Basis of Identities of the Algebra of Simplified Insertions

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Abstract

We prove that all identities of the algebra of simplified insertions on countably many generators over a field of zero characteristic follow from the right-symmetric identity. We prove that bases of the free special Jordan algebra and the special algebra of simplified insertions coincide. We construct an infinite series of relations in the algebra of simplified insertions which hold for words of length $k, k \in \mathbb{N}$.

1. Introduction

Given a free associative algebra $Ass[X]$ on a set of generators $X = \{x_1, ..., x_n, ...\}$, let's define a new operation of multiplication $*$ by the following rule:

$$x_{i_1} ... x_{i_n} * a = \sum_{k=0}^{n} x_{i_1} ... x_{i_k} a x_{i_{k+1}} ... x_{i_n} ,$$

where a is an arbitrary element in $Ass[X]$, and if $b = \sum \alpha_s u_s$, where $\alpha_s \in F$ and u_s are monomials in $Ass[X]$, then

$$b * a = \sum \alpha_s (u_s * a) .$$

The introduced operation $*$ is called *the simplified insertion operation*, which was first introduced by M. Bremner [1] and represents the algebraic formalization of the normal insertion operation in DNA computing (see [2, 3]).

Let's call the new algebra $\langle Ass[X], +, * \rangle$ the *algebra of simplified insertions* on the set of generators X and denote it by $Ass^*[X]$.

It is known that the algebra $Ass^*[X]$ satisfies the right-symmetric identity

$$(x, y, z) = (x, z, y) . \qquad (1)$$

This fact was first proved by Gerstenhaber in [4].

Using an exhaustive computational search M. Bremner [1] proved that all identities of degree 4 of the algebra $Ass^*[X]$ follow from (1). For identities of degree 5 M. Bremner constructed a relation satisfied by words of length 2 of the algebra $Ass^*[X]$.

Denote by RS the variety of right-symmetric algebras and by $RS[X]$ a free algebra in the variety RS on set of generators $X=\{x_1,...,x_n,...\}$.

We will need the description of the basis of a free right-symmetric algebra $RS[X]$ constructed by Segal [5].

Denote by W the set of all non-associative words of X and by $d(w)$ the length of $w\in W$. Let $w=uv\in W$. Then $l(w)=u$ and $z(w)=v$ are called *the left and the right factors of the word* w. If $w\in X$, we assume that $l(w)=1$, $r(w)=w$. Define the order on W by the rule: $u<v$ if $d(u)<d(v)$, or if $d(u)=d(v)$, $r(u)<r(v)$, or if $d(u)=d(v)$, $r(u)=r(v)$, $l(u)<l(v)$.

Define the Segal basis for $RS[X]$ by induction on $d(w)$. If $d(w)\le 2$, then w is a basis word. Let $d(w)=n,\ n\ge 3$, and all basis words of length less than n are already defined. Then w is a basis word if $w=x_iu,\ x_i\in X$, u is a basis word, or $w=(ab)c$, where $a,\ b,\ c,\ ab$ are basis words and $b\le c$.

For notation of non-associative words we will use the left-normed arrangement of parentheses: i.e., $v_1...v_n$ means $(...(v_1v_2)...v_{n-1})v_n$. Let's consider an arbitrary word $w\in W$. Let $w=x_ia_1...a_m$, where $x_i\in X$, $a_j\in W$. Then we call $b(w)=x_i$ *the beginning of the word* w and the number $R(w)=m$ is called the *R-length of the word* w. Assume that $R(w)=0$ if $w\in X$. E.g. if $w=x_1((x_1,x_1)x_1)(x_1x_1)$, then $d(w)=6$, $R(w)=2$.

All algebras in this work are considered over a field F of zero characteristic. Therefore, the defining identities of the variety of algebras over F can be assumed to be polylinear. We will use the general definitions and notations introduced in [6].

2. Basis of identities for the algebra $Ass^*[X]$.

Let J be an algebra over F and $A\subseteq J$. Denote the linear subspace of J generated by the set A in J by $L_J(A)$, i.e.,

$$L_J(A)=\{\sum_i \alpha_i a_i \mid \alpha_i\in F,\ a_i\in A\}.$$

Let $k_1,...,k_m$ be a set of non-zero natural numbers. Denote $L=L(k_1,...,k_m\mid y)=L_{Ass|y,X|}(y^{k_1}a_1y^{k_2}a_2...a_{m-1}y^{k_m}\mid a_1,...,a_{m-1}\in Ass[X])$.

