Right ideals in non–associative universal enveloping algebras of Lie triple systems

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Abstract

We prove that the only proper right ideal of the universal enveloping algebra of a finite–dimensional central simple Lie triple system over a field of characteristic zero is its augmentation ideal.

1 Introduction

In this paper we will assume that the base field is algebraically closed of characteristic zero.

Any Lie algebra with the trilinear product $[x, y, z] = [[x, y], z]$ can be considered as a Lie triple system. A Lie triple system (abbreviated as L.t.s.) is a vector space T equipped with a trilinear product $[x, y, z]$ satisfying the followings identities:

$$[x, x, y] = 0, \tag{1}$$

$$[x, y, z] + [y, z, x] + [z, x, y] = 0, \tag{2}$$

$$[a, b[x, y, z]] = [[a, b, x], y, z] + [x, [a, b, y], z] + [x, y, [a, b, z]]. \tag{3}$$

The first and the second identities are reminiscent of Lie algebras. The third identity says that the maps $D_{a,b} \colon x \mapsto [a, b, x]$ are derivations of T. These maps are called inner derivations and they form the Lie algebra of inner derivations $\mathrm{InnDer}(T)$. Roughly, Lie triple systems may be thought of as subspaces of Lie algebras stable under the product $[[x, y], z]$. This is due to the fact that for any Lie triple system T the vector space $L(T) = \mathrm{InnDer}(T) \oplus T$ is a Lie algebra with the product defined by $[a, b] = D_{a,b}$ and $[D_{a,b}, c] = [[a, b], c] = [a, b, c]$ for any $a, b, c \in T$, and by the condition that $\mathrm{InnDer}(T)$ a subalgebra. The map $\sigma \colon D_{a,b} + c \mapsto D_{a,b} - c$ is an automorphism of $L(T)$ with $\sigma^2 = \mathrm{Id}$. Thus, Lie triple systems are often considered as the negative eigenspaces of involutions of Lie algebras. The classification of these involutions has led to the classification

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of semisimple L.t.s. by Lister [2]. Later, Faulkner obtained Lister's classification of simple L.t.s. by means of Dynkin diagrams [1].

Given any (unital) nonassociative algebra A the generalized left alternative nucleus

$$\mathrm{LN}_{\mathrm{alt}}(A) = \{a \in A \mid (a,x,y) = -(x,a,y)\,\forall x,y \in A\},$$

where $(x,y,z) = (xy)z - x(yz)$, is a L.t.s. with the product $[a,b,c] = a(bc) - b(ac) - c(ab) + c(ba)$. In analogy with the usual universal enveloping algebras of Lie algebras, for any L.t.s. T there exists a unital algebra $U(T)$ and a monomorphism of L.t.s. $\iota\colon T \to \mathrm{LN}_{\mathrm{alt}}(U(T))$ with the additional property $\iota(a)\iota(b) - \iota(b)\iota(a) = 0$ for any $a,b \in T$. The pair $(U(T),\iota)$ is universal with respect to homomorphisms of L.t.s. $T \to \mathrm{LN}_{\mathrm{alt}}(A)$ with this property. To simplify the notation we will write a instead of $\iota(a)$. Thus,

$$[a,b,c] = a(bc) - b(ac) = a(bc) - (ab)c + (ba)c - b(ac) = -2(a,b,c)$$

in $U(T)$.

The universal enveloping algebra $U(T)$ is a nonassociative Hopf algebra or H–bialgebra (see [4, 5]) in the sense that it is a nonassociative bialgebra with a left and a right division $x\backslash y$ and x/y defined by

$$\sum x_{(1)}\backslash x_{(2)}y = \epsilon(x)y = \sum x_{(1)} \cdot x_{(2)}\backslash y,$$
$$\sum yx_{(1)}/x_{(2)} = \epsilon(x)y = \sum y/x_{(1)} \cdot x_{(2)},$$

