ROOT-GRADED LIE ALGEBRAS WITH COMPATIBLE GRADING

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Abstract

Lie algebras graded by a finite irreducible reduced root system Δ will be generalized to *predivision* (Δ, G)*-graded Lie algebras* for an abelian group G. In this paper such algebras are classified, up to central extensions, when $\Delta = \mathrm{A}_l$ for $l \geq 3$, D or E, and $G = \mathbb{Z}^n$.

Introduction

The concept of a Lie algebra over a field F of characteristic 0 graded by a finite irreducible reduced root system Δ or a Δ-graded Lie algebra was introduced by Berman and Moody [3]. It is a Lie algebra L together with a finite dimensional split simple Lie algebra $\mathfrak{g}$, a split Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the root system Δ, so that $\mathfrak{g}$ has the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \big(\oplus_{\mu \in \Delta} \ \mathfrak{g}_\mu \big)$ with $\mathfrak{h} = \mathfrak{g}_0$, satisfying the following three conditions:

(i) L contains $\mathfrak{g}$ as a subalgebra;

(ii) $L = \oplus_{\mu \in \Delta \cup \{0\}} \ L_\mu$, where $L_\mu = \{x \in L \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$; and

(iii) $L_0 = \sum_{\mu \in \Delta} \ [L_\mu, L_{-\mu}]$.

The subalgebra $\mathfrak{g}$ is called the *grading subalgebra of* L.

Berman and Moody classified Δ-graded Lie algebras, up to central extensions, when Δ has type A_l, $l \geq 2$, D or E in [3], and then Benkart and Zelmanov completed the classification for the other types in [5]. (In [7], using the connection to Jordan pairs, Δ-graded Lie algebras were classified, where $\Delta \neq \mathrm{E}_8$, F_4 or G_2. The results in [7] hold for root systems Δ of infinite rank, as well as for Lie algebras over rings.)

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Let $L = \oplus_{\mu\in\Delta\cup\{0\}} L_\mu$ be a Δ-graded Lie algebra over F and let G be an abelian group. We say that L admits a *compatible G-grading* or simply L is a *(Δ, G)-graded Lie algebra* if $L = \oplus_{g\in G} L^g$ is a G-graded Lie algebra such that $\mathfrak{g} \subset L^0$. Then we have

$$L = \bigoplus_{\mu\in\Delta\cup\{0\}} \bigoplus_{g\in G} L_\mu^g,$$

where $L_\mu^g = L_\mu \cap L^g$ (see Definition 2.5). Let $Z(L)$ be the centre of L and let

$$\{h_\mu \in \mathfrak{h} \mid \mu \in \Delta\}$$

be the set of coroots. Then L is called a *division (Δ, G)-graded Lie algebra* if for any $\mu \in \Delta$ and any $0 \neq x \in L_\mu^g$, there exists $y \in L_{-\mu}^{-g}$ such that $[x, y] \equiv h_\mu$ modulo $Z(L)$.

Let us explain the case $\Delta = \mathrm{A}_l$ for $l \geq 3$ in order to describe our motivation of this paper. By [3], an A_l-graded Lie algebra covers $psl_{l+1}(A)$ for some unital associative algebra A (see Definition 2.9). Then Berman, Gao and Krylyuk showed in [4] that the core of an extended affine Lie algebra of type A_l for $l \geq 3$ is an A_l-graded Lie algebra and covers $sl_{l+1}(\mathbb{C}_{\boldsymbol{q}})$ where $\mathbb{C}_{\boldsymbol{q}} = \mathbb{C}_{\boldsymbol{q}}[t_1^{\pm}, \dots, t_n^{\pm}]$ is a certain $\mathbb{Z}^n$-graded associative algebra, called a *quantum torus* over $\mathbb{C}$ (see §2 below). We will see that $L = sl_{l+1}(\mathbb{C}_{\boldsymbol{q}})$ is a division $(\mathrm{A}_l, \mathbb{Z}^n)$-graded Lie algebra over $\mathbb{C}$ so that $L = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{\alpha\in\mathbb{Z}^n} L_\mu^\alpha$. Moreover, L satisfies

$$(*) \qquad \dim_{\mathbb{C}} L_\mu^\alpha = 1 \quad \text{for all } \mu \in \Delta \text{ and } \alpha \in \mathbb{Z}^n.$$

Our goal is to describe division $(\mathrm{A}_l, \mathbb{Z}^n)$-graded Lie algebras without assuming $(*)$. This generalizes the core of an extended affine Lie algebra of type A_l (see Example 2.8(c)). One of the main results of the paper, which is contained in Proposition 2.13 is the following:

Result 1. *Let $l \geq 3$. Then any division (A_l, G)-graded Lie algebra covers $psl_{l+1}(P)$ where P is a division G-graded associative algebra.*

For a group G, a division G-graded algebra is defined as a G-graded algebra whose nonzero homogeneous elements are all invertible. A division G-graded associative algebra over a field F can be considered as a crossed product algebra $D * G$ for an associative division algebra D over F (see §1). Our next goal is to describe $D * \mathbb{Z}^n$. For this purpose, we introduce the following definition: A triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$ is called a *division $\mathbb{Z}^n$-grading triple* over F if

(1) D is an associative division algebra over F;
(2) $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$ is an n-tuple of F-automorphisms φ_i of D; and
(3) $\boldsymbol{q} = (q_{ij})$ is an $n \times n$ matrix over D satisfying, for all $1 \leq i, j, k \leq n$,

$$q_{ii} = 1 \quad \text{and} \quad q_{ji}^{-1} = q_{ij},$$
$$\varphi_j\varphi_i = \mathrm{I}(q_{ij})\varphi_i\varphi_j,$$
$$\varphi_k(q_{ij}) = q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{kj})q_{ki},$$

where $\mathrm{I}(q_{ij})$ is the inner automorphism of D determined by q_{ij}, i.e.,

$$\mathrm{I}(q_{ij})(d) = q_{ij} d q_{ij}^{-1} \quad \text{for } d \in D.$$

We will show that any $D * \mathbb{Z}^n$ can be constructed from a division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$.

Let us briefly explain how this works. First we consider the simplest example of $D * \mathbb{Z}^n$, namely, the ring $D[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of Laurent polynomials over D in n-variables. Note that $D[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = \oplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} Dt_{\boldsymbol{\alpha}}$ is a $\mathbb{Z}^n$-graded algebra, where $t_{\boldsymbol{\alpha}} = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, and the multiplication rule is determined by

$$t_i d = d t_i \quad \text{and} \quad t_j t_i = t_i t_j \quad \text{for all } d \in D \text{ and all } i, j.$$

Then one sees that $D * \mathbb{Z}^n$ has the same $\mathbb{Z}^n$-grading as in $D[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, i.e., $D * \mathbb{Z}^n = \oplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} Dt_{\boldsymbol{\alpha}}$ as a D-vector space. It is easily seen that the multiplication rule in $D * \mathbb{Z}^n$ determines a division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$ as follows:

$$(**) \qquad t_i d = \varphi_i(d) t_i \quad \text{and} \quad t_j t_i = q_{ij} t_i t_j, \quad \text{for all } 1 \leq i, j \leq n,$$

as the defining relations in the quantum torus $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ for $\boldsymbol{q} = (q_{ij})$.

Conversely, for a division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$, let $D_{\boldsymbol{\varphi},\boldsymbol{q}} = D_{\boldsymbol{\varphi},\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be the same $\mathbb{Z}^n$-graded D-vector space as $D[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ above. We will show that the relations $(**)$ determine an associative multiplication on $D_{\boldsymbol{\varphi},\boldsymbol{q}}$. Thus we will get the following:

Result 2. *For any division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$, there exists a crossed product $D_{\boldsymbol{\varphi},\boldsymbol{q}} = D_{\boldsymbol{\varphi},\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ such that $D_{\boldsymbol{\varphi},\boldsymbol{q}} = \oplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} Dt_{\boldsymbol{\alpha}}$ has the same $\mathbb{Z}^n$-grading as $D[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ above, and the multiplication rule is determined by $(**)$. Conversely, any crossed product $D * \mathbb{Z}^n$ is isomorphic to $D_{\boldsymbol{\varphi},\boldsymbol{q}}$ for some $\boldsymbol{\varphi}$ and $\boldsymbol{q}$ (see Theorem 3.3 for more precise statements).*

Note that if $D = F$, then $\boldsymbol{\varphi} = \mathbf{1} = (\mathrm{id}, \dots, \mathrm{id})$ and $F_{\mathbf{1},\boldsymbol{q}} = F_{\boldsymbol{q}}$ is the quantum torus.

Consequently, one gets that any division $(\mathrm{A}_l, \mathbb{Z}^n)$-graded Lie algebra for $l \geq 3$ covers $psl_{l+1}(D_{\boldsymbol{\varphi},\boldsymbol{q}})$. We will also classify division $(\Delta, \mathbb{Z}^n)$-graded Lie algebras when $\Delta = \mathrm{D}$ or E, which is simpler than the case A. Moreover, our concept of "division" can be generalized to "predivision" (see Definition 2.7). Results 1 and 2 above will be proved in this more general set-up.

The organization of the paper is as follows. In §1 we review basic concepts of graded algebras and crossed product algebras. In §2 we prove some properties of (Δ, G)-graded Lie algebras. Then predivision or division (Δ, G)-graded Lie algebras are defined. After describing some examples of them, we classify predivision (Δ, G)-graded Lie algebras for $\Delta = \mathrm{A}_l$ $(l \geq 3)$, D and E types. In §3 we classify crossed product algebras $R * \mathbb{Z}^n$. Finally in §4 we give a summary of our results.

§1 Basic Concepts

For any group G and any G-graded algebra $L = \oplus_{g\in G} L_g$, we denote

$$\operatorname{supp} L := \{g \in G \mid L_g \neq (0)\}.$$

Then we have $L = \oplus_{g\in G'} L_g$ where $G' = \langle \operatorname{supp} L\rangle$ is the subgroup of G generated by $\operatorname{supp} L$. Because of this, we will in the following always assume

$$G = \langle \operatorname{supp} L\rangle. \tag{1.1}$$

Whenever a class of algebras has a notion of invertibility, one can make the following definition:

Definition 1.2. Let G be a group. A G-graded algebra $P = \oplus_{g\in G} P_g$ is called a *predivision G-graded algebra* if P_g contains an invertible element for all $g \in \operatorname{supp} P$. Also, P is called a *division G-graded algebra* if all nonzero homogeneous elements are invertible.

One can easily check that if P is a predivision G-graded associative algebra, then $\operatorname{supp} P = G$ and P is strongly graded, i.e., $P_g P_h = P_{gh}$ for all $g, h \in G$. This is not true if P is a Jordan algebra (see [9]). Predivision G-graded associative algebras are realized as crossed product algebras, which we recall here:

Definition 1.3. Let R be a unital associative algebra over a field F and G a group. Let $R * G$ be the free left R-module with basis $\overline{G} = \{\overline{g} \mid g \in G\}$, a copy of G. Define a multiplication on $R * G$ by linear extension of

$$(r\overline{g})(s\overline{h}) = r\sigma_g(s)\tau(g,h)\overline{gh},$$

for $r, s \in R$ and $g, h \in G$, where

$$\begin{aligned} &\text{(action)} && \sigma : G \longrightarrow \operatorname{Aut}_F(R), && \text{the group of } F\text{-automorphisms of } R,\\ &\text{(twisting)} && \tau : G \times G \longrightarrow U(R), && \text{the group of units of } R, \end{aligned}$$

are arbitrary maps and $\sigma_g := \sigma(g)$. It is easily seen that $R*G$ is an algebra over F. $R*G = (R, G, \sigma, \tau)$ is called a *crossed product algebra over F* if the multiplication is associative. If there is no action or twisting, that is, if $\sigma_g = \mathrm{id}$ and $\tau(g, h) = 1$ for all $g, h \in G$, then $R * G = R[G]$ is the ordinary *group algebra*. If the action is trivial, then $R * G =: R^t[G]$ is called a *twisted group algebra*. Finally, if the twisting is trivial, then $R * G =: RG$ is called a *skew group algebra*.

