Steinberg groups for Jordan pairs

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Introduction

The present note is an extended version of the announcement [**18**]. It concerns a project dealing with the theory of classical groups and Steinberg groups over arbitrary rings from the point of view of Jordan theory. This provides a unifying framework, avoiding case-by-case arguments, for the linear elementary groups, the unitary elementary groups and the orthogonal elementary groups, see, for example, the book [**9**] by A. J. Hahn and O. T. O'Meara. We start from a Jordan pair V graded by a 3-graded root system Φ and show that the projective elementary group $\mathrm{PE}(V)$ has Φ-commutator relations in the sense of J. R. Faulkner [**7**, Ch. 1]. Since our root systems are allowed to be infinite (but locally finite, as in our monograph [**16**]), we are able to deal with the infinite elementary groups and Steinberg groups directly, that is, without having to pass to the limit. To $\mathrm{PE}(V)$ we associate a Steinberg group $\mathrm{St}(V)$, following the method of J. Tits for Kac-Moody groups [**25**]. Our main result concerns the case where Φ is irreducible of infinite rank and asserts that $\mathrm{St}(V)$ is the universal central extension of the projective elementary group. Our approach is substantially less computational than those in the literature.

In this survey we present background material on elementary groups and Jordan pairs in §1 and describe the contents of our research in §2.

§1. Elementary groups

1.1. The elementary group of a Morita context. For motivation let us start with 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with coefficients in a ring R. The *elementary group* $\mathrm{E}_2(R)$ is the subgroup of $\mathrm{GL}_2(R)$ generated by the elementary matrices

$$e_{12}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad e_{21}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \qquad (x \in R).$$

More generally, one considers the *elementary group* $\mathrm{E}_n(R) \subset \mathrm{GL}_n(R)$, generated by all $e_{ij}(x) = 1_n + xE_{ij}$, $i \neq j$, $x \in R$ [**9**, 1.2]. This can also be done with (formal) 2×2 matrices by subdividing an $n \times n$ matrix into 4 blocks, say of size $p \times p$, $p \times q$, $q \times p$, $q \times q$, with $p + q = n$. Then it is easy to see that $\mathrm{E}_n(R)$ is already generated by the matrices

$$\begin{pmatrix} 1_p & x \\ 0 & 1_q \end{pmatrix}, \qquad \begin{pmatrix} 1_p & 0 \\ y & 1_q \end{pmatrix} \qquad \big(x \in \mathrm{Mat}_{pq}(R),\ y \in \mathrm{Mat}_{qp}(R)\big).$$

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This suggests to consider right away the following more general situation: Replace the $n \times n$-matrices $\mathrm{Mat}_n(R)$ by a ring $\mathfrak{A}$ with a formal block matrix decomposition. Such a decomposition can be obtained by choosing an idempotent $e \in \mathfrak{A}$ and putting $f = 1 - e$. The Peirce decomposition of $\mathfrak{A}$ with respect to the idempotent e can then be written as

$$\mathfrak{A} = \begin{pmatrix} e\mathfrak{A}e & e\mathfrak{A}f \\ f\mathfrak{A}e & f\mathfrak{A}f \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \tag{1.1.1}$$

in other words: $\mathfrak{M} = (A, B, C, D)$ is a *Morita context*. One defines the *elementary group of* $\mathfrak{M}$ as the following subgroup of the units of $\mathfrak{A}$:

$$\mathrm{E}(\mathfrak{M}) = \left\langle \begin{pmatrix} 1_A & B \\ 0 & 1_D \end{pmatrix} \cup \begin{pmatrix} 1_A & 0 \\ C & 1_D \end{pmatrix} \right\rangle.$$

The setting of a Morita context not only captures the elementary group $\mathrm{E}_n(R)$, but also its *stable* version

$$\mathrm{E}(R) = \bigcup_{n \geqslant 2} \mathrm{E}_n(R)$$

([**9**, 1.3]) where for $p \geqslant n$ the group $E_n(R)$ sits in the upper left hand corner of $\mathrm{E}_p(R)$. Indeed, for arbitrary sets J and K define $\mathrm{Mat}_{JK}(R)$ as the $J \times K$-matrices with entries from R, almost all of them zero. Then

$$R_{JK} = \begin{pmatrix} R \cdot 1_J + \mathrm{Mat}_{JJ}(R) & \mathrm{Mat}_{JK}(R) \\ \mathrm{Mat}_{KJ}(R) & R \cdot 1_K + \mathrm{Mat}_{KK}(R) \end{pmatrix}$$

is a Morita context. If $\mathbb{N} = J \mathbin{\dot\cup} K$ is a non-trivial partition, it is straightforward to see that the stable elementary group coincides with the elementary group of the Morita context R_{JK} , thus $\mathrm{E}(R) = \mathrm{E}(R_{JK})$.

1.2. Elementary groups of special Jordan pairs. Let us now always work over an arbitrary commutative base ring k. All objects for which this makes sense are assumed to be modules over k, rings are k-algebras, and so on.