where ϵ denotes the counit and $\sum x_{(1)} \otimes x_{(2)}$ stands for the image of x under the comultiplication [6]. $U(T)$ is a coassociative, cocommutative bialgebra, the subspace of primitive elements being T, and it admits a Poincaré–Birkhoff–Witt type basis. Associativity is, however, replaced by the weaker identity

$$\sum x_{(1)}(y \cdot x_{(2)}z) = \sum x_{(1)}(yx_{(2)}) \cdot z.$$

The graded algebra $\mathrm{Gr}(T)$ associated to the coradical filtration

$$U(T) = \bigcup_{n=0}^{\infty} U(T)_n,$$

where $U(T)_n$ is the linear span of all possible products of at most n elements in T, is isomorphic to the symmetric algebra $S(T)$ on T. A map similar to the antipode of universal enveloping algebras of Lie algebras is also available since the automorphism $a \mapsto -a$ of any L.t.s. T induces an automorphism $x \mapsto S(x)$ of $U(T)$ of order 2. The left and right division are written as

$$x\backslash y = S(x)y \quad \text{and} \quad y/x = \sum S(x_{(3)}) \cdot (x_{(1)}y)S(x_{(2)}).$$

For any non-trivial L.t.s. T its universal enveloping algebra $U(T)$ is infinite–dimensional. For finite–dimensional unital algebras A the existence of embeddings $\iota\colon T \to \mathrm{LN}_{\mathrm{alt}}(A)$ with the property $\iota(a)\iota(b) - \iota(b)\iota(a) = 0\,\forall a,b \in T$, (Ado's

Theorem) was studied in [3]: it turns out that only finite–dimensional nilpotent L.t.s. may admit such embeddings. The lack of such embeddings is equivalent to the non-existence of ideals of finite codimension in $U(T)$ with trivial intersection with T. This result motivated in [3] the following conjecture, verified in several cases by direct calculations in Poincaré–Birkhoff–Witt bases:

Conjecture. *The only proper ideal of the universal enveloping algebra of a simple Lie triple system is its augmentation ideal.*

The goal of this paper is to prove this conjecture by establishing the following stronger result:

Theorem 1.1. *The only proper right ideal of the universal enveloping algebra of a finite–dimensional simple Lie triple system over a field of characteristic zero is its augmentation ideal.*

The behavior of the left ideals is, however, rather different. The Harish-Chandra isomorphism for symmetric spaces, recast in our setting, implies that the right associative nucleus of $U(T)$, each of whose elements determines a left ideal, is isomorphic as an algebra to the algebra of invariants of the restricted Weyl group of the symmetric Lie algebra $(L(T), \sigma)$. By the Theorem of Kostant and Rallis on separation of variables for isotropy representations, $U(T)$ is a free right module for the right associative nucleus. This subject will be discussed in detail elsewhere.

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2 The proof

In the course of the proof we use some results whose proofs are postponed to later sections for clarity.

Recall that for any $c \in T$ the subalgebra generated by c is associative, so c^n is a well defined element of $U(T)$. In fact, $(c^i, c^j, x) = 0$ for any $x \in U(T)$ [3].

Lemma 2.1. *For any $n \geq 0$ and $a, b, c \in T$, we have*

$$(c^n, a, b) \equiv nc^{n-1}(c, a, b) \pmod{U(T)_{n-2}}.$$

The meaning of this lemma is better understood by defining $R_{a,b} \colon T \to T$ $c \mapsto [c, a, b]$ or its extension to the whole $U(T)$

$$R_{a,b}(x) = -2(x, a, b).$$

These maps preserve the filtration of $U(T)$, so they induce corresponding maps $R^{\mathrm{gr}}_{a,b}$ of zero degree on the graded algebra $\mathrm{Gr}(U(T)) = \bigoplus_{n=0}^{\infty} U(T)_n/U(T)_{n-1}$, which is isomorphic to $S(T)$. Since the powers $\{c^n \mid c \in T, n \geq 0\}$ span $S(T)$, Lemma 2.1 says that $R^{\rm gr}_{a,b}$ is the derivation on $S(T)$ induced by the map $c \mapsto [c,a,b]$ on T.