Remark 1.4. If a crossed product algebra $R * G$ is commutative, then the action is clearly trivial, and so $R * G = R^t[G]$.

The following lemma characterizes σ and τ (see [8], Lemma 1.1 p.2). We denote by $\mathrm{I}(d)$ the inner automorphism determined by $d \in U(R)$, i.e., $\mathrm{I}(d)(r) = drd^{-1}$ for $r \in R$.

1.5. *The associativity of $R * G$ is equivalent to the following two conditions: for all $g, h, k \in G$,*

(i) $\sigma_g \sigma_h = \mathrm{I}\big(\tau(g,h)\big)\sigma_{gh}$,

(ii) $\sigma_g\big(\tau(h,k)\big)\tau(g,hk) = \tau(g,h)\tau(gh,k)$.

Remark 1.6. If R is commutative, then the action $\sigma : G \longrightarrow \mathrm{Aut}_F(R)$ becomes a group homomorphism by condition (i) in 1.5. So the action is really a "group action" in usual sense. Also, for a skew group algebra RG, the action becomes a group homomorphism for the same reason. Conversely, any group action $G \longrightarrow \mathrm{Aut}_F(R)$ defines a skew group algebra RG.

If $d : G \longrightarrow U(R)$ assigns to each element $g \in G$ a unit d_g, then $\tilde{G} = \{d_g \overline{g} \mid g \in G\}$ yields another R-basis for $R * G$ so that $R * G$ is a crossed product algebra for the new basis. One calls this a *diagonal change of basis* ([8], p.3). Any crossed product algebra has an identity element. It is of the form $1 = u\overline{e}$ for some unit u in R where e is the identity element of G ([8], Exercise 2 p.9). We can and will assume that $1 = \overline{e}$, via a diagonal change of basis, and so $\tau(g,e) = \tau(e,g) = 1$ for all $g \in G$. The embedding of R into $R * G$ is then given by $r \mapsto r\overline{e}$. Also, we have ([8], p.3)

$$r\overline{g} \quad \text{is invertible if and only if} \quad r \in U(R). \tag{1.7}$$

Now, it is clear that a crossed product algebra $R * G = \oplus_{g \in G} R\overline{g}$ is a predivision G-graded associative algebra. Conversely, suppose that $A = \oplus_{g \in G} A_g$ is a predivision G-graded associative algebra over F. Then we have $A = \oplus_{g \in G} Rx_g$ where $R = A_e$ and an invertible element $x_g \in A_g$, which exists since A is predivision graded and $\operatorname{supp} A = G$. Moreover, for $h \in G$, we have $x_g x_h = x_g x_h (x_{gh})^{-1} x_{gh}$. So we can put $\tau(g,h) := x_g x_h (x_{gh})^{-1} \in U(R)$. Then we have $x_g x_h = \tau(g,h) x_{gh}$. Also, let $I(x_g)$ be the inner automorphism determined by x_g and let $\sigma_g := I(x_g) \mid_R$. Then, σ_g is clearly an F-automorphism of R and for $r, r' \in R$,

$$(rx_g)(r'x_h) = r(x_g r' x_g^{-1}) x_g x_h = r\sigma_g(r') x_g x_h = r\sigma_g(r')\tau(g,h) x_{gh}.$$

Hence A is a crossed product algebra $R * G$ determined by these σ and τ. So the two concepts, a crossed product algebra $R * G$ and a predivision G-graded associative algebra, coincide (see [8], Exercise 2 p.18). In particular, a division G-graded associative algebra is a crossed product algebra $R * G$ where R is a division algebra.

By Remark 1.4, a predivision G-graded commutative associative algebra $Z = \oplus_{g\in G} Z_g$ (G is necessarily abelian) is a twisted group algebra $K^t[G]$ where $K := Z_e$. Moreover (see [8], Exercise 6 p.10):

1.8. *If the abelian group G is free, then Z is a group algebra $K[G]$. In particular, when $G = \mathbb{Z}^n$, Z is the algebra $K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ of Laurent polynomials for invertible elements $z_i \in Z_{\boldsymbol{\varepsilon}_i}$, $i = 1, \dots, n$, where $\{\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n\}$ is a basis of $\mathbb{Z}^n$.*

§2 Predivision (Δ, G)-graded Lie algebras

In this section F is a field of characteristic 0 and Δ is a finite irreducible reduced root system. The concept of a Δ-graded Lie algebra $L = \oplus_{\mu\in\Delta\cup\{0\}} L_\mu$ over F as a triple $(L, \mathfrak{g}, \mathfrak{h})$ has been defined in the introduction. When no confusion is likely to arise we will use the abbreviation L for $(L, \mathfrak{g}, \mathfrak{h})$. Also, we note that the centre $Z(L)$ of L is contained in L_0.

A homomorphism (resp. an isomorphism) $\varphi : L \longrightarrow L'$ of Δ-graded Lie algebras $L = (L, \mathfrak{g}, \mathfrak{h})$ and $L' = (L', \mathfrak{g}', \mathfrak{h}')$, which have the same type Δ, is called a Δ-*homomorphism* (resp. a Δ-*isomorphism*) if $\varphi(\mathfrak{g}) = \mathfrak{g}'$ and $\varphi(\mathfrak{h}) = \mathfrak{h}'$ (cf. Definition 1.20 in [3]). Then one can check that $\varphi(L_\alpha) \subset L'_\alpha$ for all $\alpha \in \Delta$, and so $\varphi(L_0) \subset L'_0$. In other words, a Δ-homomorphism is graded.

Recall that a *cover* $\tilde{L} = (\tilde{L}, \pi)$ of a Lie algebra L is an epimorphism $\pi : \tilde{L} \longrightarrow L$ of Lie algebras so that $\tilde{L}$ is perfect, i.e., $\tilde{L} = [\tilde{L}, \tilde{L}]$, and $\ker \pi$ is contained in the centre of $\tilde{L}$.

Definition 2.1. Let $\tilde{L}$ and L be Δ-graded Lie algebras. If $\pi : \tilde{L} \longrightarrow L$ is a cover and a Δ-homomorphism, $\tilde{L} = (\tilde{L}, \pi)$ is called a Δ-*cover of* L. Also, for Δ-graded Lie algebras L and L', if there exist a Δ-graded Lie algebra $\tilde{L}$ and maps $\pi : \tilde{L} \longrightarrow L$ and $\pi' : \tilde{L} \longrightarrow L'$ such that $(\tilde{L}, \pi)$ and $(\tilde{L}, \pi')$ are both Δ-covers, we say that L and L' are Δ-*isogeneous*.

Example 2.2. Let $L = (L, \mathfrak{g}, \mathfrak{h})$ be a Δ-graded Lie algebra with centre $Z(L)$. Then, for any subspace V of $Z(L)$, $L/V = (L/V, \mathfrak{g}+V, \mathfrak{h}+V)$ is a Δ-graded Lie algebra, and the canonical epimorphism $L \longrightarrow L/V$ is a Δ-cover. In particular, L and L/V are Δ-isogeneous.

We will show that if L and L' are Δ-isogeneous, then $L/Z(L)$ and $L'/Z(L')$ are Δ-isomorphic, i.e., there exists a Δ-isomorphism between them.

Lemma 2.3. *Let $\pi : \tilde{L} \longrightarrow L$ be a cover. Then $Z(\tilde{L}) = \pi^{-1}\big(Z(L)\big)$. Hence, if $\omega : L \longrightarrow L/Z(L)$ is the canonical epimorphism, we have* $\ker(\omega \circ \pi) = Z(\tilde{L})$.

Proof. For $\tilde{x} \in \tilde{L}$ we have $\tilde{x} \in \pi^{-1}\big(Z(L)\big) \Leftrightarrow \pi\big([\tilde{x}, \tilde{L}]\big) = 0 \Leftrightarrow [x, \tilde{L}] \subset \ker \pi$. Since $\ker \pi \subset Z(\tilde{L})$ and $\tilde{L}$ is perfect, it follows that $\tilde{x} \in Z(\tilde{L})$, whence $\pi^{-1}\big(Z(L)\big) \subset Z(\tilde{L})$. The other inclusion is clear. The map $\omega \circ \pi : \tilde{L} \longrightarrow L/Z(L)$ is a cover. Perfectness of L implies that $L/Z(L)$ is centreless, whence $\ker(\omega \circ \pi) = Z(\tilde{L})$. □

Corollary 2.4. *Suppose that L and L' are Δ-isogeneous. Then $L/Z(L)$ and $L'/Z(L')$ are Δ-isomorphic.*

Proof. By assumption, there exists a Δ-graded Lie algebra $\tilde{L} = (\tilde{L}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ such that $\pi : \tilde{L} = (L, \mathfrak{g}, \mathfrak{h}) \longrightarrow L$ and $\pi' : \tilde{L} \longrightarrow L' = (L', \mathfrak{g}', \mathfrak{h}')$ are both Δ-covers. Let $\omega : L \longrightarrow L/Z(L)$ and $\omega' : L' \longrightarrow L'/Z(L')$ be the canonical epimorphisms. Then, by Lemma 2.3, we have $\ker(\omega \circ \pi) = Z(\tilde{L}) = \ker(\omega' \circ \pi')$. Hence there exists the induced isomorphism

$$\begin{aligned}\varphi : L/Z(L) =& \big(L/Z(L), \mathfrak{g} + Z(L), \mathfrak{h} + Z(L)\big)\\ &\longrightarrow L'/Z(L') = \big(L'/Z(L'), \mathfrak{g}' + Z(L'), \mathfrak{h}' + Z(L')\big)\end{aligned}$$

such that $\varphi \circ \omega \circ \pi = \omega' \circ \pi'$. In particular, $\varphi\big(\mathfrak{g} + Z(L)\big) = \varphi \circ \omega \circ \pi(\tilde{\mathfrak{g}}) = \omega' \circ \pi'(\tilde{\mathfrak{g}}) = \mathfrak{g}' + Z(L')$ and similarly $\varphi\big(\mathfrak{h} + Z(L)\big) = \mathfrak{h}' + Z(L')$. Therefore, φ is a Δ-isomorphism. □

Now we define new concepts.

Definition 2.5. Let $L = (L, \mathfrak{g}, \mathfrak{h}) = \oplus_{\mu \in \Delta \cup \{0\}} L_\mu$ be a Δ-graded Lie algebra over F. Let G be an abelian group. We say that L admits a *compatible G-grading* or simply L is a *(Δ, G)-graded Lie algebra* if $L = \oplus_{g \in G} L^g$ is a G-graded Lie algebra such that $\mathfrak{g} \subset L^0$. In this case, L^g is a $\mathfrak{h}$-module for all $g \in G$ via the adjoint action. Hence we have $L^g = \oplus_{\mu \in \Delta \cup \{0\}} L^g_\mu$ where $L^g_\mu = L_\mu \cap L^g$ (see [6] Proposition 1, p.92). Therefore, $L_\mu = \oplus_{g \in G} L^g_\mu$ and

$$L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{g \in G} L^g_\mu.$$

Remark 2.6. (i) The compatible G-grading is completely determined by L^g_μ for all $\mu \in \Delta$ and $g \in G$ since $L^g_0 = \sum_{\mu \in \Delta} \sum_{g = h + k} [L^h_\mu, L^k_{-\mu}]$.