If $\mathfrak{M}$ is a Morita context as above, where now $\mathfrak{A}$ is an associative k-algebra, one does not need to take all of $(M^+, M^-) := (B, C)$ to obtain a group. Rather we can start from any pair $V = (V^+, V^-)$ of k-submodules $V^\pm \subset M^\pm$ and consider the subgroup

$$\mathrm{E}(\mathfrak{M}, V) = \left\langle \begin{pmatrix} 1 & V^+ \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 \\ V^- & 1 \end{pmatrix} \right\rangle \subset \mathrm{E}(\mathfrak{M})$$

of $\mathrm{E}(\mathfrak{M})$. Since $V^\pm$ are in particular additive subgroups of $M^\pm$, $\begin{pmatrix} 1 & V^+ \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ V^- & 1 \end{pmatrix}$ are multiplicative subgroups of $\mathfrak{A}^\times$. It turns out (see Lemma 1.3), that there are natural group-theoretic reasons to require that $V = (V^+, V^-)$ have more structure than just being a pair of submodules. To explain this, recall that the associative algebra $\mathfrak{A}$ together with the commutator $[a, b] = ab - ba$ is a Lie algebra, which will be denoted by $\mathfrak{L}$. The decomposition (1.1.1) defines a 3-*grading* of the Lie algebra $\mathfrak{L}$, i.e., a $\mathbb{Z}$-grading $\mathfrak{L} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{L}_n$ with $\mathfrak{L}_n = \{0\}$ for $|n| > 1$, namely

$$\mathfrak{L}_{-1} = \begin{pmatrix} 0 & 0 \\ M^- & 0 \end{pmatrix}, \qquad \mathfrak{L}_0 = \begin{pmatrix} e\mathfrak{A}e & 0 \\ 0 & f\mathfrak{A}f \end{pmatrix}, \quad \mathfrak{L}_1 = \begin{pmatrix} 0 & M^+ \\ 0 & 0 \end{pmatrix}.$$

We observe that (M^+, M^-) is closed under the composition

$$Q(x)y = xyx \quad (x \in M^\sigma, y \in M^{-\sigma}, \sigma = \pm). \tag{1.2.1}$$

By linearization, (M^+, M^-) is then also closed under the trilinear composition

$$\{x, y, z\} = xyz + zyx \quad (x, z \in M^\sigma, y \in M^{-\sigma}, \sigma = \pm). \tag{1.2.2}$$

For a pair $V = (V^+, V^-)$ of submodules of M we define

$$\begin{aligned}
\mathfrak{e}_{-1} &= \begin{pmatrix} 0 & 0 \\ V^- & 0 \end{pmatrix}, \qquad \mathfrak{e}_1 = \begin{pmatrix} 0 & V^+ \\ 0 & 0 \end{pmatrix}, \\
\mathfrak{e}_0 &= k \cdot \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} + k \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_D \end{pmatrix} + [\mathfrak{e}_1, \mathfrak{e}_{-1}], \\
\mathfrak{e}_i &= \{0\} \qquad \text{for } i \neq 0, \pm 1, \\
\mathfrak{e}(\mathfrak{M}, V) &= \mathfrak{e}_{-1} \oplus \mathfrak{e}_0 \oplus \mathfrak{e}_1.
\end{aligned}$$

1.3. Lemma. (a) $\mathfrak{e}(\mathfrak{M}, V)$ *is a graded Lie subalgebra of* $\mathfrak{L}$*, hence itself* 3*-graded, if and only if* V *is closed under the trilinear composition* (1.2.2) *, i.e.,* $\{V^\sigma, V^{-\sigma}, V^\sigma\} \subset V^\sigma$ *for* $\sigma = \pm$*.*

(b) $\mathfrak{e}(\mathfrak{M}, V)$ *is stable under conjugation by elements of* $\mathrm{E}(\mathfrak{M}, V)$ *if and only if* V *is closed under the* Q*-operators* (1.2.1) *, i.e.,* $Q(V^\sigma)V^{-\sigma} \subset V^\sigma$ *for* $\sigma = \pm$*.*

The condition (a) says that V is a *special linear Jordan pair* and (b) that V is a *special quadratic Jordan pair*. If they are fulfilled, we call $\mathfrak{e}(\mathfrak{M}, V)$ the *elementary Lie algebra* of $(\mathfrak{M}, V)$.

So far we have seen that any embedding of a special Jordan pair in a Morita context gives rise to an elementary group. Now it is natural to ask whether there are elementary groups for arbitrary (not necessarily special) Jordan pairs. The answer, given in 1.5, is essentially yes. For the sake of completeness, we first present some background material on Jordan pairs.

