CLASSES OF STRUCTURABLE ALGEBRAS OF SKEW-RANK 1

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Abstract. Let R be a ring such that $2, 3 \in R^\times$. We construct classes of structurable algebras over R whose residue class algebras have skew-dimension 1, which are matrix algebras or forms of matrix algebras which do not necessarily arise out of separable Jordan algebras of degree 3. As an application, we give canonical examples of structurable algebras of large dimension.

Introduction

Let k be a field of characteristic not 2 or 3. Let A be a unital nonassociative algebra over k with an involution $^-$. The pair $(A, ^-)$ is called a *structurable algebra* if

$$\{x, y, \{z, w, q\}\} - \{z, w, \{x, y, q\}\} = \{\{x, y, z\}, w, q\} - \{z, \{y, x, w\}, q\}$$

for $x, y, z, w, q \in A$, where

$$\{x, y, z\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y.$$

Structurable algebras were introduced by Allison [A1]. Examples of structurable algebras include associative and alternative algebras with involution, Jordan algebras with the trivial involution id, and tensor products of two composition algebras with involution the tensor product of their canonical involutions [A1, Theorem 13]. An analogue of the Köcher-Kantor-Tits functor gives a correspondence between a structurable algebra and a Lie algebra. Using this functor all classical simple isotropic Lie algebras can be obtained [A2].

Define $S(A, ^-) = \{a \in A \,|\, \sigma(a) = -a\}$. If $\dim_k S(A, ^-) = 0$ then the structurable algebra $(A, ^-)$ over k is a Jordan algebra [A1, p. 135]. In the introduction of [A3], Allison mentions that simple structurable algebras of skew-dimension one are in a sense those structurable algebras closest to Jordan algebras. Our two constructions will highlight a close connection to cubic Jordan algebras.

Let R be a ring such that $2, 3 \in R^\times$. In the present paper we present two classes of structurable algebras $(A, ^-)$ over R, whose residue class algebras $A(P)$ are central simple structurable algebras of skew-dimension one, i.e. $\dim_{k(P)} S(A(P), ^-) = 1$. Over a field of characteristic not 2, 3 or 5, every structurable algebra of skew-dimension one is a form of a 2×2 matrix algebra with off-diagonal entries from a cubic Jordan algebra. The first class of algebras introduced in our paper are matrix algebras of rank 4, 8 or 20 which do not necessarily arise from the generic trace and norm of separable Jordan algebras of degree 3 together with a suitable non-zero scalar as described in [A-F, p. 195] (for an earlier

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reference consult [Sp1, 2]). However, this classical construction occurs as a special case. There is a clear connection between these matrix algebras and the first Tits construction for cubic Jordan algebras over rings. Both have the same 'ingredients'. The second class of structurable algebras we define becomes a matrix algebra under a suitable ring extension. This construction is connected to the second Tits construction for cubic Jordan algebras over rings. Again, both have the same 'ingredients'. Moreover, both constructions presented here are related to and overlap with the one given in [A-F-Y], Section 7.

The contents of the paper are as follows. We briefly summarize results and terminology we need to know in Sections 1 and 2. Given a unital associative algebra B over a ring R' with involution $*$, we explain the general setup of norms, traces and adjoints defined on locally free B-modules of rank 1 in Section 3. These results we use repeatedly in the rest of the paper. Section 4 contains the first main result of the paper. It deals with matrix algebras over R and their automorphism groups. Section 5 deals with the second main result: a construction of structurable algebras of rank ≤ 20 which are forms of the matrix algebras of Section 4.

One application of our previous results will be discussed in Section 6. Looking at structurable algebras over the projective space $\mathbb{P}^n_R$, by passing to their global sections we canonically construct classes of structurable algebras $M(T, N, N^\vee)$ over R of large dimension, with highly degenerate cubic forms N, $N^\vee$.

We will repeatedly use the results and notation from [Pu1, 2] and [Pa-Sr-T]. The approach taken in [Pu1, 2] which goes back to [Ach1, 2] is different from the one in [Pa-Sr-T]. Both techniques have advantages and disadvantages. The method introduced in [Pa-Sr-T] is more functorial, while the one in [Pu1, 2] works in a much more general setting. Both were developed to generalize the first and second Tits construction of Jordan algebras of degree 3 to base rings.

For the standard terminology on Jordan algebras, the reader is referred to the books by McCrimmon [M], Jacobson [J1] and Schafer [Sch]. For recent related literature on structurable algebras, see for instance [A-F-Y], [Kr], [G].

We assume throughout the paper that R is a ring such that $2, 3 \in R^\times$.

1. Preliminaries

1.1. **Forms of higher degree over R.** Let d be a positive integer. Let M, G be two R-modules which are finitely generated projective of finite rank. When talking about maps of degree d, we will always assume that $d! \in R^\times$. A *map of degree* d over R is a map $N : M \to G$ such that $N(ax) = a^d N(x)$ for all elements $a \in R$, $x \in M$, where the map

$$\theta : M \times \cdots \times M \to G$$

defined by

$$\theta(x_1, \ldots, x_d) = \frac{1}{d!} \sum_{1 \leq i_1 < \cdots < i_l \leq d} (-1)^{d-l} N(x_{i_1} + \cdots + x_{i_l})$$

is a d-linear map over R (the range of summation of l being $1 \leq l \leq d$). Obviously, $N(x) = \theta(x, \ldots, x)$ for all $x \in M$. We canonically identify a map of degree d and its associated symmetric d-linear map θ.

If $G = R$, then a map of degree d is called a *form of degree d*, θ is called the *symmetric d-linear form* associated with N.

A form $N : M \to R$ of degree d on a locally free R-module of finite rank with full support is called *nondegenerate* if $N(P) : M(P) \to k(P)$ is nondegenerate in the classical sense for all $P \in \operatorname{Spec} R$. I.e., the residue maps $\theta' \otimes k(P)$ of the map

$$\theta' : M \to \operatorname{Hom}_R(M \otimes \cdots \otimes M, R)$$

($(d-1)$-copies of M) defined by

$$x_1 \to \theta_{x_1}(x_2 \otimes \cdots \otimes x_d) = \theta(x_1, x_2, \dots, x_d).$$

are injective for all $P \in X$. This concept of nondegeneracy is invariant under base change.

1.2. **Algebras over R.** For $P \in \operatorname{Spec} R$, let R_P be the local ring of R at P and m_P the maximal ideal of R_P. The corresponding residue class field is denoted by $k(P) = R_P/m_P$. For an R-module F the localization of F at P is denoted by F_P. The *rank* of F is defined to be $\sup\{\operatorname{rank}_{R_P} F_P \mid P \in \operatorname{Spec} R\}$. The term "$R$-algebra" always refers to nonassociative R-algebras which are unital and projective of finite constant rank as R-modules. An algebra A over R is called *separable* if $A(P)$ is a separable $k(P)$-algebra for all $P \in X$.

For an R-algebra A, an anti-automorphism $\sigma : A \to A$ of order 2 is called an *involution* on A. Define $H(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$ and $\mathrm{S}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\}$. Then $A = H(A, \sigma) \oplus \mathrm{S}(A, \sigma)$.

1.3. **Jordan algebras over R.** (see [Pu1])

Let J be an R-module. $(J, U, 1)$ with $1 \in J$ is a *(unital quadratic) Jordan algebra* over R if:

(1) The U-operator $U : J \to End_R(J), x \to U_x$ is a quadratic map;
(2) $U_1 = id_J$;
(3) $U_{U_x(y)} = U_x \circ U_y \circ U_x$ for all elements $x, y \in J$;
(4) $U_x \circ U_{y,z}(x) = U_{x,U_x(z)}(y)$ for all elements $x, y, z \in J$;
(5) for every commutative associative R-algebra R', $J \otimes_R R'$ satisfies (3) and (4).

We write J instead of $(J, U, 1)$.

An R-algebra J is called an *Albert algebra* if $J(P) = J_P \otimes k(P)$ is an Albert algebra over $k(P)$ for all $P \in X$. This terminology is compatible with the one used in [Pa-S-T], cf. [P1, Section 2].

1.4. **Cubic forms with adjoint and base point.** Let W be an R-module. Following [P-R1] or [Ach], a tripel $(N, \sharp, 1)$ is a *cubic form with adjoint and base point* on W if $N : W \to R$ is a cubic form, $\sharp : W \to W$ a quadratic map and $1 \in W$, such that

$$x^{\sharp\,\sharp} = N(x)x,$$
$$T(x^\sharp, y) = D_y N(x) \text{ for } T(x, y) = -D_x D_y \log N(1),$$
$$N(1) = 1,\ 1^\sharp = 1,$$
$$1 \times y = T(y)1 - y \text{ with } T(y) = T(y, 1),\ x \times y = (x + y)^\sharp - x^\sharp - y^\sharp$$

for all elements $x, y \in W$. Here, $D_y N(x)$ denotes the directional derivative of N in the direction y, evaluated at x. Since we assume that $2, 3 \in R^\times$, this means that the quadratic map $D_y N(x)$ is the coefficient $N(x; y)$ of the indeterminant Z in the expansion

$$N(x + Zy) = N(x) + ZN(x; y) + Z^2 N(y; x) + Z^3 N(y)$$

and that $T(x^\sharp, y) = 3N(x, x, y)$ [M, p. 200].

Let $D_x D_y N$ denote the bilinearization of the quadratic form $D_y N$ and $D_x D_y N(z) = D_y N(x, z)$. The term $D_x D_y \log N(z)$ is defined by

$$D_x D_y \log N(z) = N(z)^{-2}[N(z) D_x D_y N(z) - D_x N(z) D_y N(z)]$$

for all elements $x, y, z \in W$ with $N(z) \in R^\times$. Hence

$$T(x, y) = D_x N(1) D_y N(1) - D_x D_y N(1) = T(x)T(y) - S(x, y)$$

with $S(x, y) = 6N(x, y, 1)$. The symmetric bilinear form $T : W \times W \to R$ is called the *trace form* of W.

Every cubic form with adjoint and base point $(N, \sharp, 1)$ on a locally free R-module W of finite rank defines a unital Jordan algebra structure $\mathcal{J}(N, \sharp, 1) = (W, U, 1)$ on W via

$$U_x(y) = T(x, y)x - x^\sharp \times y$$

for all $x, y \in W$, where the identities given in [P-R1, p. 213] hold for all elements in W.

1.5. **Structurable algebras.** An *algebra with involution* is a pair $(A, ^-)$ consisting of an R-algebra A and an involution $^- : A \to A$. A *structurable algebra* is an algebra with involution $(A, ^-)$ satisfying

$$\{x, y, \{z, w, q\}\} - \{z, w, \{x, y, q\}\} = \{\{x, y, z\}, w, q\} - \{z, \{y, x, w\}, q\}$$

for all elements $x, y, z, w, q \in A$, where

$$\{x, y, z\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$$

[A1, (3) and Cor. 5]. If B is an R-submodule if A which is closed under multiplication, we call B a subalgebra of A. If, additionally, $\overline{B} = B$ then we call $(B, ^-)$ a subalgebra of $(A, ^-)$.

An *ideal* of $(A, ^-)$ is an ideal of A which is stabilized by $^-$. We call an algebra with involution $(A, ^-)$ over k *simple* if the only ideals of $(A, ^-)$ are 0 and A and *central* if

$$\mathrm{Z}(A, ^-) = \{x \in A \,|\, xy = yx \text{ for all } y \in A \text{ and } \bar{x} = x\}$$

equals k. In the following, we will investigate structurable algebras $(A, ^-)$ over R, where the residue class algebras $A(P)$ are central simple structurable algebras of skew-dimension 1, i.e. of $\dim_{k(P)} S(A(P), ^-) = 1$.

