left $\mathcal{E}$-module; suppose $h : \mathcal{W}\times\mathcal{W} \to \mathcal{E}$ is a non-degenerate Λ-graded hermitian form; and suppose $N : \mathcal{W} \to \mathcal{E}$ is a Λ-graded cubic form over $\mathcal{E}$ such that (h, N) satisfies the adjoint identity. Let*

$$\mathcal{A}(h, N) = \mathcal{E} \oplus \mathcal{W}$$

be the algebra with involution constructed from the pair (h, N) as in §7, and give $\mathcal{A}(h, N)$ a Λ-grading by extending the given Λ-gradings on $\mathcal{E}$ and $\mathcal{W}$. Then $\mathcal{A}(h, N)$ is a Λ-graded structurable algebra. Moreover, if

(i) the support Λ_- of $\mathcal{E}$ is a subgroup of Λ, and $\mathcal{E}$ is a commutative associative Λ_--torus with involution,

(ii) $\mathcal{W}$ is finely graded, $\mathrm{supp}(\mathcal{W}) \cap \Lambda_- = \emptyset$, *and* $\Lambda = \langle\Lambda_-, \mathrm{supp}(\mathcal{W})\rangle$,

(iii) $4\Lambda \subset \Gamma(\mathcal{E})$,

(iv) $0 \neq x \in \mathcal{W}^{\alpha}$ with $2\alpha \notin \Gamma(\mathcal{E})$ implies $x^{\natural} \neq 0$,

then $\mathcal{A}$ is a structurable Λ-torus. If, in addition,

(v) $N \neq 0$ and the involution $*$ on $\mathcal{E}$ is nontrivial,

then $\mathcal{A}$ is a structurable torus of class III and centre $\mathcal{Z}$.

Proof. Let $\mathcal{A} = \mathcal{A}(h, N)$. Then, by Theorem 7.2, $\mathcal{A}$ is a structurable algebra. To see that $\mathcal{A}$ is Λ-graded as an algebra with involution, it suffices to show that $\mathcal{W}^{\alpha} \diamond \mathcal{W}^{\beta} \subset \mathcal{W}^{\alpha+\beta}$ for $\alpha, \beta \in \Lambda$. This follows from $h(\mathcal{W}^{\lambda}, \mathcal{W}^{\alpha} \diamond \mathcal{W}^{\beta}) \subset \mathcal{E}^{\lambda+\alpha+\beta}$, which holds since N is graded, and from $\mathcal{W}^{\mu} = \{x \in \mathcal{W} \mid h(\mathcal{W}^{\lambda}, x) \in \mathcal{E}^{\lambda+\mu}$ for all $\lambda \in \Lambda\}$, which holds since h is nondegenerate and graded.

Now suppose that Λ_- is a subgroup of Λ satisfying (i)–(iv), and let $\Gamma = \Gamma(\mathcal{E}) = \mathrm{supp}(\mathcal{Z})$. By (i)–(ii), $\mathcal{A}$ is finely graded and $\Lambda = \langle\mathrm{supp}(\mathcal{A})\rangle$.

To show that $\mathcal{A}$ is a structurable torus, it remains to show that each homogenous element of $\mathcal{A}$ is invertible. To do this, it suffices, by Remark 3.1(i), to show that for $0 \neq x \in \mathcal{A}^\alpha$ there is $y \in \mathcal{A}^{-\alpha}$ with $xy = 1$ and $[L_x, L_y] = 0$. If $\alpha \in \Lambda_-$, we can take $y = x^{-1}$ in $\mathcal{E}$ and use the fact that $\mathcal{W}$ is a left $\mathcal{E}$-module. So we may suppose that $\alpha \notin \Lambda_-$. Then $h(x, \mathcal{W}^\beta) \neq 0$ for some $\beta \in -\alpha + \Lambda_-$, since h is nondegenerate. However, $\mathcal{E}^\gamma \mathcal{W}^\beta = \mathcal{W}^{\gamma+\beta}$, so $h(x, \mathcal{W}^\beta) \neq 0$ for all $\beta \in -\alpha + \Lambda_-$. In particular,

$$h(x, y) = 1$$

for some $y \in \mathcal{W}^{-\alpha}$. Since $0 \notin \operatorname{supp}(\mathcal{W})$, we also have $x \diamond \mathcal{W}^{-\alpha} = 0$, so

$$xy = 1.$$

If $2\alpha \in \Gamma$, then $x = zy$ for some $z \in \mathcal{E}^{2\alpha}$, so $[L_x, L_y] = L_z[L_y, L_y] = 0$. Thus, we can assume that $2\alpha \notin \Gamma$. Now, for $e \in \mathcal{E}$, we have $x(ye) = h(x, e^*y) + x \diamond (e^*y) = e$ and similarly $y(xe) = e$, so $[L_x, L_y]\mathcal{E} = 0$. Hence, it remains to show that

$$[L_x, L_y]w = 0$$

for $w \in \mathcal{W}$. Since $2\alpha \notin \Gamma$, we have $x^\natural \neq 0$ (by (iv)) and similarly $y^\natural \neq 0$. Now $x \diamond y^\natural \in \mathcal{W}^{-\alpha}$ so

$$x \diamond y^\natural = ay$$

with $a \in F$. Moreover, $a = a^* = h(x, ay) = h(x, x \diamond y^\natural) = h(y^\natural, x \diamond x) = 2h(y^\natural, x^\natural) \neq 0$. By (ADJ2), we have

$$(w \diamond y^\natural) \diamond x^\natural + (w \diamond x) \diamond (y^\natural \diamond x) = h(y^\natural, x^\natural)w + h(w, y^\natural \diamond x)x + h(w, x^\natural)y^\natural,$$

so

$$(w \diamond y^\natural) \diamond x^\natural + a(w \diamond x) \diamond y = aw + ah(w, y)x + h(w, x^\natural)y^\natural.$$

Thus

$$y \diamond (x \diamond w) - h(w, y)x = w + a^{-1}h(w, x^\natural)y^\natural - a^{-1}(w \diamond y^\natural) \diamond x^\natural. \tag{42}$$

Since $x^\natural = zy^\natural$ for some $z \in \mathcal{Z}(\mathcal{E})^{4\alpha}$ (by (iii)), we see that the right side of (42) is symmetric in x and y. Thus, the left side of (42) or equivalently $x \diamond (y \diamond w) + h(w, y)x$ is symmetric in x and y. Since

$$x(yw) = h(w, y)x + h(x, y \diamond w) + x \diamond (y \diamond w),$$

we see that $[L_x, L_y]w = 0$. So $\mathcal{A}$ is a structurable torus.

Finally, suppose that (v) also holds. Then we may choose $\sigma_0 \in S_-(\mathcal{E})$, in which case $2\sigma_0 \in \Gamma$, $S_+(\mathcal{E}) = \Gamma$, and $S_-(\mathcal{E}) = \sigma_0 + \Gamma$. Thus, $S_-(\mathcal{E})$ generates Λ_-, so our notation Λ_-, $\mathcal{E}$, $\mathcal{W}$ agrees with the notation from §4. Hence, since the map $\diamond : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ is nonzero, it follows that $\mathcal{A}$ is of class III. Also, the centre of $\mathcal{A}$ is $\mathcal{Z}$ by Lemma 7.3. $\square$

We will also need the following simple lemma.

Lemma 12.2. *Let $\mathcal{E}$ be a commutative, associative algebra with involution $*$ and let $\mathcal{W}$ be a left $\mathcal{E}$-module with cubic form N, hermitian form h, and symmetric bilinear form T. If $\psi : \mathcal{W} \to \mathcal{W}$ is a $*$-semilinear map satisfying $N(\psi(x)) = N(x)^*$ and $h(x, y) = T(x, \psi(y))$, then N is nondegenerate and (h, N) satisfies the adjoint identity if and only if T is nondegenerate and (T, N) satisfies the adjoint identity. In that case the adjoints ($\natural$ for (h, N) and $\#$ for (T, N)) are related by*

$$y^\natural = \psi^{-1}(y^\#) = \psi(y)^\#.$$

Proof. Clearly h is nondegenerate and $y^\natural$ is an adjoint for (h, N) if and only if T is nondegenerate and $y^\# = \psi(y^\natural)$ is an adjoint for (T, N). Also if those conditions hold, then, since $N = * \circ N \circ \psi$ and $\psi^* = \psi$, we have

$$\begin{aligned} T(x, y^\#) &= \partial_x(* \circ N \circ \psi)|_y = (\partial_{\psi(x)} N|_{\psi(y)})^* \\ &= (T(\psi(x), \psi(y)^\#))^* = T(x, \psi(\psi(y)^\#)), \end{aligned}$$

so $y^\# = \psi(\psi(y)^\#)$, and hence $y^\natural = \psi(y)^\#$. Thus,

$$(y^\natural)^\natural = (\psi^{-1}(y^\#))^\natural = (\psi(\psi^{-1}(y^\#)))^\# = (y^\#)^\#,$$

proving (a). □

Our goal now is to construct a pair (h, N) satisfying the assumptions of Proposition 12.1, where h and N are defined on the space of 3×3-hermitian matrices over an alternative torus with involution $\mathcal{C}$. Ultimately we will use two choices for $\mathcal{C}$, a quaternion torus $\mathcal{Q}$ and an octonion torus $\mathcal{O}$. However, it will only be at the very end of the proof of the classification theorem (Theorem 13.3) that we will see that these are the only choices. For this reason, we begin with three choices for $\mathcal{C}$, a quaternion torus $\mathcal{Q}$ and *two* octonion tori $\mathcal{O}$ and $\mathcal{O}'$.