1.4. Jordan pairs ([13]). A (quadratic) *Jordan pair* $V = (V^+, V^-)$ is a pair of k-modules together with a pair (Q_+, Q_-) of quadratic maps $Q_\sigma \colon V^\sigma \to \mathrm{Hom}_k(V^\sigma, V^{-\sigma})$ such that, defining bilinear maps $D_\sigma : V^\sigma \times V^{-\sigma} \to \mathrm{End}_k(V^\sigma)$ by

$$D_\sigma(x, y)(z) = Q_\sigma(x + z)(y) - Q_\sigma(x)(y) - Q_\sigma(z)(y),$$

the following identities hold in all base ring extensions of V:

$$\begin{aligned}
D_\sigma(x, y)Q_\sigma(x) &= Q_\sigma(x)D_{-\sigma}(y, x), \\
D_\sigma(Q_\sigma(x)y, y) &= D_\sigma(x, Q_{-\sigma}(y)x), \\
Q_\sigma(Q_\sigma(x)y) &= Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x).
\end{aligned}$$

Examples. (a) It is a useful exercise to verify that these identities do indeed hold for the special Jordan pairs $(\mathrm{Mat}_{JK}(R), \mathrm{Mat}_{KJ}(R))$ or, more generally (B, C) of the previous subsection 1.1 for $Q_+ = Q_- = Q$ defined by (1.2.1) and $D_\pm(x, y)z = \{x, y, z\} = xyz + zyx$ as in (1.2.2). One is justified to call these Jordan pairs special, since there are examples of

Jordan pairs which cannot be embedded into a Morita context, for example Jordan pairs arising from exceptional Jordan algebras, as in (b).

(b) Any quadratic Jordan algebra J with quadratic map $U : J \to \mathrm{End}_k(J)$ gives rise to a Jordan pair (J, J) with $Q_\pm = U$. For example, any associative algebra A can be viewed as a Jordan algebra A^+ with $U(a)b = aba$ and hence a fortiori to a Jordan pair (A^+, A^+). Similarly, symmetric matrices over an associative algebra B form a Jordan algebra, viz., a subalgebra of an appropriate Jordan algebra A^+, and hence provide another example of a Jordan pair.

The aforementioned examples are all special. However, if J is not special (exceptional), then so is the Jordan pair (J, J). The reader should be warned that not all Jordan pairs arise from Jordan algebras. For example $(\mathrm{Mat}_{pq}(R), \mathrm{Mat}_{qp}(R))$ is in general not isomorphic to a Jordan pair associated to a Jordan algebra. Thus Jordan pairs afford a substantial extension of the theory of Jordan algebras.

(c) The reader may be more familiar with the linear (as opposed to quadratic) version of a Jordan pair, which can be phrased completely in terms of the *Jordan triple products* $\{\cdots\}_\sigma$ defined by $\{xyz\}_\sigma = D_\sigma(x, y)z$. For simpler notation, we often drop the index σ at D, Q and the triple products which usually can be supplied from the context. Then a *linear Jordan pair* is a pair (V^+, V^-) of k-modules with trilinear maps $V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$, $(x, y, z) \mapsto \{xyz\}$, symmetric in x and z, and such that for all $u, x, z \in V^\sigma$ and $y, v \in V^{-\sigma}$,

$$\{xy\{uvz\}\} = \{\{xyu\}vz\} - \{u\{yxv\}z\} + \{uv\{xyz\}\}. \tag{1.4.1}$$

Any quadratic Jordan pair satisfies these identities. Conversely, if 2 and 3 are units in k, then they are enough to define a Jordan pair by putting $Q_\sigma(x)y = \frac{1}{2}\{xyx\}$.

A homomorphism $h\colon V \to W$ of Jordan pairs is a pair $h = (h_+, h_-)$ of k-linear maps $h_\sigma\colon V^\sigma \to W^\sigma$, $\sigma = \pm$, satisfying $h_\sigma Q(x) = Q(h_\sigma(x))h_{-\sigma}$ for $x \in V^\sigma$. It is then clear how to define automorphism. In contradistinction to Jordan algebras, Jordan pairs have natural inner automorphisms, defined in terms of the *Bergmann operators*

$$B(x, y) = \mathrm{Id} - D(x, y) + Q(x)Q(y) \quad (x \in V^+,\ y \in V^-).$$

These satisfy the (non-obvious) identity $Q(B(x, y)z) = B(x, y)Q(z)B(y, x)$. Hence, if $B(x, y)$ and $B(y, x)$ are invertible, then

$$\beta(x, y) = (B(x, y), B(y, x)^{-1}) \in \mathrm{Aut}(V), \tag{1.4.2}$$

called the *inner automorphism* defined by (x, y).

In the same vein, a *derivation of* V is a pair $(\Delta_+, \Delta_-) \in \mathrm{End}_k(V^+) \times \mathrm{End}_k(V^-)$ satisfying the quadratic version of the usual derivation identity, namely $\Delta_\sigma(Q(u)v) = \{\Delta_\sigma(u), v, u\} + Q(u)\Delta_{-\sigma}(v)$ for $\sigma = \pm$ and all $u \in V^\sigma$ and $v \in V^{-\sigma}$. Linearizing this condition in u leads to

$$[\Delta_\sigma, D(u, v)] = D(\Delta_\sigma(u), v) + D(u, \Delta_\sigma(v)), \tag{1.4.3}$$

a condition which is sufficient in case $\frac{1}{2} \in k$.

