SPECIAL IDENTITY FOR NOVIKOV-JORDAN ALGEBRAS

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ABSTRACT. A commutative algebra with identity $(a\star b)\star(c\star d)-(a\star d)\star(c\star b)=(a,b,c)\star d-(a,d,c)\star b$ is called Novikov-Jordan. Example: $K[x]$ under multiplication $a\star b=\partial(ab)$ is Novikov-Jordan. Special identity for Novikov-Jordan algebras of degree 5 is constructed. Free Novikov-Jordan algebras with q generators are exceptional for any $q\geq 1$.

1. INTRODUCTION

All algebras and vector fields are considered over a field K of characteristic 0. Let $A=(A,\circ)$ be an algebra with a vector space A and a multiplication $A\times A\to A,\quad (a,b)\mapsto a\circ b$. If f be some a (non)associative polynomial with q variables, then $f=0$ is called an *identity* on A, if $f(a_1,a_2,\ldots,a_q)=0$ for any substitution $t_i:=a_i\in A, i=1,\ldots,q$. Here expressions of the form a_ia_j one understands as $a_i\circ a_j$.

An algebra A with identities $rsym=0$ and $lcom=0$, where

$$rsym(t_1,t_2,t_3)=t_1(t_2t_3)-(t_1t_2)t_3-t_1(t_3t_2)+(t_1t_3)t_2,$$

$$lcom(t_1,t_2,t_3)=t_1(t_2t_3)-t_2(t_1t_3)$$

is called (left) Novikov [1], [6], [8]. Any Novikov algebra is Lie-admissible: $A^-=(A,[\ ,\ ])$ is a Lie algebra if A is Novikov. Here $[a,b]=a\circ b-b\circ a$ is a commutator.

Example. Denote by Os_1 an algebra with a vector space $K[x]$ and the multiplication $a\circ b=\partial(a)b$, where $\partial=\frac{\partial}{\partial x}$. Then Os_1 is Novikov. Its Lie algebra is isomorphic to Witt algebra $W_1=\{e_i:[e_i,e_j]=(j-i)e_{i+j},-1\leq i,j\}$.

Algebras with identity $tortken=0$, where

$$tortken(t_1,t_2,t_3,t_4)=$$

$$-(t_1(t_2t_3))t_4+((t_1t_2)t_3)t_4+(t_1t_2)(t_3t_4)+(t_1(t_4t_3))t_2-((t_1t_4)t_3)t_2-(t_1t_4)(t_2t_3),$$

was considered in [5]. Any Novikov algebra is Tortken-admissible: $A^+=(A,\{\ ,\ \})$ satisfies Tortken identity if A is Novikov and $\{a,b\}=$

$a\circ b+b\circ a$ is an anticommutator. In terms of associators $(a,b,c)^+ = \{a,\{b,c\}\}-\{\{a,b\},c\}$ the Tortken identity looks as following

$$\{\{a,b\},\{c,d\}\}-\{\{a,d\},\{c,b\}\}=\{(a,b,c)^+,d\}-\{(a,d,c)^+,b\}.$$

Call commutative Tortken algebras *Novikov-Jordan.*

If A is associative, then A^+ satisfies the Jordan identity

$$(t_1t_1)(t_2t_1)-((t_1t_1)t_2)t_1=0.$$

Recall that Jordan algebra B is called *special* if there exists some associative algebra A such that B is isomorphic to a subalgebra of A^+. Albert has proved that any Jordan algebra with one generator is power-associative. Shirshov proved that any free Jordan algebra with two generator is special. In case of three or more generators free Jordan algebras are no longer special. Glennie has constructed special identity (s-identity for shortly) of degree 8 and has proved that any identity of degree no more than seven is special. All of these results are well known and one can find exact references and other details for example in [3] or [9].

By analogy with the correspondence between associative and Jordan algebras, call Novikov-Jordan algebra B *special*, if there exists some Novikov algebra A such that B is isomorphic to a subalgebra of A^+. In our paper we prove that any free Novikov-Jordan algebra is exceptional and we construct one s-identity of degree 5.