We now describe these initial choices for $\mathcal{C}$. In each case $\mathcal{C}$ is a finely Λ-graded algebra with involution $^-$, where Λ is a finitely generated free abelian group. Also in each case, we let

$$M = \operatorname{supp}(\mathcal{C}), \quad \mathcal{E} = \mathcal{Z}(\mathcal{C}) \quad \text{and} \quad \Lambda_- = \operatorname{supp}(\mathcal{E}).$$

Case 1: Let $\mathcal{C} = \mathcal{Q}$, where

$$\mathcal{Q} = \operatorname{CD}(F[t_1^{\pm 1}, t_2^{\pm 1}, s^{\pm 1}], t_1, t_2)$$

with canonical involution $^-$ and canonical generators v_1, v_2 (see (3)). Let

$$\Lambda = \langle \lambda_1, \lambda_2, \sigma \rangle$$

with basis $\lambda_1, \lambda_2, \sigma$, and give $\mathcal{C}$ the Λ-grading so that

$$v_1, v_2, s \text{ have degrees } 2\lambda_1, 2\lambda_2, \sigma \text{ respectively.}$$

Then,

$$M = \langle 2\lambda_1, 2\lambda_2, \sigma \rangle, \quad \mathcal{E} = F[t_1^{\pm 1}, t_2^{\pm 1}, s^{\pm 1}] \quad \text{and} \quad \Lambda_- = \langle 4\lambda_1, 4\lambda_2, \sigma \rangle.$$

Case 2: Let $\mathcal{C} = \mathcal{O}$, where

$$\mathcal{O} = \operatorname{CD}(F[t_1^{\pm 1}, t_2^{\pm 1}, s^{\pm 1}], t_1, t_2, s)$$

with canonical involution $^-$ and canonical generators v_1, v_2, v. Let

$$\Lambda = \langle \lambda_1, \lambda_2, \lambda \rangle$$

with basis $\lambda_1, \lambda_2, \lambda$, and give $\mathcal{C}$ the Λ-grading so that

$$v_1, v_2, v \text{ have degrees } 2\lambda_1, 2\lambda_2, \lambda \text{ respectively.}$$

Then,

$$M = \langle 2\lambda_1, 2\lambda_2, \lambda \rangle, \quad \mathcal{E} = F[t_1^{\pm 1}, t_2^{\pm 1}, s^{\pm 1}] \quad \text{and} \quad \Lambda_- = \langle 4\lambda_1, 4\lambda_2, 2\lambda \rangle.$$

Case 3: Let $\mathcal{C} = \mathcal{O}'$, where

$$\mathcal{O}' = \operatorname{CD}(F[t_1^{\pm 1}, t_2^{\pm 1}, s^{\pm 1}, r^{\pm 1}], t_1, t_2, r)$$

with canonical involution $^-$ and canonical generators v_1, v_2, v. Let

$$\Lambda = \langle \lambda_1, \lambda_2, \sigma, \lambda \rangle$$

with basis $\lambda_1, \lambda_2, \sigma, \lambda$, and give $\mathcal{C}$ the Λ-grading so that

$$v_1, v_2, s, v \text{ have degrees } 2\lambda_1, 2\lambda_2, \sigma, \lambda \text{ respectively.}$$

Then,

$$M = \langle 2\lambda_1, 2\lambda_2, \sigma, \lambda \rangle, \quad \mathcal{E} = F[t_1^{\pm 1}, t_2^{\pm 1}, s^{\pm 1}, r^{\pm 1}] \quad \text{and} \quad \Lambda_- = \langle 4\lambda_1, 4\lambda_2, \sigma, 2\lambda \rangle.$$

For the remainder of this section *we assume that* $\mathcal{C} = \mathcal{Q}$, $\mathcal{O}$ *or* $\mathcal{O}'$ *and we use the notation introduced in Cases 1, 2 and 3.*

Note that in each case, $\mathcal{C}$ is a composition algebra over $\mathcal{E}$ with norm $n : \mathcal{E} \to \mathcal{E}$ defined by $n(a) = a\bar{a}$ for $a \in \mathcal{C}$ (see §2). We denote the trace on $\mathcal{C}$ by $t : \mathcal{C} \to \mathcal{E}$. Note also that $\mathcal{C}$ is an alternative M-torus with involution since, regarding $\mathcal{C}$ as an M-graded algebra with involution, we have $\mathcal{C} \simeq_{ig} \mathcal{A}(2) \otimes \mathcal{P}(1)$, $\mathcal{A}(3)$ or $\mathcal{A}(3) \otimes \mathcal{P}(1)$ in Cases 1, 2 and 3 respectively.

In each case we define an involution $*$ on $\mathcal{E}$ such that

$$t_i^* = t_i, \quad s^* = -s \quad \text{and, when } \mathcal{C} = \mathcal{O}', \quad r^* = r. \tag{43}$$

We then let

$$\mathcal{Z} = \{a \in \mathcal{E} \ : \ a^* = a\}$$

be the centre of the algebra with involution $\mathcal{E}$, and we let

$$\Gamma = \operatorname{supp}(\mathcal{Z}).$$

Then $\Gamma = \langle 4\lambda_1, 4\lambda_2, 2\sigma \rangle$, $\langle 4\lambda_1, 4\lambda_2, 4\lambda \rangle$ or $\langle 4\lambda_1, 4\lambda_2, 2\sigma, 2\lambda \rangle$ in Cases 1, 2 and 3 respectively. Thus in all cases we have

$$4\Lambda \subset \Gamma. \tag{44}$$

In some cases, we can introduce an important $*$-semilinear automorphism θ of $\mathcal{C}$:

Lemma 12.3. *Suppose that either* $\mathcal{C} = \mathcal{Q}$ *or there exists* $\iota \in F$ *with* $\iota^2 = -1$ *and* $\mathcal{C} = \mathcal{O}$. *Then there exists a unique* $*$*-semilinear* F*-algebra automorphism* θ *of* $\mathcal{C}$ *which fixes* v_1 *and* v_2*, and, if* $\mathcal{C} = \mathcal{O}$*, satisfies* $\theta(v) = \iota v$.

Proof. Uniqueness is clear. If $\mathcal{C} = \mathcal{Q} = \mathrm{CD}(\mathcal{E}, t_1, t_2)$, there is clearly an F-algebra automorphism θ of $\mathcal{C}$ extending $*$ on $\mathcal{E}$ and fixing v_i. On the other hand, if $\mathcal{C} = \mathcal{O} = \mathrm{CD}(\mathcal{E}, t_1, t_2, s)$, we can view $\mathcal{O}$ as $\mathrm{CD}(\mathcal{E}, t_1, t_2, -s)$ with canonical generators $v_1, v_2, \iota v$, so there exists an F-algebra automorphism θ of $\mathcal{C}$ extending $*$ on $\mathcal{E}$ and mapping $v_i \mapsto v_i$ and $v \mapsto \iota v$. □

As in §11 (with $K = \mathcal{E}$), let $\mathcal{H}(\mathcal{C}_3)$ be the $\mathcal{E}$-module of 3×3-hermitian matrices over $\mathcal{E}$, and let T and N denote the standard trace and norm on $\mathcal{H}(\mathcal{C}_3)$ respectively. Let

$$\lambda_3 = -\lambda_1 - \lambda_2,$$

and we use Lemma 11.2(a) to grade $\mathcal{H}(\mathcal{C}_3)$ so that N is a graded cubic form. Thus,

$$a[ij] \in \mathcal{H}(\mathcal{C}_3) \quad \text{has degree} \quad \lambda_i + \lambda_j + \mu,$$

for $a \in \mathcal{C}^\mu$, $\mu \in M$, $(i, j, k) \circlearrowleft$, and for $a \in K^\mu$, $\mu \in \Lambda_-$, $i = j$.

Although N is graded, T is not graded and needs to be modified to give a graded hermitian form. We do this in the next lemma using a $*$-semilinear vector space isomorphism $\psi : \mathcal{H}(\mathcal{C}_3) \to \mathcal{H}(\mathcal{C}_3)$. Such an isomorphism is said to be *semi-norm preserving* if $N(\psi(x)) = N(x)^*$ for all $x \in \mathcal{H}(\mathcal{C}_3)$.

In general, if $\mathcal{U}$ is an $\mathcal{E}$-module with a nondegenerate symmetric bilinear form $f : \mathcal{U} \times \mathcal{U} \to \mathcal{E}$ and $\psi : \mathcal{U} \to \mathcal{U}$ is an α-semilinear vector space isomorphism where α is an F-algebra automorphism of $\mathcal{E}$, we write $\psi^* = \varphi$ (relative to f) if $f(\psi(x), y) = \alpha(f(x, \varphi(y))$ for $x, y \in \mathcal{U}$. We note that if φ exists, it is unique and is an α^{-1}-semilinear vector space isomorphism. Moreover, if ψ_1^* and ψ_2^* exist, then $(\psi_1\psi_2)^* = \psi_2^*\psi_1^*$. Also, if $\alpha = *$, the involution on $\mathcal{E}$, then $h(x, y) = f(x, \psi(y))$ is a nondegenerate sesquilinear form, which is hermitian if and only if $\psi^* = \psi$ (relative to f).