2. Matrix algebras

The following is well-known over fields and easily extends to base rings. Let W and W' be two finitely generated projective R-modules of constant rank with cubic forms $N : W \to R$ and $N' : W' \to R$, paired by a nondegenerate bilinear form $T : W \times W' \to R$. That is, T induces R-module isomorphisms

$$T : W \to \mathrm{Hom}_R(W', R), \quad x \mapsto T(x, \cdot)$$

and

$$T : W' \to \mathrm{Hom}_R(W, R), \quad y' \mapsto T(\cdot, y').$$

We say that the triple (T, N, N') *is defined on* (W, W'). Let $x \in W$, $x' \in W'$ and define quadratic maps $\sharp : W \to W'$ and $\sharp' : W' \to W$ via

$$D_y N(x) = T(y, x^\sharp) \text{ and } D_{y'} N'(x') = T(x'^{\sharp'}, y')$$

for all elements $x, y \in W$, $x', y' \in W'$. I.e.,

$$3N(x, x, y) = T(y, x^\sharp) \text{ and } 3N'(x', x', y') = T(x'^{\sharp'}, y').$$

The triple (T, N, N') *satisfies the adjoint identities* if

$$(x^\sharp)^{\sharp'} = N(x)x \text{ and } (x'^{\sharp'})^\sharp = N'(x')x'.$$

If $N = 0$ and $N' = 0$ these identities are trivially satisfied. If $N \neq 0$ or $N' \neq 0$ then both N and N' are nonzero and (T, N, N') is called *non-trivial*.
Let (T, N, N') be a triple defined on (W, W'). Let $N(x, y, z)$ denote the trilinear form associated with N and $N'(x', y', z')$ the trilinear form associated with N'. Define symmetric bilinear maps $\times : W \times W \to W'$ and $\times' : W' \times W' \to W$ via

$$x \times y = (x + y)^\sharp - x^\sharp - y^\sharp, \quad x' \times' y' = (x' + y')^{\sharp'} - x'^{\sharp'} - y'^{\sharp'}.$$

Then

$$x^\sharp = \frac{1}{2} x \times x, \quad x'^{\sharp'} = \frac{1}{2} x' \times' x',$$
$$N(x, y, z) = T(x, y \times z), \quad N'(x', y', z') = T(x' \times' y', z').$$

Theorem 1. *Suppose the triple (T, N, N') satisfies the adjoint identities. Then the R-module*

$$M(T, N, N') = \begin{bmatrix} R & W \\ W' & R \end{bmatrix}.$$

becomes a structurable algebra over R with multiplication given by

$$\begin{bmatrix} a & x \\ x' & b \end{bmatrix} \begin{bmatrix} c & y \\ y' & d \end{bmatrix} = \begin{bmatrix} ac + T(x, y') & ay + dx + x' \times' y' \\ cx' + by' + x \times y & bd + T(y, x') \end{bmatrix}$$

and involution

$$\overline{\begin{bmatrix} a & x \\ x' & b \end{bmatrix}} = \begin{bmatrix} b & x \\ x' & a \end{bmatrix}.$$

For the symmetric elements, we have $S(M(T, N, N')) \cong R\, s_0$ for the global section

$$s_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the residue class algebras $A(P) = A_P \otimes k(P)$ are central simple structurable algebras of skew-dimension 1 over $k(P)$.

Proof. The defining identity of a structurable algebra is satisfied as in the classical case over fields. For $P \in X$, we have

$$M(T,N,N')_P \cong M(T_P,N_P,N'_P)$$

and

$$M(T,N,N')_P \otimes_{R_P} k(P) \cong M(T(P),N(P),N'(P)),$$

is a central simple structurable algebra. Moreover, $S(M(T,N,N')) = R\,s_0$ for the global section s_0 as in [A-F, p. 195]. □

Let

$$u = \begin{bmatrix} a & x \\ x' & b \end{bmatrix}$$

and

$$v = \begin{bmatrix} c & y \\ y' & d \end{bmatrix}$$

with $a,b,c,d \in R$ and $x,y \in W$, $x',y' \in W'$.

Remark 2. Suppose the triple (T,N,N') satisfies the adjoint identities. Then $q : M(T,N,N') \to R$,

$$q(x) = 4aN(x) + 4bN'(x') - 4T({x'}^{\sharp'}, x^\sharp) + (ab - T(x,x'))^2,$$

is a quartic form such that $q(1) = 1$ (see also [Kr]). For each $P \in \operatorname{Spec} R$, the quartic residue class form $q \otimes_R k(P)$ over $k(P)$ is 2-round [Pu3].

Lemma 3. *Let $(N,\sharp,1)$ be a cubic form with adjoint and base point on W such that $T(x,y) = -D_xD_y\log N(1)$ is nondegenerate. Let $\mu \in R^\times$.*
(i) $(\mu T, \mu N, \mu^2 N)$ is a non-trivial triple defined on (W,W) which satisfies the adjoint identities.
(ii) Let $(\widetilde{N},\widetilde{\sharp},\widetilde{1})$ be a cubic form with adjoint and base point on $\widetilde{W}$, such that $\widetilde{T}(x,y) = -D_xD_y\log\widetilde{N}(1)$ is nondegenerate and $f : W \to \widetilde{W}$ an R-module isomorphism such that $\widetilde{N}(f(x)) = N(x)$, $f(x)^{\widetilde{\sharp}} = f(x^\sharp)$ and $f(1) = \widetilde{1}$. Then

$$M(\mu T, \mu N, \mu^2 N) \cong M(\mu\widetilde{T}, \mu\widetilde{N}, \mu^2\widetilde{N}).$$

Proof. (i) This follows from 1.4.
(ii) f induces a Jordan algebra isomorphism $f : J(N,\sharp,1) \to J(\widetilde{N},\widetilde{\sharp},\widetilde{1})$, which yields the assertion. □

Over a field k, any non-trivial triple (T,N,N') satisfying the adjoint identities which is defined on a k-vector space W of dimension larger than 2, is isomorphic to $(\mu T, \mu N, \mu^2 N)$ for a suitable $\mu \in k^\times$ and a separable Jordan algebra over k of degree 3 with reduced trace T and norm N and $M(T,N,N') \cong M(\mu T, \mu N, \mu^2 N)$ [Sp1, 2]. In Section 4 we will see that this is no longer true over rings.

3. The setup

The results in this section are due to Achammer [Ach] and can be found with proofs in [Pu1] as part of the Tits process for Jordan algebras over locally ringed spaces.

Remark 4. The results of this section, apart from Remark 5 (ii), hold for arbitrary rings, the restriction that $2, 3 \in R'^\times$ can be omitted. In that case, we work with the more general notion of a quadratic and cubic map viewed as certain polynomial laws between R-modules, see [R]. If $2, 3 \notin R^\times$ this definition is equivalent to the one given in 1.1.

3.1. Let R' be a ring, $X = \operatorname{Spec} R'$ and B a unital associative R'-algebra. Write $B^\times$ for the sheaf of units of B. Let $\mathrm{Pic}_l B$ denote the set of isomorphism classes of locally free left B-modules of rank 1. $\mathrm{Pic}_l B$ is a pointed set. By non-commutative Čech-cohomology [Mi, III, 4.6], we canonically identify $\mathrm{Pic}_l B = \check{H}^1(X, B^\times)$. Let $* : R' \to R'$ be an involution on R' and $*_B$ an involution on B which extends $*$, that means $*_B|_{R'} = *$.

Let $(N_B, \sharp_B, 1)$ be a cubic form with adjoint and base point on B such that

(1) $B^+ = J(N_B, \sharp_B, 1)$ with 1 the unit element in B, and

$$xyx = T_B(x, y)x - x^{\sharp_B} \times_B y$$

for x, y in B;

(2) $N_B(xy) = N_B(x)N_B(y)$ for all x, y in B.

(3) $N(x^{*_B}) = N(x)^{*_B}$ for $x \in B$

These identities imply that

(4) $(xy)^{\sharp_B} = y^{\sharp_B} x^{\sharp_B}$;

(5) $(x^{*_B})^{\sharp_B} = (x^{\sharp_B})^{*_B}$;

(6) $T_B(x, y) = T_B(xy)$

for all $x, y \in B$. Because of (2) and (4), the maps

$$N_B : B^\times \to R'^\times, \quad \sharp_B : B^\times \to B^{\mathrm{op}\times}$$

are morphisms of groups. Using the natural identifications

$$\mathrm{Pic}_l B = \check{H}^1(X, B^\times), \quad \mathrm{Pic}_l B^{\mathrm{op}} = \check{H}^1(X, B^{\mathrm{op}\times})$$

as pointed sets, the group morphisms

$$N_B : B^\times \to R'^\times, \quad \sharp_B, *_{\mathcal{B}} : B^\times \to B^{\mathrm{op}\times}$$

induce morphisms

$$N_B : \mathrm{Pic}_l B \to \operatorname{Pic} R', \quad P \to N_B(P),$$
$$\sharp_B : \mathrm{Pic}_l B \to \mathrm{Pic}_l B^{\mathrm{op}}, \quad P \to P^{\sharp_B},$$
$$*_{\mathcal{B}} : \mathrm{Pic}_l B \to \mathrm{Pic}_l B^{\mathrm{op}}, \quad P \to P^{*_{\mathcal{B}}}$$

of pointed sets.

Let $P \in \mathrm{Pic}_l B$ and F be a right B-module. A quadratic map $g : P \to F$ in the category of R'-modules is called *multiplicative* if

$$g(bv) = g(v)b^{\sharp_B}$$

for all elements $b \in B$, $v \in P$.
Let E be an R'-module. A cubic map $f : P \to E$ in the category of R'-modules is called *multiplicative* if

$$f(bw) = N_B(b)f(w)$$

for all $b \in B$, $w \in P$.

A multiplicative quadratic map $\sharp : P \to P^{\sharp_B}$ is called an *adjoint* on P, if $\sharp$ is universal in the category of multiplicative quadratic maps on P.

A multiplicative cubic map $N : P \to N_B(P)$ is called a *norm* on P, if N is universal in the category of multiplicative cubic maps on P. Norms on P always exist and are unique up to an invertible factor in R'.

Remark 5. (i) In order to be able to construct non-trivial (i.e., non-free) locally free B-modules P, we observe that $\operatorname{Pic} R'$ acts on $\check{H}^1(X, B^\times)$ via $(L, P) \to L \otimes P$ and $N_B(L \otimes P) \cong L^3 \otimes N_B(P)$. In particular, if $L \in \operatorname{Pic} R'$ has order 3, then $L \otimes P$ is a left B-module of rank one which admits a norm $N : P \to R'$.
(ii) If B is an Azumaya algebra over R' and R' satisfies that $2, 3 \in R'^\times$, we can also take the point of view of [K-O-S] described and also used for instance in [Pa-Sr-T, p. 16]: Then we can define a reduced norm functor $\mathcal{N} :_A \mathrm{Mod} \to_{R'} \mathrm{Mod}$ which associates to every locally free B-module P a locally free R'-module $\mathcal{N}(P)$ such that

- $\mathcal{N}(B) = R'$,
- the map $\mathcal{N}_P : P = \operatorname{Hom}_B(B, P) \to \operatorname{Hom}_R(\mathcal{N}(B), \mathcal{N}(P)) = \mathcal{N}(P)$ induced by functoriality, satisfies

  $$\mathcal{N}_P(bw) = N_B(b)\mathcal{N}_P(w)$$

  for all $b \in B$, $w \in P$,
- if P has rank 1 then $\mathcal{N}(P)$ is invertible and $\mathcal{N}(B)$ is the norm of B.