Let $F(q)$ be free Novikov-Jordan algebra with q generators $t_1,t_2,\ldots,t_q$. Define a polynomial *besken* of degree 5 with two variables by

$$besken(t_1,t_2)=$$

$$(((t_1t_1)t_1)t_2)t_2+(((t_1t_2)t_2)t_1)t_1+2(((t_1t_1)t_2)t_2)t_1$$
$$+2(((t_1t_2)t_1)t_1)t_2-3(((t_1t_1)t_2)t_1)t_2-3(((t_1t_2)t_1)t_2)t_1.$$

The aim of our paper is to prove that $besken=0$ is identity for any Novikov-Jordan algebra of the form A^+, where A is Novikov, but it is not identity on the free Novikov algebra $F(q)$ for any $q\geq 1$. More exactly, we establish that it has a consequence of degree 7 that is not identity on $F(1)$, and therefore $besken=0$ is not identity on any free Novikov algebra.

Theorem 1.1. *$besken=0$ is s-identity for Novikov-Jordan algebras. For any $q\geq 1$ it is not an identity for free Novikov-Jordan algebra $F(q)$.*

For any Novikov algebra A and its Novikov-Jordan algebra A^+,

$$besken(a,b)=$$

$$2(a,(a,a,b)^+,b)^+ + \{(a,\{a,a\},b)^+,b\} + \{(b,\{b,a\},a)^+,a\}.$$

Another formulation of the identity $besken(a,b)=0$:

$$(a,(a,a,b)^+,b)^+ + \{(a,\{a,a\},b)^+,b\} = -(b,(b,a,a)^+,a)^+ - \{(b,\{b,a\},a)^+,a\}.$$

Define a multilinear commutative polynomial $besken'$ of degree 5 by

$$besken'(t_1,t_2,t_3,t_4,t_5) =$$

$$-t_4t_1t_3t_5t_2 + t_4t_2t_3t_5t_1 + t_5t_1t_2t_3t_4 - t_5t_1t_3t_4t_2 - t_5t_2t_1t_3t_4 + t_5t_2t_3t_4t_1$$
$$-2t_3t_1t_4t_2t_5 + 2t_3t_1t_4t_5t_2 + 2t_3t_2t_4t_1t_5 - 2t_3t_2t_4t_5t_1 + 2t_4t_1t_5t_2t_3 - 2t_4t_2t_5t_1t_3$$
$$+3t_4t_1t_2t_3t_5 - 3t_4t_2t_1t_3t_5 + 4t_4t_2t_1t_5t_3 - 4t_4t_1t_2t_5t_3.$$

Here we use left-normed bracketing. For instance, $t_1t_2t_3t_4t_5$ means $(((t_1t_2)t_3)t_4)t_5$. One can establish that $besken'=0$ is also s-identity of Novikov-Jordan algebras and it is equivalent (up to the identities $tortken=0$ and $com=0$) to the identity $besken=0$. For example,

$$besken'(t_2,t_1,t_1,t_1,t_2) = besken(t_1,t_2).$$

2. Basis of free Novikov algebra

In [4] are given constructions of a basis of free Novikov algebra in terms of r-elements and in terms of rooted trees. In this section we give an algorithm how to construct such a basis in terms of Young diagrams.

Recall that Young diagram is a set of boxes (we denote them by bullets) with non-increasing numbers of boxes in each row. Rows and columns are numerated from top to bottom and from left to right. Let n be the number of rows and r_i be the number of boxes in i-th row. The total number of boxes, $r_1+\cdots+r_n$, is called order of Young diagram.

To construct Novikov diagram, we need to complement Young diagram by "a nose". Namely, we need to add the first row by one more box.

$$\begin{array}{cccccc} \bullet & \cdots & \bullet & \bullet & \bullet & \\ \bullet & \cdots & \bullet & \bullet & & \\ \vdots & \cdots & \vdots & \vdots & & \\ \bullet & \cdots & \bullet & & & \end{array} \quad \mapsto \quad \begin{array}{cccccc} \bullet & \cdots & \bullet & \bullet & \bullet & \circ \\ \bullet & \cdots & \bullet & \bullet & & \\ \vdots & \cdots & \vdots & \vdots & & \\ \bullet & \cdots & \bullet & & & \end{array}$$

The number of boxes in Novikov diagram is called its order. So, difference between orders of Novikov diagram and corresponding Young diagram is equal to 1.