Lemma 1. *Let* $b_i,\ i\in I$ *be an arbitrary basis of* $Ass[X]$. *Then the set* $B=\{y^{k_1}b_{i_1}y^{k_2}b_{i_2}...b_{i_{m-1}}y^{k_m}\}$, *where* $(i_1,...,i_{m-1})$ *runs over all different sets in* I^{m-1}, *is a basis of* L.

Proof. Let $A=\overset{m-1}{\underset{i=1}{\otimes}} Ass[X]$ be a tensor product of linear spaces over F. Consider the mapping $\varphi: A\to L$ defined by the rule $\varphi(a_1\otimes...\otimes a_{m-1})=y^{k_1}a_1y^{k_2}a_2...a_{m-1}y^{k_m}$. It is easy to note that φ is an isomorphism. Consequently, the set B is a basis of L, what proves the lemma.

Lemma 2. *Let* $C=\{f_i, i\in I\}$ *be a set of all polylinear monomials in* $x_1,...,x_n$ *from the Segal basis of degree* $\le n$. *Then for any* $s>n$ *the set* $C|_{x_1=y_1^s,...,x_n=y_n^s}=\{f_i|_{x_1=y_1^s,...,x_n=y_n^s}, f_i\in C\}$ *is linearly independent in* $Ass^*[Y]$.

Proof. We proceed by induction on n. For $n=1,2$ the induction assertion is obvious. Let the induction assertion be valid for all polylinear monomials in $x_1,...,x_{n-1}$ of degree $\le n-1$ from the Segal basis. Suppose the contrary; i.e., assume that the set $C|_{x_1=y_1^s,...,x_n=y_n^s}$ is linearly dependent in $Ass^*[Y]$ for some $s>n$. Fix a number s. We will write $\overline{f}=f|_{x_1=y_1^s,...,x_n=y_n^s}$ for short. Then there exists a non-trivial linear combination of elements of C, $f=\sum_{i\in J}\alpha_i f_i,\ \alpha_i\ne 0$ for any $i\in J\subseteq I$, such that $\overline{f}=\sum_{i\in J}\alpha_i\overline{f_i}=0$. Let $R(f)=\max_{i\in J}R(f_i)=m$. It is obvious that $m<n<s$. Among the summands $f=\sum_{i\in J}\alpha_i f_i,\ \alpha_i\ne 0$ choose those having R-length m and equal beginning. Without loss of generality we may assume that all these summands start with x_1. Then

$$f=\sum_{(i_1,...,i_m)\in I_1}\alpha_{i_1...i_m}x_1a_{i_1}...a_{i_m}+\sum_{i\in I_2}\beta_i g_i\ , \tag{2}$$

where for any $(i_1,...,i_m)\in I_1,\ i\in I_2$ the scalars $\alpha_{i_1...i_m},\ \beta_i$ differ from zero; $x_1a_{i_1}...a_{i_m}$ and g_i are linearly independent elements of the Segal basis either $b(g_1)=x_1$ and $R(g_1)<m$ or $b(g_i)\ne x_1$.

Consider the set $D=\{a_{i_k}\}$ of all elements a_{i_k} constituting the expression $x_1a_{i_1}...a_{i_m}$ of the first sum (2) for all $(i_1,...,i_m)\in I_1$. By construction the set D consists of linearly independent polylinear monomials of $x_2,...,x_n$ of degree $\le n-1$ from the Segal basis. By the induction assumption the set $D|_{x_2=y_2^s,...,x_n=y_n^s}$ is linearly independent in the algebra $Ass^*[X]$. By our assumption

$$\overline{f}=\sum_{(i_1,...,i_m)\in I_1}\alpha_{i_1...i_m}y_1^s*\overline{a_{i_1}}*...*\overline{a_{i_m}}+\sum_{i\in I_2}\beta_i\overline{g_i}=0 \tag{3}$$

By the above remarks, the elements of the form $y_1^{s-m}\overline{a}_{j_1}y_1\overline{a}_{j_1}...\overline{a}_{j_m}y_1$, where $a_{j_s}\in D$ enter into the first sum (3) only. Consequently,

$$\sum_{(i_1,...,i_m)\in I_1}\alpha_{i_1...i_m}\sum_{\sigma\in S_m}(y_1^{s-m}\overline{a}_{i_{\sigma(1)}}y_1\overline{a}_{i_{\sigma(2)}}...\overline{a}_{i_{\sigma(m)}}y_1)=0\,.$$

It follows from the Lemma 1 that $\alpha_{i_1...i_m}=0$ for any $(i_1,...,i_m)\in I$. This makes a contradiction. The lemma is proved.