The unifying feature of simple L.t.s. that we use in our proof is recorded in the following theorem:

Theorem 2.2. *Let T be a finite–dimensional simple L.t.s. over a field of characteristic zero. Then* $\mathrm{End}(T)$ *is the Lie algebra generated by the maps $R_{a,b}\colon c \mapsto [c,a,b]$ $(a,b \in T)$.*

Any finite–dimensional simple L.t.s. contains a two–dimensional subsystem $S_2 = \mathrm{span}\langle e,f\rangle$ with the product given by

$$[e,f,e] = 2e \quad \text{and} \quad [e,f,f] = -2f.$$

We can reduce the proof of Theorem 1.1 to the case $T = S_2$ as follows. Any nonzero right ideal I of $U(T)$ is filtered by $I_n = I \cap U(T)_n$ so it gives rise to a (two–sided) graded ideal

$$I^{\rm gr} = \bigoplus_{n=0}^{\infty} (I_n + U(T)_{n-1})/U(T)_{n-1} \cong \bigoplus_{n=0}^{\infty} I_n/I_{n-1}$$

that is stable under the derivations $R^{gr}_{a,b}$ $\forall a,b \in T$. By Theorem 2.2, the action of these operators on $\mathrm{Gr}(T)$ can be identified with the natural action of $\mathrm{End}(T)$ on $S(T)$. This module decomposes as direct sum of irreducible modules $S(T) \cong \oplus_{n=0}^{\infty} V(n\lambda_1)$ where $V(n\lambda_1)$ corresponds to the homogeneous polynomials of degree n. Since I^{gr} is a submodule as well as a graded ideal then there must exist N such that

$$I^{gr} = \bigoplus_{n=N}^{\infty} U(T)_n/U(T)_{n-1}.$$

In terms of I we get the decomposition

$$U(T) = U(T)_{N-1} \oplus I \tag{4}$$

which shows that the codimension of I is finite. Therefore, given a subsystem $S_2 = \mathrm{span}\langle e,f\rangle \subseteq T$, the right ideal $I \cap U(S_2) \subseteq U(S_2) \subseteq U(T)$ has finite codimension in $U(S_2)$. Assuming the truth of Theorem 1.1 in the case of S_2, we get that $I \cap T \neq 0$. Since T is simple this intersection generates all T under the action of the operators $R_{a,b}$. Thus, I is either the augmentation ideal or the whole $U(T)$ as desired.

3 Proof of Lemma 2.1

The main tool in computing products in $U(T)$ is the operator identity

$$L_{ax+xa} = L_a L_x + L_x L_a \tag{5}$$

for all $a \in T$ and $x \in U(T)$ [3]. To use it we first observe that

$$\begin{aligned}
ac^n &= ac \cdot c^{n-1} - (a, c, c^{n-1}) = ca \cdot c^{n-1} - (a, c, c^{n-1}) \\
&= c \cdot ac^{n-1} + (c, a, c^{n-1}) - (a, c, c^{n-1}) = c \cdot ac^{n-1} - 2(a, c, c^{n-1}) \\
&= -2(a, c, c^{n-1}) - 2c \cdot (a, c, c^{n-2}) - \cdots - 2c^{n-2}(a, c, c) + c^n a,
\end{aligned}$$

so

$$c^n a = \frac{1}{2}(ac^n + c^n a) + \sum_{i=0}^{n-2} c^i (a, c, c^{n-1-i}).$$

The associator (c^n, a, b) is then written as

$$(c^n, a, b) = \frac{1}{2} L_{ac^n + c^n a}(b) + \sum_{i=0}^{n-2} L_{c^i (a,c,c^{n-1-i})}(b) - L_{c^n} L_a(b)$$