A key lemma in our construction and classification of structurable tori of class III(c) is the following.

Lemma 12.4. *Suppose that $\psi : \mathcal{H}(\mathcal{C}_3) \to \mathcal{H}(\mathcal{C}_3)$ is a $*$-semilinear vector space isomorphism and set $h(x, y) := T(x, \psi(y))$ for $x, y \in \mathcal{H}(\mathcal{C}_3)$. Then ψ is semi-norm preserving and h is a graded hermitian form if and only if the following conditions hold:*

(i) *Either $\mathcal{C} = \mathcal{Q}$ or there exists $\iota \in F$ with $\iota^2 = -1$ and $\mathcal{C} = \mathcal{O}$,*

(ii) *$\psi : \sum_i a_i[ii] + \sum_{(i,j,k)\circlearrowleft} x_i[jk] \to \sum_i a_i^* n(w_i)[ii] + \sum_{(i,j,k)\circlearrowleft} (w_j\theta(x_i))\bar{w}_k[jk]$, where $w_i \in \mathcal{C}^{2\lambda_i}$ with $w_1w_2w_3 = 1$ and θ is the $*$-semilinear F-algebra automorphism of $\mathcal{C}$ which fixes v_1 and v_2, and satisfies $\theta(v) = \iota v$ in the case $\mathcal{C} = \mathcal{O}$.*

Proof. Before beginning the proof, we need some preliminaries about triality triples. (See for example [J1] for a slightly more general concept in the finite dimensional case.)

Let α be an automorphism of $\mathcal{E}$. A *triality triple* of α-semisimilarities of $\mathcal{C}$ is a triple $(\varphi_1, \varphi_2, \varphi_3)$ of α-semilinear bijections from $\mathcal{C}$ to $\mathcal{C}$ such that for some invertible $c_i \in \mathcal{E}$ we have

$$n(\varphi_i(x_i)) = c_i\alpha(n(x_i)), \tag{45}$$

$$t(\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)) = \alpha(t(x_1x_2x_3)) \tag{46}$$

for $x_i \in \mathcal{C}$. In that case, the elements c_i are called the *multipliers* of the triple. If $\alpha = id$, we say that φ_i is a *similarity*. Since $t(abc)$ is invariant under cyclic permutations, any cyclic permutation of a triality triple is a triality triple.

If $\varphi : \mathcal{H}(\mathcal{C}_3) \to \mathcal{H}(\mathcal{C}_3)$ is an α-semilinear bijection which stabilizes the spaces $\mathcal{E}[ii]$, $\mathcal{C}[jk]$ for $(i, j, k) \circlearrowleft$, we can write

$$\varphi : \sum_i a_i[ii] + \sum_{(i,j,k)\circlearrowleft} x_i[jk] \to \sum_i \alpha(a_i)c_i^{-1}[ii] + \sum_{(i,j,k)\circlearrowleft} \varphi_i(x_i)[jk] \tag{47}$$

for some α-semilinear bijections $\varphi_i : \mathcal{C} \to \mathcal{C}$ and some $0 \neq c_i \in \mathcal{E}$. We claim that $N(\varphi(x)) = \alpha(N(x))$ for $x \in \mathcal{H}(\mathcal{C}_3)$ if and only if $(\varphi_1, \varphi_2, \varphi_3)$ is a triality triple with multipliers c_i. Indeed, (45) and (46) follow from $N(\varphi(x)) = \alpha(N(x))$ for $x = 1[ii] + x_i[jk]$ and $x = \sum_{(i,j,k)\circlearrowleft} x_i[jk]$, respectively. The converse is immediate.

Triality triples of similarities can be created using invertible elements of $\mathcal{C}$. Indeed, it is obvious that (L_a, L_b, L_c) is a triality triple of similarities if $a, b, c \in \mathcal{E}$ with $abc = 1$ (see §2 for this notation). Also, if $x \in \mathcal{C}$ is invertible, then the right (or left) Moufang identity and the associativity of $t(abc)$ shows that $t((ax)(x^{-1}bx^{-1})(xc)) = t(abc)$, so $(R_x, L_{x^{-1}}R_{x^{-1}}, L_x)$ is a triality triple. Moreover, if $xyz = 1$ in $\mathcal{C}$, then

$yz = n(x)^{-1}\bar{x}$ and we can compute

$$\begin{aligned}\bar{z}^{-1}(by)\bar{z}^{-1} &= n(z)^{-2}(zb)(yz) = n(x)^{-1}n(z)^{-2}(zb)\bar{x},\\ \bar{z}(y^{-1}cy^{-1}) &= n(y^{-2})((\bar{z}\bar{y})c)\bar{y} = n(x)^{-1}n(y)^{-2}(xc)\bar{y}.\end{aligned}$$

Thus,

$$\begin{aligned}(\mathrm{id}, L_{n(x)n(z)^2}, L_{n(x)n(y)^2})(R_{\bar{z}}, L_{\bar{z}^{-1}}R_{\bar{z}^{-1}}, L_{\bar{z}})(L_y, R_y, L_{y^{-1}}R_{y^{-1}})&\\ = (R_{\bar{z}}L_y, R_{\bar{x}}L_z, R_{\bar{y}}L_x) \quad (48)&\end{aligned}$$

is a triality triple of similarities.

To begin the proof, we recall that T is not graded using the given grading on $\mathcal{H}(\mathcal{C}_3)$. But we obtain a second grading on $\mathcal{H}(\mathcal{C}_3)$ by letting

$$a[ij] \in \mathcal{H}(\mathcal{C}_3) \quad \text{have degree} \quad -\lambda_i - \lambda_j + \mu,$$

for $a \in \mathcal{C}^\mu$, $\mu \in M$, $(i,j,k)\circlearrowleft$, and for $a \in K^\mu$, $\mu \in \Lambda_-$, $i = j$. Denoting $\mathcal{H}(\mathcal{C}_3)$ with this grading by $\mathcal{H}(\mathcal{C}_3)'$, we see that $T : \mathcal{H}(\mathcal{C}_3) \times \mathcal{H}(\mathcal{C}_3)' \to \mathcal{E}$ is graded. Since T is nondegenerate, we see that h is graded if and only if $\psi : \mathcal{H}(\mathcal{C}_3) \to \mathcal{H}(\mathcal{C}_3)'$ is graded; that is

$$\psi : \mathcal{H}(\mathcal{C}_3)^{\lambda_i+\lambda_j+\mu} \to \mathcal{H}(\mathcal{C}_3)'^{\lambda_i+\lambda_j+\mu} = \mathcal{H}(\mathcal{C}_3)'^{-\lambda_i-\lambda_j+\mu'} = \mathcal{H}(\mathcal{C}_3)^{\lambda_i+\lambda_j+\mu'}$$

for $\mu \in M$, $(i,j,k)\circlearrowleft$, and for $\mu \in \Lambda_-$, $i = j$, where $\mu' = 2(\lambda_i+\lambda_j)+\mu$. In particular, this implies that ψ stabilizes $\mathcal{E}[ii]$ and $\mathcal{C}[ij]$ for $(i,j,k)\circlearrowleft$. Thus, ψ is semi-norm preserving and h is a graded sesquilinear form if and only if ψ has the form (47) for a triality triple (ψ_1, ψ_2, ψ_3) of $*$-semisimilarities with $\psi_i(\mathcal{C}^\mu) = \mathcal{C}^{-2\lambda_i+\mu}$ and multipliers $c_i \in \mathcal{E}^{-4\lambda_i}$. If these conditions hold, let $u_i = \psi_i(1) \in \mathcal{C}^{-2\lambda_i}$. We see that $\bar{u}_i = -u_i$ and $n(u_i) = c_i$ by (45). Also, since $(u_1u_2)u_3 \in \mathcal{C}^0 = F1$, (46) shows that $u_1u_2u_3 = 1$. Let $w_i = \bar{u}_i^{-1} = -u_i^{-1} \in \mathcal{C}^{2\lambda_i}$ so $w_1w_2w_3 = 1$ and $\bar{w}_i = -w_i$. We know that $\eta_i = -R_{w_k}L_{w_j}$ for $(i,j,k)\circlearrowleft$ defines a triality triple (η_1, η_2, η_3) of similarities with multipliers $n(w_j)n(w_k) = n(w_i^{-1}) = c_i$ and hence a norm preserving map η. We see that $\eta_i(1) = -w_jw_k = -w_i^{-1} = u_i$. Thus, $\eta^{-1}\psi$ is semi-norm preserving and given by the triality triple $(\theta_1, \theta_2, \theta_3)$ where $\theta_i = \eta_i^{-1}\psi_i$. Clearly, $\theta_i(1) = 1$, so θ_i has multiplier 1. Thus, $t(\theta_i(x)) = t(x)$ and θ_i commutes with $-$. Now

$$\begin{aligned}n(ab,c)^* = t(ab\bar{c})^* &= t(\theta_1(a)\theta_1(b)\theta_3(\bar{c})) = t(\theta_1(a)\theta_1(b)\overline{\theta_3(c)})\\ &= n(\theta_1(a)\theta_2(b), \theta_3(c)) = n(\theta_3^{-1}(\theta_1(a)\theta_2(b)), c)^*,\end{aligned}$$

so $\theta_3(ab) = \theta_1(a)\theta_2(b)$. Since $\theta_i(1) = 1$, we see that $\theta_1 = \theta_2 = \theta_3$ is a $*$-semilinear automorphism θ of $\mathcal{C}$. Since $\eta_i(\mathcal{C}^\mu) = \mathcal{C}^{-2\lambda_i+\mu} = \psi_i(\mathcal{C}^\mu)$, we see that θ is graded. We have shown that if ψ is semi-norm preserving and h is a graded sesquilinear form, then ψ is of the form (ii) for some graded $*$-semilinear automorphism θ and some $w_i \in \mathcal{C}^{2\lambda_i}$ with $w_1w_2w_3 = 1$. Conversely, if ψ is of the form (ii) for such a choice of θ and w_i, then (ψ_1, ψ_2, ψ_3) is a triality triple of $*$-semisimilarities with $\psi_i(\mathcal{C}^\mu) = \mathcal{C}^{-2\lambda_i+\mu}$ and multipliers $c_i \in \mathcal{E}^{-4\lambda_i}$, so ψ is semi-norm preserving and h is a graded sesquilinear form.