Let us given an alphabet (ordered set) Ω. To construct Novikov tableau on Ω we need to feel Novikov diagrams by elements of Ω. Denote by $a_{i,j}$ an element of Ω in the box (i,j), that is the cross of i-th row by j-th column. The feeling rule is the following

- $a_{i,1} \geq a_{i+1,1}$, if $r_i = r_{i+1}, i = 1, 2, \ldots, n-1$.
- the sequence $a_{n,2} \cdots a_{n,r_n} a_{n-1,2} \cdots a_{n-1,r_{n-1}} \cdots a_{1,2} \cdots a_{1,r_1+1}$ is non-decreasing.

In particular, all boxes beginning from the second place in each row are labeled by non-decreasing elements of alphabet. Denote by R_n a set of Novikov tableaux labeled by Ω of order n.

Let $F(\Omega)$ be free Novikov algebra generated by Ω. Let $F_n(\Omega)$ be its subspace generated by basic elements of degree n. Correspond to any Novikov tableaux

$$\begin{array}{cccccc}
a_{1,1} & \cdots & \cdots & a_{1,r_1-1} & a_{1,r_1} & a_{1,r_1+1} \\
a_{2,1} & \cdots & a_{2,r_2-1} & a_{2,r_2} & & \\
\vdots & \cdots & \vdots & \vdots & & \\
a_{n,1} & \cdots & a_{n,r_n} & & &
\end{array}$$

an element

$$X = X_n \circ (X_{n-1} \circ (\cdots \circ (X_2 \circ X_1) \cdots)),$$

(right-normed bracketing) where

$$X_i = (\cdots((a_{i,1} \circ a_{i,2}) \circ a_{i,3}) \cdots \circ a_{i,r_i-1}) \circ a_{i,r_i+\delta_{i,1}}$$

(left-normed bracketing). All basic elements of free Novikov algebra $F(\Omega)$ are obtained by this way. In particular, $\dim F_n(\Omega) = |R_n|$.

As an example let us construct basic elements of degree 5 of free Novikov algebra that have three a's and two b's. These basic elements will be used in the proof of lemma 3.1.

Young diagrams of order 4:

$$\begin{array}{l} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \qquad \begin{array}{l} \bullet\ \bullet \\ \bullet \\ \bullet \end{array} \qquad \begin{array}{l} \bullet\ \bullet\ \bullet \\ \bullet \end{array} \qquad \begin{array}{l} \bullet\ \bullet \\ \bullet\ \bullet \end{array} \qquad \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet \end{array}$$

Novikov diagrams of order 5:

$$\begin{array}{l} \bullet\ \circ \\ \bullet \\ \bullet \\ \bullet \end{array} \qquad \begin{array}{l} \bullet\ \bullet\ \circ \\ \bullet \\ \bullet \end{array} \qquad \begin{array}{l} \bullet\ \bullet\ \bullet\ \circ \\ \bullet \end{array} \qquad \begin{array}{l} \bullet\ \bullet\ \circ \\ \bullet\ \bullet \end{array} \qquad \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ \circ \end{array}$$

Novikov tableaux of order 5 of type $aaabb$:

$$E_1 = \begin{array}{ll} b & b \\ a & \\ a & \\ a & \end{array}, \quad E_2 = \begin{array}{ll} b & a \\ b & \\ a & \\ a & \end{array},$$

$$E_3 = \begin{array}{lll} a & b & b \\ a & & \\ a & & \end{array}, \quad E_4 = \begin{array}{lll} b & a & b \\ a & & \\ a & & \end{array}, \quad E_5 = \begin{array}{lll} a & a & b \\ b & & \\ a & & \end{array},$$

$$E_6 = \begin{array}{lll} b & a & a \\ b & & \\ a & & \end{array}, \quad E_7 = \begin{array}{lll} a & a & a \\ b & & \\ b & & \end{array},$$

$$E_8 = \begin{array}{llll} a & a & b & b \\ a & & & \end{array}, \quad E_9 = \begin{array}{llll} b & a & a & b \\ a & & & \end{array}, \quad E_{10} = \begin{array}{llll} a & a & a & b \\ b & & & \end{array}, \quad E_{11} = \begin{array}{llll} b & a & a & a \\ b & & & \end{array},$$

$$E_{12} = \begin{array}{lll} a & b & b \\ a & a & \end{array}, \quad E_{13} = \begin{array}{lll} b & a & b \\ a & a & \end{array}, \quad E_{14} = \begin{array}{lll} b & a & a \\ b & a & \end{array},$$

$$E_{15} = \begin{array}{lllll} a & a & a & b & b \end{array}, \quad E_{16} = \begin{array}{lllll} b & a & a & a & b \end{array}.$$