that by (5) gives

$$\begin{aligned}
(c^n, a, b) &= \frac{1}{2}[L_a, L_{c^n}](b) + \sum_{i=0}^{n-2} L_{c^i (a,c,c^{n-1-i})}(b) \\
&= \frac{1}{2} \sum_{i=0}^{n-1} L_c^i [L_a, L_c] L_c^{n-1-i}(b) + \sum_{i=0}^{n-2} L_{c^i (a,c,c^{n-1-i})}(b)
\end{aligned}$$

where we have used that $L_{c^n} = L_c^n$. Since $D_{a,b} = [L_a, L_b]$ is a derivation of $U(T)$ [3] we get

$$\begin{aligned}
(c^n, a, b) &= \frac{1}{2} \sum_{i=0}^{n-1} c^i D_{a,c}(c^{n-1-i} b) + \sum_{i=0}^{n-2} c^i (a, c, c^{n-1-i}) \cdot b \\
&= \frac{1}{2} \left(\sum_{i=0}^{n-1} c^i \cdot D_{a,c}(c^{n-1-i}) b + c^i \cdot c^{n-1-i} [a, c, b] \right) \\
&\quad + \sum_{i=0}^{n-2} c^i (a, c, c^{n-1-i}) \cdot b \\
&= \frac{n}{2} c^{n-1} [a, c, b] + \frac{1}{2} \sum_{i=0}^{n-2} c^i \cdot D_{a,c}(d^{n-1-i}) b \\
&\quad - \frac{1}{2} \sum_{i=0}^{n-2} c^i D_{a,c}(c^{n-1-i}) \cdot b \\
&= \frac{n}{2} c^{n-1} [a, c, b] - \frac{1}{2} \sum_{i=0}^{n-2} (c^i, D_{a,c}(c^{n-1-i}), b). \qquad (6)
\end{aligned}$$

The derivation $D_{a,c}$ preserves the filtration of $U(T)$ so $(c^i, D_{a,c}(c^{n-1-i}), b) \in U(T)_{n-2}$ and $(c^n, a, b) \equiv \frac{n}{2} c^{n-1}[a, c, b] = -nc^{n-1}(a, c, b) = nc^{n-1}(c, a, b)$ modulo $U(T)_{n-2}$.

4 The two–dimensional case

The case $T = S_2$ is simple and illustrative because we can perform computations in the Poincaré–Birkhoff–Witt basis $\{e^i f^j \mid i, j \geq 0\}$. In this case Theorem 2.2 encodes the following correspondence between maps and coordinate matrices in the basis $\{e, f\}$:

$$R_{e,e} \equiv \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, R_{e,f} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, R_{f,e} \equiv \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } R_{f,f} \equiv \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}.$$

Fix I a nonzero right ideal of $U(S_2)$. By (4), I contains an element of the form $e^N + \alpha_1 e^{N-1} + \cdots + \alpha_{N-1} e + \alpha_N$. Moreover, since (6) implies that $R_{f,e}(e^n) = -2(e^n, f, e) = 2ne^n$ then I must contain the power e^N, although $e^{N-1} \notin I$. The result is a consequence of the following proposition which shows that in this situation $N = 1$ or $N = 0$, so I is either the augmentation ideal or the whole $U(S_2)$.

Proposition 4.1. *In $U(S_2)$ we have that $(e^n, f, f)e = ne^n f - n(n-1)e^{n-1}$ $\forall n \geq 0$.*

Proof. The cases $n = 0$ and $n = 1$ are obvious, so we assume that $n \geq 2$. On one hand, $(e^n, f, f)e$ is an eigenvector of $D_{e,f}$ with eigenvalue $2n - 2$, so it is a linear combination of $\{e^{n-1+i} f^i \mid i \geq 0\}$. Since it also belongs to $U(S_2)_{n+1}$, we deduce that $(e^n, f, f)e = \alpha e^n f + \beta e^{n-1}$ for some $\alpha, \beta \in F$. On the other hand, by (6) $(e^n, f, f)e = ne^{n-1} f \cdot e = ne^n f - n[e, e^{n-1} f]$ and, by Lemma 24 in [3], this is congruent to $ne^n f - n(n-1)e^{n-1}$ modulo $U(S_2)_{n-3}$. Therefore, $\alpha = n$ and $\beta = -n(n-1)$. □