So we assume for the rest of the proof that ψ has the form (ii) for some graded $*$-semilinear automorphism θ and some $w_i \in \mathcal{C}^{2\lambda_i}$ with $w_1w_2w_3 = 1$. We then have

$$\psi_i = \eta_i\theta, \quad \text{where } \eta_i = -R_{w_k}L_{w_j}$$

for $(i,j,k) \circlearrowleft$, and we set $u_i = \psi_i(1) = -w_j w_k = -w_i^{-1}$ as above. It remains to show that under these conditions $\psi^* = \psi$ if and only if (i) holds and θ is as in (ii).

First note, that $n(w_i) \in \mathcal{E}^{4\lambda_i} = Ft_i$ for $i = 1,2$, and $n(w_3) \in \mathcal{E}^{4\lambda_3} = F(t_1t_2)^{-1}$. Thus, $n(w_i)^* = n(w_i)$, so $\psi^* = \psi$ (relative to T) if and only if $\psi_i^* = \psi_i$ (relative to n) for $i = 1,2,3$. Now $n(\theta(x), y) = n(x, \theta^{-1}(y))^*$ so $\theta^* = \theta^{-1}$. Also, $L_w^* = L_{\bar{w}}$ and $R_w^* = R_{\bar{w}}$, so $\eta_i^* = -L_{w_j}R_{w_k}$. Hence

$$\begin{aligned}\psi_i^* = \psi_i &\iff \theta^*\eta_i^* = \eta_i\theta \iff \theta^{-1}\eta_i^* = \eta_i\theta \\ &\iff \theta^2 = \theta\eta_i^{-1}\theta^{-1}\eta_i^* \iff \theta^2 = L^{-1}_{\theta(w_j)}R^{-1}_{\theta(w_k)}L_{w_j}R_{w_k}.\end{aligned}$$

Now $\eta_i^*(1) = -w_jw_k = u_i = \eta_i(1)$. So if $\psi_i^* = \psi_i$, then $\theta^{-1}\eta_i^*(1) = \eta_i\theta(1)$, which implies that $\theta^{-1}(u_i) = u_i$ and hence $\theta(w_i) = w_i$. Thus setting

$$\beta_i = L_{w_j^{-1}}R_{w_k^{-1}}L_{w_j}R_{w_k}$$

for $(i,j,k) \circlearrowleft$, we see that $\psi^* = \psi$ if and only if $\theta(w_i) = w_i$ and $\theta^2 = \beta_i$ for $i = 1,2,3$.

If $\mathcal{C} = Q$ and $(i,j,k) \circlearrowleft$, then $\beta_i(x) = w_j^{-1}w_jxw_kw_k^{-1} = x$ for $x \in \mathcal{C}$. If $\mathcal{C} = \mathcal{O}$ or $\mathcal{O}'$, we have $\mathcal{C} = \mathrm{CD}(\mathcal{E}, t_1, t_2) \oplus v\,\mathrm{CD}(\mathcal{E}, t_1, t_2)$, with multiplication given in (2) (with $v^2 = s$ or r respectively). Moreover, each w_i is in $\mathrm{CD}(\mathcal{E}, t_1, t_2)$ since $w_i \in \mathcal{C}^{2\lambda_i} = Fv_i$, and hence, for $x \in \mathrm{CD}(\mathcal{E}, t_1, t_2)$, we have $\beta_i(x) = x$ and

$$\beta_i(vx) = v(w_j^{-1}w_k^{-1}w_jw_kx) = -v(w_j^{-1}w_k^{-1}w_kw_jx) = -vx.$$

Thus, $\beta_i = \mathrm{id}$ if $\mathcal{C} = \mathcal{Q}$, whereas β_i is the automorphism with $\beta_i(v_1) = v_1$, $\beta_i(v_2) = v_2$ and $\beta_i(v) = -v$ if $\mathcal{C} = \mathcal{O}$ or $\mathcal{O}'$. In particular, $\beta := \beta_i$ is independent of i. Since $Fw_i = Fv_i$, we have shown that $\psi^* = \psi$ if and only if $\theta(v_i) = v_i$, for $i = 1,2$, and $\theta^2 = \beta$. Since θ is graded so $\theta(v) \in Fv$, this is equivalent to $\theta(v_i) = v_i$ and $\theta(v) = \iota v$ with $\iota^2 = -1$ if $\mathcal{C} = \mathcal{O}$ or $\mathcal{O}'$. However, if $\mathcal{C} = \mathcal{O}'$, then $v^2 = r \in \mathcal{E}$, so $r^* = \theta(v^2) = -r$, a contradiction. $\square$

We can now use Proposition 12.1 and Lemma 12.4 to give a construction of structurable tori of class III(c). In this construction we make a choice of $w_i \in \mathcal{C}^{2\lambda_i}$ with $w_1w_2w_3 = 1$. We note that there always exists such a choice since we may take $w_i = v_i$, where $v_3 = (v_1v_2)^{-1}$.

Theorem 12.5. *Let $\mathcal{E} = F[t_1^{\pm 1}, t_2^{\pm 1}, s^{\pm 1}]$ with the involution $*$ such that $t_i^* = t_i$ and $s^* = -s$, and let $\mathcal{Z}$ be the centre of this algebra with involution. Suppose that either*

1. *$\mathcal{C} = \mathcal{Q} = \mathrm{CD}(\mathcal{E}, t_1, t_2)$ with canonical generators v_1, v_2 and canonical involution $^-$, and $\mathcal{C}$ is Λ-graded with v_1, v_2, s of degrees $2\lambda_1, 2\lambda_2, \sigma$ respectively, where Λ has basis $\lambda_1, \lambda_2, \sigma$; or*
2. *There exists $\iota \in F$ with $\iota^2 = -1$, $\mathcal{C} = \mathcal{O} = \mathrm{CD}(\mathcal{E}, t_1, t_2, s)$ with canonical generators v_1, v_2, v and canonical involution $^-$, and $\mathcal{C}$ is Λ-graded with v_1, v_2, v of degrees $2\lambda_1, 2\lambda_2, \lambda$ respectively, where Λ has basis $\lambda_1, \lambda_2, \lambda$.*

Let $\mathcal{H}(\mathcal{C}_3)$ be the $\mathcal{E}$-module of hermitian 3×3-matrices over $\mathcal{C}$, set $\lambda_3 = -\lambda_1 - \lambda_2$, and grade $\mathcal{H}(\mathcal{C}_3)$ by Λ with $a[ij]$ of degree $\lambda_i + \lambda_j + \mu$ if $a \in \mathcal{C}^\mu$. Choose $w_i \in \mathcal{C}^{2\lambda_i}$ with $w_1w_2w_3 = 1$ and define $\psi : \mathcal{H}(\mathcal{C}_3) \to \mathcal{H}(\mathcal{C}_3)$ by

$$\psi\Big(\sum_i a_i[ii] + \sum_{(i,j,k)\circlearrowleft} x_i[jk]\Big) = \sum_i a_i^* n(w_i)[ii] + \sum_{(i,j,k)\circlearrowleft} (w_j\theta(x_i))\bar{w}_k[jk],$$

where θ is the $*$-semilinear F-algebra automorphism of $\mathcal{C}$ which fixes v_1 and v_2, and satisfies $\theta(v) = \iota v$ in the case $\mathcal{C} = \mathcal{O}$. Let T and N be the standard trace and norm forms on $\mathcal{H}(\mathcal{C}_3)$ and define $h : \mathcal{H}(\mathcal{C}_3) \times \mathcal{H}(\mathcal{C}_3) \to \mathcal{E}$ by

$$h(x, y) = T(x, \psi(y)).$$

Let $\mathcal{A}(h, N) = \mathcal{E} \oplus \mathcal{H}(\mathcal{C}_3)$ with product and involution given (as in (18)) by

$$(a, x)(b, y) = (ab + h(x, y), ay + b^*x + x \diamond y) \quad \text{and} \quad (a, u)^* = (a^*, u),$$

where $x \diamond y = \psi(x) \times \psi(y)$ and $\times$ is defined by (35). Give $\mathcal{A}(h, N)$ the Λ-grading extending the gradings on $\mathcal{E}$ and $\mathcal{H}(\mathcal{C}_3)$. Then $\mathcal{A}(h, N)$ is a structurable Λ-torus of class III(c) with centre $\mathcal{Z}$. Moreover, up to isograded isomorphism, $\mathcal{A}(h, N)$ depends only on $\mathcal{C}$ and not on the choice of w_i or ι.