(ii) Let $\operatorname{supp} L_\mu := \{g \in G \mid L^g_\mu \neq (0)\}$. Recall $\operatorname{supp} L = \{g \in G \mid L^g \neq (0)\}$ as defined in the beginning of §1. If $g \in \operatorname{supp} L$, then $L^g_0 \neq (0)$ or there exists some $\mu \in \Delta$ such that $L^g_\mu \neq (0)$. If $L^g_\mu \neq (0)$, we have $g \in \operatorname{supp} L_\mu$. If $L^g_0 \neq (0)$, then $g = h + k \in \operatorname{supp} L_\mu + \operatorname{supp} L_{-\mu}$ for some $\mu \in \Delta$ and $h, k \in G$ by (i). Thus since $0 \in \operatorname{supp} L_{-\mu}$, we obtain

$$\operatorname{supp} L \subset \bigcup_{\mu \in \Delta} (\operatorname{supp} L_\mu + \operatorname{supp} L_{-\mu}).$$

Definition 2.7. Let $L = (L, \mathfrak{g}, \mathfrak{h})$ be a (Δ, G)-graded Lie algebra with centre $Z(L)$ and let

$$\{h_\mu \in \mathfrak{h} \mid \mu \in \Delta\}$$

be the set of coroots. Then L is called *predivision* if

(pd) for any $\mu \in \Delta$ and any $L_\mu^g \neq (0)$, there exist $x \in L_\mu^g$ and $y \in L_{-\mu}^{-g}$ such that $[x, y] \equiv h_\mu$ modulo $Z(L)$;

and *division* if

(d) for any $\mu \in \Delta$ and any $0 \neq x \in L_\mu^g$, there exists $y \in L_{-\mu}^{-g}$ such that $[x, y] \equiv h_\mu$ modulo $Z(L)$.

Note that (d) implies (pd), i.e., 'division' $\Longrightarrow$ 'predivision'. If $\dim_F L_\mu^g \leq 1$ for all $\mu \in \Delta$ and $g \in G$, then two concepts, 'predivision' and 'division', coincide.

Example 2.8. (a) A Δ-graded Lie algebra is a predivision (Δ, G_0)-graded algebra for the trivial group $G_0 = \{0\}$.

(b) The core of an extended affine Lie algebra of reduced type Δ with nullity n is a division (Δ, Λ)-graded Lie algebra over $\mathbb{C}$, where Λ is a free abelian group of rank n. Indeed, it is known that such a core is a Δ-graded Lie algebra over $\mathbb{C}$, say $L = \oplus_{\mu \in \Delta \cup \{0\}} L_\mu$, and each L_μ has a decomposition $L_\mu = \oplus_{\delta \in \Lambda} L_\mu^\delta$, where Λ is defined as the group generated by the isotropic roots δ (we use the notation L_μ^δ instead of $L_{\mu+\delta}$ which is normally used in the theory of extended affine Lie algebras). It turns out that Λ is a lattice of rank n with $\langle \operatorname{supp} L \rangle = \Lambda$ (for details see [2]). Let $L^\delta := \oplus_{\mu \in \Delta \cup \{0\}} L_\mu^\delta$. Then the grading subalgebra $\mathfrak{g}$ is contained in L^0 so that $L = \oplus_{\delta \in \Lambda} L^\delta$ gives a compatible Λ-grading. Thus L is a (Δ, Λ)-graded Lie algebra.

We recall one of the basic properties of extended affine Lie algebras (see [1]): For any $\mu \in \Delta$, $\delta \in \Lambda$ and any $0 \neq e_\mu^\delta \in L_\mu^\delta$, there exist some $f_\mu^\delta \in L_{-\mu}^{-\delta}$ and $h_\mu^\delta \in L_0^0$ such that $\langle e_\mu^\delta, f_\mu^\delta, h_\mu^\delta \rangle$ is an sl_2-triplet, and in particular $[e_\mu^\delta, f_\mu^\delta] = h_\mu^\delta$.

One can check that $h_\mu - h_\mu^\delta \in Z(L)$ for all coroots $h_\mu = h_\mu^0$ of $\mathfrak{g}$. Therefore L is a division (Δ, Λ)-graded Lie algebra. We note that $\dim_{\mathbb{C}} L_\mu^\delta \leq 1$ for all $\mu \in \Delta$ and $\delta \in \Lambda$, which is also one of the basic properties of extended affine Lie algebras.

(c) Let $Z = \oplus_{g \in G} Z_g$ be a G-graded commutative associative algebra over F and let $\mathfrak{g} = \mathfrak{h} \oplus \big(\oplus_{\mu \in \Delta} \ \mathfrak{g}_\mu \big)$ be a finite dimensional split simple Lie algebra over F of type Δ with the set $\{h_\mu \in \mathfrak{h} \mid \mu \in \Delta\}$ of coroots. Then $L := \mathfrak{g} \otimes_F Z$ is a (Δ, G)-graded Lie algebra. In fact, $L = \oplus_{\mu \in \Delta \cup \{0\}} (\mathfrak{g}_\mu \otimes_F Z)$ for $\mathfrak{g}_0 = \mathfrak{h}$ is a Δ-graded Lie algebra with grading subalgebra $\mathfrak{g} = \mathfrak{g} \otimes 1$. We put $L^g := \mathfrak{g} \otimes_F Z_g$ for all $g \in G$. Then $\operatorname{supp} L = \operatorname{supp} Z$ and $L = \oplus_{g \in G} L^g$ is a G-graded Lie algebra with $\mathfrak{g} \subset L^0$, i.e, the G-grading is compatible. Hence L is a

(Δ, G)-graded Lie algebra. We call the compatible G-grading of $L = \mathfrak{g} \otimes_F Z$ the *natural compatible G-grading obtained from the G-grading of Z.*

Suppose that $Z = \oplus_{g \in G} K\overline{g}$ is a crossed product commutative algebra over F. Let $e \in \mathfrak{g}_\mu$ and $f \in \mathfrak{g}_{-\mu}$ such that $[e, f] = h_\mu$. Then $e \otimes \overline{g} \in L^g_\mu$, $f \otimes \overline{g}^{-1} \in L^{-g}_{-\mu}$ and

$$[e \otimes \overline{g}, f \otimes \overline{g}^{-1}] = [e, f] \otimes \overline{g}\, \overline{g}^{-1} = h_\mu \otimes 1 = h_\mu$$

for all $g \in G$, and so L is a predivision (Δ, G)-graded Lie algebra over F. Note that $Z(L) = (0)$. Also, if K is a field, then L is a division (Δ, G)-graded Lie algebra.

Suppose that $\tilde{L} = (\tilde{L}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) = \oplus_{g \in G} \tilde{L}^g$ is a (Δ, G)-graded Lie algebra and that $\pi : \tilde{L} \longrightarrow L$ is a cover of a Lie algebra L. Then $L = \big(L, \pi(\tilde{\mathfrak{g}}), \pi(\tilde{\mathfrak{h}})\big)$ becomes a Δ-graded Lie algebra so that $(\tilde{L}, \pi)$ is a Δ-cover of L. Moreover, if $\ker \pi$ is G-graded, then L admits the induced compatible G-grading $L = \oplus_{g \in G} \pi(\tilde{L}^g)$. In particular, since the centre $Z(\tilde{L})$ is always G-graded, $\tilde{L}/Z(\tilde{L})$ is a (Δ, G)-graded Lie algebra.

Definition 2.9. Let P be a unital associative algebra over F and let $\mathfrak{gl}_{l+1}(P)$ be the Lie algebra consisting of all $(l+1) \times (l+1)$ matrices over P under the commutator product $(l \geq 1)$. Let $e_{ij}(a) \in \mathfrak{gl}_{l+1}(P)$ whose (i, j)-entry is a and the other entries are all 0. We define $sl_{l+1}(P)$ as the subalgebra of $\mathfrak{gl}_{l+1}(P)$ generated by $e_{ij}(a)$ for all $a \in P$ and $1 \leq i \neq j \leq l+1$. The centre $Z\big(sl_{l+1}(P)\big)$ of $sl_{l+1}(P)$ consists of $\sum_{i=1}^{l+1} e_{ii}(a)$ for $a \in [P, P] \cap Z(P)$ where $[P, P]$ is the span of all commutators in P and $Z(P)$ is the centre of P. We define $psl_{l+1}(P)$ as $sl_{l+1}(P)/Z\big(sl_{l+1}(P)\big)$.

It is well-known that $sl_{l+1}(P)$ is an A_l-graded Lie algebra (see [3]): Denote $\{e_{ij}(b) \mid b \in B\}$ by $e_{ij}(B)$ for any subset $B \subset P$. Let

$$sl_{l+1}(F) = \mathfrak{h} \oplus \bigoplus_{1 \leq i \neq j \leq l+1} e_{ij}(F1) \subset sl_{l+1}(P),$$

be the split simple Lie algebra over F of type A_l where $\mathfrak{h}$ is the Cartan subalgebra consisting of diagonal matrices of $sl_{l+1}(F)$. Let $\varepsilon_i : \mathfrak{h} \longrightarrow F$ be the projection onto the (i, j)-entry for $i = 1, \ldots, l+1$, and $\Delta := \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, which is a root system of type A_l. Then

$$sl_{l+1}(P) = L_0 \oplus \left(\bigoplus_{\varepsilon_i - \varepsilon_j \in \Delta} e_{ij}(P) \right),$$

where $L_0 = \sum_{\varepsilon_i - \varepsilon_j \in \Delta} [e_{ij}(P), e_{ji}(P)]$, is an A_l-graded Lie algebra with grading subalgebra $sl_{l+1}(F)$. Let $Z := Z\big(sl_{l+1}(P)\big)$. We can and will identify $sl_{l+1}(F) + Z$ with $sl_{l+1}(F)$ and $e_{ij}(P) + Z$ with $e_{ij}(P)$, and so

$$psl_{l+1}(P) = (L_0/Z) \oplus \left(\bigoplus_{\varepsilon_i - \varepsilon_j \in \Delta} e_{ij}(P) \right)$$

is also an A_l-graded Lie algebra with the same grading subalgebra $sl_{l+1}(F)$.

Example 2.10. Let $L = sl_{l+1}(P)$ be the A_l-graded Lie algebra over F with grading subalgebra $sl_{l+1}(F)$ described above. If $P = \oplus_{g\in G} P_g$ is a G-graded algebra, then L admits a compatible G-grading. Indeed, let

$$L^g := \{\sum_{i,j} e_{ij}(P_g) \mid \sum_{i,j} e_{ij}(P_g) \subset L\}.$$

Then $L = \oplus_{g\in G} L^g$, and it is a G-graded Lie algebra with $sl_{l+1}(F) \subset L^0$. Note that $\operatorname{supp} L \supset \operatorname{supp} P$, and so $\langle \operatorname{supp} L\rangle = G$. Also, $psl_{l+1}(P)$ admits the induced compatible G-grading. We call the compatible G-grading of L or $psl_{l+1}(P)$ the *natural compatible G-grading obtained from the G-grading of P*. This grading is the unique G-grading so that

$$L^g_{\varepsilon_i-\varepsilon_j} = e_{ij}(P_g) = psl_{l+1}(P)^g_{\varepsilon_i-\varepsilon_j} \quad \text{for all } \varepsilon_i - \varepsilon_j \in \Delta \text{ and } g \in G.$$

If $P = \oplus_{g\in G} R\overline{g}$ is a crossed product algebra, then

$$[e_{ij}(\overline{g}), e_{ji}(\overline{g}^{-1})] = e_{ii}(1) - e_{jj}(1) = [e_{ij}(1), e_{ji}(1)] = h_{\varepsilon_i-\varepsilon_j}$$

for all $g \in G$. Thus L and $psl_{l+1}(P)$ with the natural compatible G-gradings from the G-grading of P are predivision (A_l, G)-graded Lie algebras over F. Also, if R is a division algebra, then the (A_l, G)-graded Lie algebras L and $psl_{l+1}(P)$ are division.

Lemma 2.11. (i) *Let P be a unital associative algebra. Suppose that $l \geq 2$ and that the A_l-graded Lie algebra $psl_{l+1}(P)$ described above admits a predivision (resp. division) compatible G-grading. Then P is a predivision (resp. division) G-graded algebra, and the G-grading of $psl_{l+1}(P)$ is the natural compatible G-grading obtained from the G-grading of P.*

(ii) *Let Z be a unital commutative associative algebra. Suppose that the Δ-graded Lie algebra $\mathfrak{g} \otimes_F Z$ described in Example 2.8(c) admits a predivision (resp. division) compatible G-grading. Then Z is a predivision (resp. division) G-graded algebra, and the G-grading of $\mathfrak{g} \otimes_F Z$ is the natural compatible G-grading obtained from the G-grading of Z.*

Proof. (i): By assumption, $psl_{l+1}(P) = psl_{l+1}(P)_0 \oplus \big(\oplus_{\varepsilon_i-\varepsilon_j\in\Delta} e_{ij}(P)\big)$ admits a predivision (resp. division) compatible G-grading, say

$$psl_{l+1}(P) = psl_{l+1}(P)_0 \oplus \big(\oplus_{\varepsilon_i-\varepsilon_j\in\Delta} \ \oplus_{g\in G} e_{ij}(P)^g\big).$$

Let

$$P^{ij}_g := \{p \in P \mid e_{ij}(p) \in e_{ij}(P)^g\} \quad \text{for } i \neq j.$$

We claim that $P^{ij}_g = P^{rs}_g$ for all $\varepsilon_r - \varepsilon_s \in \Delta$.

