# Remarks on Holger P. Petersson's "Idempotent 2-by-2 matrices"

**Ottmar Loos**
*Föhrenweg 10, A-6094 Axams, Austria*
*email:* ottmar.loos@uibk.ac.at

## Introduction

This note grew out of H. P. Petersson's recent preprint [**4**], in particular, his Theorem 7.3. Let $\mathbf{X}$ be the scheme of elementary idempotent 2-by-2 matrices over a commutative ring $k$. There is a natural projection $\pi$ from $\mathbf{X}$ to the projective line $\mathbf{P}_1$. The standard open covering $\mathfrak{U}$ of $\mathbf{P}_1$ by two affine lines pulls up to an open covering $\mathfrak{V}$ of $\mathbf{X}$. We show that the groups $\mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1)$ and $\mathrm{Pic}_{\mathfrak{V}}(\mathbf{X})$ of all line bundles which are trivial over $\mathfrak{U}$ and $\mathfrak{V}$ are isomorphic to the group $\mathbf{Z}(k)$ of locally constant maps $\mathrm{Spec}(k) \to \mathbb{Z}$. The universal line bundle $\mathbf{L}$ on $\mathbf{X}$ introduced in [**4**, Sect. 7] is the pull-back of the tautological bundle of $\mathbf{P}_1$ and represents one of the two generators of $\mathbb{Z} \subset \mathbf{Z}(k)$.

## 1. Open coverings of $\mathbf{P}_1$ and $\mathbf{X}$

**1.1. Notations.** We follow the notations used in [**4**]. For a $k$-module $M$, let $M_{\mathbf{a}}$ denote the $k$-functor $R \mapsto M \otimes R$ ($R \in k$-**alg**) and $M_{\mathbf{u}}$ the subfunctor $M_{\mathbf{u}}(R) = \{x \in M_{\mathbf{a}}(R) : x \text{ is unimodular}\}$. If $M$ is finitely generated and projective then $M_{\mathbf{a}}$ is affine with affine algebra the symmetric algebra over the dual $M^*$ of $M$, and $M_{\mathbf{u}}$ is a quasi-affine finitely presented $k$-scheme, open in $M_{\mathbf{a}}$. In particular, $k_{\mathbf{a}}^n$ is affine $n$-space over $k$ and $k_{\mathbf{u}}(R) = R^\times$ is the set of units of $R$.

**1.2. The projective line.** Recall from [**2**, I, §1, 3.4] that the projective line $\mathbf{P}_1$ over $k$ is the functor

$$\mathbf{P}_1(R) = \{L \subset R^2 : L \text{ is a direct summand of rank } 1\} \quad (R \in k\text{-}\mathbf{alg}).$$

If $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$ is a unimodular vector, we write as usual $R \cdot x = (x_1 \colon x_2) \in \mathbf{P}_1(R)$. In general, not every $L \in \mathbf{P}_1(R)$ is free, so $\{(x_1 \colon x_2) : x \text{ unimodular}\}$ will be a proper subset of $\mathbf{P}_1(R)$. However, equality holds if $R$ is a field. Define open subschemes $\mathbf{U}_i \subset \mathbf{P}_1$ by

$$\mathbf{U}_i(R) = \{(x_1 \colon x_2) : x_i \in R^\times\}.$$

Since $(rx_1 \colon rx_2) = (x_1 \colon x_2)$ for all $r \in R^\times$, this means

$$\mathbf{U}_1(R) = \{(1 \colon t) : t \in R\}, \quad \mathbf{U}_2(R) = \{(t \colon 1) : t \in R\},$$

and in fact, the maps $t \mapsto (1 \colon t)$ and $t \mapsto (t \colon 1)$ are isomorphisms $\varphi_i \colon k_{\mathbf{a}} \xrightarrow{\cong} \mathbf{U}_i$. The subschemes $\mathbf{U}_1, \mathbf{U}_2$ form an open affine covering of $\mathbf{P}_1$ in the sense of [**2**, I, §1, 3.10], i.e., for every field $F \in k$-**alg**, we have $\mathbf{P}_1(F) = \mathbf{U}_1(F) \cup \mathbf{U}_2(F)$. The intersection $\mathbf{U}_{12} = \mathbf{U}_1 \cap \mathbf{U}_2$ is isomorphic to $k_{\mathbf{u}}$; more precisely, the restrictions $\varphi_i'$ of $\varphi_i$ to $k_{\mathbf{u}}$ are isomorphisms $k_{\mathbf{u}} \cong \mathbf{U}_{12}$, and

$$({\varphi_2'}^{-1} \circ \varphi_1')(\lambda) = \lambda^{-1}, \tag{1}$$

for all $\lambda \in R^\times$, $R \in k$-**alg**.