The definition of the Bergmann operator suggests that the left multiplications $D(x, y)$ are infinitesimal versions of Bergmann operators. By general philosophy, they should therefore be derivations. This is indeed the case: Appropriate linearizations of the Jordan pair identities show that

$$\delta(x, y) = (D(x, y), -D(y, x))$$

is a derivation, naturally called the *inner derivation* defined by $(x, y) \in V^+ \times V^-$. We denote by $\mathrm{Inder}(V)$ the *inner derivation algebra*, spanned by all inner derivations. Note that the identity (1.4.3) implies that $\mathrm{Inder}(V)$ is an ideal of the Lie algebra $\mathrm{Der}(V) \subset \mathrm{End}\, V^+ \times \mathrm{End}\, V^-$ of all derivations of V.

1.5. The projective elementary group of a Jordan pair. As indicated in 1.2, we now construct, for an arbitrary Jordan pair V, an analogue of the elementary group of a special Jordan pair.

First of all, the elementary Lie algebra $\mathfrak{e}(\mathfrak{M}, V)$ of 1.2 has an abstract counterpart, namely the Tits-Kantor-Koecher algebra $\mathfrak{g}(V)$ (Tits [**23**], Koecher [**11**, **12**] for Jordan algebras, Kantor [**10**], Meyberg [**19**, **20**] for Jordan triple systems): Put

$$\mathfrak{g}_0 = k \cdot (\mathrm{Id}_{V^+}, -\mathrm{Id}_{V^-}) + \mathrm{Inder}(V) \quad \text{and} \quad \mathfrak{g}_{\pm 1} = V^{\pm} \qquad \text{(as } k\text{-modules)}.$$

One can define a multiplication on

$$\mathfrak{g} = \mathfrak{g}(V) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

in such a way that $\mathfrak{g}$ becomes a 3-graded Lie algebra, as follows:

- the Lie product of elements of $\mathfrak{g}_0$ is the usual commutator of maps,
- $[\Delta, x] = \Delta_+(x)$ and $[\Delta, y] = \Delta_-(y)$ for $\Delta = (\Delta_+, \Delta_-) \in \mathfrak{g}_0$ and $(x, y) \in V$,
- $[x, y] = -\delta(x, y)$ for $(x, y) \in V$.

With this product, $(\mathrm{ad}\, x)^3 = 0$ for $x \in \mathfrak{g}_{\pm 1}$, and one can therefore "exponentiate" $\mathrm{ad}\, x$ for $x \in \mathfrak{g}_{\pm 1}$ (even when 2 is not invertible in k!), by defining for $x, z \in \mathfrak{g}_{\pm 1}$, $h \in \mathfrak{g}_0$, $y \in \mathfrak{g}_{\mp 1}$:

$$e^{\mathrm{ad}\, x} \cdot z = z, \quad e^{\mathrm{ad}\, x} \cdot h = h + [x, h],$$
$$e^{\mathrm{ad}\, x} \cdot y = y + [x, y] + Q_x y.$$

The Jordan identities guarantee that

$$\exp_{\pm}(x) := e^{\mathrm{ad}\, x} \in \mathrm{Aut}(\mathfrak{g}).$$

Finally we define the *projective elementary group of* V by

$$\mathrm{PE}(V) = \Big\langle \exp_+(V^+) \cup \exp_-(V^-) \Big\rangle,$$

a subgroup of the automorphism group of $\mathfrak{g}$, see Faulkner [**8**], Loos [**14**], Bertram-Neeb [**2**, **3**]. These groups have nice properties, for example:

- $\exp_\sigma \colon V^\sigma \to \mathrm{PE}(V)$ is an injective homomorphism,
- when V is special, $\mathrm{PE}(V) \cong \mathrm{E}(\mathfrak{M}, V)/Z_0$, where Z_0 is the intersection of the centre of $\mathrm{E}(\mathfrak{M}, V)$ with the diagonal $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ of the Morita context associated to the special V,
- Simplicity Theorem: Similarly to known results about classical groups (Dieudonné [**6**], or see [**9**]), Chevalley groups ([**5**], or see [**22**]) and algebraic groups (Tits [**24**]), the following theorem holds:

Theorem 1 (Loos [15]). *Let V be a simple non-degenerate Jordan pair with descending chain condition for principal inner ideals. Then* $\mathrm{PE}(V)$ *is simple with exactly three exceptions, namely the one-dimensional Jordan pairs over $\mathbb{F}_2$ and $\mathbb{F}_3$ and the symmetric 2×2 matrices over $\mathbb{F}_2$.*

Here the term *simple* has the usual meaning. A Jordan pair is *nondegenerate* if $Q(x) = 0$ implies $x = 0$. The *descending chain condition for principal inner ideals* is a standard Artinian-like condition in the theory of Jordan pairs. For example, for the Jordan pair associated to the Jordan algebra A^+, A an associative algebra, the conditions in the theorem mean that A is simple Artinian. The three exceptions in the theorem are the symmetric respectively alternating groups $\mathfrak{S}_3$, $\mathfrak{A}_4$, and $\mathfrak{S}_6$.

§2. Steinberg groups

2.1. Standard example: The Steinberg groups of a ring. Let us return to the elementary group $\mathrm{E}_n(R)$ of a ring R, $n \geqslant 3$, defined in 1.1. The matrices $e_{ij}(x)$ $(i \neq j)$ satisfy the following relations:

$$\text{(E1)} \qquad e_{ij}(x)e_{ij}(y) = e_{ij}(x+y),$$

$$\text{(E2)} \qquad \big(e_{ij}(x),\, e_{lm}(y)\big) = 1 \qquad (j \neq l,\ i \neq m),$$

$$\text{(E3)} \qquad \big(e_{ij}(x),\, e_{jl}(y)\big) = e_{il}(xy) \qquad (i, j, l \neq).$$

Here

$$\big(a, b\big) = aba^{-1}b^{-1}$$

denotes the group commutator.