In general, it is well-known that for any distinct $\alpha, \beta \in \Delta = \mathrm{A}_l$, $l \geq 2$, D or E, there exists a sequence $\alpha_1, \dots, \alpha_t \in \Delta$ so that $\alpha_1 = \alpha$, $\alpha_t = \beta$ and $\alpha_{i+1} - \alpha_i \in \Delta$ for $i = 1, \dots, t-1$.

Now, it is enough to show that $P_g^{ij} \subset P_g^{rs}$. Let $p \in P_g^{ij}$. We apply the above for $\alpha = \varepsilon_i - \varepsilon_j$ and $\beta = \varepsilon_r - \varepsilon_s$. For $p \in P_g^{ij}$,

$$[\cdots[[e_{ij}(p), e_{\alpha_2-\alpha_1}(1)], e_{\alpha_3-\alpha_2}(1)], \dots, e_{\alpha_t-\alpha_{t-1}}(1)] = \pm e_{\alpha_t}(p) = \pm e_{rs}(p) \in e_{rs}(P)^g$$

since $[e_{ij}(p), e_{kl}(1)] = \delta_{jk} e_{il}(p) - \delta_{li} e_{kj}(p)$ and $e_{\alpha_{i+1}-\alpha_i}(1) \in L^0_{\alpha_{i+1}-\alpha_i}$. Hence $p \in P_g^{rs}$ and our claim is settled.

Thus one can write $P_g = P_g^{ij}$ and $P = \oplus_{g \in G} P_g$. Since, for $p \in P_g$ and $q \in P_h$ $(g, h \in G)$,

$$[e_{ij}(p), e_{jk}(q)] = e_{ik}(pq) \in e_{ik}(P)^{g+h} \quad \text{for } i \neq k,$$

we have $pq \in P_{g+h}$. Also, one can see that $\operatorname{supp} L \subset \operatorname{supp} P + \operatorname{supp} P$ (see Remark 2.6(ii)), and so $\langle \operatorname{supp} P \rangle \supset \langle \operatorname{supp} L \rangle = G$, whence $\langle \operatorname{supp} P \rangle = G$. Therefore, P is a G-graded algebra. Note that $e_{ij}(P)^g = e_{ij}(P_g)$ for all $\varepsilon_i - \varepsilon_j \in \Delta$ and $g \in G$, and hence the G-grading is natural (see Remark 2.6(i)).

By (pd), for any $\varepsilon_i - \varepsilon_j \in \Delta$ and any $g \in \operatorname{supp} P$, there exist $e_{ij}(p) \in e_{ij}(P_g)$ and $e_{ji}(q) \in e_{ji}(P_{-g})$ such that

$$[e_{ij}(p), e_{ji}(q)] = [e_{ij}(1), e_{ji}(1)] + z \quad \text{for some } z \in Z\big(sl_{l+1}(P)\big).$$

Hence $e_{ii}(pq) - e_{jj}(qp) = e_{ii}(1) - e_{jj}(1) + \sum_{k=1}^{l+1} e_{kk}(a)$ for some $a \in P$, and so $a = 0$ and $pq = qp = 1$, i.e., p is invertible. Also, p is invertible in $P \Leftrightarrow p$ is invertible in P^+. Therefore, $P = \oplus_{g \in G} P_g$ is a predivision G-graded associative algebra. The statement for 'division' can be shown in the same manner.

(ii): Let $Z_g := \{z \in Z \mid \mathfrak{g} \otimes z \subset (\mathfrak{g} \otimes_F Z)^g\}$. Then $Z = \oplus_{g \in G} Z_g$ becomes a G-graded algebra. The rest can be shown in the same manner as in (i). $\square$

Definition 2.12. Let $\tilde{L} = \oplus_{g \in G} \tilde{L}^g$ and $L = \oplus_{g \in G} L^g$ be (Δ, G)-graded Lie algebras and suppose that $\pi : \tilde{L} \longrightarrow L$ is a Δ-cover. If $L^g = \pi(\tilde{L}^g)$ for all $g \in G$, then $\tilde{L} = (\tilde{L}, \pi)$ is called a *(Δ, G)-cover of L*. Also, for (Δ, G)-graded Lie algebras L and L', if there exist a (Δ, G)-graded Lie algebra $\tilde{L}$ and maps $\pi : \tilde{L} \longrightarrow L$ and $\pi' : \tilde{L} \longrightarrow L'$ such that $(\tilde{L}, \pi)$ and $(\tilde{L}, \pi')$ are both (Δ, G)-covers, we say that L and L' are *(Δ, G)-isogeneous*.

It is clear using Lemma 2.3 that if $\tilde{L}$ is a (Δ, G)-cover of L, then

$$\tilde{L} \text{ is is predivision (resp. division)} \quad \Longleftrightarrow \quad L \text{ is predivision (resp. division)}.$$

Also, by the proof of Corollary 2.4, if L and L' are (Δ, G)-isogeneous, then $L/Z(L)$ and $L'/Z(L')$ are *(Δ, G)-isomorphic*, i.e., there exists a Δ-isomorphism which is also G-graded between them. In particular, $\tilde{L}/Z(\tilde{L})$ and $L/Z(L)$ above are (Δ, G)-isomorphic.

Proposition 2.13. (i) *Let $l \geq 3$. Then a predivision (resp. division) (A_l, G)-graded Lie algebra L over F is an (A_l, G)-cover of $psl_{l+1}(P)$ admitting the natural compatible G-grading obtained from the G-grading of a predivision (resp. division) G-graded associative algebra P over F. Hence $L/Z(L)$ and $psl_{l+1}(P)$ are (Δ, G)-isomorphic.*

(ii) *Let $\Delta = D$ or E and let $\mathfrak{g}$ be a finite dimensional split simple Lie algebra L over F of type Δ. Then a predivision (resp. division) (Δ, G)-graded Lie algebra over F is a (Δ, G)-cover of $\mathfrak{g} \otimes_F Z$ admitting the natural compatible G-grading obtained from the G-grading of a predivision (resp. division) G-graded commutative associative algebra Z over F. Hence $L/Z(L)$ and $\mathfrak{g} \otimes_F Z$ are (Δ, G)-isomorphic.*

Proof. For (i), let L be a predivision (A_l, G)-graded Lie algebra over F. Berman and Moody showed in [3] that L is A_l-isogeneous to $\big(sl_{l+1}(P), sl_{l+1}(F)\big)$ (the Steinberg Lie algebra $st_{l+1}(P)$ serves as an A_l-cover of L and $sl_{l+1}(P)$). Hence, by Corollary 2.4, $L/Z(L)$ is A_l-isomorphic to $psl_{l+1}(P)$. Thus $\big(psl_{l+1}(P), sl_{l+1}(F)\big)$ admits a compatible G-grading via the A_l-isomorphism from the compatible G-grading of $L/Z(L)$ induced by the compatible G-grading of L. Therefore, the statement follows from Lemma 2.11.

(ii): Let L be a predivision (Δ, G)-graded Lie algebra over F. Berman and Moody showed in [3] that L is a Δ-cover of $\mathfrak{g} \otimes_F Z$. Thus the statement follows from Lemma 2.11. $\square$

In this paper we will classify predivision $(\Delta, \mathbb{Z}^n)$-graded Lie algebras for $\Delta = \mathrm{A}_l$, $l \geq 3$, D or E, up to central extensions. By Proposition 2.13, it remains to classify crossed product algebras $R * \mathbb{Z}^n$. We determine such algebras as a generalization of *quantum tori*. Namely, let $\boldsymbol{q} = (q_{ij})$ be an $n \times n$ matrix over F such that

$$q_{ii} = 1 \quad \text{and} \quad q_{ji} = q_{ij}^{-1}.$$

The *quantum torus* $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ *determined by* $\boldsymbol{q}$ is defined as the associative algebra over F with $2n$ generators $t_1^{\pm 1}, \ldots, t_n^{\pm 1}$, and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad \text{and} \quad t_j t_i = q_{ij} t_i t_j$$

for all $1 \leq i, j \leq n$. Quantum tori are characterized as predivision $\mathbb{Z}^n$-graded associative algebras whose homogeneous spaces are all 1-dimensional (see [4]). Note that $F_{\boldsymbol{q}}$ is commutative $\iff \boldsymbol{q} = \mathbf{1}$ whose entries are all 1, i.e., $F_{\mathbf{1}} = F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is the algebra of Laurent polynomials. Also, a quantum torus is a twisted group algebra $F^t[\mathbb{Z}^n]$.

§3 Classification of $R * \mathbb{Z}^n$

Throughout this section F is an arbitrary field and G is an arbitrary group. For a G-graded algebra $S = \oplus_{g \in G} S_g$ over F in general, we denote by $\mathrm{GrAut}_F(S)$ the group of graded automorphisms of S, i.e.,

$$\mathrm{GrAut}_F(S) := \{\sigma \in \mathrm{Aut}_F(S) \mid \sigma(S_g) = S_g \text{ for all } g \in G\}.$$

Lemma 3.1. *Let $R*G = (R, G, \sigma, \tau)$ be a crossed product algebra over F and $(R*G)*M = (R*G, M, \eta, \xi)$ a crossed product algebra over F for a group M, an action η and a twisting ξ. Suppose that $\eta(M) \subset GrAut_F(R*G)$ and that $\xi(m,l) \in U(R)$ for all $m, l \in M$. Then, $(R*G)*M$ is a crossed product algebra $R*(G\times M) = \big(R, (G\times M), \sigma', \tau'\big)$ over F for some action σ' and twisting τ'.*

Proof. We have

$$(R*G)*M = \oplus_{m\in M}\ (R*G)\overline{m} = \oplus_{m\in M}\ (\oplus_{g\in G}\ R\overline{g})\overline{m} = \oplus_{(g,m)\in G\times M}\ R\overline{gm}$$

as free R-modules, where $\overline{gm} = \overline{g}\ \overline{m}$. We define $\eta_m = \eta(m) \mid_{R1}$ an F-automorphism of R for every $m \in M$. Also for $h \in G$, $\overline{h}$ is a unit in $R*G$ (see 1.6). Since η_m is a graded automorphism of $R*G$ by our first assumption, $\eta(m)(\overline{h}) = d_{m,h}\overline{h}$ for some $d_{m,h} \in U(R)$. Therefore, for $r\overline{gm} \in R\overline{gm}$ and $s\overline{hl} \in R\overline{hl}$, we have

$$\begin{aligned}
(r\overline{gm})(s\overline{hl}) &= r\overline{g}\eta(m)(s\overline{h})\overline{m}\overline{l}\\
&= r\overline{g}\eta_m(s)\eta(m)(\overline{h})\xi(m,l)\overline{ml}\\
&= r\overline{g}\eta_m(s)d_{m,h}\overline{h}\xi(m,l)\overline{ml}\\
&= r\overline{g}\eta_m(s)d_{m,h}\sigma_h\big(\xi(m,l)\big)\overline{hml} \quad \text{(by our second assumption)}\\
&= r\sigma_g\eta_m(s)\sigma_g(d_{m,h})\sigma_{gh}\big(\xi(m,l)\big)\overline{g}\overline{hml}\\
&= r\sigma_g\eta_m(s)\sigma_g(d_{m,h})\sigma_{gh}\big(\xi(m,l)\big)\tau(g,h)\overline{gh}\ \overline{ml}.
\end{aligned}$$

Thus we have the action

$$\sigma' : G\times M \longrightarrow \mathrm{Aut}_F R \quad \text{by} \quad \sigma'_{(g,m)} = \sigma_g\eta_m,$$

and the twisting $\tau' : (G\times M)\times(G\times M) \longrightarrow U(R)$ by

$$\tau'\big((g,m),(h,l)\big) = \sigma_g(d_{m,h})\sigma_{gh}\big(\xi(m,l)\big)\tau(g,h).$$

Since the crossed product algebra $(R*G)*M$ is associative, we get

$$(R*G)*M = R*(G\times M) = (R, G\times M, \sigma', \tau'). \quad \square$$

A triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ where R is a unital associative algebra over F,

$$\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$$

is an n-tuple of F-automorphisms φ_i of R, and $\boldsymbol{q} = (q_{ij})$ is an $n \times n$ matrix over R satisfying

(G1) $$q_{ii} = 1 \quad \text{for } 1 \le i \le n \quad \text{and} \quad q_{ji}^{-1} = q_{ij} \quad \text{for } 1 \le i < j \le n,$$

(G2) $$\varphi_j \varphi_i = \mathrm{I}(q_{ij}) \varphi_i \varphi_j \quad \text{for } 1 \le i < j \le n,$$

(G3) $$\varphi_k(q_{ij}) = q_{jk} \varphi_j(q_{ik}) q_{ij} \varphi_i(q_{kj}) q_{ki} \quad \text{for } 1 \le i < j < k \le n,$$

is called a $\mathbb{Z}^n$*-grading triple over* F, and a *division* $\mathbb{Z}^n$*-grading triple over* F if R is a division algebra. It follows easily from (G1)-(G3) that

$$\text{these equations hold for all } i, j, k \text{ satisfying } 1 \le i, j, k \le n.$$

For a $\mathbb{Z}^n$-grading triple, we introduce several notations and prove some identities.