Now if $N : P \to R'$ is a map such that $N(bw) = N_B(b)N(w)$ for all $b \in B$, $w \in P$ and such that the values of N generate the unit ideal in R', then there exists a unique isomorphism $\mu : \mathcal{N}_P \to R'$ such that $N = \mu\mathcal{N}_P$ [Pa-Sr-T, 1.1].

Hence we can view a norm $N : P \to R$ on a locally free left B-module P of rank 1 in our terminology also as given by an isomorphism $\mu : \mathcal{N}(P) \to R'$ of R'-modules and vice versa.

3.2. We canonically identify left B^{op}-modules and right B-modules. For $P \in \operatorname{Pic}_l B$, let $P^\vee$ denote the locally free right B-module $\operatorname{Hom}_B(P, B)$. Let $\langle\, , \,\rangle : P \times P^\vee \to B$ denote the canonical map $\langle u, u^\vee \rangle = u^\vee(u)$.

Suppose that $N_B(P) \cong R'$ and let $N : P \to R'$ be a norm on P. Then

$$P^{\sharp_B} \cong P^\vee, \quad P^{\vee \sharp_B} \cong P$$

and $N_B(P^\vee) \cong R'$. There exists a uniquely determined norm $N^\vee : P^\vee \to R'$ and uniquely determined adjoints $\sharp : P \to P^\vee$ and $\check{\sharp} : P^\vee \to P$ such that

(7) $\langle w, w^\sharp \rangle = N(w)1$;
(8) $\langle \check{w}^{\check{\sharp}}, \check{w} \rangle = N^\vee(\check{w})1$;
(9) $w^{\sharp\check{\sharp}} = N(w)w$

for all $w \in P$, $\check{w} \in P^\vee$. Moreover,

(10) $\check{w}^{\check{\sharp}\sharp} = N^\vee(\check{w})\check{w}$;
(11) $\langle w, \check{w}\rangle^{\sharp_B} = \langle \check{w}^{\check{\sharp}}, w^\sharp\rangle$;
(12) $N_B(\langle w, w^\sharp\rangle) = N(w)N^\vee(\check{w})$;
(13) $D_{w'}N(w) = T_B(\langle w', w^\sharp\rangle)$;
(14) $D_{\check{w}'}N^\vee(\check{w}) = T_B(\langle \check{w}^{\check{\sharp}}, \check{w}'\rangle)$;
(15) $\langle w, \check{w}\rangle w = T_B(\langle w, \check{w}\rangle)w - w^\sharp \check{\times} \check{w}$

for all $w, w' \in P$, $\check{w}, \check{w}' \in P^\vee$ where $\check{\times} : P^\vee \times P^\vee \to P$ denotes the bilinear map associated to the quadratic map $\check{\sharp}$.

For a right B-module F, let $\overline{F}$ denote the left B-module obtained by defining a new scalar multiplication on F via

$$b \cdot w = wb^{*_B}$$

for $b \in B$, $w \in F$ [K, I, (2.1)]. Any morphism of right B-modules $f : F \to E$ induces a morphism of left B-modules $\overline{f} : \overline{F} \to \overline{E}$. (An analogous argument holds for left B-modules.)

Let P be a locally free left B-module of rank 1. An isomorphism of left B-modules $j : P \to \overline{P^{*_B}}$ is called an *involution* on P.

3.3. A pair (A, R) consisting of a subring R of R' and an R-submodule A of B is called B-*ample* if

(16) $R \subset H(R', *_B)$,
(17) $rr^{*_B} \in R$ for $r \in R'$,
(18) $A \subset H(B, *_B)$,
(19) $1 \in A$,
(20) $bab^* \in A$ for $a \in A$, $b \in B$,
(21) $N_B(A) \subset R$; i.e. $N_B|_A : A \to R$ is a cubic form over R,
(22) $A^{\sharp_B} \subset A$; i.e., $\sharp_B|_A : A \to A$ is a quadratic map over R.

Equation (21) implies that $T_B(A, A) \subset R$. If $2 \in R^\times$,

$$(H(B, *_B), H(R', *_B))$$

is the only B-ample pair.

Let (A, R) be B-ample and P be a locally free left B-module of rank 1 with $N_B(P) \cong R'$. If $P^{*_B} \cong P^\vee$ and $N_B(P) \cong R'$, then a pair $(N, *)$ with $N : P \to R'$ a norm on P and an involution $* : P \to \overline{P^\vee}$ on P is called A-*admissible* if

$$\langle w, w^*\rangle \in A \text{ and } N_B(\langle w, w^*\rangle) = N(w)N(w)^{*_B}$$

for $w \in P$.
P is called A-*admissible* if there is a norm $N : P \to R'$ and a nondegenerate $*_B$-sesquilinear form $h : P \times P \to B$ (i.e., $h(aw, bv) = ah(w, v)b^{*_B}$ and h induces an isomorphism $j_h : P \to \overline{P^\vee}$ of left B-modules) such that

$$h(w, w) \in A \text{ and } N_B(h(w, w)) = N(w)N(w)^{*_B}$$

for $w \in P$. Note that $P^\vee \cong P^{*_B}$ and that therefore j_h (denoted $*$ from now on) is an involution on P which is A-admissible.

Let $(N_B, \sharp_B, 1)$ be B-admissible, (A, R) B-ample, P an A-admissible B-module and $(N, *)$ A-admissible. Then $\check{*} = \overline{*^{-1}} : P^\vee \to \overline{P}$ is an involution on P and

(23) $w^{*\check{*}} = w$;
(24) $\check{w}^{\check{*}*} = \check{w}$;
(25) $\check{N}(w^*) = N(w)^{*_B}$;
(26) $w^{*\check{\sharp}} = w^{\sharp\check{*}}$;
(27) $\check{w}^{\check{\sharp}*} = \check{w}^{\check{*}\sharp}$;
(28) $\langle w, \check{w}\rangle^{*_B} = \langle \check{w}^{\check{*}}, w^*\rangle$;
(29) $\langle w, w^*\rangle w^{*\check{\sharp}} = N(w)^{*_B} w$

for all $w \in P$, $\check{w} \in P^\vee$.

4. Matrix algebras

4.1. Let B be a *separable* unital associative R-algebra. Let $(N_B, \sharp_B, 1)$ be a cubic form with adjoint and base point on B satisfying equations (1) and (2). Let $P \in \mathrm{Pic}_l B$ such that $N_B(P) \cong R$ and let $N : P \to R$ be a norm on P. Let $N^\vee : P^\vee \to R$ be the uniquely determined norm and $\sharp : P \to P^\vee$, $\check{\sharp} : P^\vee \to P$ be the uniquely determined adjoints satisfying equations (7), (8), (9). Let $\times : P^\vee \times P^\vee \to P$ denote the bilinear map associated to the quadratic map $\sharp$ and $\check{\times} : P^\vee \times P^\vee \to P$ the bilinear map associated to the quadratic map $\check{\sharp}$.

Define $T : P \times P^\vee \to R$ via

$$T(w, \check{w}) = T_B(\langle w, \check{w}\rangle).$$

Theorem 6. *For any $\mu \in R^\times$, the triple $(\mu T, \mu N, \mu^2 N^\vee)$ satisfies the adjoint identities.*

Proof. Since B is separable, T_B is a nondegenerate symmetric bilinear form, hence so is T. $(T, N, N^\vee)$ is a non-trivial triple defined on $(P, P^\vee)$ and the norms $N : P \to R$, $N^\vee : P^\vee \to R$ and adjoints $\sharp : P \to P^\vee$, $\check{\sharp} : P^\vee \to P$ satisfy the identities

$$D_{w'} N(w) = T_B(\langle w', w^\sharp\rangle) \text{ and } D_{\check{w}'} N^\vee(\check{w}) = T_B(\langle \check{w}^{\check{\sharp}}, \check{w}'\rangle)$$

by equations (13) and (14). By equations (9) and (10), $(T, N, N^\vee)$ satisfies the adjoint identitites.

Moreover, an easy calculation shows that for any $\mu \in R^\times$, the adjoint belonging to μN is $\mu\,\sharp$ and $(\mu N)^\vee = \mu^2 N^\vee$, $(\mu\,\sharp)^\vee = \mu^2\,\sharp$. The assertion again follows involving equations (9), (10), (13) and (14). □

Corollary 7. *For any $\mu \in R^\times$,*

$$M(\mu T, \mu N, \mu^2 N^\vee) = \begin{bmatrix} R & P \\ P^\vee & R \end{bmatrix}$$

is a structurable algebra over R.

If we want to make it clear that the map N is defined on the locally free left B-module P, we also sometimes use the notation N_P. Analogously, we then also write $T_P : P \times P^\vee \to R$ instead of $T : P \times P^\vee \to R$.

Remark 8. This construction generalizes the classical construction by Springer [Sp1, 2]: B itself canonically is a globally free left B-module of rank 1 denoted ${}_BB$ and μN_B, $\mu \in R^\times$, is a norm on ${}_BB$. We have $({}_BB)^\vee = B_B$. So choose $P = {}_B B$. The adjoint belonging to μN_B is $\mu \sharp_B$ and $(\mu N_B)^\vee = \mu^2 N_B$, $(\mu \sharp_B)^\vee = \mu^2 \sharp_B$. Moreover, $\langle w, \check{w}\rangle = w\check{w}$, hence $T(w,\check{w}) = \mu T_B(w,\check{w})$ and we obtain the classical matrix algebra

$$M(\mu T_B, \mu N_B, \mu^2 N_B) = \begin{bmatrix} R & B \\ B & R \end{bmatrix}.$$

4.2. Let B and D be Azumaya algebras over R. We take the point of view of [Pa-Sr-T], see also Remark 5 (b). Let $g : B \to D$ be an algebra isomorphism and $\widetilde{g} : P \to Q$ a g-semilinear isomorphism of R-modules. Then there is an R-linear isomorphism $\mathcal{N}(\widetilde{g}) : P \to Q$ such that $\mathcal{N}(\widetilde{g})\mathcal{N}_P = \mathcal{N}_Q\widetilde{g}$. The map $\mathcal{N}(\widetilde{g})$ is constructed by descent, see [Pa-Sr-T, p. 16].

Proposition 9. *Let R be a domain. Let B, D be two Azumaya algebras over R with properties as described in 4.1. Let P, Q be locally free left modules of rank 1 over B and D, respectively, with norms $N_P : P \to R$ and $N_Q : Q \to R$ given by isomorphisms $\mu : \mathcal{N}(P) \to R$ and $\nu : \mathcal{N}(Q) \to R$ of R-modules, respectively. Let $g : B \to D$ be an algebra isomorphism and $\widetilde{g} : P \to Q$ a g-semilinear isomorphism of R-modules such that*

$$\mu = \nu \circ \mathcal{N}(\widetilde{g}).$$

Then the map

$$S(g) : \begin{bmatrix} R & P \\ P^\vee & R \end{bmatrix} \to \begin{bmatrix} R & Q \\ Q^\vee & R \end{bmatrix}$$

between $M(T_P, N_P, N_P^\vee)$ and $M(T_Q, N_Q, N_Q^\vee)$ given by

$$S(g) : \begin{bmatrix} a & w \\ \check{w} & b \end{bmatrix} \to \begin{bmatrix} g(a) & \widetilde{g}(w) \\ (\widetilde{g}^\vee)^{-1}(\check{w}) & g(b) \end{bmatrix}$$

is an isomorphism of structurable algebras.