5 Proof of Theorem 2.2

Let T be a finite–dimensional simple L.t.s. and $L = L(T) = \mathrm{InnDer}(T) \oplus T$ — its standard embedding Lie algebra. The results of the previous section allow us to assume that $\dim T \geq 3$. By [2] $L(T)$ is a semisimple Lie algebra so its Killing form $K(\, , \,)$ is nondegenerate. The corresponding automorphism σ of order 2 preserves the Killing form so $\mathrm{InnDer}(T)$ and T are orthogonal and the restriction of $K(\, , \,)$ to these subspaces is nondegenerate. The adjoint of $R_{a,b} = \mathrm{ad}_b \, \mathrm{ad}_a |_T$ relative to $K(\, , \,)$ is $R_{b,a}$ and

$$2 \, \mathrm{tr}(R_{a,b}) = 2 \, \mathrm{tr}(\mathrm{ad}_b \, \mathrm{ad}_a |_T) = K(a, b).$$

As an $\mathrm{InnDer}(T)$–module, T is either irreducible or it decomposes into a direct sum of two irreducible submodules $T = T_1 \oplus T_2$ with the further restriction that $[T_1, T_1] = 0 = [T_2, T_2]$. In the latter case the center $Z(\mathrm{InnDer}(T))$ of $\mathrm{InnDer}(T)$ is one–dimensional and it is spanned by the map acting as the identity Id on T_1 and as $-$ Id on T_2. The skew–symmetry of this map with respect to the Killing form implies that T_1 and T_2 are isotropic relative to the Killing form, so $T_2 \cong T_1^*$, the dual module of T_1.

One important property of simple L.t.s. that we will use is that for any $0 \neq x_\eta \in T_\eta$ with η a nonzero weight of T relative to some Cartan subalgebra of $\mathrm{InnDer}(T)$ there exists $x_{-\eta} \in T_{-\eta}$ such that $\mathrm{span}\langle x_\eta, x_{-\eta}, [x_\eta, x_{-\eta}]\rangle$ is a three–dimensional Lie algebra isomorphic to $sl(2)$ [1]. In particular, $[x_\eta, x_{-\eta}, x_\eta] \neq 0$.

Let $\mathcal{L}$ be the Lie algebra generated by $\{R_{a,b} \mid a, b \in T\}$. We will use the notation

$$\tau_{x,y} \colon z \mapsto K(y,z)x, \quad \sigma_{x,y} = \tau_{x,y} + \tau_{y,x}, \text{ and } \lambda_{x,y} = \tau_{x,y} - \tau_{y,x}.$$

Assume that T is a $\mathrm{InnDer}(T)$–irreducible module. Fix μ to be the highest weight of T relative to a basis of a root system of $\mathrm{InnDer}(T)$ and $0 \neq x_\mu \in T_\mu$. For any weight η of T and $x_\eta \in T_\eta$, $[x_\eta, x_\mu, x_\mu] \in T_{2\mu+\eta}$ so it vanishes if $\eta \neq -\mu$. Since in case that $\eta = -\mu$ we have already noticed that $[x_{-\mu}, x_\mu, x_\mu] \neq 0$, then R_{x_μ,x_μ} is a nonzero multiple of τ_{x_μ,x_μ}. Therefore, τ_{x_μ,x_μ} as well as $\tau_{x_{-\mu},x_{-\mu}}$ and $\lambda_{x_\mu,x_{-\mu}} = [\tau_{x_\mu,x_\mu}, \tau_{x_{-\mu},x_{-\mu}}]$ belong to $\mathcal{L}$.