The (linear) *Steinberg group* $\mathrm{St}_n(R)$ is defined as the abstract group presented by generators $e_{ij}(x)$, $1 \leqslant i \neq j \leqslant n$ and the relations (E1)–(E3) above. Similarly, the *stable Steinberg group* $\mathrm{St}(R)$ of R is defined using the same relations but allowing infinitely many generators $e_{ij}(x)$, $i, j \in \mathbb{N}$, $i \neq j$.

We have already seen that $\mathrm{E}_n(R)$ can be generated by the $p \times q$ and $q \times p$ matrices over R which form a Jordan pair and that $E(R)$ has a similar description using the special Jordan pair R_{JK}. There should therefore also be a Jordan pair approach to $\mathrm{St}_n(R)$ and $\mathrm{St}(R)$. More generally, one should be able to define Steinberg groups for arbitrary Jordan pairs. This is our next aim which we will achieve in 2.5.

2.2. Groups with commutator relations. First, it is useful to re-interpret the relations (E1)–(E3) of 2.1 and also the definition of Steinberg groups in a more abstract way as follows.

We start with $\mathrm{E}_n(R)$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the standard basis of $\mathbb{R}^n$ and let Φ be the set of all $\varepsilon_i - \varepsilon_j$, $i \neq j$, i.e., the usual realization of the root system A_{n-1}. For $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$ let

$$U_\alpha := e_{ij}(R) \subset G := \mathrm{E}_n(R).$$

Then (E1) says in particular that U_α is a subgroup of G. For a subset $\Sigma \subset \Phi$ let U_Σ be the subgroup of G generated by all U_α, $\alpha \in \Sigma$, and for $\alpha, \beta \in \Phi$ let

$$\big(\alpha, \beta\big) = \Phi \cap \big(\mathbb{N}_+\alpha + \mathbb{N}_+\beta\big),$$

the so-called open root interval between α and β. The relations (E2) and (E3) imply in particular the commutator relations

$$\text{(CR)} \qquad \big(U_\alpha,\, U_\beta\big) \subset U_{(\alpha,\beta)}$$

for every *nilpotent pair* (α, β) in Φ which by definition means that $p\alpha + q\beta \neq 0$ for all $p.q \in \mathbb{N}_+$. (These relations are somewhat weaker than (E1)–(E3), since for example (E3) gives a precise formula for the commutator of two elementary matrices while (CR) just states an inclusion between subsets.)

This approach to interpreting the relations (E1)–(E3) also works for the stable elementary group $\mathrm{E}(R)$. All we have to do, is to replace the finite root system A_{n-1} by the "infinite root system" $\mathrm{A}_\infty^+ = \{\varepsilon_i - \varepsilon_j : i, j \in \mathbb{N},\ i \neq j\} \subset \mathbb{R}^{(\infty)}$ — an example of an infinite but locally finite root system.

Recall from [**16**] that a subset Φ of a real vector space X is a *locally finite root system* if it satisfies the same axioms as finite root systems (see e.g. [**4**]), except that the finiteness condition is replaced by *local finiteness*: The intersection of Φ with every finite-dimensional subspace of X is finite. In particular, any finite root system is a locally-finite root system. Locally finite root systems are direct sums of irreducible locally finite root systems, and irreducible locally finite root systems can be classified: They are either finite or the infinite analogues of the classical root systems of type $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ and BC.

Our interpretation of the relations (E1)–(E3) leads us to the following general definition:

Let Φ be a locally finite root system. A *group with commutator relations of type* Φ is a group G together with a family $(U_\alpha)_{\alpha\in\Phi}$ of subgroups, called *root groups*, which generate G and for which (CR) holds. In case Φ is not reduced, we also require that $\beta = n\alpha$ for $n \in \mathbb{N}$ implies $U_\beta \subset U_\alpha$.

Examples. As we have seen above, the elementary linear groups $\mathrm{E}_n(R)$ and $\mathrm{E}(R)$ are groups with commutator relations for root systems of type A. The elementary unitary groups of [**9**, 5.3] are groups with commutator relations for root systems of type C.