Proof. The proof goes along similar lines as the one of [Pa-Sr-T, 1.3]. It suffices to show the assertion that $S(g)$ is an isomorphism after a faithfully flat base change of R. Hence assume w.l.o.g. that $P = Be$ is free. Then $Q = De'$ with $e' = \widetilde{g}(e)$. Let $\mu_e = \mu(\mathcal{N}(e))$ and $\nu_{e'} = \nu(\mathcal{N}(e'))$. Then $\mu_e = \nu_{e'}$.

In other words, norms $N_P = \alpha_e N_B$ and $N_Q = \beta_{e'} N_D$ are determined by μ_e and $\nu_{e'}$ with $\alpha_e, \beta_{e'} \in R^\times$ such that $g(\alpha_e) = \beta_{e'}$. Hence $g : B \to D$ induces a map $S(g) : M(\alpha_e T_B, \alpha_e N_B, \alpha_e^2 N_B) \to M(\beta_{e'} T_D, \beta_{e'} N_D, \beta_{e'}^2 N_D)$ defined by

$$S(g) : \begin{bmatrix} a & c \\ d & b \end{bmatrix} \to \begin{bmatrix} g(a) & g(c) \\ g(d) & g(b) \end{bmatrix}.$$

This is an isomorphism of structurable algebras (in particular, it is compatible with the involutions). Define maps

$$F : M(\alpha_e T_B, \alpha_e N_B, \alpha_e^2 N_B) \to M(T_P, N_P, N_P)$$

and

$$H : M(\beta_{e'} T_D, \beta_{e'} N_D, \beta_{e'}^2 N_D) \to M(T_Q, N_Q, N_Q)$$

via

$$F(\begin{bmatrix} a & c \\ d & b \end{bmatrix}) = \begin{bmatrix} a & ce \\ e^{\vee}d & b \end{bmatrix}.$$

and

$$H(\begin{bmatrix} a & c \\ d & b \end{bmatrix}) = \begin{bmatrix} a & ce' \\ e'^{\vee}d & b \end{bmatrix}.$$

A straighforward calculation shows that these are isomorphisms of structurable algebras. Since we also have

$$H \circ S(g) = S(g) \circ F,$$

this shows that $S(g)$ is an isomorphism of structurable algebras. □

Theorem 10. *(a) Let $J = J(B, P, N)$ be a first Tits construction with norm N_J, trace T_J and adjoint $\sharp_J$. Then the structurable algebras*

$$M(T_B, N_B, N_B^{\vee}) = \begin{bmatrix} R & B \\ B & R \end{bmatrix}$$

and

$$M(T, N, N^{\vee}) = \begin{bmatrix} R & P \\ P^{\vee} & R \end{bmatrix}$$

are subalgebras of the structurable algebra

$$M(T_J, N_J, N_J^{\vee}) = \begin{bmatrix} R & B \oplus P \oplus P^{\vee} \\ B \oplus P \oplus P^{\vee} & R \end{bmatrix}.$$

over R, which are stable under the involution of $M(T_J, N_J, N_J)$.
(b) Let $J = J(B, H(B, *_B), P, N, *) = H(B, *_B) \oplus P$ be a Tits process with norm N_J, trace T_J and adjoint $\sharp_J$. Then the structurable algebra

$$M(T_B, N_B, N_B) = \begin{bmatrix} R & H(B, *_B) \\ H(B, *_B) & R \end{bmatrix}.$$

is a subalgebra of the structurable algebra

$$M(T_J, N_J, N_J) = \begin{bmatrix} R & H(B, *_B) \oplus P \\ H(B, *_B) \oplus P & R \end{bmatrix}.$$

over R, which is stable under the involution of $M(T_J, N_J, N_J)$.

For details on the first Tits construction and the Tits process, see [Pu1, 3.7, 5.1], [P-R1].

Proof. (a) We have $B^+ = J(N_B, \sharp_B, 1)$. As an R-module, $J = J(B, P, N) \cong B \oplus P \oplus P^{\vee}$ and

$$\begin{aligned} &1_J = (1, 0, 0), \\ &N_J(a, w, \check{w}) = N_B(a) + N(w) + \check{N}(\check{w}) - T_B(a, \langle w, \check{w} \rangle) \\ &(a, w, \check{w})^{\sharp_J} = (a^{\sharp_B} - \langle w, \check{w} \rangle, \check{w}^{\check{\sharp}} - aw, w^{\sharp} - \check{w}a) \end{aligned}$$

for $a \in B$, $w \in P$, $\check{w} \in P^{\vee}$. $(N_J, \sharp_J, 1_J)$ is a cubic form with adjoint and base point on J and with trace form

$$T_J((a, w, \check{w}), (c, v, \check{v})) = T_B(a, c) + T_B(\langle w, \check{v} \rangle) + T_B(\langle v, \check{w} \rangle).$$

$J(A,P,N)$ is the induced Jordan algebra $J(N_J,\sharp_J,1_J)$. B^+ identifies canonically with a subalgebra of $J(B,P,N)$.
Since the multiplication in $M(T_J,N_J,N_J)$ is given by

$$\begin{bmatrix} a & x \\ x' & b \end{bmatrix}\begin{bmatrix} c & y \\ y' & d \end{bmatrix} = \begin{bmatrix} ac+T_J(x,y') & ay+dx+x'\times_J y' \\ cx'+by'+x\times_J y & bd+T_J(y,x') \end{bmatrix}$$

with $x\times_J y=(x+y)^{\sharp_J}-x^{\sharp_J}-y^{\sharp_J}$, restricting the multiplication and a straightforward calculation yields the assertion.
(b) As an R-module, $J=J(B,P,N)\cong A\oplus P$. We have

$$1_J=(1,0)\in H^0(X,\widetilde{\mathcal{J}}),$$

$$\begin{aligned} N_J(a,w) &= N_{\mathcal{B}}(a)+N(w)+\check{N}(w^*)-T_{\mathcal{B}}(a,\langle w,w^*\rangle) \\ &= N_{\mathcal{B}}(a)+N(w)+N(w)^{*_{\mathcal{B}}}-T_{\mathcal{B}}(a,\langle w,w^*\rangle), \\ (a,w)^{\sharp_J} &= (a^{\sharp_{\mathcal{B}}}-\langle w,w^*\rangle, w^{*\check{\sharp}}-aw) \\ T_J((a,w),(c,v)) &= T_{\mathcal{B}}(a,c)+T_{\mathcal{B}}(\langle w,v^*\rangle)+T_{\mathcal{B}}(\langle v,w^*\rangle) \end{aligned}$$

for $a,c\in H(B,*_B)$ and $v,w\in P$. $(N_J,\sharp_J,1_J)$ is a cubic form with adjoint and base point on J and with trace form T_J. Again, restricting the multiplication of $M(T_J,N_J,N_J)$ yields the assertion. □

Remark 11. For any Albert algebra J, the residue class algebras of $M(T_J,N_J,N_J)$ are isomorphic to a form of the 56-dimensional irreducible module for the split simple Lie algebra of type E_7 over $k(P)$ [A-F, p. 195].

Lemma 12. *(a) For $\mu\in R^\times$,*

$$M(\mu T,\mu N,\mu^2N^\vee)\cong M(\mu^2T^\vee,\mu^2N^\vee,N).$$

(b) For $b\in B^\times$,

$$M(\mu T,\mu N,\mu^2N^\vee)\cong M(\mu N_B(b)T^\vee,\mu N_B(b)N^\vee,\mu^2N_B(b^2)N).$$

Proof. (a) Define

$$g:\begin{bmatrix} R & P \\ P^\vee & R \end{bmatrix}\to\begin{bmatrix} R & P^\vee \\ P & R \end{bmatrix},$$

$$g(\begin{bmatrix} a & w \\ \check{w} & b \end{bmatrix})=\begin{bmatrix} b & \check{w} \\ \mu^{-1}w & a \end{bmatrix}.$$

Then g is an algebra isomorphism which is compatible with the involutions.
(b) Let $\epsilon=N_B(b)$. Define

$$G:\begin{bmatrix} R & P \\ P^\vee & R \end{bmatrix}\to\begin{bmatrix} R & P^\vee \\ P & R \end{bmatrix}$$

,

$$G(\begin{bmatrix} a & w \\ \check{w} & b \end{bmatrix})=\begin{bmatrix} b & \check{w}\epsilon \\ \epsilon^{-1}w & a \end{bmatrix}.$$

Then G is an algebra isomorphism which is compatible with the involutions. □

Example 13. Let $B = R$, $N_B(a) = a^3$ and $a^{\sharp_B} = a^2$. Then $J(N, \sharp, 1) = R^+$ and the associated trace form is $T_B(a, b) = 3ab$. Let L be a projective R-module of rank 1 (also called an *invertible* R-module) and order 3. If $\beta : L \otimes L \otimes L \to R$ is an isomorphism then

$$N(w) = \beta(w \otimes w \otimes w)$$

defines a norm on L. Let $\mu \in R^\times$ and $T : L \times L^\vee \to R$,

$$T(w, \check{w}) = \langle w, \check{w} \rangle.$$

There exists a uniquely determined norm $\check{N} : L^\vee \to R$ and uniquely determined adjoints $\sharp : L \to L^\vee$ and $\check{\sharp} : L^\vee \to L$ such that $\langle w, w^\sharp \rangle = N(w)1$; $\langle \check{w}^{\check{\sharp}}, \check{w} \rangle = \check{N}(\check{w})1$ and $w^{\sharp \check{\sharp}} = N(w)w$ for $w \in L$, $\check{w} \in L^\vee$ and

$$M(\mu T, \mu N, \mu^2 N^\vee) = \begin{bmatrix} R & L \\ L^\vee & R \end{bmatrix}$$

is a structurable algebra over R of rank 4.

Example 14. Let $B = R \times R \times R$. Then $B^+ = J(N_B, \sharp_B, 1)$ and B satisfies equations (1) and (2). Since

$$\check{H}^1(X, B^\times) = \operatorname{Pic} R \times \operatorname{Pic} R \times \operatorname{Pic} R,$$

every locally free left B-module P of rank one satisfies $P \cong L \oplus M \oplus S$ with invertible R-modules L, M and S. Thus $N_B(P) \cong R$ iff $L \otimes M \otimes S \cong R$. Choose an isomorphism $\alpha : L \otimes M \otimes S \to R$, then $N(x, y, z) = \alpha(x \otimes y \otimes z)$ defines a norm on P and

$$\begin{aligned}
&(x, y, z)^\sharp = (y \otimes z, z \otimes x, x \otimes y),\\
&(\check{x}, \check{y}, \check{z})^{\sharp'} = (\check{y} \otimes \check{z}, \check{z} \otimes \check{x}, \check{x} \otimes \check{y}),\\
&N^\vee(\check{x}, \check{y}, \check{z}) = \check{\alpha}^{-1}(\check{x} \otimes \check{y} \otimes \check{z}),\\
&T((x, y, z), (\check{x}, \check{y}, \check{z})) = \langle (x, y, z), (\check{x}, \check{y}, \check{z}) \rangle = \langle x, \check{x} \rangle + \langle y, \check{y} \rangle + \langle z, \check{z} \rangle.
\end{aligned}$$

for $(x, y, z) \in P$, $(\check{x}, \check{y}, \check{z}) \in P^\vee$, see [Pu1, Example 8]. Here, we canonically identify $L \otimes M \cong S^\vee$ etc. Then

$$M(T, N, N^\vee) = \begin{bmatrix} R & L \oplus M \oplus S \\ L^\vee \oplus M^\vee \oplus S^\vee & R \end{bmatrix}.$$

is a structurable algebra over R of rank 8.