To these Novikov tableaux correspond the following elements

$$e_1 = a(a(a(bb))), \quad e_2 = a(a(b(ba))),$$
$$e_3 = a(a((ab)b)), \quad e_4 = a(a((ba)b)), \quad e_5 = a(b((aa)b)),$$
$$e_6 = a(b((ba)a))), \quad e_7 = b(b((aa)a)),$$
$$e_8 = a(((aa)b)b), \quad e_9 = a(((ba)a)b), \quad e_{10} = b(((aa)a)b), \quad e_{11} = b(((ba)a)a)),$$
$$e_{12} = (aa)((ab)b), \quad e_{13} = (aa)((ba)b), \quad e_{14} = (ba)((ba)a),$$
$$e_{15} = (((aa)a)b)b, \quad e_{16} = (((ba)a)a)b.$$

Then the set $\{e_i = e_i(a,b) : i = 1, \dots, 16\}$ consists of basis of free Novikov algebra in degree 5 of type $aaabb$.

3. Identities of degree 5

Lemma 3.1. *Any identity of Novikov algebra Os_1 of type aaabb follows from identities $com = 0$ and $tortken = 0$.*

Proof. As we have shown in section 2 there are 16 basic elements $e_i = e_i(a,b), i = 1,\dots,16,$ of Novikov algebra of type *aaabb.*

Suppose that

$$\sum_{i=1}^{16} \lambda_i e_i(a,b) = 0, \quad \lambda_i \in K,$$

for any $a, b \in Os_1$.

Let M be a set of 5-typles $(i_1, i_2, i_3, i_4, i_5)$ with conditions $0 \leq i_1, i_2, i_3, i_4, i_5 \leq 4, i_1 \leq i_2 \leq i_3, i_4 \leq i_5, i_1 + i_2 + i_3 + i_4 + i_5 = 4$. Then M has the following 16 elements

$$(0,0,0,0,4),(0,0,0,1,3),(0,0,0,2,2),(0,0,1,0,3),$$
$$(0,0,1,1,2),(0,0,2,0,2),(0,0,2,1,1),(0,0,3,0,1),$$
$$(0,0,4,0,0),(0,1,1,0,2),(0,1,1,1,1),(0,1,2,0,1),$$
$$(0,1,3,0,0),(0,2,2,0,0),(1,1,1,0,1),(1,1,2,0,0).$$

Notice that any basic element $e_i(a,b)$ is a linear combination of monoms of the form $\partial^{i_1}(a)\partial^{i_2}(a)\partial^{i_3}(a)\partial^{i_4}(b)\partial^{i_5}(b)$, where $(i_1, i_2, i_3, i_4, i_5) \in M$. Set $a' = \partial(a), a'' = \partial^2(a)$, etc. Thus, $\sum_{i=1}^{16} \lambda_i e_i$ is a linear combination of the following monoms

$$a^3\, b\, b^{(4)}, a^3\, b'\, b^{(3)}, a^3\, b''^2, a^2\, a'\, b\, b^{(3)}, a^2\, a'\, b'\, b'', a^2\, a''\, b\, b'', a^2\, a''\, b'^2, a^2\, a^{(3)}\, b\, b',$$
$$a^2\, a^{(4)}\, b^2, a\, a'^2\, b\, b'', a\, a'^2\, b'^2, a\, a'\, a''\, b\, b', a\, a'\, a^{(3)}\, b^2, a\, a''^2\, b^2, b\, b'\, a'^3, a'^2\, a''\, b^2.$$