Theorem 1. *All identities of the algebra $Ass^*[X]$ follow from (1).*

Proof. Let $f=f(x_1,...,x_n)$ is a polylinear homogenous identity in $Ass^*[X]$. Decompose f by the Segal basis $f=\sum_{i\in I}\alpha_i f_i$. Choose an arbitrary $s>n$. Then $\overline{f}=f|_{x_1=y_1^s,...,x_n=y_n^s}=0$ in the algebra $Ass^*[Y]$. It follows from the Lemma 2 that $\alpha_i=0$ for any $i\in I$. The theorem is proved.

3. Special algebras of simplified insertions.

Recall that the free special Jordan algebra $SJ[X]$ is a subalgebra of the algebra $Ass[X]^{(+)}$ generated by the set X with respect to the operation $a\circ b=\frac{1}{2}(ab+ba)$, where $a,b\in Ass[X]$.

Similarly, we will call the subalgebra of $Ass^*[X]$ generated by the set X the *special algebra of simplified insertions* on the set of generators X and will denote it by $SI[X]$. In this section we will show that the linear spaces $SJ[X]$ and $SI[X]$ coincide; i.e., $SJ[X]=SI[X]$, and will prove that all identities of $SI[X]$ follow from (1).

Lemma 3. *The following relations are valid in the algebra $Ass^*[X]$:*

$$(x_1...x_n)*y=(x_1...x_n)\circ y+\sum_{i=1}^{n}x_1...(x_i\circ y)...x_n \tag{4}$$

for any $y\in Ass^*[X]$;

$$f(x_1...x_n)*y=\sum_{i=1}^{n}f(x_1,...,x_i\circ y,...,x_n)+f(x_1,...,x_n)\circ y \tag{5}$$

for any polylinear polynomial $f(x_1,...,x_n)\in SJ[X]$ and any $y\in Ass^*[X]$.

Proof. (4) We proceed by induction on n. It is obvious that

$$x_1*y=yx_1+x_1y=x_1\circ y+x_1\circ y\,.$$

Now, by the definition of the operation $*$ and by the induction assumption,

$$(x_1\ldots x_n)*y=(x_1\ldots x_{n-1})*yx_n+x_1\ldots x_n y=$$
$$=\sum_{i=1}^{n-1}x_1\ldots(x_i\circ y)\ldots x_{n-1}x_n+(x_1\ldots x_{n-1})\circ yx_n+x_1\ldots x_n y=$$
$$=\sum_{i=1}^{n-1}x_1\ldots(x_i\circ y)\ldots x_n+\frac{1}{2}yx_1\ldots x_n+\frac{1}{2}x_1\ldots x_{n-1}yx_n+x_1\ldots x_n y=$$
$$=\sum_{i=1}^{n}x_1\ldots(x_i\circ y)\ldots x_n+(x_1\ldots x_n)\circ y.$$

The relation (5) follows immediately from (4). This proves the lemma.

Theorem 2. *The linear spaces $SJ[X]$ and $SI[X]$ coincide.*

Proof. It suffices to prove that the subspaces $SJ[X]$ and $SI[X]$ of polylinear polynomials coincide.

1. $SI[X]\subseteq SJ[X]$. Let's consider a homogeneous polylinear monomial $f(x_1,...,x_n)\in SI[X]$ and prove that $f\in SJ[X]$ by induction on n. The assertion is obvious for $n=1,2$. Let $f=g*h$, where g and h are homogenous polylinear polynomials in $SI[X]$. By the induction assumption $g,h\in SJ[X]$. If $g\in X$, then $g*h=2g\circ h\in SJ[X]$. Let $g=g(x_1,...,x_k)$, where $1<k<n$. Then

$$g(x_1,...,x_k)*h\underset{(5)}{=}\sum_{i=1}^{k}g(x_1,...,x_i\circ h,...,x_k)+g(x_1,...,x_k)\circ h\in SJ[X].$$

2. $SJ[X]\subseteq SI[X]$. We argue as in item 1. Let $f\in SJ[X]$ and $f=g(x_1,...,x_k)\circ h$, where $g=g(x_1,...,x_k)$, $h\in SI[X]$. Then

$$f=g\circ h\underset{(5)}{=}g*h-\sum_{i=1}^{k}g(x_1,...,x_i\circ h,...,x_k).$$

By the induction assumption, $x_i\circ h\in SI[X]$ for $i=1,...,k$. Therefore, $f\in SI[X]$. This proves the theorem.