This algebra can be viewed as a generalization of the split quartic Cayley algebra denoted by $M(R^3)$ in [A3, p. 1276], since we obtain $M(R^3)$ by simply choosing $P =_B B$ and $N = N_B$.

If k has characteristic not 2 or 3 and $J = k \times k \times k$, any form of a 2×2-matrix algebra $M(\mu T_J, \mu N_J, \mu^2 N_J)$ is isomorphic to a "diagonal isotope" of an algebra $\mathrm{Cay}(B, \eta) = B \oplus vB$ obtained by the Cayley-Dickson process from a 4-dimensional commutative associative algebra B and a scalar $\eta \in k^\times$ [A3, 9.1]. Quartic Cayley algebras are used in the construction of non-split Lie algebras over k of type D_4.

Example 15. Let D be a quaternion algebra over R and $B = R \times D$, $N_B((a, x)) = a n_D(x)$ for all $a \in R$, $x \in D$, n_D the norm of D. B is a separable unital associative R-algebra and $B^+ = (N_B, \sharp_B, 1)$ with $(N_{\mathcal{B}}, \sharp_{\mathcal{B}}, 1)$ a cubic form with adjoint and base point satisfying equations (1) and (2). Every left B-module P of rank one with $N_B(P) \cong R$ satisfies $P \cong L \oplus P_0$, where L is an invertible R-module and $P_0 \in \operatorname{Pic}_l D$ satisfies $N_D(P_0) \cong L^\vee$. Let

$N' : P_0 \to L^\vee$ be a norm. A norm on P is then given by $N_P((m,u)) = \langle a, N'(u)\rangle$ for $m \in L$, $u \in P_0$. Define $T_P : P \times P^\vee \to R$ as usual via $T_P(w, \check{w}) = T_B(\langle w, \check{w}\rangle)$.

If $D = \mathrm{End}_R(E)$ then every locally free left B-module P of rank one with $N_B(P) \cong R$ has the form

$$P \cong ((\det E)^\vee \otimes \det F) \oplus (E \otimes F^\vee),$$

where F is a projective R-module of constant rank 2. Choose norms $N_P : P \to R$, $N_P^\vee : P^\vee \to R$ and adjoints $\sharp : P \to P^\vee$, $\check{\sharp} : P^\vee \to P$ satisfying equations (7), (8) (9). Then

$$M(T_P, N_P, N_P^\vee) = \begin{bmatrix} R & ((\det E)^\vee \otimes \det F) \oplus (E \otimes F^\vee) \\ (\det E \otimes (\det F)^\vee) \oplus (E^\vee \otimes F) & R \end{bmatrix}.$$

is a structurable algebra over R of rank 12.

Example 16. For every finitely generated projective R-module E of constant rank 3, $B = \mathrm{End}_R(E)$ is an Azumaya algebra of rank 9 and $B^+ = J(N_B, \sharp_B, 1)$ with N_B the usual determinant, $\sharp_B$ the usual adjoint. The locally free left B-modules of rank 1 all have the form

$$P = E \otimes F^\vee = \mathrm{Hom}_R(F, E),$$

where F is another finitely generated projective R-module of constant rank 3. Since

$$N_B(P) \cong \det E \otimes \det F^\vee,$$

we have $N_B(P) \cong R$ iff there exists an isomorphism $\alpha : \det E \to \det F$. Fixing such an isomorphism, there exists a unique cubic form $N : P \to R$ with

$$(\alpha \circ \det)(g) = N(g)\mathrm{id}_{\det F}$$

for $g \in P$. N is a norm on P. Moreover,

$$P^\vee = E^\vee \otimes F = \mathrm{Hom}_R(E, F),$$
$$\langle g, f\rangle = g \circ f \text{ for } g \in P, f \in P^\vee.$$

The adjoint $\sharp : P \to P^\vee$ of N satisfies

$$g \circ g^\sharp = N(g)\mathrm{id}_E,$$

see [Ach, 4.2] or [Pu1, Example 9]. Moreover, $T : P \times P^\vee \to R$,

$$T(g, f) = T_B(g \circ f),$$

where T_B is the usual trace of B. Then

$$M(T, N, N^\vee) = \begin{bmatrix} R & Hom_R(F, E) \\ Hom_R(E, F) & R \end{bmatrix}.$$

is a structurable algebra over R of rank 20.

4.3. Let R be a domain. Let B be a unital associative separable R-algebra. Let $(N_B, \sharp_B, 1)$ be a cubic form with adjoint and base point on B satisfying equations (1) and (2). Let $P \in \mathrm{Pic}_l B$ such that $N_B(P) \cong R$ and let $N : P \to R$ be a norm on P. Let $N^\vee : P^\vee \to R$ be the uniquely determined norm and $\sharp : P \to P^\vee$, $\check{\sharp} : P^\vee \to P$ be the uniquely determined adjoints satisfying (6) – (8). Let $T : P \times P^\vee \to R$, $T(w, \check{w}) = T_B(\langle w, \check{w} \rangle)$.

Let $P' \in \mathrm{Pic}_l B$ such that $N_B(P') \cong R$ and let $N' : P' \to R$ be a norm on P'. Let $N'^\vee : P'^\vee \to R$ be the uniquely determined norm and $\sharp' : P' \to P'^\vee$, $\check{\sharp}' : P'^\vee \to P'$ be the uniquely determined adjoints satisfying (6) – (8). Let $T' : P' \times P'^\vee \to R$, $T(w', \check{w}') = T_B(\langle w', \check{w}' \rangle)$.

As in [G, 2.8 (2)] we obtain:

Lemma 17. *Let $f : M(\mu T, \mu N, \mu^2 N^\vee) \to M(\mu' T', \mu' N', \mu'^2 N'^\vee)$ be an algebra isomorphism. Then there is a bijective R-linear map $\varphi : P \to P'$ such that*

$$N'(\varphi(w)) = \frac{\mu}{\mu'} N(w),$$

i.e. $(P, \frac{\mu}{\mu'} N) \cong (P', N')$ or a bijective R-linear map $\varphi : P \to P'^\vee$ such that

$$N'^\vee(\varphi(w)) = \frac{\mu}{\mu'^2} N(w)$$

for all $w \in P$, i.e. $(P, \frac{\mu}{\mu'} N) \cong (P'^\vee, N'^\vee)$.

Proof. The element

$$s_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is a global section which is skew-symmetric, i.e.

$$\overline{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since f respects the involutions, $f(s_0)$ must also be skew-symmetric, hence $f(s_0)^2 = f(s_0^2) = 1$ which implies that $f(s_0) = \pm s_0$.

Suppose that $f(s_0) = s_0$. Then f restricted to the diagonal matrices is the identity and so

$$f(\begin{bmatrix} a & w \\ \check{w} & b \end{bmatrix}) = \begin{bmatrix} a & \varphi(w) \\ \varphi'(\check{w}) & b \end{bmatrix}$$

for two R-linear maps $\varphi : P \to P'$, $\varphi' : P^\vee \to P'^\vee$. Since f is an algebra isomorphism, $\mu' T'(\varphi(w), \varphi'(\check{w})) = \mu T(w, \check{w})$. Let $\varphi_0 : P^\vee \to P^\vee$ be the unique R-linear map such that

$$T'(\varphi(w), \varphi_0(\check{w})) = T(w, \check{w})$$

for all $w \in P$, $\check{w} \in P^\vee$. Then $\varphi' = \frac{\mu}{\mu'} \varphi_0$. Looking at the lower left corner, we obtain

$$\varphi(w) \times \varphi(v) = \varphi'(w \times v) = \frac{\mu}{\mu'} \varphi_0(w \times v).$$

By equations (5) and (6),

$$N(w)1 = \frac{1}{3} T(w, w^\sharp) = \frac{1}{6} T(w, w \times w),$$

$$N^\vee(\check{w})1 = \frac{1}{3} T(\check{w}^{\check{\sharp}}, \check{w}) = \frac{1}{6} T(\check{w} \times' \check{w}, \check{w}).$$

Hence
$$T'(\varphi(w), \varphi_0(\check{w})) = T(w, \check{w})$$
implies
$$N'(\varphi(w))1 = \frac{1}{6}T'(\varphi(w), \varphi(w) \times \varphi(w)) = \frac{1}{6}T'(\varphi(w), \frac{\mu}{\mu'}\varphi_0(w \times w)) = \frac{\mu}{\mu'}\frac{1}{6}T(w, w \times w) = \frac{\mu}{\mu'}N(w).$$
Therefore $\varphi : P \to P'$ is a linear map such that
$$N'(\varphi(w)) = \frac{\mu}{\mu'}N(w)$$
for all $w \in P$, implying that it is a norm similarity on P with multiplier $\frac{\mu}{\mu'}$.

Now suppose $f(s_0) = -s_0$. Then we look at the isomorphism
$$g \circ f : M(\mu T, \mu N, \mu^2 N^\vee) \to M(\mu'^2 T'^\vee, \mu'^2 N'^\vee, N')$$
which satisfies $g \circ f(s_0) = s_0$, where
$$g : M(\mu T, \mu N, \mu^2 N^\vee) \to M(\mu^2 T^\vee, \mu^2 N^\vee, N),$$
$$g(\begin{bmatrix} a & w \\ \check{w} & b \end{bmatrix}) = \begin{bmatrix} b & \check{w} \\ \mu^{-1}w & a \end{bmatrix}$$
as in Lemma 12. Then $g \circ f(s_0) = s_0$ and we are back in the first case. So there is an R-linear map $\varphi : P \to P'^\vee$ such that
$$N'^\vee(\varphi(w)) = \frac{\mu}{\mu'^2}N(w)$$
for all $w \in P$. □

If the algebra has rank 8, the automorphism group of $M(\mu T, \mu N, \mu^2 N^\vee) \otimes k(P)$ is isomorphic to $\mathrm{Sl}_2 \times \mathrm{Sl}_2 \times \mathrm{Sl}_2$, if it has rank 20, it is isomorphic to Sl_6. As in [G, 2.9] it can be shown:

Theorem 18. *The automorphism group of the structurable algebra $M(\mu T, \mu N, \mu^2 N^\vee)$ is isomorphic to the semi-direct product of $\mathbb{Z}/2$ and the group of bijective norm isometries of P.*

5. Forms of matrix algebras

5.1. We use a slightly refined version of the setup of Section 3 by assuming that B is a unital *separable* associative algebra over the ring R'. Let $* : R' \to R'$ be an involution on R' and $*_B$ an involution on B such that $*_B|_{R'} = *$. Let $(N_B, \sharp_B, 1)$ be a cubic form with adjoint and base point on B satisfying identities (1), (2), (3) and let $(H(B, *_B), H(R', *_B))$ be a B-ample pair. Define $R = H(R', *_B)$. Let $P \in \mathrm{Pic}_l\, B$ such that $N_B(P) \cong R'$ and such that there is a nondegenerate *hermitian* form $h : P \times P \longrightarrow B$ satisfying
$$h(w, w) \in H(B, *_B) \text{ and } N_B(h(w, w)) = N(w)N(w)^{*_B}$$
for $w \in P$. Denote the $H(B, *_B)$-admissible involution $j_h : P \to \overline{P^\vee}$ on P induced by h by $*$. Let $N : P \to R'$ be a norm on P. Let $N^\vee : P^\vee \to R'$ be the uniquely determined norm

and $\sharp : P \to P^\vee$, $\check{\sharp} : P^\vee \to P$ be the uniquely determined adjoints satisfying equations (7), (8), (9). We point out that we can also write

$$\langle u, v^* \rangle = h(u,v), \quad v^* = j_h(v) \text{ and } \check{v}^{\check{*}} = j_h^{-1}(\check{v})$$

for $j_h : P \to \overline{P^\vee}$ induced by h.