Notations.

(N1) $$\boldsymbol{\varepsilon}_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n,$$

i.e., the i-th coordinate is 1 and the others are 0.

(N2) $$q_{ij}^{(m)} := \begin{cases} q_{ij} \varphi_i(q_{ij}) \varphi_i^2(q_{ij}) \cdots \varphi_i^{m-1}(q_{ij}) = \prod_{l=0}^{m-1} \varphi_i^l(q_{ij}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \varphi_i^{-1}(q_{ji}) \varphi_i^{-2}(q_{ji}) \cdots \varphi_i^{m}(q_{ji}) = \prod_{l=-1}^{m} \varphi_i^l(q_{ji}), & \text{if } m < 0, \end{cases}$$

$$\text{and} \quad q_{ij}^{-(m)} := (q_{ij}^{(m)})^{-1}.$$

For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ and $k = 0, 1, 2, \dots, n$,

(N3) $$\varphi^{(\boldsymbol{\alpha})_k} := \begin{cases} \mathrm{id}, & \text{if } k = 0, 1 \\ \varphi_1^{\alpha_1} \cdots \varphi_{k-1}^{\alpha_{k-1}}, & \text{if } k > 1, \end{cases}$$

$$\text{and} \quad \varphi^{\boldsymbol{\alpha}} := \varphi_1^{\alpha_1} \cdots \varphi_n^{\alpha_n}.$$

(N4) $$q_{\boldsymbol{\varepsilon}_1, \boldsymbol{\alpha}} := 1 \quad \text{and} \quad q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\alpha}} := \prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)}) \quad \text{for } j > 1.$$

(N5) $$q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\alpha}}^{(m)} := \begin{cases} \prod_{l=m-1}^{0} \varphi_j^l(q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\alpha}}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \prod_{l=m}^{-1} \varphi_j^l(q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\alpha}}^{-1}), & \text{if } m < 0. \end{cases}$$

(N6) $$q_{\boldsymbol{\alpha}, \boldsymbol{\beta}} := \prod_{j=n}^{1} \varphi^{(\boldsymbol{\alpha})_j}(q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\beta}}^{(\alpha_j)}).$$

Lemma 3.2. *For $m \in \mathbb{Z}$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we have*

$$\varphi_i^{-m}(q_{ij}^{-(m)}) = q_{ij}^{(-m)}, \tag{1}$$

$$\varphi_j \varphi_i^m = \mathrm{I}(q_{ij}^{(m)}) \varphi_i^m \varphi_j, \tag{2}$$

$$\varphi_j \varphi^{(\boldsymbol{\alpha})_i} = \begin{cases} \mathrm{I}\big(\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\big) \varphi^{(\boldsymbol{\alpha})_i} \varphi_j & \textit{for } j \geq i, \\ \mathrm{I}\big(\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\big) \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i} & \textit{for } j < i, \end{cases} \tag{3}$$

$$q_{ij}^{(m+1)} = q_{ij} \varphi_i(q_{ij}^{(m)}) \quad \textit{and} \quad q_{ij}^{-(m+1)} = \varphi_i(q_{ij}^{-(m)}) q_{ji}, \tag{4}$$

$$\varphi_k(q_{ij}^{(m)}) = q_{jk} \varphi_j(q_{ik}^{(m)}) q_{ij}^{(m)} \varphi_i^m(q_{kj}) q_{ik}^{-(m)}. \tag{5}$$

Proof. For (1), we have from (N2),

$$q_{ij}^{-(m)} = \begin{cases} \varphi_i^{m-1}(q_{ji}) \cdots \varphi_i(q_{ji}) q_{ji} = \prod_{l=m-1}^{1} \varphi_i^l(q_{ji}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \varphi_i^m(q_{ij}) \cdots \varphi_i^{-2}(q_{ij}) \varphi_i^{-1}(q_{ij}) = \prod_{l=m}^{-1} \varphi_i^l(q_{ij}), & \text{if } m < 0. \end{cases}$$

So we get

$$\varphi_i^{-m}(q_{ij}^{-(m)}) = \begin{cases} \varphi_i^{-1}(q_{ji}) \cdots \varphi_i^{-m}(q_{ji}) = \prod_{l=-1}^{-m} \varphi_i^l(q_{ji}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ q_{ij} \varphi_i(q_{ij}) \cdots \varphi_i^{-m-1}(q_{ij}) = \prod_{l=1}^{-m-1} \varphi_i^l(q_{ij}), & \text{if } m < 0, \end{cases}$$

which is exactly $q_{ij}^{(-m)}$.

For (2), the case $m = 0$ is clear. Assume that $m > 0$. Put $q := q_{ij}$ for simplicity. Then we have

$$\begin{aligned} \varphi_j \varphi_i^m &= \varphi_j \varphi_i^{m-1} \varphi_i \\ &= \mathrm{I}(q^{(m-1)}) \varphi_i^{m-1} \varphi_j \varphi_i \quad \text{by induction on } m \\ &= \mathrm{I}(q^{(m-1)}) \varphi_i^{m-1} \mathrm{I}(q) \varphi_i \varphi_j \quad \text{by (G2)} \\ &= \mathrm{I}(q^{(m-1)}) \mathrm{I}\big(\varphi_i^{m-1}(q)\big) \varphi_i^m \varphi_j \\ &= \mathrm{I}(q^{(m)}) \varphi_i^m \varphi_j. \end{aligned}$$

Also, $(\varphi_j \varphi_i^m)^{-1} = (\mathrm{I}(q_{ij}^{(m)}) \varphi_i^m \varphi_j)^{-1}$ for $m > 0$, and so

$$\varphi_i^{-m} \varphi_j^{-1} = \varphi_j^{-1} \varphi_i^{-m} (\mathrm{I}(q_{ij}^{-(m)}) = \varphi_j^{-1} \mathrm{I}\big(\varphi_i^{-m}(q_{ij}^{-(m)})\big) \varphi_i^{-m} = \varphi_j^{-1} \mathrm{I}(q_{ij}^{(-m)}) \varphi_i^{-m},$$

by (1). Hence we get $\varphi_j \varphi_i^{-m} = \mathrm{I}(q_{ij}^{(-m)}) \varphi_i^{-m} \varphi_j$, and (2) holds for all $m \in \mathbb{Z}$.

For (3), when $j \geq i$, using (2), we have

$$\begin{aligned}
\varphi_j \varphi_i^{(\boldsymbol{\alpha})_i} &= \varphi_j \varphi_1^{\alpha_1} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \varphi_j \varphi_2^{\alpha_2} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \mathrm{I}(q_{2j}^{(\alpha_2)}) \varphi_2^{\alpha_2} \varphi_j \varphi_3^{\alpha_3} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&\cdots\cdots \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \mathrm{I}(q_{2j}^{(\alpha_2)}) \varphi_2^{\alpha_2} \mathrm{I}(q_{3j}^{(\alpha_3)}) \varphi_3^{\alpha_3} \cdots \mathrm{I}(q_{i-1,j}^{(\alpha_{i-1})}) \varphi_{i-1}^{\alpha_{i-1}} \varphi_j \\
&= \mathrm{I}(\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})) \varphi^{(\boldsymbol{\alpha})_i} \varphi_j. \quad \text{(Note } \varphi^{(\boldsymbol{\alpha})_0} = \mathrm{id} \text{ when } i = 1\text{)}
\end{aligned}$$

When $j < i$, we have

$$\begin{aligned}
\varphi_j \varphi_i^{(\boldsymbol{\alpha})_i} &= \varphi_j \varphi_1^{\alpha_1} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \varphi_j \varphi_2^{\alpha_2} \cdots \varphi_j^{\alpha_j} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&\cdots\cdots \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \cdots \mathrm{I}(q_{j-1,j}^{(\alpha_{j-1})}) \varphi_{j-1}^{\alpha_{j-1}} \mathrm{I}(q_{jj}^{(\alpha_j)}) \varphi_j^{\alpha_j} \varphi_j \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \cdots \mathrm{I}(q_{j-1,j}^{(\alpha_{j-1})}) \varphi_{j-1}^{\alpha_{j-1}} \varphi^{\alpha_j+1} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}(\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})) \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}. \quad \text{(Note } \varphi^{(\boldsymbol{\alpha})_0} = \mathrm{id} \text{ when } j = 1\text{)}
\end{aligned}$$

For the first formula of (4), the case $m = 0$ is clear. We put $q := q_{ij}$, $p := q^{-1}$ and $\varphi := \varphi_i$ for simplicity. For $m > 0$, we have

$$\begin{aligned}
q^{(m+1)} &= q\varphi(q)\varphi^2(q) \cdots \varphi^m(q) \\
&= q\varphi\big(q\varphi(q) \cdots \varphi^{m-1}(q)\big) = q\varphi(q^{(m)}).
\end{aligned}$$

For $m = -1$, we have $q^{(-1+1)} = 1$, while $q\varphi(q^{(-1)}) = q\varphi\varphi^{-1}(p) = 1$. For $m < -1$, we have

$$\begin{aligned}
q^{(m+1)} &= \varphi^{-1}(p)\varphi^{-2}(p) \cdots \varphi^{m+1}(p) \\
&= qp\varphi^{-1}(p)\varphi^{-2}(p) \cdots \varphi^{m+1}(p) \\
&= q\varphi\big(\varphi^{-1}(p)\varphi^{-2}(p) \cdots \varphi^m(p)\big) = q\varphi(q^{(m)}).
\end{aligned}$$

The second formula follows from the first since $q_{ij}^{-(m+1)} = (q_{ij}^{(m+1)})^{-1}$.