Proof. Let $\mathcal{A} = \mathcal{A}(h, N) = \mathcal{E} \oplus \mathcal{W}$, where $\mathcal{W} = \mathcal{H}(\mathcal{C}_3)$. By Lemmas 12.2 and 12.4, all of the assumptions in the first sentence of Proposition 12.1 are satisfied. So $\mathcal{A}$ is a Λ-graded structurable algebra. To show that $\mathcal{A}$ is a structurable torus of class III, we check conditions (i)–(v) of Proposition 12.1. We note first that $\mathcal{W}$ is finely graded by Λ and the support of $\mathcal{W}$ is disjoint from Λ_-. Indeed, $\mathcal{C}$ is finely graded by M, so the three spaces $\mathcal{C}[ij]$ are finely graded by the three cosets $-\lambda_k + M$ where $(i, j, k) \circlearrowleft$, and the three spaces $\mathcal{E}[ii]$ are finely graded by the three cosets $2\lambda_i + \Lambda_-$ which lie in M. Also, since $\lambda_i \in \operatorname{supp}(\mathcal{W})$ and (if $\mathcal{C} = \mathcal{O}$) $\lambda_i + \lambda \in \operatorname{supp}(\mathcal{W})$, we have $\Lambda = \langle \Lambda_-, \operatorname{supp}(\mathcal{W}) \rangle$. Next, we have seen in (44) that $4\Lambda \subset \Gamma$. Also, if $0 \neq x \in \mathcal{W}^\alpha$ with $2\alpha \notin \Gamma$, then $\alpha \in -\lambda_k + M$ for some k and $x = a[ij]$ where $(i, j, k) \circlearrowleft$. Thus, $x^\natural = \psi^{-1}(x^\#) = -n(a)^* n(w_k)^{-1}[kk] \neq 0$. The other conditions in (i)–(v) of Proposition 12.1 clearly hold, so $\mathcal{A}$ is a structurable torus of class III with centre $\mathcal{Z}$.

To see that $\mathcal{A}$ is of class III(c), let $\bar{\Lambda} = \Lambda/\Lambda_-$ and $\mathcal{I}$ be as in §8. The points in $\mathcal{I}$ of order 2 are $2\bar{\lambda}_i$, $1 \leq i \leq 3$, and the points in $\mathcal{I}$ of order 4 are $-\bar{\lambda}_i + \bar{\mu}$, $1 \leq i \leq 3$, $\bar{\mu} \in \bar{M}$. In particular, $\mathcal{I}$ has a point of order 4, so $\mathcal{A}$ is not of class III(b). Assume for contradiction that $\mathcal{I}$ is of class III(a). Then, by Corollary 8.4(a), $\mathcal{I}$ is a star with centre $2\bar{\lambda}_i$ for some i. But if $(i, j, k) \circlearrowleft$, then $2\bar{\lambda}_i$ is not collinear with $-\bar{\lambda}_k$ since $1[ii] \in \mathcal{W}^{2\lambda_i}$, $1[ij] \in \mathcal{W}^{-\lambda_k}$ and $(1(ii)) \diamond (1[ij]) \in \mathcal{E}[ii] \times \mathcal{C}[ij] = 0$ (by (35)). This contradiction shows that $\mathcal{A}$ is of class III(c).

For the last statement, suppose we use the v_i and (if $\mathcal{C} = \mathcal{O}$) a fixed choice of ι to construct θ, ψ, h and $\mathcal{A} = \mathcal{A}(h, N)$. Suppose that θ', ψ', h' and $\mathcal{A}' = \mathcal{A}(h', N)$ are constructed from some w_i and (if $\mathcal{C} = \mathcal{O}$) the same ι. Now there is a graded automorphism φ of $\mathcal{C}$ with $\varphi(v_i) = w_i$ and (if $\mathcal{C} = \mathcal{O}$) $\varphi(v) = v$. Also $\theta' = \theta$, so $\varphi\theta = \theta'\varphi$. Hence, $\varphi R_{\bar{v}_k} L_{v_j} \theta = R_{\bar{w}_k} L_{w_j} \theta' \varphi$. Also, $n(\varphi(a)) = \varphi(n(a))$ for $a \in \mathcal{O}$. Thus, if we define φ on $x \in \mathcal{H}(\mathcal{C}_3)$ by letting φ act on the entries of x, we have $\varphi \circ \psi = \psi' \circ \varphi$. So $N(\varphi(x)) = \varphi(N(x))$, $T(\varphi(x), \varphi(y)) = \varphi(T(x, y))$ and $h'(\varphi(x), \varphi(y)) = \varphi(h(x, y))$. Hence, the map $(a, x) \mapsto (\varphi(a), \varphi(x))$ is a graded isomorphism of $\mathcal{A}$ onto $\mathcal{A}'$.

Now suppose $\mathcal{C} = \mathcal{O}$, suppose θ, ψ, h and $\mathcal{A}$ are constructed from v_i and ι as above, and suppose θ', ψ', h' and $\mathcal{A}'$ are constructed from v_i and $-\iota$. Let φ be the isograded automorphism of $\mathcal{O}$ with $\varphi(v_i) = v_i$ and $\varphi(v) = v^{-1}$ associated with the automorphism of Λ fixing λ_i and with $\lambda \to -\lambda$. Now $\theta'(v) = -\iota v$ implies $\theta'(v^{-1}) = \iota v^{-1}$, so $\varphi\theta = \theta'\varphi$. Hence, $\varphi R_{\bar{v}_k} L_{v_j} \theta = R_{\bar{v}_k} L_{v_j} \theta' \varphi$. The other calculations are the same as before and the map $(a, x) \mapsto (\varphi(a), \varphi(x))$ is an isograded isomorphism of $\mathcal{A}$ onto $\mathcal{A}'$. $\square$

Definition 12.6. Suppose that we have the assumptions of Theorem 12.5. Let

$$\mathcal{A}(\mathcal{H}(\mathcal{C}_3)) := \mathcal{A}(h, N)$$

be the structurable torus of class III(c) constructed in the theorem using the choice $w_i = v_i$ for $1 \le i \le 3$, and (if $\mathcal{C} = \mathcal{O}$) a fixed choice of ι. We call $\mathcal{A}(\mathcal{H}(\mathcal{C}_3))$ the *torus based on* $\mathcal{H}(\mathcal{C}_3)$.

We have thus constructed two structurable tori of class III(c): $\mathcal{A}(\mathcal{H}(\mathcal{Q}_3))$ and, if $-1 \in (F^\times)^2$, $\mathcal{A}(\mathcal{H}(\mathcal{O}_3))$. Note that under the inclusion map we have

$$\mathcal{A}(\mathcal{H}(\mathcal{Q}_3)) \simeq_{ig} \mathcal{A}(\mathcal{H}(\mathcal{O}_3))^{\langle \lambda_1, \lambda_2, 2\lambda \rangle}.$$

13. Classification of tori of class III(c)

In this final section, we classify structurable tori of class III(c).

Lemma 13.1. *Let Λ_1 be a subgroup of a free abelian group Λ of rank n and let Λ_2 be a subgroup of Λ_1 with $[\Lambda_1 : \Lambda_2] = 2$ and $4\Lambda \subset \Lambda_2$. If $v_1, \dots, v_r \in \Lambda/\Lambda_1$ and $2v_1, \dots, 2v_r$ is a basis for the $\mathbb{Z}_2$-vector space $2(\Lambda/\Lambda_1)$, then there is a basis $\lambda_1, \dots, \lambda_n$ for Λ such that*

(i) $v_i = \lambda_i + \Lambda_1$, $1 \le i \le r$,
(ii) $\Lambda_1 = \langle 4\lambda_1, \dots, 4\lambda_r, k_{r+1}\lambda_{r+1}, \dots, k_n\lambda_n \rangle$,
(iii) $\Lambda_2 = \langle 4\lambda_1, \dots, 4\lambda_r, 2k_{r+1}\lambda_{r+1}, k_{r+2}\lambda_{r+2} \dots, k_n\lambda_n \rangle$

with $k_i = 1$ or 2 for $r + 1 \le i \le n$.