Theorem 3. *All identities of the algebra $SI[X]$ follow from (1).*

Proof. Since the permutations $x_1=y_1^s,...,x_n=y_n^s$ are elements of $SI[Y]$, the proof of the present theorem repeats that of the Theorem 1. The theorem is proved.

4. The Dirichlet relations in the algebra $Ass^*[X]$.

As it is mentioned in the introduction (Section 1), using an exhaustive computational search, M. Bremner [1] constructed a relation of degree 5 which holds for all words of length 2 of the algebra $Ass^*[X]$ and is not a consequence of (1):

$$I(v,w,x,y,z)=$$

$$\begin{aligned}
&-z(y(x(wv))) + (xw)(z(yv)) + (yw)(z(xv)) + (yx)(z(wv)) + (zw)(y(xv)) \\
&+ (zx)(y(wv)) + (zy)(x(wv)) - (zw)((yx)v) - (zx)((yw)v) - (zy)((xw)v) \\
&+ (y(xw))(zv) + (z(xw))(yv) + (z(yw))(xv) + (z(yx))(wv) - ((xw)y)(zv) \\
&- ((xw)z)(yv) - ((yw)x)(zv) - ((yw)z)(xv) - ((yx)w)(zv) - ((yx)z)(wv) \\
&- ((zw)x)(yv) - ((zw)y)(xv) - ((zx)w)(yv) - ((zx)y)(wv) - ((zy)w)(xv) \\
&- ((zy)x)(wv) + (z(y(xw)))v - ((xw)(zy))v - ((yw)(zx))v - ((yx)(zw))v \\
&- ((y(xw))z)v - ((z(xw))y)v - ((z(yw))x)v - ((z(yx))w)v + (((xw)y)z)v \\
&+ (((xw)z)y)v + (((yw)x)z)v + (((yw)z)x)v + (((yx)w)z)v + (((yx)z)w)v \\
&+ (((zw)x)y)v + (((zw)y)x)v + (((zx)w)y)v + (((zx)y)w)v + (((zy)w)x)v \\
&+ (((zy)x)w)v = 0. \qquad (6)
\end{aligned}$$

By the Theorem 1, the relation (6) is not an identity in the algebra $Ass^*[X]$.

In particular, by the Lemma 2, $I(v,w,x,y,z)\neq 0$ for words of length 6. The relation (6) has a rather complicated structure, 46 summands, and a hardly noticeable symmetry. In [1] a natural question was brought up: if there are any other relations of type (6) in the algebra $Ass^*[X]$.

In this section we will construct an infinite series of relations which hold for all words of the algebra $Ass^*[X]$ of length $k, k\in\mathbb{N}$, which do not follow from the right-symmetric identity.

The algorithm for construction of these relations is connected with the jocular Dirichlet principle: it is impossible to place $(n+1)$ rabbits in n cages so that each cage contains only one rabbit.

Formalize the Dirichlet allocation algorithm. Let $a=x_1...x_n$ be a monomial in $Ass[X]$ and $r_1,...,r_k,\ k\le n+1$ are the variables to be allocated. We need to allocate $r_1,...,r_k$ in $n-1$ blocks $\lfloor\ \rfloor x_1\lfloor\ \rfloor x_2\lfloor\ \rfloor...\lfloor\ \rfloor x_n\lfloor\ \rfloor$ so that the blocks $\lfloor\ \rfloor$ each contain no more than one variable. We will drop the generators $x_1,...,x_n$ for short. Define the allocation operator $T(r_1,...,r_k),\ k\le n+1$ by the following rule:

$$aT(r_1)=\sum_{i=1}^{n+1}\lfloor\ \rfloor...\underset{i}{\lfloor r_1\rfloor}...\lfloor\ \rfloor,$$

....

$$aT(r_1,...,r_k)=\sum_{\sigma\in S_k}\ \sum_{1\le i_1<i_2<...<i_k\le n+1}\lfloor\ \rfloor...\lfloor r_{\sigma(1)}\rfloor...\lfloor r_{\sigma(k)}\rfloor...\lfloor\ \rfloor, \qquad (7)$$

....

$$aT(r_1,...,r_{n+1})=\sum_{\sigma\in S_{n+1}} r_{\sigma(1)}x_1r_{\sigma(2)}...x_nr_{\sigma(n+1)}.$$

Denote by R_a, $a \in Ass^*[X]$ the right multiplication operator in the algebra $Ass^*[X]$, i.e.

$$\forall b \in Ass^*[X] \quad bRa = b * a.$$

Denote the algebra of right multiplications in $Ass^*[X]$ by $R(Ass^*[X])$.