2.3. Steinberg groups defined by groups with commutator relations. Let $\bar{G}$ be a group with subgroups $(\bar{U}_\alpha)_{\alpha\in\Phi}$ and commutator relations as in 2.2, and let $(G, (U_\alpha))$ be another such group. Naturally, a morphism $\varphi\colon (G, U_\alpha) \to (\bar{G}, \bar{U}_\alpha)$ is just a group homomorphism preserving root groups: $\varphi(U_\alpha) \subset \bar{U}_\alpha$ for all $\alpha \in \Phi$. For our purposes the following more restrictive type of morphism is important. A *strong morphism* is a morphism $\varphi\colon (G, U_\alpha) \to (\bar{G}, \bar{U}_\alpha)$ with the property that

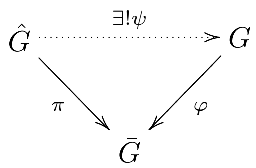
$$\varphi\colon U_{[\alpha,\beta]} \to \bar{U}_{[\alpha,\beta]} \quad \text{is bijective, for all nilpotent pairs } (\alpha, \beta),$$

where we put $\big[\alpha, \beta\big] := \{\alpha\} \cup \big(\alpha, \beta\big) \cup \{\beta\}$. Since the pair (α, α) is in particular nilpotent and $\big[\alpha, \alpha\big] = \{\alpha\}$ or $\{\alpha, 2\alpha\}$, a strong morphism satisfies

$$\varphi\colon U_\alpha \to \bar{U}_\alpha \quad \text{is bijective for all } \alpha \in \Phi.$$

Roughly speaking, this means that G has "the same" generators and commutator relations as $\bar{G}$. Adapting an argument of Tits [**25**], one can prove that there is a largest such group G, more precisely:

Theorem 2. *Let Φ be a locally finite root system and let $(\bar{G}, (\bar{U}_\alpha)_{\alpha\in\Phi})$ be a group with Φ-commutator relations. Then there exists a group $(\hat{G}, \hat{U}_\alpha)_{\alpha\in\Phi}$ with Φ-commutator relations and a strong morphism $\pi\colon \hat{G} \to \bar{G}$ such that every strong morphism $\varphi\colon G \to \bar{G}$ is obtained from π by taking a quotient:*



$(\hat{G}, \pi)$ is uniquely determined up to unique isomorphism by this property.

We call $\hat{G}$ the *Steinberg group of* $\bar{G}$. This is justified since for example the classical Steinberg group $\mathrm{St}_n(R) = \hat{G}$ is obtained in this way from the elementary group $\bar{G} = \mathrm{E}_n(R)$. Similarly, the stable Steinberg group $\mathrm{St}(R)$ is $\hat{G}$ for $\bar{G} = \mathrm{E}(R)$. The same approach works for the unitary Steinberg groups, as for example defined in [**9**, 5.5].

Remarks. (a) In categorical language , $\pi: \hat{G} \to \bar{G}$ is an initial object in the category of groups "over $\bar{G}$", defined in an evident way. One constructs $\hat{G}$ as a suitable inductive limit. It can also be defined via a presentation.

(b) It is obvious that the notion of a group with commutator relations makes sense for (much) more general types of root systems, and in fact, we work in this generality. Then it may very well happen that the root interval $\big(\alpha, \beta\big)$ is infinite and one must define nilpotent pairs with the additional requirement that $\big(\alpha, \beta\big)$ be finite. The structure of these nilpotent pairs is explored in [**17**]. In particular it is shown there that for Φ a Kac-Moody root system the nilpotent pairs are precisely those that enter in Tits' construction of Kac-Moody groups.

(c) Our approach to Steinberg groups greatly simplifies the description of the defining relations which have been used elsewhere in the literature. This is most evident in the case of unitary Steinberg groups, where several different types of generators and many relations are used, see for example [**1**, **9**].

2.4. Commutator relations for $\mathrm{PE}(V)$**.** We wish to construct Steinberg groups by way of Theorem 2 in case $\bar{G} = \mathrm{PE}(V)$. This group has two generating abelian subgroups $\bar{U}^{\pm} = \exp_{\pm}(\mathfrak{g}_{\pm 1}) \cong V^{\pm}$ satisfying the A_1-commutator relations . But this is not very interesting; the root system A_1 is much too small, and the resulting Steinberg group is simply the free product of the additive groups V^+ and V^-. Therefore, $\bar{G}$ should satisfy commutator relations for bigger root systems, but in a way compatible with the fact that $\bar{G}$ contains the subgroups $\bar{U}^{\pm}$, coming from the Jordan pair V. The key to this is to consider 3-graded root systems, in analogy to 3-graded Lie algebras.

A 3-*grading* ([**21**], [**16**, §§17, 18]) of a locally finite root system Φ is a decomposition $\Phi = \Phi_{-1} \dot\cup \Phi_0 \dot\cup \Phi_1$ such that

(i) if $\alpha \in \Phi_i$, $\beta \in \Phi_j$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi_{i+j}$; in particular, then $i + j \in \{-1, 0, 1\}$,

(ii) every $\mu \in \Phi_0$ can be written (not uniquely) as $\mu = \alpha - \beta$ with $\alpha, \beta \in \Phi_1$,

(iii) $\Phi_{-1} = -\Phi_1$.

Remark. A 3-grading is uniquely determined by Φ_1. It therefore makes sense to denote a 3-graded root system by (Φ, Φ_1). Such 3-gradings exist for all locally finite irreducible reduced root systems except for the finite root systems of type G_2, F_4 or E_8; in general, there are several non-isomorphic 3-gradings on a given Φ.