Proof. Since $4\Lambda \subset \Lambda_1$, it follows from the fundamental theorem for finitely generated abelian groups that there is a basis $\lambda_1, \dots, \lambda_n$ for Λ with

$$\Lambda_1 = \langle k_1\lambda_1, \dots, k_n\lambda_n \rangle$$

where $k_i = 1$, 2, or 4. Assume that l is an integer such that $1 \le l \le r$ and $v_i = \lambda_i + \Lambda_1$ for $i < l$. Since $4\lambda_i \in \Lambda_1$, we can write $v_l = \sum_{i=1}^n m_i\lambda_i + \Lambda_1$ with $-1 \le m_i \le 2$. Now

$$2v_l = \sum_{i=1}^{l-1} m_i 2v_i + (\sum_{i=l}^{n} 2m_i\lambda_i + \Lambda_1)$$

so $\sum_{i=l}^n 2m_i\lambda_i \notin \Lambda_1$. We can rearrange $\lambda_l, \dots, \lambda_n$ to assume that $2m_l\lambda_l \notin \Lambda_1$; i.e., $m_l = \pm 1$. Thus, $2\lambda_l \notin \Lambda_1$ and $k_l = 4$. Replacing λ_l by $\lambda_l' = \sum_{i=1}^n m_i\lambda_i$ gives a basis B' for Λ with $v_l = \lambda_l' + \Lambda_1$. Since $4\lambda_l \equiv \pm 4\lambda_l'$ modulo the subgroup generated by $k_i\lambda_i$ with $i \ne l$, we still have $\Lambda_1 = \langle k_1\lambda_1, \dots, k_l\lambda_l', \dots, k_n\lambda_n \rangle$ using the basis B'. By induction on l, we see that there is a basis for Λ satisfying (i) and $\Lambda_1 = \langle k_1\lambda_1, \dots, k_n\lambda_n \rangle$. Since $2v_1, \dots, 2v_r$ is a basis for $2(\Lambda/\Lambda_1)$, we have also have (ii).

Clearly, $k_i\lambda_i = 4\lambda_i \in \Lambda_2$ if $i \le r$. Since some $k_i\lambda_i \notin \Lambda_2$, we can rearrange $\lambda_{r+1}, \dots, \lambda_n$ so that $k_{r+1}\lambda_{r+1} \notin \Lambda_2$ and $k_{r+1} \ge k_j$ if $k_j\lambda_j \notin \Lambda_2$. Suppose $k_j\lambda_j \notin \Lambda_2$ with $j > r + 1$. Set $\lambda_j' = k_{r+1}\lambda_{r+1} + \lambda_j$ if $k_j = 1$, and set $\lambda_j' = \lambda_{r+1} + \lambda_j$ if $k_j = 2$ (so $k_{r+1} = 2$). In either case, replacing λ_j by λ_j' gives a basis B' for Λ and $k_j\lambda_j' = k_{r+1}\lambda_{r+1} + k_j\lambda_j$. Moreover, $k_j\lambda_j' \in (\Lambda_1\backslash\Lambda_2) + (\Lambda_1\backslash\Lambda_2) \subset \Lambda_2$, since$[\Lambda_1 : \Lambda_2] = 2$. We see that we can choose a basis for Λ satisfying (i) and (ii) with $k_i\lambda_i \in \Lambda_2$ for all $i \ne r + 1$. Now $\Lambda_2' := \langle 4\lambda_1, \dots, 4\lambda_r, 2k_{r+1}\lambda_{r+1}, k_{r+2}\lambda_{r+2} \dots, k_n\lambda_n \rangle \subset \Lambda_2$ has $[\Lambda_1 : \Lambda_2'] = 2$, so $\Lambda_2 = \Lambda_2'$, showing (iii). $\square$

Assume for the rest of the section that $\mathcal{A}$ is a structurable Λ-torus of class III(c). We use the notation of §4 and §8. In particular, $\mathfrak{I}$ denotes the incidence geometry associated with $\mathcal{A}$.

Lemma 13.2. *If $\mathcal{A}$ is a structurable Λ-torus of class III(c), then there is a basis $\lambda_1, \lambda_2, \eta_0, \eta_1, \ldots, \eta_n$ of Λ with $n \geq 0$ such that*

(i) $\Lambda_- = \langle 4\lambda_1, 4\lambda_2, k_0\eta_0, k_1\eta_1, \ldots, k_n\eta_n\rangle$ *with $k_i = 1$ or 2,*
(ii) $\Gamma = \langle 4\lambda_1, 4\lambda_2, 2k_0\eta_0, k_1\eta_1, \ldots, k_n\eta_n\rangle$,
(iii) *if we set $\lambda_3 = -\lambda_1 - \lambda_2$ and $M = \langle 2\lambda_1, 2\lambda_2, \eta_0, \eta_1, \ldots, \eta_n\rangle$, then the points of $\mathfrak{I}$ are $2\bar\lambda_i$ and all $\bar\alpha_i$ with $\alpha_i \in \lambda_i + M$, and the lines are $[2\bar\lambda_1, 2\bar\lambda_2, 2\bar\lambda_3]$, $[\bar\alpha_i, \bar\alpha_i, 2\bar\lambda_i]$ and $[\bar\alpha_1, \bar\alpha_2, \bar\alpha_3]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$.*

Proof. Let $\bar\lambda_1$ be a point of order 4. By Lemma 8.3 (b) and (a), if $\bar\delta \neq 2\bar\lambda_1$ is any point of order 2, then $\bar\delta$ is collinear with some point on $[\bar\lambda_1, \bar\lambda_1, 2\bar\lambda_1]$ which must be $2\bar\lambda_1$. Since $\mathfrak{I}$ is not a star, Corollary 8.4(c) shows that there is some $\bar\lambda_2$ of order 4 which is not collinear with $2\bar\lambda_1$. Since $\bar\lambda_2$ is collinear with $2\bar\lambda_2$ by Lemma 8.3(a), we see that $2\bar\lambda_2 \neq 2\bar\lambda_1$. Since $\left|2\bar\lambda_2\right| = 2$, this shows that $2\bar\lambda_2$ is collinear with $2\bar\lambda_1$. Thus, $[2\bar\lambda_1, 2\bar\lambda_2, 2\bar\lambda_3]$ is a line where $\lambda_3 = -\lambda_1 - \lambda_2$.

If $\bar\alpha$ is a point of order 4, then $\bar\alpha$ is collinear with some $2\bar\lambda_i$ by Lemma 8.3(b), and thus $2\bar\alpha = 2\bar\lambda_i$; i.e., $\bar\alpha = \bar\lambda_i + \bar\mu$ with $2\bar\mu = 0$ Define

$$M_i = \{\mu \in \Lambda : \bar\lambda_i + \bar\mu \text{ is a point and } 2\bar\mu = 0\},$$

so the points of order 4 are all $\bar\alpha$ with $\alpha \in \lambda_i + M_i$ for some $1 \leq i \leq 3$. Clearly,

$$2M_i \subset \Lambda_- \subset M_i.$$

If $\alpha \in \lambda_i + M_i$ and $\beta \in \lambda_j + M_j$ with $i \neq j$, then $2\bar\alpha = 2\bar\lambda_i \neq 2\bar\lambda_j = 2\bar\beta$, so $\bar\beta$ must be collinear with $\bar\alpha$ on $[\bar\alpha, \bar\alpha, 2\bar\alpha]$. This shows that if $\bar\alpha_i \in \bar\lambda_i + M_i$, $i = 1, 2, 3$, and $\alpha_1 + \alpha_2 + \alpha_3 = 0$, then $[\bar\alpha_1, \bar\alpha_2, \bar\alpha_3]$ is a line. In particular, $[\bar\lambda_1, \bar\lambda_2, \bar\lambda_3]$ is a line. If $\bar\delta$ is a point of order 2, then $\bar\delta$ is collinear with some $\bar\lambda_i$, so $\bar\delta = 2\bar\lambda_i$. Thus, the points are $2\bar\lambda_i$ and all $\bar\alpha_i$ with $\alpha_i \in \lambda_i + M_i$.

Next, if $[\bar\alpha, \bar\beta, \bar\gamma]$ is a line with $\alpha, \beta \in \lambda_i + M_i$, then $2\bar\gamma = -2(\bar\alpha + \bar\beta) = -4\bar\lambda_i = 0$. Since $\bar\gamma$ has order 2, we see that $\bar\gamma = 2\bar\alpha = 2\bar\lambda_i$. Also, $\bar\alpha + \bar\beta = -\bar\gamma = 2\bar\alpha$ shows that $\bar\beta = \bar\alpha$. Thus, the only lines are $[2\bar\lambda_1, 2\bar\lambda_2, 2\bar\lambda_3]$, $[\bar\alpha_i, \bar\alpha_i, 2\bar\lambda_i]$ and $[\bar\alpha_1, \bar\alpha_2, \bar\alpha_3]$ with $\alpha_i \in \lambda_i + M_i$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

If $\alpha \in \lambda_i + M_i$ and $\beta \in \lambda_j + M_j$ with $i \neq j$, then $-(\alpha + \beta) \in \lambda_k + M_k$ where $\{i, j, k\} = \{1, 2, 3\}$. Thus, $M_i + M_j \subset -M_k$. Since $0 \in M_j$, we see $M_i \subset -M_k$ so $M_i = -M_k = M_j$ is a subgroup M.

Since $\bar\Lambda = \Lambda/\Lambda_-$ is generated by the points of $\mathfrak{I}$ (by (ST3)), we see $\bar\Lambda = \left\langle \bar\lambda_1, \bar\lambda_2, \bar M\right\rangle$. Now $2M \subset \Lambda_-$ and $2\bar\lambda_2 \neq 2\bar\lambda_1$ shows that $v_i = 2\bar\lambda_i$, $i = 1, 2$, is a basis for $2(\Lambda/\Lambda_-)$. Thus, we may apply Lemma 13.1 to $\Gamma \subset \Lambda_- \subset \Lambda$ to rechoose λ_1, λ_2 and get a basis $\lambda_1, \lambda_2, \eta_0, \ldots, \eta_n$ for Λ with $n \geq 0$ satisfying (i) and (ii).