For (5), the case $m=0$ is clear. Assume that $m>0$. Then we have

$$\begin{aligned}
&\varphi_k(q_{ij}^{(m)})\\
&\qquad = \varphi_k(q_{ij})\varphi_k\varphi_i(q_{ij}^{(m-1)}) \quad \text{by (4)}\\
&\qquad = \varphi_k(q_{ij})q_{ik}\varphi_i\varphi_k(q_{ij}^{(m-1)})q_{ki} \quad \text{by (G2)}\\
&\qquad = q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{kj})q_{ki}q_{ik}\varphi_i\big(q_{jk}\varphi_j(q_{ik}^{(m-1)})q_{ij}^{(m-1)}\varphi_i^{m-1}(q_{kj})(q_{ik}^{-(m-1)})\big)q_{ki}
\end{aligned}$$

by (G3) and induction on m

$$\begin{aligned}
&= q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i\varphi_j(q_{ik}^{(m-1)})\varphi_i(q_{ij}^{(m-1)})\varphi_i^m(q_{kj})\varphi_i(q_{ik}^{-(m-1)})q_{ki}\\
&= q_{jk}\varphi_j(q_{ik})q_{ij}q_{ji}\varphi_j\varphi_i(q_{ik}^{(m-1)})q_{ij}\varphi_i(q_{ij}^{(m-1)})\varphi_i^m(q_{kj})q_{ik}^{-(m)}
\end{aligned}$$

by (G2) and (3)

$$\begin{aligned}
&= q_{jk}\varphi_j(q_{ik})\varphi_j\varphi_i(q_{ik}^{(m-1)})q_{ij}^{(m)}\varphi_i^m(q_{kj})q_{ik}^{-(m)} \quad \text{by (4)}\\
&= q_{jk}\varphi_j(q_{ik}^{(m)})q_{ij}^{(m)}\varphi_i^m(q_{kj})q_{ik}^{-(m)} \quad \text{by (4).}
\end{aligned}$$

Also, one has $\big(\varphi_k(q_{ij}^{(m)})\big)^{-1}=\big(q_{jk}\varphi_j(q_{ik}^{(m)})q_{ij}^{(m)}\varphi_i^m(q_{kj})q_{ik}^{-(m)}\big)^{-1}$ for $m>0$, and so $\varphi_k(q_{ji}^{-(m)}) = q_{ik}^{(m)}\varphi_i^m(q_{jk})q_{ij}^{-(m)}\varphi_j(q_{ik}^{-(m)})q_{kj}$. Applying φ_i^{-m} in both hands, we get

$$\begin{aligned}
\varphi_i^{-m}\varphi_k(q_{ij}^{-(m)}) &= \varphi_i^{-m}\big(q_{ik}^{(m)}\varphi_i^m(q_{jk})q_{ij}^{-(m)}\varphi_j(q_{ik}^{-(m)})q_{kj}\big)\\
&= \varphi_i^{-m}(q_{ik}^{(m)})q_{jk}q_{ij}^{(-m)}\varphi_i^{-m}\varphi_j(q_{ik}^{-(m)})\varphi_i^{-m}(q_{kj}) \quad \text{by (1).}
\end{aligned}$$

Then, by (1) and (2), we have

$$I(q_{ik}^{-(-m)})\varphi_k(q_{ij}^{(-m)}) = q_{ik}^{-(-m)}q_{jk}q_{ij}^{(-m)}I(q_{ij}^{-(-m)})\varphi_j(q_{ik}^{(-m)})\varphi_i^{-m}(q_{kj}),$$

and we obtain

$$\varphi_k(q_{ij}^{(-m)}) = q_{jk}\varphi_j(q_{ik}^{(-m)})q_{ij}^{(-m)}\varphi_i^{-m}(q_{kj})q_{ik}^{-(-m)} \quad \text{for } m>0.$$

Hence, (5) holds for all $m\in\mathbb{Z}$. $\square$

Now we are ready to state our theorem.

Theorem 3.3. *Let $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ be a $\mathbb{Z}^n$-grading triple and let $R_{\boldsymbol{\varphi},\boldsymbol{q}} := \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} Rt_{\boldsymbol{\alpha}}$ be a free left R-module with basis $\{t_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{Z}^n\}$. Then there exists a unique associative multiplication on $R_{\boldsymbol{\varphi},\boldsymbol{q}}$ such that, for $t_i := t_{\boldsymbol{\varepsilon}_i}$, $i = 1, \ldots, n$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $r \in R$,*

$$t_{\boldsymbol{\alpha}} = t_1^{\alpha_1} \cdots t_n^{\alpha_n}, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i r = \varphi_i(r) t_i \quad \text{and} \quad t_j t_i = q_{ij} t_i t_j. \tag{3.4}$$

Moreover, for $rt_{\boldsymbol{\alpha}}, r't_{\boldsymbol{\beta}} \in R_{\boldsymbol{\varphi},\boldsymbol{q}}$, we have

$$rt_{\boldsymbol{\alpha}} r' t_{\boldsymbol{\beta}} = r\varphi^{\boldsymbol{\alpha}}(r') q_{\boldsymbol{\alpha},\boldsymbol{\beta}} t_{\boldsymbol{\alpha}+\boldsymbol{\beta}},$$

where $\varphi^{\boldsymbol{\alpha}}$ and $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are defined in (N3) and (N6). In particular, $R_{\boldsymbol{\varphi},\boldsymbol{q}}$ is a crossed product algebra $R * \mathbb{Z}^n$ with

$$\begin{array}{llll} (\textit{action}) & \sigma : \mathbb{Z}^n \longrightarrow Aut_F(R) & \textit{by} & \sigma(\boldsymbol{\alpha}) = \varphi^{\boldsymbol{\alpha}} \\ (\textit{twisting}) & \tau : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow U(R) & \textit{by} & \tau(\boldsymbol{\alpha}, \boldsymbol{\beta}) = q_{\boldsymbol{\alpha},\boldsymbol{\beta}}. \end{array}$$

Conversely, for any crossed product algebra $R * \mathbb{Z}^n$, there exists a $\mathbb{Z}^n$-grading triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ such that $R * \mathbb{Z}^n = R_{\boldsymbol{\varphi},\boldsymbol{q}}$.

Proof. We first consider a crossed product algebra $R * \mathbb{Z}$. Let $t := \overline{1} \in R * \mathbb{Z}$. Then, t^m is a unit in $R\overline{m}$ for all $m \in \mathbb{Z}$. Using the diagonal basis change, one can obtain an R-basis $\{t^m \mid m \in \mathbb{Z}\}$. So we have $t^m t^l = t^{m+l}$ for all $m, l \in \mathbb{Z}$. Hence, $R * \mathbb{Z} = R\mathbb{Z}$ is a skew group algebra. Let ψ be the action of 1, i.e., $t(r1) = \psi(r)t$ for $r \in R$. (Note that $1 = \overline{0}$.) Then the action of m is ψ^m, i.e.,

$$t^m(r1) = \psi^m(r)t^m.$$

Conversely, it is clear that any F-automorphism ψ of R determines a skew group algebra $R\mathbb{Z}$ by the action $m \mapsto \psi^m$ (see Remark 1.3). We denote this $R\mathbb{Z}$ by $R[t; \psi]$.

Let $R^{(1)} := R[t_1; \psi_1]$ where $\psi_1 = \varphi_1$. Let ψ_2 be a graded F-automorphism ψ_2 of $R^{(1)}$ and $R^{(2)} := R^{(1)}[t_2; \psi_2]$. Then, by Lemma 3.1, we get $R^{(2)} = (R\mathbb{Z})\mathbb{Z} = R * \mathbb{Z}^2$. Repeating this process n times, one can construct $R * \mathbb{Z}^n$ inductively. Namely, for a crossed product algebra $R^{(k-1)} = R * \mathbb{Z}^{k-1}$, if we specify an F-graded automorphism ψ_k of $R^{(k-1)}$, then

$$R^{(k)} := R^{(k-1)}[t_k; \psi_k] = R * \mathbb{Z}^k,$$

and we obtain $R^{(n)} = R * \mathbb{Z}^n$. Thus, our task is to specify ψ_k on $R^{(k-1)}$ and to show that ψ_k is a graded F-automorphism where $k \geq 2$. We note that

$$\{t_1^{\alpha_1} \cdots t_{k-1}^{\alpha_{k-1}} \mid (\alpha_1, \ldots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}\}$$

is a basis of the free R-module $R^{(k-1)}$. For convenience, we put

$$t^{(\boldsymbol{\alpha})_k} = t_1^{\alpha_1} \cdots t_{k-1}^{\alpha_{k-1}},$$

and define an F-linear transformation ψ_k on $R^{(k-1)}$ by

$$\psi_k(rt^{(\boldsymbol{\alpha})_k}) = \varphi_k(r) \left[\prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) \right] t^{(\boldsymbol{\alpha})_k} \quad \text{for} \quad r \in R,$$

which is clearly graded. If $\psi_k(rt^{(\boldsymbol{\alpha})_k}) = 0$, then $\varphi_k(r) = 0$, and hence $r = 0$, and so ψ_k is injective. Since

$$\psi_k \left(\varphi_k^{-1} \left(r \left[\prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) \right]^{-1} \right) t^{(\boldsymbol{\alpha})_k} \right) = rt^{(\boldsymbol{\alpha})_k},$$

ψ_k is surjective. Therefore, ψ_k is an F-linear graded isomorphism on $R^{(k-1)}$. So it remains to prove that ψ_k is a homomorphism. For this purpose, we use a well-known fact.

3.5. *Let A and B be unital associative algebras over F and f a F-linear map from A into B. Let $\{t_i\}_{i \in I}$ be a generating set of the F-algebra A. Then, f is a homomorphism if and only if $f(t_i y) = f(t_i) f(y)$ for all $i \in I$ and $y \in A$. Moreover, if $\{t_i^{\pm 1}\}_{i \in I}$ is a generating set of A, then f is a homomorphism if and only if $f(t_i y) = f(t_i) f(y)$ and $f(t_i^{-1}) = f(t_i)^{-1}$ for all $i \in I$ and $y \in A$.*

We have a generating set $R \cup \{t_1^{\pm 1}, \ldots, t_{k-1}^{\pm 1}\}$ of $R^{(k-1)}$ over F, and

$$\begin{aligned} \psi_k(t_j^{-1}) &= q_{jk}^{(-1)} t_j^{-1} = \varphi_j^{-1}(q_{kj}) t_j^{-1} \\ &= (t_j \varphi_j^{-1}(q_{jk}))^{-1} = (q_{jk} t_j)^{-1} = \psi_k(t_j)^{-1}. \end{aligned}$$

So, by 3.5, we only need to show that, for all $r, r' \in R$ and $1 \leq j \leq k-1$,

(A) $$\psi_k(rr't^{(\boldsymbol{\alpha})_k}) = \psi_k(r) \psi_k(r't^{(\boldsymbol{\alpha})_k}),$$

(B) $$\psi_k(t_j rt^{(\boldsymbol{\alpha})_k}) = \psi_k(t_j) \psi_k(rt^{(\boldsymbol{\alpha})_k}).$$

For (A), we have

$$\begin{aligned} \psi_k(rr't^{(\boldsymbol{\alpha})_k}) &= \varphi_k(rr') \prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) t^{(\boldsymbol{\alpha})_k} \\ &= \varphi_k(r) \varphi_k(r') \prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(d_{ik}^{(\alpha_i)}) t^{(\boldsymbol{\alpha})_k} \\ &= \psi_k(r) \psi_k(r't^{(\boldsymbol{\alpha})_k}). \end{aligned}$$

For (B), we first note that there is the embedding of $R^{(j)}$ into $R^{(k-1)}$ for $1 \le j \le k-1$, and so

$$t_j t^{(\boldsymbol{\alpha})_j} = \psi_j(t^{(\boldsymbol{\alpha})_j})t_j = \left[\prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)})\right] t^{(\boldsymbol{\alpha})_j} t_j.$$

Thus we have

$$\begin{aligned}
\psi_k(t_j r t^{(\boldsymbol{\alpha})_k}) &= \psi_k\big(\varphi_j(r) t_j t^{(\boldsymbol{\alpha})_k}\big) \\
&= \psi_k\big(\varphi_j(r)\psi_j(t^{(\boldsymbol{\alpha})_j}) t_j^{\alpha_j+1} \cdots t_{k-1}^{\alpha_{k-1}}\big) \\
&= \psi_k\left(\varphi_j(r)\left[\prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)})\right] t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k}\right) \\
&= \varphi_k\varphi_j(r)\left[\prod_{i=1}^{j-1} \varphi_k\varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)})\right]\left[\prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i+\delta_{ij})})\right] t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k} \\
:&= ABC t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k},
\end{aligned}$$

where $A = \varphi_k\varphi_j(r)$, $B = \prod_{i=1}^{j-1} \varphi_k\varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)})$ and $C = \prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i+\delta_{ij})})$. First of all, we have

$$A = \varphi_k\varphi_j(r) = q_{jk}\varphi_j\varphi_k(r)q_{kj} \quad \text{by (G2)}.$$