For any map $d \in so(T)$ skew–symmetric with respect to the Killing form we have that $[d, \tau_{x,y}] = \tau_{d(x),y} + \tau_{x,d(y)}$ so by using root vectors of positive roots, and due to the irreducibility of T, we easily get that $\lambda_{x_\mu,y} \in \mathcal{L}$ for any $y \in T$. From this we can obtain that $so(T) = \mathrm{span}\langle \lambda_{x,y} \mid x, y \in T\rangle \subseteq \mathcal{L}$.

The traceless map $0 \neq R_{x_\mu,x_\mu} \in \mathcal{L}$ is symmetric with respect to the Killing form. By the usual decomposition of $sl(T)$ into the direct sum of two irreducible submodules (symmetric and skew–symmetric maps) with respect to the action of $so(T)$ by commutation it follows that $sl(T) \subseteq \mathcal{L}$. Since $2\,\mathrm{tr}(R_{x,x}) = K(x,x)$ then $\mathcal{L}$ also contains the identity, so $\mathcal{L} = \mathrm{End}(T)$.

Let us assume now that T is not irreducible. We have

$$\mathrm{End}(T) = \mathrm{End}(T)_0 \oplus \mathrm{End}(T)_1$$

with

$$\mathrm{End}(R)_0 = \{f \in \mathrm{End}(T) \mid f(T_j) \subseteq T_j, j = 1,2\}$$

(even maps) and

$$\mathrm{End}(R)_1 = \{f \in \mathrm{End}(T) \mid f(T_j) \subseteq T_{3-j}, j = 1,2\}$$

(odd maps).

Fix μ to be the highest weight of T_1 with respect to a basis of a root system of $\mathrm{InnDer}(T)$, $0 \neq x_\mu \in (T_1)_\mu$ and $0 \neq x_{-\mu} \in (T_2)_{-\mu}$ (recall that $T_2 \cong T_1^*$). Since $[T_1, T_1] = 0$ and $T_2 \cong (T_1)^*$ then $[x_\eta, x_\mu, x_\mu] = 0$ for any weight $\eta \neq -\mu$. As before, this implies that $\lambda_{x_\mu,x_{-\mu}} \in \mathcal{L}$ which ultimately gives that

$$so(T)_0 := so(T) \cap \mathrm{End}(T)_0 = \lambda_{T_1,T_2} \subseteq \mathcal{L}\,.$$

The dimension of T_1 is ≥ 2, so under commutation $\mathrm{End}(T)_0$ is the sum of three irreducible $so(T)_0$–modules: $so(T)_0$, the traceless even symmetric maps and the scalar maps. As above $\{R_{x,y} + R_{y,x} \mid x \in T_1, y \in T_2\}$ is not contained inside $sl(T)$, so $\mathrm{Id} \in \mathcal{L}$. In order to prove that $\mathrm{End}_0(T) \subseteq \mathcal{L}$ we only have to check that $\mathcal{L}_0 \neq so(T)_0 \oplus F\,\mathrm{Id}$. By the contrary, if $R_{x,y} + R_{y,x} \subseteq F\,\mathrm{Id}$ for any $x \in T_1, y \in T_2$

then taking traces we obtain that $R_{x,y} + R_{y,x} = \frac{K(x,y)}{\dim T_1}\,\mathrm{Id}$. Given $x' \in T_1$, by (2)

$$\frac{K(x,y)}{\dim T_1}x' = [x', x, y] + [x', y, x] = [x', y, x] = [x, y, x'] = \frac{K(x', y)}{\dim T_1}x$$

which contradicts that $\dim T_1 \geq 2$. Therefore, $\mathrm{End}_0(T) \subseteq \mathcal{L}$.

$\mathrm{End}_1(T)$ decomposes as the direct sum of two irreducible submodules under the action of $\mathrm{End}_0(T)$ by commutation, namely those maps that kill T_1 and those that kill T_2. Since R_{x_μ, x_μ} and $R_{x_{-\mu}, x_{-\mu}}$ are nonzero elements of these types then $\mathrm{End}_1(T) \subseteq \mathcal{L}$, so $\mathcal{L} = \mathrm{End}(T)$ as desired.

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