It remains to show that

$$M = \langle 2\lambda_1, 2\lambda_2, \eta_0, \ldots, \eta_n\rangle. \tag{49}$$

Since $-\bar\lambda_i$ is a point, we have $2\lambda_i \in M$. Set $N = \langle \eta_0, \eta_1, \ldots, \eta_n\rangle$. If $\mu \in M$, write $\mu = l_1\lambda_1 + l_2\lambda_2 + \eta$ with $\eta \in N$. Since $2\mu \in \Lambda_-$, we see that l_i is even, so $l_i\lambda_i, \eta \in M$. Thus, $M = \langle 2\lambda_1, 2\lambda_2, M \cap N\rangle$. On the other hand, $\Lambda_- \subset M$ and $\bar\Lambda = \left\langle \bar\lambda_1, \bar\lambda_2, \bar M\right\rangle$ show $\Lambda = \langle \lambda_1, \lambda_2, M\rangle = \langle \lambda_1, \lambda_2, M \cap N\rangle$. Thus, $M \cap N = N$ giving (49). $\square$

Theorem 13.3. *Suppose that $\mathcal{A}$ is a structurable Λ-torus of class III(c). Then $\mathcal{A}$ is isograded isomorphic to either*

$$\mathcal{A}(\mathcal{H}(\mathcal{Q}_3)) \otimes \mathcal{P}(r) \quad \text{or, if } -1 \in (F^\times)^2, \quad \mathcal{A}(\mathcal{H}(\mathcal{O}_3)) \otimes \mathcal{P}(r)$$

for some $r \geq 0$. (See Theorem 12.5 and Definition 12.6.)

Proof. Recall that we are using the notation from §4 and §8. We make one exception to this and denote the hermitian form and cubic form defined in §8 by $\tilde{h}$ and $\tilde{N}$ respectively, rather that h and N. Thus, by Proposition 8.2, $\tilde{h} : \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ is a Λ-graded hermitian form, $\tilde{N} : \mathcal{W} \to \mathcal{E}$ is a Λ-graded cubic form, $(\tilde{h}, \tilde{N})$ satisfies the adjoint identity, and

$$\mathcal{A} = \mathcal{A}(\tilde{h}, \tilde{N}).$$

as Λ-graded algebras.

Assume now that we have chosen a basis $\lambda_1, \lambda_2, \eta_0, \dots, \eta_n$ for Λ satisfying the properties in Lemma 13.2, and, as in that lemma, we set $\lambda_3 = -\lambda_1 - \lambda_2$ and $M = \langle 2\lambda_1, 2\lambda_2, \eta_0, \eta_1, \dots, \eta_n \rangle$. We can rearrange this basis for Λ so that $k_i = 2$ for $1 \leq i \leq n'$ and $k_i = 1$ for $n' < i \leq n$. Let $\Lambda' = \langle \lambda_1, \lambda_2, \eta_0, \eta_1, \dots, \eta_{n'} \rangle$ and $\Lambda'' = \langle \eta_{n'+1}, \dots, \eta_n \rangle \subset \Gamma$. Thus, by Lemma 5.2, $\mathcal{A} \simeq_\Lambda \mathcal{A}^{\Lambda'} \otimes \mathcal{A}^{\Lambda''} \simeq_{ig} \mathcal{A}^{\Lambda'} \otimes \mathcal{P}(r)$, where $r = n - n'$. Replacing $\mathcal{A}$ by $\mathcal{A}^{\Lambda'}$, we can assume that $k_i = 2$ for $1 \leq i \leq n$. Thus, Λ has basis $\lambda_1, \lambda_2, \eta_0, \eta_1, \dots, \eta_n$,

$$M = \langle 2\lambda_1, 2\lambda_2, \eta_0, \eta_1, \dots, \eta_n \rangle, \quad \Lambda_- = \langle 4\lambda_1, 4\lambda_2, k_0\eta_0, 2\eta_1, \dots, 2\eta_n \rangle, \tag{50}$$

$$\Gamma = \langle 4\lambda_1, 4\lambda_2, 2k_0\eta_0, 2\eta_1, \dots, 2\eta_n \rangle, \tag{51}$$

where $n \geq 0$ and $k_0 = 1$ or 2.

Note that since $[2\bar{\lambda}_i, 2\bar{\lambda}_j, 2\bar{\lambda}_k]$ and $[-\bar{\lambda}_i, -\bar{\lambda}_i, 2\bar{\lambda}_i]$ are lines by Lemma 13.2(iii), we have

$$\mathcal{W}^{-2\lambda_i} \diamond \mathcal{W}^{-2\lambda_j} \neq 0 \quad \text{and} \quad \mathcal{W}^{-\lambda_i} \diamond \mathcal{W}^{-\lambda_i} \neq 0 \tag{52}$$

for $\{i, j, k\} = \{1, 2, 3\}$.

Choose $0 \neq u_i \in \mathcal{W}^{-\lambda_i}$ for $i = 1, 2$. Let $f_i = -u_i^\natural \in \mathcal{W}^{-2\lambda_i}$ for $i = 1, 2$. By (52), we have $f_1 \diamond f_2 \neq 0$, and so we may choose $f_3 \in \mathcal{W}^{-2\lambda_3}$ with $\tilde{h}(f_3, f_1 \diamond f_2) = 1$. Let $e_i = f_j \diamond f_k \in \mathcal{W}^{2\lambda_i}$ for $\{i, j, k\} = \{1, 2, 3\}$ in which case we have $h(f_i, e_i) = 1$. Since $4\lambda_i \in \Lambda_-$, we have $f_i^\natural = 0$. Thus, (ADJ2) gives

$$e_i \diamond e_j = (f_j \diamond f_k) \diamond (f_i \diamond f_k) = \tilde{h}(f_k, f_i \diamond f_j) f_k = f_k$$

so

$$\tilde{h}(e_1, e_2 \diamond e_3) = \tilde{h}(f_2 \diamond f_3, f_1) = 1^* = 1.$$

Since $\mathcal{W}^{2\lambda_i + \Lambda_-} = \mathcal{E} e_i$, the description of points in Lemma 13.2(iii) shows that

$$\mathcal{W} = \mathcal{E} e_1 \oplus \mathcal{E} e_2 \oplus \mathcal{E} e_3 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$$

where $\mathcal{W}_i := \mathcal{W}^{-\lambda_i + M}$. Moreover, since $\tilde{h}(\mathcal{W}^{\alpha_1}, \mathcal{W}^{\alpha_2}, \mathcal{W}^{\alpha_3}) = 0$ unless $[\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3]$ is a line, the description of lines in Lemma 13.2(iii) shows that, if $x = \sum_i a_i e_i + \sum_i x_i$ with $a_i \in \mathcal{E}$ and $x_i \in \mathcal{W}_i$, then

$$\tilde{N}(x) = \frac{1}{6} \tilde{h}(x, x \diamond x) = a_1 a_2 a_3 + \sum_i a_i \tilde{h}(e_i, x_i^\natural) + \tilde{h}(x_1, x_2 \diamond x_3).$$

Since $e_i \diamond x_j = 0$ for $i \neq j$, (ADJ3) yields

$$(e_j \diamond x_j) \diamond (e_k \diamond x_k) = -(e_j \diamond e_k) \diamond (x_j \diamond x_k), \tag{53}$$

$$(e_j \diamond e_k) \diamond (e_i \diamond x_i) = \tilde{h}(e_i, e_j \diamond e_k) x_i = x_i. \tag{54}$$

We define $\varphi : \mathcal{W} \to \mathcal{W}$ by $\varphi(ae_i) = a^* f_i$ for $a \in \mathcal{E}$ and $\varphi(x_i) = -e_i \diamond x_i$ for $x_i \in \mathcal{W}_i$. Since $\mathcal{W}^{2\lambda_i + \Lambda_-} = \mathcal{W}^{-2\lambda_i+\Lambda_-} = \mathcal{E} f_i$ and $\mathcal{W}^{-\lambda_i+M} = \mathcal{W}^{\lambda_i+M} = \mathcal{W}_i$, we see φ is a $*$-semilinear vector space isomorphism by (54). We next show that

$$\tilde{N}(\varphi(x)) = \tilde{N}(x)^*. \tag{55}$$

for $x \in \mathcal{W}$. Clearly

$$\tilde{h}(a_1^* f_1, a_2^* f_2 \diamond a_3^* f_3) = (a_1 a_2 a_3)^*.$$

Since $(e_i \diamond x_i)^\natural = \tilde{h}(e_i, x_i^\natural) e_i$ by (ADJ4), we also have

$$\tilde{h}(a_i^* f_i, (-e_i \diamond x_i)^\natural) = \tilde{h}(a_i^* f_i, \tilde{h}(e_i, x_i^\natural) e_i) = (a_i \tilde{h}(e_i, x_i^\natural))^*.$$

Finally using (53) and (54), we have

$$\begin{aligned}\tilde{h}(-e_1 \diamond x_1, &(-e_2 \diamond x_2) \diamond (-e_3 \diamond x_3)) = \tilde{h}(e_1 \diamond x_1, (e_2 \diamond e_3) \diamond (x_2 \diamond x_3)) \\ &= \tilde{h}(x_2 \diamond x_3, (e_2 \diamond e_3) \diamond (e_1 \diamond x_1)) = \tilde{h}(x_2 \diamond x_3, x_1) = (\tilde{h}(x_1, x_2 \diamond x_3))^*.\end{aligned}$$

This shows (55) as claimed.