Secondly, by Lemma 3.2(3) and (5), we have

$$\begin{aligned}
&\varphi_k\varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)}) \\
&= \left[\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]\varphi^{(\boldsymbol{\alpha})_i}\varphi_k(q_{ij}^{(\alpha_i)})\left[\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(d_{lk}^{(\alpha_l)})\right]^{-1} \\
&= \left[\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]\varphi^{(\boldsymbol{\alpha})_i}\big(q_{jk}\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}\varphi_i^{\alpha_i}(q_{kj})q_{ik}^{-(\alpha_i)}\big)\left[\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]^{-1}.
\end{aligned}$$

Note that

$$\varphi^{(\boldsymbol{\alpha})_i}(q_{ki}^{-(\alpha_i)})\left[\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]^{-1} = \left[\prod_{l=1}^{i} \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]^{-1}$$
$$\text{and} \quad \varphi^{(\boldsymbol{\alpha})_i}\varphi_i^{\alpha_i}(q_{kj}) = \varphi^{(\boldsymbol{\alpha})_{i+1}}(q_{kj}).$$

So we have

$$\begin{aligned}
&\big(\varphi_k\varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)})\big) \\
&= \left[\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]\varphi^{(\boldsymbol{\alpha})_i}(q_{jk})\varphi^{(\boldsymbol{\alpha})_i}\big(\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}\big)\varphi^{(\boldsymbol{\alpha})_{i+1}}(q_{kj})\left[\prod_{l=1}^{i} \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]^{-1}.
\end{aligned}$$

Thus, after cancellations, we get

$$
\begin{aligned}
B &= \prod_{i=1}^{j-1} \varphi_k \varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)}) \\
&= q_{jk}\left[\prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}\big(\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}\big)\right]\varphi^{(\boldsymbol{\alpha})_j}(q_{kj})\left[\prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\right]^{-1}.
\end{aligned}
$$

Thirdly, we have

$$
\begin{aligned}
C &= \prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i+\delta_{ij})}) \\
&= \left[\prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\right]\varphi^{(\boldsymbol{\alpha})_j}(q_{jk}^{(\alpha_j+1)}) \prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)}) \\
&= \left[\prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\right]\varphi^{(\boldsymbol{\alpha})_j}\big(q_{jk}\varphi_j(q_{jk}^{(\alpha_j)})\big) \prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)}),
\end{aligned}
$$

by Lemma 3.2(4). Consequently, after cancellations and noting that $q_{ii} = 1$, we obtain

$$
\begin{aligned}
\psi_k(t_j r t^{(\boldsymbol{\alpha})_k}) &= ABCt^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k} \\
&= q_{jk}\varphi_j\varphi_k(r)\left[\prod_{i=1}^{j} \varphi^{(\boldsymbol{\alpha})_i}\big(\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}\big)\right]\left[\prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)})\right]t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k}. \qquad (*)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\psi_k(t_j)\psi_k(rt^{(\boldsymbol{\alpha})_k}) &= q_{jk}t_j\varphi_k(r)\left[\prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\right]t^{(\boldsymbol{\alpha})_k} \\
&= q_{jk}\varphi_j\left[\varphi_k(r)\prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\right]t_j t^{(\boldsymbol{\alpha})_k} \\
&= q_{jk}\varphi_j\varphi_k(r)\left[\prod_{i=1}^{k-1} \varphi_j\varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\right]\left[\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\right]t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k}.
\end{aligned}
$$

We rewrite $D := \prod_{i=1}^{k-1} \varphi_j\varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})$. To find an expression for D, we use the following lemma:

Lemma 3.6. *Let A be a unital associative algebra, $a_0 = 1, a_1, \dots, a_k \in A$ units and $b_1, \dots, b_k \in A$. Then we have*

$$\prod_{i=1}^{k} \Big(I\big(\prod_{l=1}^{i-1} a_l\big)(b_i) \Big) = \big(\prod_{i=1}^{k} b_i a_i\big) b_k \big(\prod_{l=1}^{k-1} a_l\big)^{-1}. \tag{1}$$

$$\prod_{i=j+1}^{k} \Big(I\big(\prod_{l=1}^{j-1} a_l\big)(b_i) \Big) = I\big(\prod_{l=1}^{j-1} a_l\big)\big(\prod_{i=j+1}^{k} b_i\big). \tag{2}$$

Proof. (1) is straightforward and (2) is obvious. □

By Lemma 3.2(3), we have, for $i \leq j$,

$$\varphi_j \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) = \mathrm{I}\Big(\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)}) \Big) \big(\varphi^{(\boldsymbol{\alpha})_i} \varphi_j (q_{ik}^{(\alpha_i)}) \big).$$

So, by Lemma 3.6(1), we get using $q_{jj} = 1$ that

$$\prod_{i=1}^{j} \varphi_j \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) = \Big[\prod_{i=1}^{j} \varphi^{(\boldsymbol{\alpha})_i} \big(\varphi_j (q_{ik}^{(\alpha_i)})(q_{ij}^{(\alpha_i)}) \big) \Big] \Big[\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)}) \Big]^{-1}.$$

By Lemma 3.2(3), we have, for $j < i$,

$$\varphi_j \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) = \mathrm{I}\Big(\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)}) \Big) \big(\varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)}) \big).$$

So, by Lemma 3.6(2), we get

$$\prod_{i=j+1}^{k-1} \varphi_j \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) = \mathrm{I}\Big(\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)}) \Big) \Big(\prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)}) \Big).$$

Hence we get

$$\begin{aligned} D &= \prod_{i=1}^{k-1} \varphi_j \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) \\ &= \Big[\prod_{i=1}^{j} \varphi^{(\boldsymbol{\alpha})_i} \big(\varphi_j (q_{ik}^{(\alpha_i)}) q_{ij}^{(\alpha_i)} \big) \Big] \Big[\prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)}) \Big] \Big[\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)}) \Big]^{-1}. \end{aligned}$$

Consequently, we obtain

$$\psi_k(t_j)\psi_k(rt^{(\boldsymbol{\alpha})_k})$$
$$= q_{jk}\varphi_j\varphi_k(r)\Big[\prod_{i=1}^{j} \varphi^{(\boldsymbol{\alpha})_i}\big(\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}\big)\Big]\Big[\prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)})\Big]t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k},$$

which is exactly $(*)$. Hence we have shown (B) and constructed a crossed product algebra $R * \mathbb{Z}^k = R^{(k)}$ for $k = 1, \dots, n$ from $(R, \boldsymbol{\varphi}, \boldsymbol{q})$.

Let us put $R_{\boldsymbol{\varphi},\boldsymbol{q}} := R^{(n)} = \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} Rt_{\boldsymbol{\alpha}}$ where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $t_{\boldsymbol{\alpha}} = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$. Since $\psi_k \mid_R = \varphi_k$ for $k = 1, \dots, n$, we have $t_i r = \varphi_i(r)t_i$. Also, we have $t_j t_i = \psi_j(t_i)t_j = q_{ij}t_it_j$ for $1 \le i < j \le n$, and so $t_jt_i = q_{ij}t_it_j$ for all $1 \le i, j \le n$. Hence, our $R_{\boldsymbol{\varphi},\boldsymbol{q}}$ satisfies (3.4). The uniqueness of the multiplication on $R_{\boldsymbol{\varphi},\boldsymbol{q}}$ is clear since $R \cup \{t_1^{\pm1}, \dots, t_n^{\pm1}\}$ is a generating set of $R_{\boldsymbol{\varphi},\boldsymbol{q}}$.

Now, one can easily check that $\psi_j^{\alpha_j}(t^{(\boldsymbol{\beta})_j}) = q_{\boldsymbol{\varepsilon}_j,\boldsymbol{\beta}}^{(\alpha_j)} t^{(\boldsymbol{\beta})_j}$. So for $rt_{\boldsymbol{\alpha}}, r't_{\boldsymbol{\beta}} \in R_{\boldsymbol{\varphi},\boldsymbol{q}}$, we get

$$\begin{aligned}
rt_{\boldsymbol{\alpha}}r't_{\boldsymbol{\beta}} &= r\varphi^{\boldsymbol{\alpha}}(r')t_{\boldsymbol{\alpha}}t_{\boldsymbol{\beta}}\\
&= r\varphi^{\boldsymbol{\alpha}}(r')t^{(\boldsymbol{\alpha})_n}t_n^{\alpha_n}t^{(\boldsymbol{\beta})_n}t_n^{\beta_n}\\
&= r\varphi^{\boldsymbol{\alpha}}(r')t^{(\boldsymbol{\alpha})_n}\psi_n^{\alpha_n}(t^{(\boldsymbol{\beta})_n})t_n^{\alpha_n+\beta_n}\\
&= r\varphi^{\boldsymbol{\alpha}}(r')t^{(\boldsymbol{\alpha})_n}q_{\boldsymbol{\varepsilon}_n,\boldsymbol{\beta}}^{(\alpha_n)}t^{(\boldsymbol{\beta})_n}t_n^{\alpha_n+\beta_n}\\
&= r\varphi^{\boldsymbol{\alpha}}(r')\varphi^{(\boldsymbol{\alpha})_n}(q_{\boldsymbol{\varepsilon}_n,\boldsymbol{\beta}}^{(\alpha_n)})t^{(\boldsymbol{\alpha})_n}t^{(\boldsymbol{\beta})_n}t_n^{\alpha_n+\beta_n}\\
&\cdots\cdots\\
&= r\varphi^{\boldsymbol{\alpha}}(r')\varphi^{(\boldsymbol{\alpha})_n}(q_{\boldsymbol{\varepsilon}_n,\boldsymbol{\beta}}^{(\alpha_n)})\cdots\varphi^{(\boldsymbol{\alpha})_2}(q_{\boldsymbol{\varepsilon}_2,\boldsymbol{\beta}}^{(\alpha_2)})t_1^{\alpha_1+\beta_1}\cdots t_n^{\alpha_n+\beta_n}\\
&= r\varphi^{\boldsymbol{\alpha}}(r')q_{\boldsymbol{\alpha},\boldsymbol{\beta}}t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}.
\end{aligned}$$

Conversely, for any crossed product algebra $R * \mathbb{Z}^n = (R, \mathbb{Z}^n, \tau, \sigma) = \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} R\overline{\boldsymbol{\alpha}}$, we take a new R-basis $\{t_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{Z}^n\}$ of $R * \mathbb{Z}^n$ where $t_{\boldsymbol{\alpha}} = \overline{\boldsymbol{\varepsilon}}_1{}^{\alpha_1} \cdots \overline{\boldsymbol{\varepsilon}}_n{}^{\alpha_n}$. We set $q_{ij} := \tau(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i)$ for $1 \le i \le j \le n$, $q_{ji} := q_{ij}^{-1}$ and $\varphi_i := \sigma_{\boldsymbol{\varepsilon}_i}$. Note that $\tau(\boldsymbol{\varepsilon}_i, \boldsymbol{\varepsilon}_j) = 1$ for $i \le j$. Then one can check that the triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ is a $\mathbb{Z}^n$-grading triple:

(G1) is clear. Let $t_i := \overline{\boldsymbol{\varepsilon}}_i$ for $i = 1, \dots, n$. Then, for $i \le j$ and $r \in R$, we have $t_jt_ir = \varphi_j\varphi_i(r)t_jt_i = \varphi_j\varphi_i(r)q_{ij}t_it_j$ and $t_jt_ir = q_{ij}t_it_jr = q_{ij}\varphi_i\varphi_j(r)t_it_j$. Hence, $\varphi_j\varphi_i(r)q_{ij} = q_{ij}\varphi_i\varphi_j(r)$, i.e., (G2) holds. For $i \le j \le k$, we have $t_kt_jt_i = t_kq_{ij}t_it_j = \varphi_k(q_{ij})q_{ik}t_it_kt_j = \varphi_k(q_{ij})q_{ik}\varphi_i(q_{jk})t_it_jt_k$ and $t_kt_jt_i = q_{jk}t_jt_kt_i = q_{jk}\varphi_j(q_{ik})t_jt_it_k = q_{jk}\varphi_j(q_{ik})q_{ij}t_it_jt_k$. Hence, $\varphi_k(q_{ij})q_{ik}\varphi_i(q_{jk}) = q_{jk}\varphi_j(q_{ik})q_{ij}$, i.e., (G3) holds.