Remark 19. Let us compare this with the setup used in [Pa-Sr-T, p. 22] for their general Tits construction. We slightly adjust their notation here, because there would be some ambiguity with ours otherwise. There R is required to be a domain and B an Azumaya algebra of degree 3 over a quadratic étale algebra R'/R, with involution $*_B$ such that $*_B$ restricts to the non-trivial automorphism $*$ of R'/R. Let $P \in \mathrm{Pic}_l\, B$ such that there exists a nondegenerate $*_B$-hermitian form $b : P \times P \to R'$. Regarding b as an isomorphism $b : P \to \overline{P^\vee}$, $b(u)(v) = b(v,u)$, its discriminant $\mathrm{disc}(b) : \mathcal{N}(P) \times \mathcal{N}(P) \to R'$ is a rank 1 hermitian form over $(R, *)$ such that

$$\mathrm{disc}(b)(\mathcal{N}(u), \mathcal{N}(w)) = N_B(h(u,w))$$

for all $u, w \in P$. It is assumed that this form is trivial. Let

$$\mu : (\mathcal{N}(P), \mathrm{disc}(b)) \to (R', \langle 1 \rangle)$$

be an isomorphism of hermitian spaces and let

$$\nu = (\mu^\vee)^{-1} : (\mathcal{N}(P)^\vee, \mathrm{disc}(b)) \to (R', \langle 1 \rangle).$$

The connection with our scenario is the following: our norm $N : P \to R'$ is given by the choice of the hermitian form $b : P \times P \to R'$ via $N(u) = b(u,u)$ and $N^\vee : P^\vee \to R'$ is given by ν as explained in Remark 5 (b). The bilinearization $\times$ of the quadratic map $\sharp$ corresponds with the map ϕ in [Pa-Sr-T, p. 22], our bilinearization $\check{\times}$ with their ϕ_* and their $b^{-1} : P^\vee \to P$ is our $\check{*}$.

On the other hand, we have

$$b(u,v) = T_B(\langle u, v^* \rangle) = T_B(h(u,v)).$$

Theorem 20. *The R-module $S(B, *_B, P, N, h) = R' \oplus P$ together with the multiplication*

$$(a,u)(b,v) = (ab + T_B(\langle u, v^* \rangle), b^{*_B} u + av + (u \times v)^{\check{*}})$$

and the involution

$$\overline{(a,u)} = (\bar{a}, u)$$

for $a, b \in R'$, $u, v \in P$ becomes an R-algebra which is a form of the structurable algebra $M(T, N, N^\vee)$ from Corollary 7 and thus is a structurable algebra over R.

Proof. Define a map

$$F : S(B, *_B, P, N, h)_{R'} = S(B, *_B, P, N, h) \otimes_R R' \longrightarrow M(T, N, N^\vee)$$

via

$$F((a,u) \otimes 1) = (a, a^{*_B}, u, u^*),$$

$$F(1 \otimes r) = (r, r^{*_B}, 0, 0)$$

for $a, r \in R'$, $u \in P$. F is an R'-linear bijection. To show that F is multiplicative, it suffices to check that it is multiplicative on $S(B, *_B, P, N, h)$, since $S(B, *_B, P, N, h)$ is generated by $S(B, *_B, P, N, h)_{R'}$ over R': For $(a, u), (b, v) \in R' \oplus P$ we have

$$\begin{aligned}&F((a,u)(b,v)) = F(ab + T_B(\langle u, v^* \rangle), b^{*_B}u + av + (u \times v)^{\check{*}}) = \\ &(ab + T_B(\langle u, v^* \rangle),\ a^{*_B}b^{*_B} + T_B(\langle u, v^* \rangle)^{*_B}, \\ &b^{*_B}u + av + (u \times v)^{\check{*}}, bu^* + a^{*_B}v^* + (u \times v)^{\check{*}})^*)\end{aligned}$$

which equals

$$\begin{aligned}&F(a,u)F(b,v) = (a, a^{*_B}, u, u^*)(b, b^{*_B}, v, v^*) = \\ &(ab + T_B(\langle u, v^* \rangle), a^{*_B}b^{*_B} + T_B(\langle v, u^* \rangle), \\ &av + b^{*_B}u + u^* \times v^*, bu^* + a^{*_B}v^* + u \times v),\end{aligned}$$

since $((u \times v)^{\check{*}})^* = u \times v$ and $u^* \check{\times} v^* = (u \times v)^{\check{*}}$ by equations (26), (27). (If B is an Azumaya algebra of degree 3, this is proved also in [Pa-S-T, p. 24]; the argument presented there goes through in our more general setting, as well.) Moreover, h is hermitian, so using (28) we obtain

$$T_B(\langle v, u^* \rangle) = T_B(\langle (v^*)^{\check{*}}, u^* \rangle) = T_B(\langle u, v^* \rangle^{*_B}) = T_B(\langle u, v^* \rangle)^{*_B}.$$

□

Remark 21. (i) Let (c, μ) be an *admissible scalar*, i.e. $\mu \in R^\times$, $c \in H(B, *_B)^\times$ and $N_B(c) = \mu\mu^*$ [P-R2, p. 246].

Let $P =_B B$ be the globally free left B-module of rank 1, then P is $H(B, *_B)$-admissible and the $H(B, *_B)$-admissible pairs are given by the pairs $(\mu N_B, c*_B)$ with (c, μ) an admissible scalar. The adjoint belonging to μN_B is $\mu\sharp_B$, $(\mu N_B)^\vee = \mu^{-1}N_B$ $(\mu\sharp_B)^\vee = \mu^{-1}\sharp_B$. Moreover, $\langle w, \check{w} \rangle = w\check{w}$ and $h : B \times B \to B$ is given by $h(w, \check{w}) = \mu wc\check{w}^{*_B}$. Then $S(B,_B B, c*_B, \mu N_B, h) = R' \oplus B$ together with the multiplication

$$(a, u)(b, v) = (ab + \mu T_B(u, cv^{*_B}), b^{*_B}u + av + c\mu^*(u \times v)^{*_B})$$

and the involution

$$\overline{(a, u)} = (\bar{a}, u)$$

for $a, b \in R'$, $u, v \in B$ is a structurable algebra over R, which is a form of the structurable algebra $M(\mu T_B, \mu N_B, \mu^2 N_B)$. For simplicity, we denote $S(B,_B B, c*_B, \mu N_B, h)$ also by $S(B, *_B, c, \mu)$.
(ii) Let K be a separable quadratic field extension of k. Let B be a central simple K-algebra of degree 3 with a unitary involution $*_B$ such that $*_B|_K = *$ is the non-trivial automorphism of K/k. Let (c, μ) be an admissible scalar. As in (i), we obtain the structurable algebra $S(B, *_B, c, \mu) = K \oplus B$ over k with the involution $\overline{(a, u)} = (\bar{a}, u)$ for $a, b \in K$, $u, v \in B$.

Remark 22. Our construction overlaps with the construction in [A-F-Y, Section 7]: their commutative associative algebra $\mathcal{E}$ with involution $*$ is our R', and $\mathcal{W}$ is our $_BB$. The algebra $S(B,_B B, *_B, , N_B, h)$ described in Remark 21 is identical to the algebra $\mathcal{A}(h, N_B)$ with $h(u, v) = T_B(u, v^{*_B})$. The construction in [A-F-Y, Section 7] is much more general than ours in the sense that it allows for the structurable algebra to be of infinite dimension. As the authors mention, the requirement that the hermitian form $h : \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ has to be weakly nondegenerate, i.e. induce an injection $j_h : u \to h(\ \ , u)$ of $\mathcal{W}$ to its dual space $\mathrm{Hom}_{\mathcal{E}}(\mathcal{W},\mathcal{E})$, guarantees that an adjoint is unique, if it exists. Nondegeneracy in our stronger sense guarantees the existence and uniqueness of an adjoint.

Our construction explicitly describes how to obtain a nondegenerate hermitian form in certain cases, e.g. for central simple algebras over fields, thus proving the existence of h.

For an admissible scalar (c,μ) we know that, for any $d\in B^\times$, also $(dcd^{*_B},\mu N_B(d))$ is an admissible scalar.

Lemma 23. *(i) Let $d\in B^\times$. Then*

$$S(B,*_B,c,\mu)\cong S(B,*_B,dcd^*,\mu N_B(d))$$

via $(a,u)\to(a,ud)$.
(ii) For any structurable algebra $S(B,*_B,c,\mu)$ there is an isomorphic structurable algebra $S(B,*_B,c,\mu)$ such that $N(c')=1=\mu'\mu'^*$.

Proof. (1) is a tedious calculation, similar to the one to prove the corresponding statement for cubic Jordan algebras which arise from a Tits construction.
Take $d=\mu^{-1}c$ in (1) to prove (2).

Theorem 24. *As in 4.1, let R be a ring and B be a separable unital associative R-algebra. Let $(N_B,\sharp_B,1)$ be a cubic form with adjoint and base point on B satisfying equations (1) and (2). Let $P\in\mathrm{Pic}_l B$ such that $N_B(P)\cong R$ and let $N:P\to R$ be a norm on P. Let $N^\vee:P^\vee\to R$ be the uniquely determined norm and $\sharp:P\to P^\vee$, $\check{\sharp}:P^\vee\to P$ be the uniquely determined adjoints satisfying equations (7), (8), (9). Define*

$$R'=R\times R,\quad (r,r')^{*_C}=(r,r'),$$

$$C=B\oplus B^{op},\quad (a,b)^{*_C}=(b,a),\quad 1_C=(1,1)$$

$$N_C(a,b)=(N_B(a),N_B(b)),\quad (a,b)^{\sharp_C}=(a^{\sharp_B},b^{\sharp_B})$$

$$R^0=\{(r,r)\,|\,r\in R\},\quad C^0=\{(a,a)\,|\,a\in B\},\quad P^0=P\oplus P^\vee,$$

$$N^0(w,\check{w})=(N(w),N^\vee(\check{w}))\ \text{and}\ (w,\check{w})^{*^0}=(\check{w},w)$$

for $a,b\in B$, $r,r'\in R$ and $w\in P$, $\check{w}\in P^\vee$. Let h^0 denote the nondegenerate hermitian form induced by the isomorphism $*^0:P^0\to P^0$. Then R', C, $*_C$, $(N_C,\sharp_C,1_C)$, (C^0,R^0), P^0 and $(N^0,*^0)$ satisfy the assumptions in 5.1 and

$$S(C,C^0,P^0,N^0,h^0)\cong M(T,N,N^\vee)$$

with $T:P\times P^\vee\to R$, $T(w,\check{w})=T_B(\langle w,\check{w}\rangle)$.

The proof is a straightforward but tedious calculation. The non-trivial parts are done in [Ach1, 2.25] as part of the proof of the corresponding statement for the first Tits construction being viewed as a special case of the Tits process, see [Pu1, 5.1]. Hence also in our setting the above result implies that we can view the matrix algebra construction $M(T,N,N^\vee)$ as a special case of the construction $S(B,*_B,P,N,h)$.

5.2. Let R be a domain and let R' be a quadratic étale ring extension of R. Let $(B, *_B)$, $(D, *_D)$ be two Azumaya algebras over R' with properties as described in 5.1. We use the setup from [Pa-Sr-T, p. 23].