Define $\tilde{T}(x, y) := \tilde{h}(x, \varphi(y))$. If $y = \sum_i b_i e_i + \sum_i y_i$ with $b_i \in \mathcal{E}$ and $y_i \in \mathcal{W}_i$, then

$$\tilde{T}(x, y) = \sum_i \tilde{h}(a_i e_i, b_i^* f_i) + \sum_i \tilde{h}(x_i, -e_i \diamond y_i) = \sum_i a_i b_i - \sum_i \tilde{h}(e_i, x_i \diamond y_i),$$

so $\tilde{T}$ is symmetric. Now Lemma 12.2 (with $\psi = \varphi^{-1}$) shows that $\tilde{T}$ is nondegenerate and $(\tilde{T}, \tilde{N})$ satisfies the adjoint identity with $x^\# = \varphi^{-1}(x^\natural) = \varphi(x)^\natural$. For $x_i \in \mathcal{W}_i$, set

$$n_i(x_i) = -\tilde{h}(e_i, x_i^\natural) \in \mathcal{E}.$$

We now have

$$\begin{aligned}\tilde{T}(x, y) &= \sum_i a_i b_i + \sum_i n_i(x_i, y_i), \\ \tilde{N}(x) &= a_1 a_2 a_3 - \sum_i a_i n_i(x_i) + \tilde{h}(x_1, x_2 \diamond x_3).\end{aligned}$$

Also $n_i(u_i) = -\tilde{h}(e_i, u_i^\natural) = \tilde{h}(e_i, f_i) = 1$ for $i = 1, 2$. So we may apply Lemma 11.1 (with the identification of $\mathcal{E}e_1 \oplus \mathcal{E}e_2 \oplus \mathcal{E}e_3$ with $\mathcal{E}^3$ and with $\tau(x_1, x_2, x_3) = \tilde{h}(x_1, x_2 \diamond x_3)$) to get a composition algebra $\mathcal{C}$ over $\mathcal{E}$ and $\mathcal{E}$-linear isomorphisms $\eta_i : \mathcal{C} \to \mathcal{W}_i$ such that $\eta_i(1) = u_i$ (with $u_3 = u_1 \times u_2$) and such that the $\mathcal{E}$-linear isomorphism $\eta : \mathcal{H}(\mathcal{C}_3) \to \mathcal{W}$ given by

$$\eta(\sum_i a_i[ii] + \sum_{(i,j,k)\circlearrowleft} x_i[jk]) = \sum_i a_i e_i + \eta_i(x_i),$$

satisfies

$$\tilde{T}(\eta(x), \eta(y)) = T(x, y) \quad \text{and} \quad \tilde{N}(\eta(x)) = T(x, y). \tag{56}$$

We next use η^{-1} to transfer the grading of $\mathcal{W}$ to a grading of $\mathcal{H}(\mathcal{C}_3)$, in which case η is a graded map. Then, by Lemma 11.2(a), $\mathcal{C}$ has a Λ-grading as an algebra with involution over $\mathcal{E}$ and as an $\mathcal{E}$-module so that $\mathcal{H}(\mathcal{C}_3)$ has the grading given by (41).

Next, since $\eta_3(\mathcal{C}) = \mathcal{W}_3 = \mathcal{W}^{-\lambda_3+M}$, we see that $\mathcal{C}$ is finely graded with support M. Also, $-n(\mathcal{C}^\mu) = T(1[33], (\mathcal{C}^\mu[12])^\#) = \tilde{h}(e_3, (\mathcal{W}_3^\mu)^\natural) \neq 0$, so $\mathcal{C}$ is an alternative M-torus with involution.

We now know that $\mathcal{C}$ can be built from $\mathcal{E}$ by iterating the Cayley-Dickson process. To be more precise, suppose M' is a subgroup of Λ such that $\Lambda_- \subset M' \subset M$, $\mu \in M \setminus M'$ and $0 \neq x \in \mathcal{C}^\mu$. Then, $n(\mathcal{C}^{M'}, x) \subset \mathcal{E}^{M'+\mu} = 0$. Using this fact along with the fact that nonzero homogeneous elements of $\mathcal{C}$ are invertible, the standard argument (see for example [Mc, p. 164]) shows that $\mathcal{C}^{\langle M',\,\mu\rangle} = \mathcal{C}^{M'} \oplus x\mathcal{C}^{M'}$ is a graded subalgebra of $\mathcal{C}$ that can be identified with $\mathrm{CD}(\mathcal{C}^{M'}, x^2)$. If we do this process repeatedly, starting with $M' = \Lambda_-$, we see that $\mathcal{C}$ can be obtained from $\mathcal{E}$ by a finite number of iterations of the Cayley-Dickson process. Moreover, since $\mathcal{C}$ is alternative, the number of iterations is necessarily at most 3 [Mc, p. 162].

Since M/Λ_- has exponent 2, it follows that $|M/\Lambda_-| \leq 2^3 = 8$. Thus, by (50), we have one of the following cases:

$$(1)\ n = 0 \text{ and } k_0 = 1; \quad (2)\ n = 0 \text{ and } k_0 = 2; \quad (3)\ n = 1 \text{ and } k_0 = 1.$$

In all three cases, we choose $0 \neq v_i \in \mathcal{C}^{2\lambda_i}$ and set $t_i = v_i^2 \in \mathcal{E}^{4\lambda_i}$ for $i = 1, 2$.

In Case 1, we have $M = \langle 2\lambda_1, 2\lambda_2, \eta_0\rangle$ and $\Lambda_- = \langle 4\lambda_1, 4\lambda_2, \eta_0\rangle$. We choose $0 \neq s \in \mathcal{E}^{\eta_0}$. Then $\mathcal{E} = F[t_1^{\pm1}, t_2^{\pm1}, s^{\pm1}]$ and, by the argument above, we may identify $\mathcal{C} = \mathrm{CD}(\mathcal{E}, t_1, t_2) = \mathcal{Q}$ with canonical generators v_1, v_2 (writing $\sigma = \eta_0$ to conform with the notation in §12).

In Case 2, we have $M = \langle 2\lambda_1, 2\lambda_2, \eta_0\rangle$ and $\Lambda_- = \langle 4\lambda_1, 4\lambda_2, 2\eta_0\rangle$. We choose $0 \neq v \in \mathcal{C}^{\eta_0}$ and set $s = v^2 \in \mathcal{E}^{2\eta_0}$. Then $\mathcal{E} = F[t_1^{\pm1}, t_2^{\pm1}, s^{\pm1}]$ and we may identify $\mathcal{C} = \mathrm{CD}(\mathcal{E}, t_1, t_2, s) = \mathcal{O}$ with canonical generators v_1, v_2, v (writing $\lambda = \eta_0$).

In Case 3, we have $M = \langle 2\lambda_1, 2\lambda_2, \eta_0, \eta_1\rangle$ and $\Lambda_- = \langle 4\lambda_1, 4\lambda_2, \eta_0, 2\eta_1\rangle$. We choose $0 \neq s \in \mathcal{E}^{\eta_0}$ and $0 \neq v \in \mathcal{C}^{\eta_1}$, and we set $r = v^2 \in \mathcal{E}^{2\eta_1}$. Then $\mathcal{E} = F[t_1^{\pm1}, t_2^{\pm1}, s^{\pm1}, r^{\pm1}]$ and we may identify $\mathcal{C} = \mathrm{CD}(\mathcal{E}, t_1, t_2, r) = \mathcal{O}'$ with canonical generators v_1, v_2, v (writing $\sigma = \eta_0$, $\lambda = \eta_1$).

One sees, using (50) and (51), that the involution $*$ on $\mathcal{E}$ coincides in each case with the involution defined by (43) in §12.

If we now define $\psi : \mathcal{H}(\mathcal{C}_3) \to \mathcal{H}(\mathcal{C}_3)$ by $\psi = \eta^{-1}\varphi^{-1}\eta$ and $h : \mathcal{H}(\mathcal{C}_3)\times\mathcal{H}(\mathcal{C}_3) \to \mathcal{E}$ by

$$h(x, y) = T(x, \psi(y)) = \tilde{T}(\eta(x), \varphi^{-1}\eta(y)) = \tilde{h}(\eta(x), \eta(y)), \tag{57}$$

then h is a graded hermitian form. Also, by (55) and (56), ψ is semi-norm preserving. Therefore, by Lemma 12.4, Case 3 does not occur, and in Case 2 we have $-1 \in (F^\times)^2$. Moreover, the map ψ is determined, as in Lemma 12.4(ii), by some choice of $w_i \in \mathcal{C}^{2\lambda_i}$ and some choice of ι with $\iota^2 = -1$. Let $\mathcal{A}(h, N)$ be the structurable torus constructed as in Theorem 12.5 using these choices. Then, it follows using (56) and (57) that the map $(a, x) \mapsto (a, \eta(x))$ from $\mathcal{A}(h, N) = \mathcal{E} \oplus \mathcal{H}(\mathcal{C}_3)$ onto $\mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ is an isomorphism of graded algebras with involution. Hence, by the final statement in Theorem 12.5, we have $\mathcal{A} \simeq_{ig} \mathcal{A}(\mathcal{H}(\mathcal{C}_3))$. $\square$

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