More exactly,

$$\sum_{i=1}^{16} \lambda_i e_i(a,b) =$$

$$(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_8 + \lambda_9 + \lambda_{10} + \lambda_{12} + \lambda_{13} + \lambda_{15} + \lambda_{16})\, a'^3 bb'$$
$$+ (\lambda_2 + \lambda_6 + \lambda_7 + \lambda_{11} + \lambda_{14})\, aa'^2 b'^2 + (\lambda_3 + 3\lambda_8 + \lambda_{12} + 7\lambda_{15})\, a'^2 a'' b^2$$
$$+ (\lambda_5 + \lambda_8 + \lambda_9 + 4\lambda_{10} + \lambda_{12} + \lambda_{13} + 4\lambda_{15} + 4\lambda_{16})\, aa'bb'a'' + (\lambda_7 + \lambda_{11})\, a^2 b'^2 a''$$
$$+ (\lambda_{12} + 4\lambda_{15})\, aa''^2 b^2 + (\lambda_4 + 3\lambda_9 + \lambda_{13} + 7\lambda_{16})\, aa'^2 bb''$$
$$+ (\lambda_6 + 3\lambda_{11} + 2\lambda_{14})\, a^2 a' b' b'' + (\lambda_{13} + 4\lambda_{16})\, a^2 a'' bb''$$
$$+ \lambda_{14} a^3 b''^2 + (\lambda_8 + 6\lambda_{15})\, aa' a^{(3)} b^2 + (\lambda_{10} + \lambda_{15} + \lambda_{16})\, a^2 a^{(3)} bb'$$
$$+ (\lambda_9 + 6\lambda_{16})\, a^2 a' bb^{(3)} + \lambda_{11} a^3 b' b^{(3)} + \lambda_{15} a^2 a^{(4)} b^2 + \lambda_{16} a^3 bb^{(4)}.$$

So, we obtain the following system of linear equations

$$\lambda_{11} = 0, \lambda_7 + \lambda_{11} = 0, \lambda_{14} = 0, \lambda_2 + \lambda_6 + \lambda_7 + \lambda_{11} + \lambda_{14} = 0,$$

$$\lambda_6+3\lambda_{11}+2\lambda_{14}=0, \lambda_{15}=0, \lambda_{12}+4\lambda_{15}=0, \lambda_8+6\lambda_{15}=0,$$
$$\lambda_3+3\lambda_8+\lambda_{12}+7\lambda_{15}=0, \lambda_{16}=0, \lambda_{10}+\lambda_{15}+\lambda_{16}=0,$$
$$\lambda_1+\lambda_3+\lambda_4+\lambda_5+\lambda_8+\lambda_9+\lambda_{10}+\lambda_{12}+\lambda_{13}+\lambda_{15}+\lambda_{16}=0, \lambda_{13}+4\lambda_{16}=0,$$
$$\lambda_5+\lambda_8+\lambda_9+4\lambda_{10}+\lambda_{12}+\lambda_{13}+4\lambda_{15}+4\lambda_{16}=0, \lambda_9+6\lambda_{16}=0,$$
$$\lambda_4+3\lambda_9+\lambda_{13}+7\lambda_{16}=0.$$

It is easy to see that this system has trivial solution: $\lambda_i=0$, for all $i=1,\ldots,16$. This means that any identity of Novikov algebra of type *aaabb* follows from commutativity and Tortken identities.

Lemma 3.2. *For any* $a,b\in Os_1$,

$$2(a,(a,a,b)^+,b)^+ + \{(a,\{a,a\},b)^+,b\} + \{(b,\{b,a\},a)^+,a\}=0.$$

Proof. Recall that $\{a,b\}=(ab)'$ and

$$(a,b,c)^+ = \{a,\{b,c\}\}-\{\{a,b\},c\} = \{b,[c,a]\} = (abc'-a'bc)'.$$

Here and below we use notations $a'=\partial(a), a''=\partial^2(a), a^{(3)}=\partial^3(a)$, etc.

Therefore,

$$(a,a,b)^+ =$$
$$-b\,a'^2 + a\,a'\,b' - a\,b\,a'' + a^2\,b'',$$

and

$$(a,(a,a,b)^+,b)^+ =$$
$$(b^2\,a'^3 - 2\,a\,b\,a'^2\,b' + a^2\,a'\,b'^2 + a\,b^2\,a'\,a'' - a^2\,b\,b'\,a'' - a^2\,b\,a'\,b'' + a^3\,b'\,b'')'.$$

Further,

$$(a,\{a,a\},b)^+ =$$
$$-2\,b\,a'^3 + 2\,a\,a'^2\,b' + 2\,a^2\,b'\,a'' + 2\,a^2\,a'\,b'' - 4\,a\,b\,a'\,a'',$$

and

$$\{(a,\{a,a\},b)^+,b\} =$$
$$(-2\,b^2\,a'^3 + 2\,a\,b\,a'^2\,b' - 4\,a\,b^2\,a'\,a'' + 2\,a^2\,b\,b'\,a'' + 2\,a^2\,b\,a'\,b'')'.$$