Define $D(x_1,...,x_m) \in R(Ass^*[X])$ by recursion on m:

$$D(x_1) = R_{x_1},$$

$$D(x_1,...,x_m) = D(x_1,...,x_{m-1})R_{x_m} - \sum_{i=1}^{m-1} D(x_1,...,x_i * x_m,...,x_{m-1}). \tag{8}$$

Lemma 4. *The following relation is valid in the algebra* $Ass^*[X]$*:*

$$aD(y_1,...,y_k) = aT(y_1,...,y_k), \tag{9}$$

where $1 \le k \le n+1$ *;* $a, y_1,..., y_k \in Ass^*[X]$ *and* $\deg(a) = n$.

Proof. It suffices to verify (9) in the case when $a = x_1...x_n$ is a monomial in $Ass^*[X]$. Let's proceed by induction on k. For $k = 1$ from (7) and (8) we obtain $aT(y_1) = a * y_1 = aD(y_1)$. Let this assertion is valid for $k-1$, where $1 < k-1 < n+1$. Then

$$aD(y_1,...,y_{k-1})R_{y_k} = aT(y_1,...,y_{k-1}) * y_k \underset{(7)}{=}$$

$$= (\sum_{\sigma \in S_{k-1}} \sum_{1 \le i_1 < ... < i_{k-1} \le n+1} (\lfloor\ \rfloor ... \lfloor y_{\sigma(1)} \rfloor ... \lfloor y_{\sigma(k-1)} \rfloor ... \lfloor\ \rfloor)) * y_k =$$

$$= \sum_{\sigma \in S_k} \sum_{1 \le i_1 < ... < i_k \le n+1} (\lfloor\ \rfloor ... \lfloor y_{\sigma(1)} \rfloor ... \lfloor y_{\sigma(k)} \rfloor ... \lfloor\ \rfloor) - \sum_{i=1}^{k-1} aT(y_1,..., y_i * y_k,..., y_{k-1}) \underset{(7)}{=}$$

$$= aT(y_1,...,y_k) - \sum_{i=1}^{k-1} aD(y_1,..., y_i * y_k,..., y_{k-1}).$$

Hence, $aD(y_1,...,y_k) = aT(y_1,...,y_k)$. The lemma is proved.

Theorem 4. *The following relations are valid in the algebra* $Ass^*[X]$:

$$aD(y_1,...,y_{n+1}) * y_{n+2} = \sum_{i=1}^{n+1} aD(y_1,..., y_i * y_{n+2},..., y_{n+1}) \tag{10}$$

for all $a, y_1,..., y_{n+2} \in Ass^*[X]$ *and* $\deg(a) = n$.

Proof. It suffices to verify (10) in the case when $a = x_1...x_n$ is a monomial in $Ass^*[X]$. We have

$$aD(y_1,...,y_{n+1})R_{y_{n+2}} \underset{(9)}{=} aT(y_1,...,y_{n+1}) * y_{n+2} \underset{(7)}{=}$$

$$= (\sum_{\sigma\in S_{n+1}} y_{\sigma(1)}x_1 y_{\sigma(2)}x_2 ... x_n y_{\sigma(n+1)}) * y_{n+2} =$$

$$= \sum_{i=1}^{n+1} \sum_{\sigma\in S_{n+1}} (y_{\sigma(1)}x_1 ... (y_{\sigma(i)} * y_{n+2})x_i ... x_n y_{\sigma(n+1)}) \underset{(7)}{=}$$

$$= \sum_{i=1}^{n+1} aT(y_1,...,y_i * y_{n+2},...,y_{n+1}) \underset{(9)}{=} \sum_{i=1}^{n+1} aD(y_1,...,y_i * y_{n+2},...,y_{n+1}).$$

The theorem is proved.

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