Finally, it is clear that $R*\mathbb{Z}^n = \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} Rt_{\boldsymbol{\alpha}}$ satisfies (3.4). Therefore, we obtain $R*\mathbb{Z}^n = R_{\boldsymbol{\varphi},\boldsymbol{q}}$. □

Thus the following is clear:

Corollary 3.7. *Let $(D, \boldsymbol{\varphi}, \boldsymbol{q})$ be a division $\mathbb{Z}^n$-grading triple. Then, $D_{\boldsymbol{\varphi},\boldsymbol{q}}$ is a division $\mathbb{Z}^n$-graded algebra. Conversely, for any division $\mathbb{Z}^n$-graded algebra A, there exists a division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$ such that $A = D_{\boldsymbol{\varphi},\boldsymbol{q}}$.*

Remark. What we have shown in Theorem 3.3 can be written in the following way:

Let $B := \{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n\}$ and $C := \{(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i) \mid 1 \le i < j \le n\}$. Suppose that maps

$$\sigma : B \longrightarrow \mathrm{Aut}_F(R) \quad \text{and} \quad \tau : C \longrightarrow U(R)$$

satisfy

(a) $$\sigma_{\boldsymbol{\varepsilon}_j}\sigma_{\boldsymbol{\varepsilon}_i} = \mathrm{I}\big(\tau(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i)\big)\sigma_{\boldsymbol{\varepsilon}_i}\sigma_{\boldsymbol{\varepsilon}_j} \quad \text{and}$$

(b) $$\sigma_{\boldsymbol{\varepsilon}_k}\big(\tau(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i)\big)\tau(\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_i)\sigma_{\boldsymbol{\varepsilon}_i}\big(\tau(\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_j)\big) = \tau(\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_j)\sigma_{\boldsymbol{\varepsilon}_j}\big(\tau(\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_i)\big)\tau(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i)$$

for all $1 \le i < j < k \le n$. Then there exist unique action $\tilde{\sigma} : \mathbb{Z}^n \longrightarrow \mathrm{Aut}_F(R)$ and twisting $\tilde{\tau} : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow U(R)$ such that $\tilde{\sigma}\mid_B = \sigma$, $\tilde{\tau}\mid_C = \tau$ and

(c) $$\tilde{\tau}(\alpha_1\boldsymbol{\varepsilon}_1 + \cdots + \alpha_i\boldsymbol{\varepsilon}_i, \alpha_j\boldsymbol{\varepsilon}_j + \cdots + \alpha_n\boldsymbol{\varepsilon}_n) = 1 \quad \text{for all } 1 \le i \le j \le n.$$

Conversely, for any crossed product algebra $R * \mathbb{Z}^n$, we can use the diagonal basis change so that the action and twisting satisfy (a), (b) and (c).

In a certain situation, the condition (G3) for a $\mathbb{Z}^n$-grading triple is not needed.

Lemma 3.8. *Let R be a unital associative algebra over F, $\boldsymbol{\varphi} = \big(I(d_1), \ldots, I(d_n)\big)$ an n-tuple of inner automorphisms φ_i of R for some $d_1, \ldots, d_n \in U(R)$ and $\boldsymbol{q} = (q_{ij})$ an $n \times n$ matrix over F. Suppose that a triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ satisfies (G1) and (G2). Then, $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ is a $\mathbb{Z}^n$-grading triple.*

Proof. We only need to check (G3). By (G1) and (G2), we have, for all $1 \le i, j \le n$, $I(d_j)I(d_i) = I(q_{ij})I(d_i)I(d_j)$. So for all $r \in R$, $d_jd_ird_i^{-1}d_j^{-1} = q_{ij}d_id_jrd_j^{-1}d_i^{-1}q_{ji}$ and hence $rd_i^{-1}d_j^{-1}q_{ij}d_id_j = d_i^{-1}d_j^{-1}q_{ij}d_id_jr$, i.e., $d_i^{-1}d_j^{-1}q_{ij}d_id_j =: c_{ij}$ is in the centre of R. Note that $c_{ji}^{-1} = c_{ij}$. Thus we have

$$q_{ij} = c_{ij}[d_j, d_i]$$

for all i, j, where $[d_j, d_i] = d_j^{-1}d_i^{-1}d_jd_i$. Using this identity, we get (G3): for all $1 \le i < j < k \le n$,

$$\begin{aligned} &q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{kj})q_{ki} \\ =&c_{jk}[d_k, d_j]d_jc_{ik}[d_k, d_i]d_j^{-1}c_{ij}[d_j, d_i]d_ic_{kj}[d_j, d_k]d_i^{-1}c_{ki}[d_i, d_k] \\ =&d_kc_{ij}[d_j, d_i]d_k^{-1} = \varphi_k(q_{ij}). \quad \square \end{aligned}$$

By this lemma, if R is a finite dimensional central simple associative algebra, the defining identities of a $\mathbb{Z}^n$-grading triple are just (G1) and (G2).

Remark 3.9. (1) For a $\mathbb{Z}^n$-grading triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$, if $\boldsymbol{\varphi} = \mathbf{1} := (\mathrm{id}, \dots, \mathrm{id})$, then the crossed product algebra $R_{\mathbf{1},\boldsymbol{q}}$ has the trivial action by Theorem 3.3. So, $R_{\mathbf{1},\boldsymbol{q}} = R^t[\mathbb{Z}^n]$ is a twisted group algebra.

(2) For a $\mathbb{Z}^n$-grading triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$, if $\boldsymbol{q} = \mathbf{1}_n = \mathbf{1} := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$, then a crossed product algebra $R_{\boldsymbol{\varphi},\mathbf{1}}$ has the trivial twisting by Theorem 3.3. So, $R_{\boldsymbol{\varphi},\mathbf{1}} = R\mathbb{Z}^n$ is a skew group algebra.

(3) By (G2), $(R, \boldsymbol{\varphi}, \mathbf{1})$ is a $\mathbb{Z}^n$-grading triple if and only if

$$(*) \qquad \varphi_j \varphi_i = \varphi_i \varphi_j \quad \text{for all } i, j.$$

Finally, we give some examples.

Example. (1) Let $F_{\boldsymbol{q}}$ be an arbitrary quantum torus and R an arbitrary associative algebra. Then it is easy to see that $R \otimes_F F_{\boldsymbol{q}}$ is a predivision $\mathbb{Z}^n$-graded associative algebra (division $\mathbb{Z}^n$-graded if R is a division algebra) and is isomorphic to $R_{\mathbf{1},\boldsymbol{q}}$. Note also if R is a field, then this example becomes a quantum torus over R. Conversely, for a division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$, if $\boldsymbol{\varphi} = \mathbf{1}$, then $I(q_{ij}) = \mathrm{id}$ for all q_{ij}, by (G2). Hence q_{ij} is in the centre of D, say K, and we can show that $D_{\mathbf{1},\boldsymbol{q}} \cong D \otimes_K K_{\boldsymbol{q}}$. Therefore, $D_{\boldsymbol{\varphi},\boldsymbol{q}}$ is a tensor product with D and some quantum torus if and only if $\boldsymbol{\varphi} = \mathbf{1}$.

(2) Let $Q = \langle \boldsymbol{i}, \boldsymbol{j} \rangle$ be a quaternion algebra over a field, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the standard generators, $\boldsymbol{\varphi} = \boldsymbol{\varphi}_3 = \big(\mathrm{I}(\boldsymbol{i}), \mathrm{I}(\boldsymbol{j}), \mathrm{I}(\boldsymbol{ij})\big)$ and $\mathbf{1} = \mathbf{1}_3$. Then one can easily check $(*)$ in Remark 3.9(3), and hence $Q_{\boldsymbol{\varphi},\mathbf{1}}$ is a predivision $\mathbb{Z}^3$-graded associative algebra.

(3) Let $K = \mathbb{Q}(\zeta_5)$ be a cyclotomic extension of $\mathbb{Q}$ (the field of rational numbers) where $\zeta := \zeta_5$ is a primitive 5th root of unity, and φ the automorphism of K defined by $\varphi(\zeta) = \zeta^2$. Let $\boldsymbol{\varphi} = (\varphi, \varphi^2, \varphi^3)$ and

$$\boldsymbol{q} = \begin{pmatrix} 1 & \zeta & \zeta^2 \\ \zeta^{-1} & 1 & \zeta^{-1} \\ \zeta^3 & \zeta & 1 \end{pmatrix}.$$

Then one can easily check that $(K, \boldsymbol{\varphi}, \boldsymbol{q})$ is a division $\mathbb{Z}^3$-grading triple, and hence $K_{\boldsymbol{\varphi},\boldsymbol{q}}$ is a division $\mathbb{Z}^3$-graded associative algebra over $\mathbb{Q}$.

(4) Let $\mathbb{H} = \langle \boldsymbol{i}, \boldsymbol{j} \rangle$ be Hamilton's quaternion over $\mathbb{R}$ (the field of real numbers), i.e., the unique quaternion division algebra over $\mathbb{R}$. Put $\boldsymbol{k} := \boldsymbol{ij}$. Let $\boldsymbol{\varphi} = \big(\mathrm{I}(d_1), \mathrm{I}(d_2), \mathrm{I}(d_3)\big)$ where

$d_1 = 1 + \boldsymbol{i}$, $d_2 = 1 + \boldsymbol{j}$ and $d_3 = 1 + \boldsymbol{k}$. We put $q_{ij} = 2[d_j, d_i]$ for $1 \leq i < j \leq 3$, $q_{ji} = q_{ij}^{-1}$ and $q_{ii} = 1$. Then, $(\mathbb{H}, \boldsymbol{\varphi}, \boldsymbol{q})$ satisfies (G1) and (G2), and

$$\boldsymbol{q} = \begin{pmatrix} 1 & 1 - \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} & 1 - \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} \\ (1 - \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})^{-1} & 1 & 1 - \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} \\ (1 - \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})^{-1} & (1 - \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})^{-1} & 1 \end{pmatrix}.$$

By Lemma 3.8, this is a division $\mathbb{Z}^3$-grading triple and hence $\mathbb{H}_{\boldsymbol{\varphi},\boldsymbol{q}}$ is a division $\mathbb{Z}^3$-graded associative algebra over $\mathbb{R}$.

§4 Conclusion

By 1.8, Example 2.8(c), Example 2.10, Proposition 2.13, Theorem 3.3 and Corollary 3.7, one can summarize our results as follows:

Corollary. (i) *Any predivision (resp. division) $(A_l, \mathbb{Z}^n)$-graded Lie algebra over F for $l \geq 3$ is an $(A_l, \mathbb{Z}^n)$-cover of $psl_{l+1}(R_{\boldsymbol{\varphi},\boldsymbol{q}})$ for some (resp. division) $\mathbb{Z}^n$-grading triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ over F. Conversely, any $psl_{l+1}(R_{\boldsymbol{\varphi},\boldsymbol{q}})$ for $l \geq 1$ is a predivision (resp. division) $(A_l, \mathbb{Z}^n)$-graded Lie algebra over F.*

(ii) *Any predivision (resp. division) $(\Delta, \mathbb{Z}^n)$-graded Lie algebra over F for $\Delta = D$ or E is a $(\Delta, \mathbb{Z}^n)$-cover of $\mathfrak{g} \otimes_F K[z_1^{\pm}, \ldots, z_n^{\pm}]$ where $\mathfrak{g}$ is a finite dimensional split simple Lie algebra over F of type D or E and K is a unital commutative associative algebra over F (resp. K is a field extension of F). Conversely, for any finite dimensional split simple Lie algebra $\mathfrak{g}$ over F of any type Δ, $\mathfrak{g} \otimes_F K[z_1^{\pm}, \ldots, z_n^{\pm}]$ is a predivision (resp. division) $(\Delta, \mathbb{Z}^n)$-graded Lie algebra over F.* $\square$

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