Similarly,

$$(b,\{b,a\},a)^+ =$$
$$2\,b\,a'^2\,b' - 2\,a\,a'\,b'^2 + 2\,b^2\,a'\,a'' - 2\,a^2\,b'\,b'',$$

and

$$\{(b,\{b,a\},a)^+,a\} =$$
$$(2\,a\,b\,a'^2\,b' - 2\,a^2\,a'\,b'^2 + 2\,a\,b^2\,a'\,a'' - 2\,a^3\,b'\,b'')'.$$

We see that

$$2(a,(a,a,b)^+,b)^+ + \{(a,\{a,a\},b)^+,b\} + \{(b,\{b,a\},a)^+,a\}=0.$$

Lemma 3.3. *Let A be any Novikov algebra. Then for any $a, b \in A$,*

$$2(a,(a,a,b)^+,b)^+ + \{(a,\{a,a\},b)^+,b\} + \{(b,\{b,a\},a)^+,a\} = 0.$$

In particular $besken = 0$ is identity for any Novikov algebra. In other words, it is an identity for any special Novikov-Jordan algebra.

Proof. Follows from lemma 3.1, 3.2.

Corollary 3.4. *Define the polynomial $jetken$ of degree 7 with one variable by the following rule*

$$jetken(t) = -((((tt)t)(tt))t)t - (((tt)t)(tt))(tt)$$
$$-2(((tt)(tt))(tt))t - 2((((tt)t)t)t)(tt) + 3((((tt)t)t)(tt))t + 3(((tt)(tt))t)(tt).$$

Then $jetken = 0$ is an identity for any special Novikov-Jordan algebra.

Proof. Notice that $jetken(t) = besken(t, tt)$.

4. Free basis of Novikov-Jordan algebra of degree ≤ 7

In this section we construct a basis of free Novikov-Jordan algebra with one generator of degree no more than 7.

Denote by $\{\ ,\ \}$ a multiplication in free commutative algebra. Free commutative algebra with one generator a has the basic with $1, 1, 1, 2, 3, 6, 11$ elements correspondingly until degree 7. Namely, one can get the following basic elements

$$a,$$

$$\{a,a\},$$

$$\{\{a,a\},a\},$$

$$\{\{\{a,a\},a\},a\}, \quad \{\{a,a\},\{a,a\}\},$$

$$\{\{\{\{a,a\},a\},a\},a\}, \ \{\{\{a,a\},\{a,a\}\},a\}, \ \{\{\{a,a\},a\},\{a,a\}\},$$

$$\{\{\{\{\{a,a\},a\},a\},a\},a\}, \ \{\{\{\{a,a\},\{a,a\}\},a\},a\}, \ \{\{\{\{a,a\},a\},\{a,a\}\},a\},$$
$$\{\{\{\{a,a\},a\},a\},\{a,a\}\}, \quad \{\{\{a,a\},\{a,a\}\},\{a,a\}\}, \quad \{\{\{a,a\},a\},\{\{a,a\},a\}\},$$

$$\{\{\{\{\{\{a,a\},a\},a\},a\},a\},a\}, \quad \{\{\{\{\{a,a\},\{a,a\}\},a\},a\},a\},$$
$$\{\{\{\{\{a,a\},a\},\{a,a\}\},a\},a\}, \quad \{\{\{\{\{a,a\},a\},a\},\{a,a\}\},a\},$$
$$\{\{\{\{a,a\},\{a,a\}\},\{a,a\}\},a\}, \quad \{\{\{\{a,a\},a\},\{\{a,a\},a\}\},a\},$$

$$\{\{\{\{\{a,a\},a\},a\},a\},\{a,a\}\},\quad \{\{\{\{a,a\},\{a,a\}\},a\},\{a,a\}\},$$
$$\{\{\{\{a,a\},a\},\{a,a\}\},\{a,a\}\},\{\{\{\{\{a,a\},a\},a\},\{\{a,a\},a\}\},$$
$$\{\{\{a,a\},\{a,a\}\},\{\{a,a\},a\}\}.$$