It turns out that 3-graded root systems are precisely the correct index sets for gradings of Jordan pairs and lead to the desired commutator relations for the projective elementary group. Given a 3-graded root system (Φ, Φ_1) and a Jordan pair V, we define a (Φ, Φ_1)-*grading* Γ *of* V as a decomposition

$$\Gamma : V = \bigoplus_{\alpha \in \Phi_1} V_\alpha$$

where the $V_\alpha = (V_\alpha^+, V_\alpha^-)$ are pairs of submodules and the direct sum is to be understood component-wise, such that for all $\alpha, \beta, \gamma \in \Phi_1$ and $\sigma \in \{+, -\}$ the following multiplication rules hold:

$$Q(V_\alpha^\sigma)V_\beta^{-\sigma} \subset V_{2\alpha-\beta}^\sigma, \qquad \{V_\alpha^\sigma, V_\beta^{-\sigma}, V_\gamma^\sigma\} \subset V_{\alpha-\beta+\gamma}^\sigma,$$
$$\{V_\alpha^\sigma, V_\alpha^{-\sigma}, V_\beta^\sigma\} = 0 \qquad \text{for} \quad \alpha \perp \beta.$$

Observe that the roots $2\alpha - \beta$ and $\alpha - \beta + \gamma$ in the first condition either lie in Φ_1 or are not roots at all. In the latter case the first condition becomes $Q(V_\alpha^\sigma)V_\beta^{-\sigma} = 0$ or $\{V_\alpha^\sigma, V_\beta^{-\sigma}, V_\gamma^\sigma\} = 0$. The notation $\alpha \perp \beta$ in the second condition means of course that the roots α and β are orthogonal.

Example. The special Jordan pair $(V^+, V^-) = (\mathrm{Mat}_{pq}(R), \mathrm{Mat}_{qp}(R))$ considered earlier has a (Φ, Φ_1)-grading with $\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leqslant i \neq j \leqslant n\} = \mathrm{A}_{n-1}$ for $n = p + q$ and

$$\Phi_1 = \{\varepsilon_i - \varepsilon_{p+j} : 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q\}.$$

Indeed, the root subspaces $V_\alpha = (V_\alpha^+, V_\alpha^-)$, $\alpha \in \Phi_1$, are given by

$$V^+_{\varepsilon_i - \varepsilon_{p+j}} = RE_{i,j}\,, \quad V^-_{\varepsilon_i - \varepsilon_{p+j}} = RE_{j,i}.$$

This immediately extends to the infinite setting: The Jordan pair $(\mathrm{Mat}_{JK}(R), \mathrm{Mat}_{KJ}(R)$ has a (Φ, Φ_1)-grading with $\Phi_1 = \{\varepsilon_j - \varepsilon_k : j \in J,\ k \in K\}$ of type A, where J and K are disjoint sets.

Theorem 3. *Assume the Jordan pair V has a (Φ, Φ_1)-grading with root subspaces V_α. Then the projective elementary group $\bar{G} = \mathrm{PE}(V)$ has Φ-commutator relations with the following root groups:*

$$\bar{U}_{\pm\alpha} = \exp_\pm(V_\alpha^\pm) \qquad \text{for } \alpha \in \Phi_1,$$
$$\bar{U}_\mu = \Big\langle \bigcup \{\beta(V_\alpha^+, V_\beta^-) : \alpha - \beta = \mu,\ \alpha, \beta \in \Phi_1\} \Big\rangle \qquad \text{for } 0 \neq \mu \in \Phi_0.$$

Here $\beta(x, y)$ is the inner automorphism defined in (1.4.2).

2.5. Steinberg groups for Jordan pairs. Combining Theorem 2 and Theorem 3, we now define: The *Steinberg group* $\mathrm{St}(V, \Gamma)$ of a Jordan pair V with a (Φ, Φ_1)-grading Γ is the Steinberg group of $\mathrm{PE}(V)$, considered as a group with Φ-commutator relations.

It is a classical result that the stable Steinberg group $\mathrm{St}(R)$ of a ring R is the universal central extension of the stable elementary group $\mathrm{E}(R)$ (the kernel of the canonical homomorphism being the second K-group $K_2(R)$). Hence it is natural to ask whether similar results are true for $\mathrm{St}(V, \Gamma)$. This is indeed the case, but one needs to make stronger assumptions on V than just a (Φ, Φ_1)-grading. In the classical cases of linear and unitary Steinberg groups, R is always a ring with unit element. This is essential and yields so-called Weyl elements in the elementary groups. For example, in $\mathrm{E}_2(R)$ the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a Weyl element. More generally, every $r \in R^\times$ yields the Weyl element

$$w(r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix} \in \mathrm{E}_2(R).$$

In the Jordan pair case, one uses idempotents to construct Weyl elements. Here an *idempotent* in V is a pair $e = (e^+, e^-) \in V^+ \times V^-$ such that $Q(e^+)e^- = e^+$ and $Q(e^-)e^+ = e^-$. Similar to the associative or Jordan algebra case, an idempotent induces a *Peirce decomposition* $V = V_2(e) \oplus V_1(e) \oplus V_0(e)$. For example, if J is a unital Jordan algebra with associated Jordan pair $V = (J, J)$ and $u \in J$ is a unit then $e = (u, u^{-1})$ is an idempotent of V with $V = V_2(e)$.