Let $P \in \mathrm{Pic}_l\, B$ such that there exists a nondegenerate $*_B$-hermitian form $b : P \times P \to R'$ respectively $b' : Q \times Q \to R'$ with trivial discriminant. Let $\mu : (\mathcal{N}(P), \mathrm{disc}(b)) \to (R', \langle 1 \rangle)$ respectively $\nu : (\mathcal{N}(Q), \mathrm{disc}(b')) \to (R', \langle 1 \rangle)$ be a trivialization.

Let $f : (B, *_B) \to (D, *_D)$ be an isomorphism of R'-algebras with unitary involutions and $\widetilde{f} : (P, b) \to (Q, b')$ an f-semilinear isomorphism of hermitian spaces. There is an isomorphism

$$\mathcal{N}(\widetilde{f}) : (\mathcal{N}(P), \mathrm{disc}(b)) \to (\mathcal{N}(Q), \mathrm{disc}(b'))$$

of hermitian spaces such that $\mathcal{N}(\widetilde{f})\mathcal{N}_P = \mathcal{N}_Q \widetilde{f}$. The map $\mathcal{N}(\widetilde{f})$ is constructed by descent, analogously as in [Pa-Sr-T, p. 16].

Proposition 25. *Let R be a domain. Let $(B, *_B)$ and $(D, *_D)$ be two Azumaya algebras over R' with properties as described in 5.1. Let P, Q be locally free left modules of rank 1 over B and D, respectively. Let (P, b), respectively (Q, b'), be a hermitian $(B, *_B)$-space, respectively a hermitian $(D, *_D)$-space, with a trivialization*

$$\mu : (\mathcal{N}(P), \mathrm{disc}(b)) \to (R', \langle 1 \rangle)$$

respectively with a trivialization

$$\nu : (\mathcal{N}(Q), \mathrm{disc}(b')) \to (R', \langle 1 \rangle).$$

Let $f : (B, *_B) \to (D, *_D)$ be an isomorphism of R'-algebras with unitary involutions and $\widetilde{f} : (P, b) \to (Q, b')$ an f-semilinear isomorphism of hermitian spaces such that

$$\mu = \nu \circ \mathcal{N}(\widetilde{f}).$$

The norms $N_P : P \to R'$ and $N_Q : Q \to R'$ are given by the nondegenerate $*_B$-hermitian form $b : P \times P \to R'$ respectively $b' : Q \times Q \to R'$. Then the map

$$S(f) : R \oplus P \to R \oplus Q$$

between $S(B, *_B, P, N_P, h)$ and $S(D, *_D, Q, N_Q, h)$ given by

$$S(f) : (a, w) \to (f(a), \widetilde{f}(w))$$

is an isomorphism of structurable algebras.

Proof. The proof goes along similar lines as the one of [Pa-Sr-T, 2.2], i.e. as the proof of Proposition 9. □

Example 26. Let R be a domain and let R' be a quadratic étale ring extension of R. Let $L \in {}_3\mathrm{Pic}\, R'$, then there exists a nondegenerate cubic form $N : L \to R'$ defined via $N(x) = \alpha(x \otimes x \otimes x)$, where we just choose an isomorphism $\alpha : L \otimes L \otimes L \to R'$. Let $\star$ be the involution on R' induced by the nontrivial automorphism of R'/R. Suppose that L also carries a nondegenerate hermitian form $h : L \times L \to R'$ such that

$$N'_R(h(w, w)) = N(w)N(w)^{*_B}.$$

For every $L \in \operatorname{Pic} R'$ of order 3 such that $L^\star \cong L^\vee$, we obtain a structurable algebra

$$S(R', \star, L, N, h) = R' \oplus L$$

over R of rank 4.

Example 27. Let R' be a quadratic étale R-algebra with canonical involution $^-$. Let E be a cubic étale R-algebra such that $E^+ = J(N_E, \sharp_E, 1)$. Choose the commutative associative R'-algebra $B = E \otimes_R R'$ and $*_B = \mathrm{id}_E \otimes {}^-$. Then $H(B, *_B) = E \otimes 1$ can be identified with E. We view $B = E_{R'}$ as an algebra over R' to see that norm and adjoint of E canonically extend to B and satisfy (1), (2). We have $h(x, y) = T_E(x, y^{*_B})$ for $x, y \in B$, T_E denoting the canonical extension of T_E to B. Then

$$S(B, *_B, B, N_B, h) = R' \oplus E \otimes_R R'$$

is a structurable algebra of rank 8.

Recall also that the étale Tits process

$$J = J(B, H(B, *_B), P, N, *) = E \oplus E \otimes_R R',$$

which yields a cubic Jordan algebra of rank 9, uses the same ingredients, see [P-T], [P-R2, p. 248].

Example 28. (a) In the situation of Example 27, let $E = R \times R \times R$. Then $B = E \otimes R' \cong R' \times R' \times R'$. Every left B-module P of rank one satisfies $P \cong L \oplus M \oplus S$ with invertible R'-modules L, M and S, and $N_B(P) \cong R'$ if and only if $L \otimes M \otimes S \cong R'$. Every isomorphism $\alpha : L \otimes M \otimes S \to R'$ defines a norm on P via $N(x, y, z) = \alpha(x \otimes y \otimes z)$.
(b) In the situation of Example 27, let $R' = R \times R$ be a split quadratic étale algebra with exchange involution $^-$. Then $B = E \oplus E$ with exchange involution and $H^1(X, B) = H^1(X, E) \times H^1(X, E)$. Hence any $P \in H^1(X, B)$ is of the type $P \cong P_1 \oplus P_2$ with $P_i \in H^1(X, E)$.

A tedious calculation, similar to the proof of Theorem 24, shows that in this case $S(B, *_{\mathcal{B}}, P, N, h)$ is a 2×2-matrix algebra.

Also the étale Tits process $J(B, H(B, *_B), P, N, *) \cong E \oplus P_1 \oplus P_1^\vee$ with $P_1 \in H^1(X, \mathcal{E}) = \operatorname{Pic}_l E$ becomes an étale first Tits construction starting with E.

6. Structurable algebras of large rank

6.1. **Structurable algebras over locally ringed spaces.** Let $(X, \mathcal{O}_X)$ be a locally ringed space such that $2, 3 \in H^0(X, \mathcal{O}_X^\times)$. For $P \in X$ let $\mathcal{O}_{P,X}$ be the local ring of $\mathcal{O}_X$ at P and m_P the maximal ideal of $\mathcal{O}_{P,X}$. The corresponding residue class field is denoted by $k(P) = \mathcal{O}_{P,X}/m_P$. For an $\mathcal{O}_X$-module $\mathcal{F}$ the stalk of $\mathcal{F}$ at P is denoted by $\mathcal{F}_P$. $\mathcal{F}$ is *locally free of finite rank*, if for each $P \in X$ there is an open neighborhood $U \subset X$ of P such that $\mathcal{F}|_U = \mathcal{O}_U^r$ for some integer $r \geq 0$. The *rank* of $\mathcal{F}$ is defined to be $\sup\{\operatorname{rank}_{\mathcal{O}_{P,X}} \mathcal{F}_P \mid P \in X\}$. An "$\mathcal{O}_X$-algebra" (or "algebra over X") is a nonassociative $\mathcal{O}_X$-algebras which is unital and locally free of finite constant rank as $\mathcal{O}_X$-module. An algebra $\mathcal{A}$ over $\mathcal{O}_X$ is called *separable* if $\mathcal{A}(P)$ is a separable $k(P)$-algebra for all $P \in X$. Recall that an associative $\mathcal{O}_X$-algebra $\mathcal{A}$ is called an *Azumaya algebra* if $\mathcal{A}_P \otimes_{\mathcal{O}_{P,X}} k(P)$ is a central simple algebra over $k(P)$ for all

$P \in X$ [K]. For an $\mathcal{O}_X$-algebra $\mathcal{A}$, an anti-automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ of order 2 is called an *involution* on $\mathcal{A}$. Define $H(\mathcal{A}, \sigma) = \{a \in \mathcal{A} \,|\, \sigma(a) = a\}$ and $\mathrm{S}(\mathcal{A}, \sigma) = \{a \in \mathcal{A} \,|\, \sigma(a) = -a\}$. Then $\mathcal{A} = H(\mathcal{A}, \sigma) \oplus \mathrm{S}(\mathcal{A}, \sigma)$.

6.2. An *algebra with involution* is a pair $(\mathcal{A}, ^-)$ consisting of an $\mathcal{O}_X$-algebra $\mathcal{A}$ and an involution $^- : \mathcal{A} \to \mathcal{A}$. A *structurable algebra* is an algebra with involution $(\mathcal{A}, ^-)$ satisfying

$$\{xy\{zwq\}\} - \{zw\{xyq\}\} = \{\{xyz\}wq\} - \{z\{yxw\}q\}$$

for all sections $x, y, z, w \in \mathcal{A}$, where

$$\{xyz\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y.$$

If $\mathcal{B}$ is an $\mathcal{O}_X$-submodule if $\mathcal{A}$ which is closed under multiplication, we call $\mathcal{B}$ a subalgebra of $\mathcal{A}$. If, additionally, $\overline{\mathcal{B}} = \mathcal{B}$ then we call $(\mathcal{B}, ^-)$ a subalgebra of $(\mathcal{A}, ^-)$.

There is a canonical equivalence between the category of structurable algebras over the affine scheme $Z = \operatorname{Spec} R$, for which the algebras are locally free as $\mathcal{O}_X$-modules and the category of structurable algebras over R which are finitely generated projective as R-modules, given by the global section functor $(A, ^-) \longrightarrow (H^0(Z, A), H^0(Z, ^-))$ and the functor $(A, ^-) \longrightarrow (\widetilde{A}, \widetilde{^-})$.

The setup given in 4.1. adapts without any problems to the setting of locally ringed spaces, for this we refer the reader to [Pu1, Section 3]. The same is obviously true for the construction of the 2×2-matrix algebra $M(T, N, N^\vee)$, as well as for 1.1, 1.3 and 1.4 (see [Pu1] or [Pu2]).

6.3. The results of the previous sections have applications to structurable algebras over fields as well. We will now construct classes of structurable algebras over a field which are of large rank by passing to the global sections of structurable algebras over projective space. We will consider the projective space over an arbitrary ring, though, since our considerations hold not just over $\mathbb{P}^n_k$.

Let $X = \mathbb{P}^n_R$ be the n-dimensional projective space over R, that is $X = \operatorname{Proj} S$ where $S = R[t_0, \dots, t_n]$ is the polynomial ring in $n+1$ variables over R, equipped with the canonical grading $S = \oplus_{m \geq 0} S_m$. We have $\operatorname{rank} S_m = \binom{m+n}{n}$. We know that $\mathcal{O}_X(m)$ is a locally free $\mathcal{O}_X$-module of rank one for each $m \in \mathbb{Z}$ and

$$H^0(X, \mathcal{O}_X(m)) = S_m \text{ for } m \geq 0,$$

$$H^0(X, \mathcal{O}_X(m)) = 0 \text{ for } m < 0.$$

Example 29. Let $\mathcal{F} = \mathcal{O}_X(m_1) \oplus \mathcal{O}_X(m_2) \oplus \mathcal{O}_X(m_3)$, then $\mathcal{B} = \mathcal{E}nd_X(\mathcal{F})$ is an Azumaya algebra over X of constant rank 9. Hence $\mathcal{B}$ is a separable unital associative $\mathcal{O}_X$-algebra and $\mathcal{B}^+ = (N_{\mathcal{B}}, \sharp_{\mathcal{B}}, 1)$ with $(N_{\mathcal{B}}, \sharp_{\mathcal{B}}, 1)$ a cubic form with adjoint and base point satisfying equations (1), (2), see [Pu1]. We have

$$\mathcal{B} = \begin{bmatrix} \mathcal{O}_X & \mathcal{O}_X(a) & \mathcal{O}_X(b) \\ \mathcal{O}_X(-a) & \mathcal{O}_X & \mathcal{O}_X(b-a) \\ \mathcal{O}_X(-b) & \mathcal{O}_X(a-b) & \mathcal{O}_X \end{bmatrix}$$

with $a = m_1 - m_2$, $b = m_1 - m_3$, the right hand side being equipped with the usual matrix multiplication. $B = H^0(X, \mathcal{B})$ is a unital associative R-algebra of degree 3 and $H^0(X, \mathcal{B}^+) = (N_0, \sharp_0, 1)$ with $(N_0, \sharp_0, 1)$ a cubic form with adjoint and base point on B satisfying equations (1), (2), where we put $N_0 = N(X) : H^0(X, \mathcal{B}) \to H^0(X, \mathcal{O}_X)$, $\sharp_0 = \sharp_{\mathcal{B}}(X) : H^0(X, \mathcal{B}) \to H^0(X, \mathcal{B})$ and $T_0 = T(X) : H^0(X, \mathcal{B}) \times H^0(X, \mathcal{B}) \to H^0(X, \mathcal{O}_X)$.