Suppose now that the multiplication $\{\ ,\ \}$ is not only commutative, but also Tortken. The condition $tortken(t_1,t_2,t_3,t_4)=0$ is trivial if at least 3 parameters among t_1,t_2,t_3,t_4 are equal. Therefore for commutative Tortken algebra with 1 generator a, the Tortken identity gives nontrivial conditions if its degree is at least 6. We have 1 condition in degree 6

$$tortken(a,a,\{a,a\},\{a,a\})=0$$

and 3 conditions in degree 7:

$$tortken(a,a,\{a,a\},\{\{a,a\},a\})=0,$$
$$tortken(a,a,\{\{a,a\},a\},\{a,a\})=0,$$

and the consequence of the condition in degree 6,

$$\{a,tortken(a,a,\{a,a\},\{a,a\})\}=0$$

For example, $\{\{\{a,a\},a\},\{\{a,a\},a\}\}$ is a linear combination of other elements of degree 5.

We see that basic for free Novikov-Jordan algebra with one generator a until degree 7 has 1,1,1,2,3,5,7 elements correspondingly. One can choose basic elements of free commutative Tortken algebra just the same as in commutative case until degree 5 and the following 5 elements in degree 6

$$\{\{\{\{\{a,a\},a\},a\},a\},a\},\quad \{\{\{\{a,a\},\{a,a\}\},a\},a\},$$
$$\{\{\{\{a,a\},a\},\{a,a\}\},a\},\quad \{\{\{\{a,a\},a\},a\},\{a,a\}\},$$
$$\{\{\{a,a\},\{a,a\}\},\{a,a\}\},$$

and the following 8 elements in degree 7

$$X_1=\{\{\{\{a,a\},a\},\{a,a\}\},\{a,a\}\},$$
$$X_2=\{\{\{\{a,a\},\{a,a\}\},a\},\{a,a\}\},$$
$$X_3=\{\{\{\{\{a,a\},a\},a\},a\},\{a,a\}\},$$
$$X_4=\{\{\{\{a,a\},\{a,a\}\},\{a,a\}\},a\},$$
$$X_5=\{\{\{\{\{a,a\},a\},a\},\{a,a\}\},a\},$$
$$X_6=\{\{\{\{\{a,a\},a\},\{a,a\}\},a\},a\},$$
$$X_7=\{\{\{\{\{a,a\},\{a,a\}\},a\},a\},a\},$$
$$X_8=\{\{\{\{\{\{a,a\},a\},a\},a\},a\},a\}.$$

Proof of theorem 1.1 We see that

$$jetken(a) =$$
$$-\{\{\{\{a,a\},a\},\{a,a\}\},a\},a\} - \{\{\{\{a,a\},a\},\{a,a\}\},\{a,a\}\}$$
$$-2\{\{\{\{a,a\},\{a,a\}\},\{a,a\}\},a\} - 2\{\{\{\{\{a,a\},a\},a\},a\},\{a,a\}\}$$
$$+3\{\{\{\{\{a,a\},a\},a\},\{a,a\}\},a\} + 3\{\{\{\{a,a\},\{a,a\}\},a\},\{a,a\}\}.$$

By corollary 3.4 $jetken = 0$ is an identity for any special Novikov-Jordan algebra. Thus, the condition $jetken(a) = 0$ gives us a linear dependence condition for the basic elements $X_i, i = 1, 2, \dots, 7,$ of free Novikov-Jordan algebra with one generator. So, $jetken = 0$ is not an identity for free Novikov algebra with one generator. Therefore, $besken = 0$ is not identity for free Novikov algebra with q generators for any $q > 0$.

Remark. It will be interesting to find direct deduction of the identity $besken = 0$ for special Novikov-Jordan algebras from identities $com = 0$ and $besken = 0$ that does not depend from characteristic $p = char\,K$. One can check that the algebra $(K[x], \Box_k)$ constructed in [5],

$$a\Box_k b = \partial(\partial^{(p^k-1}(a)\partial^{p^k-1}(b)),$$

does not satisfy the identity $besken = 0$.

Question 1. There exists an exceptional simple Jordan algebra. It is the algebra of 3×3 hermitian matrices with entries in the Octonian algebra. Is there exists a simple exceptional Novikov-Jordan algebra?

Question 2. Is it true that all identities of Novikov-Jordan algebra Os_1^+ follows from identities $com = 0$, $tortken = 0$ and $besken = 0$?

We have checked that all identities of degree ≤ 5 of the algebra Os_1^+ follows from these three identities. We have checked also that any identity of Novikov algebra of degree ≤ 7 follows from identities $com = 0$ and $tortken = 0$.

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