Definition. A (Φ, Φ_1)-grading Γ of V is called an *idempotent grading* if there exists a family $\mathcal{E} = (e_\alpha)_{\alpha \in \Phi_1}$ of idempotents with the following properties:

(i) $e_\alpha \in V_\alpha$,

(ii) for all $\alpha, \beta \in \Phi_1$, we have $V_\beta \subset V_{\langle \beta, \alpha^\vee \rangle}(e_\alpha)$.

The symbol $\langle \beta, \alpha^\vee \rangle$ in (ii) is the usual Cartan integer. For $\alpha, \beta \in \Phi_1$ we always have $\langle \beta, \alpha^\vee \rangle \in \{0, 1, 2\}$ so that $V_{\langle \beta, \alpha^\vee \rangle}(e_\alpha)$ makes sense (and is the corresponding Peirce space of e_α).

Actually, it suffices to require the existence of idempotents only for roots $\alpha \in \Phi_1^{id}$ where, for Φ irreducible,

$$\Phi_1^{id} = \left\{ \begin{array}{ll} \Phi_1 & \text{if } \Phi \text{ is simply-laced} \\ \{\text{long roots}\} & \text{if } \Phi \text{ is of type } B \\ \{\text{short roots}\} & \text{if } \Phi \text{ is of type } C \end{array} \right\}.$$

This generalization allows us to realize some of the unitary groups of [**9**] as Steinberg groups of appropriate Jordan pairs. The link to the Weyl elements $w(r)$ above comes from the fact that an idempotent e_α gives rise to the Weyl element

$$w_\alpha = \exp_+(e_\alpha^+)\exp_-(e_\alpha^-)\exp_+(e_\alpha^+)$$

in $\bar{G} = \mathrm{PE}(V)$ and with a similar definition in any group G over $\bar{G}$. The importance of Weyl elements lies in the fact that they lift the action of the reflection in the root α to the group G, providing a powerful computational tool.

Then our main results are as follows.

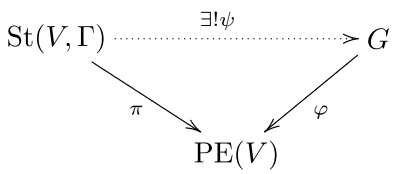
Theorem 4. *Let Γ be an idempotent (Φ, Φ_1)-grading of a Jordan pair V with Φ irreducible and different from* A_1 *and* B_2.

(a) $\mathrm{St}(V, \Gamma)$ *is perfect, i.e., equal to its commutator group.*

(b) *If Φ has rank $\geqslant 5$ then* $\mathrm{St}(V, \Gamma)$ *covers uniquely all central extensions of* $\mathrm{PE}(V)$.

(c) *If Φ has infinite rank then* $\mathrm{St}(V, \Gamma)$ *is the universal central extension of* $\mathrm{PE}(V)$.

Remarks. (a) The term "covers" means the following: If $\varphi\colon G \to \mathrm{PE}(V)$ is any central extension of abstract groups (G need not have commutator relations!) then there exists a unique group homomorphism $\psi\colon \mathrm{St}(V, \Gamma) \to G$ making the diagram



commutative. Observe that this does not yet mean that $\mathrm{St}(V, \Gamma)$ itself is a central extension of $\mathrm{PE}(V)$.

(b) Parts (b) and (c) of this theorem were known before in the following cases:

(i) Φ of type A: Then V is a rectangular matrix pair over an associative algebra R, $\mathrm{PE}(V)$ is the usual projective elementary group of R, and $\mathrm{St}(V, \Gamma)$ coincides with the usual linear Steinberg group $\mathrm{St}_n(R)$ for $n = 1 + \operatorname{rank} \Phi$. In this case, the theorem follows from [**9**, 1.4.12 and 1.4.13].

(ii) Φ is of type C and V is a hermitian matrix pair. In this case, the projective elementary group of V coincides with the usual projective elementary unitary group, while $\mathrm{St}(V, \Gamma)$ is the usual unitary Steinberg group. In case $\operatorname{rank} \Phi = \infty$, the theorem is due to Sharpe and Bak [**9**, 5.5.10]. The case $5 \leqslant \operatorname{rank} \Phi < \infty$ does not seem to be explicitly stated in [**9**], but it follows from a suitably modified version of the proof of [**9**, 5.5.10], see [**9**, 5.5.11].

In addition to the known cases (i) and (ii) above, the theorem applies to new types of Steinberg groups, like Steinberg groups of groups with commutator relations of type E_6 and E_7 or of type Φ with an uncountable Φ. We mention that the theorem is not true if Φ has rank $\leqslant 4$.

2.6. Concluding remarks. Our approach has the advantage of being substantially less computational than anything else available in the literature, where special cases of this theorem are proved case-by-case. It avoids to a large part such case distinctions and introduces two new techniques into the area of Steinberg groups: the elegant combinatorics of 3-graded root systems and the powerful methods of Jordan pairs.

In proving Theorem 4 we establish a detailed structure theory for the groups $\mathrm{PE}(V)$ and their central coverings. These results are of interest beyond the realm of Steinberg groups. For example, we expect that they will pave the way for applications of Jordan pair techniques in algebraic group theory.

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