(1) If $a, b > 0$ and $b - a > 0$ then

$$H^0(X, \mathcal{B}) = \begin{bmatrix} R & S_a & S_b \\ 0 & R & S_{b-a} \\ 0 & 0 & R \end{bmatrix}$$

has rank

$$3 + \binom{a+n}{n} + \binom{b+n}{n} + \binom{(b-a)+n}{n}.$$

(2) If $a = b > 0$ then

$$H^0(X, \mathcal{B}) = \begin{bmatrix} R & S_a & S_a \\ 0 & R & R \\ 0 & R & R \end{bmatrix}$$

has odd rank $5 + 2\binom{a+n}{n}$.

In (1), we obtain

$$N_0(\begin{bmatrix} c & f_a & f_b \\ 0 & d & f_{b-a} \\ 0 & 0 & e \end{bmatrix}) = cde,$$

$$T_0(\begin{bmatrix} c & f_a & f_b \\ 0 & d & f_{b-a} \\ 0 & 0 & e \end{bmatrix}, \begin{bmatrix} m & g_a & g_b \\ 0 & n & g_{b-a} \\ 0 & 0 & s \end{bmatrix}) = cm + dn + es$$

and in (2),

$$N_0(\begin{bmatrix} c & f_a & g_a \\ 0 & d & e \\ 0 & m & n \end{bmatrix}) = c(dn - em),$$

$$T_0(\begin{bmatrix} c & f_a & g_a \\ 0 & d & e \\ 0 & m & n \end{bmatrix}, \begin{bmatrix} q & h_a & l_a \\ 0 & r & s \\ 0 & t & u \end{bmatrix}) = cq + dr + nu.$$

The maps N_0, T_0 and $\sharp_0$ satisfy the adjoint identities by construction. In each case, the structurable R-algebra $M(T_0, N_0, N_0)$ is an R-subalgebra of the structurable algebra $M(T_E, N_E, N_E)$ over S, $E = \mathrm{Mat}_3(S)$, and has

(1) rank $8 + 2[\binom{a+n}{n} + \binom{b+n}{n} + \binom{(b-a)+n}{n}]$,
(2) rank $12 + 4\binom{a+n}{n}$.

Example 30. Let

$$\mathcal{D} = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{O}_X(m))$$

be the split quaternion algebra over X with norm $n_{\mathcal{D}}$ =det as defined in [P2, 2.11], $m \geq 0$ an integer. Let $\mathcal{B} = \mathcal{O}_X \oplus \mathcal{D}$ and $N_{\mathcal{B}}((x_1, x_2)) = x_1 n_{\mathcal{D}}(x_2)$ for all sections x_1 in $\mathcal{O}_X$, x_2 in $\mathcal{D}$. $\mathcal{B}$ is a separable unital associative $\mathcal{O}_X$-algebra and $\mathcal{B}^+ = (N_{\mathcal{B}}, \sharp_{\mathcal{B}}, 1)$ with $(N_{\mathcal{B}}, \sharp_{\mathcal{B}}, 1)$ a cubic form with adjoint and base point satisfying equations (1), (2). We get

$$B = H^0(X, \mathcal{B}) = R \oplus H^0(X, \mathcal{D}) = R \oplus \begin{bmatrix} R & S_m \\ 0 & R \end{bmatrix}.$$

B is a unital associative R-algebra of degree 3 and $B^+ = (N_0, \sharp_0, 1)$ with $(N_0, \sharp_0, 1)$ a cubic form with adjoint and base point on B satisfying equations (1), (2), where we put $N_0 = N(X) : H^0(X, \mathcal{B}) \to H^0(X, \mathcal{O}_X)$,

$$N_0((a, \begin{bmatrix} b & f_m \\ 0 & c \end{bmatrix})) = abc,$$

$\sharp_0 = \sharp_{\mathcal{B}}(X) : H^0(X, \mathcal{B}) \to H^0(X, \mathcal{O}_X)$,

$$(a, \begin{bmatrix} b & f_m \\ 0 & c \end{bmatrix})^{\sharp_0} = (bc, \begin{bmatrix} ab(c-1) & -af_m \\ 0 & ac(b-1) \end{bmatrix}),$$

and the global section $T_0 = T(X)$ of the symmetric bilinear map $T(x, y) = T_{\mathcal{B}}(xy)$ are given by

$$T_0((a, \begin{bmatrix} b & f_m \\ 0 & c \end{bmatrix}), (d, \begin{bmatrix} e & g_m \\ 0 & h \end{bmatrix})) = ad + be + ch,$$

cf. [Pu1, Exercise 1 (ii)]. B is an R-subalgebra of the associative S-algebra $F = S \oplus \mathrm{Mat}_2 S$ and of rank

$$3 + \binom{m+n}{n}.$$

The maps N_0, T_0 and $\sharp_0$ satisfy the adjoint identities by construction. The structurable R-algebra $M(T_0, N_0, N_0)$ is an R-subalgebra of the structurable algebra $M(T_F, N_F, N_F)$ over S, and has rank

$$8 + 2\binom{m+n}{n}.$$

If $n = 1$ then $\mathrm{rank}_R M(T_0, N_0, N_0) = 10 + 2m$.

Example 31. Let $\mathcal{B} = \mathcal{O}_X \times \mathcal{O}_X \times \mathcal{O}_X$. Then $\mathcal{B}^+ = \mathcal{J}(N_{\mathcal{B}}, \sharp_{\mathcal{B}}, 1)$ and, analogously as in Example 14, every left $\mathcal{B}$-module $\mathcal{P} \cong \mathcal{O}_X(l) \oplus \mathcal{O}_X(m) \oplus \mathcal{O}_X(-l-m)$ of rank one satisfies $N_{\mathcal{B}}(\mathcal{P}) \cong \mathcal{O}_X$ (see also [Pu1, Example 8] for more details). Choose an isomorphism $\alpha : \mathcal{O}_X(l) \otimes \mathcal{O}_X(m) \otimes \mathcal{O}_X(-l-m) \to \mathcal{O}_X$, then $N(x, y, z) = \alpha(x \otimes y \otimes z)$ defines a norm on $\mathcal{P}$ and

$$T((x, y, z), (\check{x}, \check{y}, \check{z})) = \langle (x, y, z), (\check{x}, \check{y}, \check{z}) \rangle = \langle x, \check{x} \rangle + \langle y, \check{y} \rangle + \langle z, \check{z} \rangle.$$

Moreover, the adjoints are given by

$$(x, y, z)^{\sharp} = (y \otimes z, z \otimes x, x \otimes y),$$
$$(\check{x}, \check{y}, \check{z})^{\widetilde{\sharp}} = (\check{y} \otimes \check{z}, \check{z} \otimes \check{x}, \check{x} \otimes \check{y})$$

and

$$\check{N}(\check{x},\check{y},\check{z}) = \check{\alpha}^{-1}(\check{x}\otimes\check{y}\otimes\check{z}),$$

for (x,y,z) in $\mathcal{P}$, $(\check{x},\check{y},\check{z})$ in $\mathcal{P}^\vee$. The structurable algebra

$$M(T,N,N^\vee) = \begin{bmatrix} \mathcal{O}_X & \mathcal{O}_X(l)\oplus\mathcal{O}_X(m)\oplus\mathcal{O}_X(-l-m) \\ \mathcal{O}_X(-l)\oplus\mathcal{O}_X(-m)\oplus\mathcal{O}_X(l+m) & \mathcal{O}_X \end{bmatrix}$$

over X has global sections

$$A = H^0(X, M(T,N,N^\vee)) = \begin{bmatrix} R & S_l\oplus S_m \\ S_{l+m} & R \end{bmatrix}.$$

These global sections are an R-algebra with involution, with the algebra multiplication in A given by

$$\begin{bmatrix} a & f_l\oplus f_m \\ f_{l+m} & a' \end{bmatrix}\begin{bmatrix} b & g_l\oplus g_m \\ g_{l+m} & b' \end{bmatrix} = \begin{bmatrix} ab & (ag_l+b'f_l)+(ag_m+b'f_m) \\ bf_{l+m}+a'g_{l+m}+f_lg_m+f_mg_l & a'b' \end{bmatrix}$$

and the involution by

$$\overline{\begin{bmatrix} a & f_l\oplus f_m \\ f_{l+m} & a' \end{bmatrix}} = \begin{bmatrix} a' & f_l\oplus f_m \\ f_{l+m} & a \end{bmatrix}.$$

Here, $a,a',b,b'\in R$ and the f's and g's are homogeneous polynomials with subscrips indicating their degrees.

Note that this multiplication closely resembles the one described in [P2, 3.8]. It is analogously defined as the one on the global sections of the split octonion algebra $\mathrm{Zor}(\mathcal{T},\alpha)$ with $\mathcal{T}=\mathcal{O}_X(l)\oplus\mathcal{O}_X(m)\oplus\mathcal{O}_X(-l-m)$, the only difference is the term $f_lg_m - f_mg_l$ instead of $f_lg_m+f_mg_l$ above, which generalizes the usual vector product $\times$ used in the multiplication of a split octonion algebra.

Note also that the maps T, $\check{\sharp}$ on the global sections become trivial, i.e. with $T_0 = T(X)$, $\check{\times}_0 = \check{\times}(X)$ and $\times_0 = \times(X)$, we have

$$T_0 : (S_l\oplus S_m)\times S_{l+m}\to R, \quad T_0 = 0,$$

$$\check{\times}_0 : S_{l+m}\times S_{l+m}\to (S_l\oplus S_m), \quad f_{l+m}\,\check{\times}_0\, g_{l+m} = 0$$

and

$$\times_0 : (S_l\oplus S_m)\times(S_l\oplus S_m)\to S_{l+m}, \quad (f_l\oplus f_m)\times_0(g_l\oplus g_m) = f_lg_m+f_mg_l.$$

Indeed, $(T,N,N^\vee)=(0,0,0)$. $(A,\bar{\ })$ is an R-subalgebra of the classical split Cayley algebra

$$M(S^3) = \begin{bmatrix} R & S^3 \\ S^3 & R \end{bmatrix}.$$

over S and free of rank

$$2+\binom{l+n}{n}+\binom{m+n}{n}+\binom{(l+m)+n}{n}.$$

It is not difficult to generate more examples of this nature, all of them arising canonically as the global sections of some structurable algebra over projective n-space (or over a curve of genus zero or one).

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