**1.3. The morphism $\pi\colon \mathbf{X} \to \mathbf{P}_1$ and the subschemes $\mathbf{V}_i$ of $\mathbf{X}$.** There is an obvious morphism $\pi\colon \mathbf{X} \to \mathbf{P}_1$ given by

$$\pi(c) = \mathrm{Im}(c), \quad (c \in \mathbf{X}(R),\ R \in k\text{-}\mathbf{alg}),$$

and since by definition, any $L \in \mathbf{P}_1(R)$ admits a complementary submodule $L'$ and the decomposition $R^2 = L \oplus L'$ determines a unique $c \in \mathbf{X}(R)$, it is clear that $\pi(R)\colon \mathbf{X}(R) \to \mathbf{P}_1(R)$ is surjective, for all $R \in k$-**alg**. The fibre of $\pi$ over $L = \pi(c) \in \mathbf{P}_1(R)$ consists of all idempotents $c' \in \mathbf{X}(R)$ with $\mathrm{Im}(c') = \mathrm{Im}(c)$, equivalently, of all line bundles $L'$ such that $R^2 = L \oplus L'$, or of all splittings $\sigma$ of the exact sequence

$$0 \longrightarrow L \longrightarrow R^2 \underset{\sigma}{\overset{\mathrm{can}}{\rightleftarrows}} R^2/L \longrightarrow 0\,,$$

i.e., $\mathrm{can} \circ \sigma = \mathrm{Id}$. After fixing (non-canonically!) one complement of $L$, this set may be identified with $\mathrm{Hom}(R^2/L, L)$. Now $R^2/L \cong L^*$ by [**4**, Lemma 5.2], so we see that the fibre of $\pi$ over $L$ is an affine space with associated module of translations $\mathrm{Hom}(L^*, L) \cong L^{\otimes 2}$. Let us put

$$\mathbf{V}_i = \pi^{-1}(\mathbf{U}_i) \subset \mathbf{X}.$$

For $c = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{X}(R)$ let $z_i(c)$ be the $i$-th row of $c$. Then

$$c \in \mathbf{V}_i(R) \quad \Longleftrightarrow \quad z_i(c) \text{ is unimodular.} \tag{1}$$

Indeed, $\mathrm{Im}(c) = \pi(c) = R\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} + R\begin{pmatrix} \beta \\ \delta \end{pmatrix}$. Hence $\pi(c) \in \mathbf{U}_1(R)$ implies there exist $r, s \in R$ such that $r\alpha + s\beta = 1$, so $z_1(c)$ is unimodular. Conversely, let this be the case and put $\lambda := r\gamma + s\delta$. Then $\gamma = (r\alpha + s\beta)\gamma = \alpha(r\gamma + s\delta)$ (because $\beta\gamma = \alpha\delta$) $= \alpha\lambda$, so $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$, and similarly the second column of $c$ is a multiple of $\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$, showing $\mathrm{Im}(c) = R \cdot \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \in \mathbf{U}_1(R)$. The proof for the case $i = 2$ is analogous.

**1.4. Lemma.** (a) *The $\mathbf{V}_i$ are open subschemes covering $\mathbf{X}$.*

(b) *The maps*

$$\psi_1\colon k_{\mathbf{a}}^2 \to \mathbf{V}_1, \quad (\lambda, \beta) \mapsto \begin{pmatrix} 1 - \lambda\beta & \beta \\ \lambda(1 - \lambda\beta) & \lambda\beta \end{pmatrix}, \tag{1}$$

$$\psi_2\colon k_{\mathbf{a}}^2 \to \mathbf{V}_2, \quad (\mu, \gamma) \mapsto \begin{pmatrix} \mu\gamma & \mu(1 - \mu\gamma) \\ \gamma & 1 - \mu\gamma \end{pmatrix}, \tag{2}$$

*are isomorphisms making the diagrams*

$$\begin{array}{ccc} k_{\mathbf{a}}^2 & \xrightarrow[\cong]{\psi_i} & \mathbf{V}_i \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ k_{\mathbf{a}} & \xrightarrow[\varphi_i]{\cong} & \mathbf{U}_i \end{array} \tag{3}$$

*commutative.*

(c) *The intersection $\mathbf{V}_{12} := \mathbf{V}_1 \cap \mathbf{V}_2$ is the open subscheme of all $c \in \mathbf{X}(R)$ for which both rows are unimodular. We have $\psi_i^{-1}(\mathbf{V}_{12}) = k_{\mathbf{u}} \times k_{\mathbf{a}}$. The $\psi_i$ restrict to isomorphisms $\psi_i'\colon k_{\mathbf{u}} \times k_{\mathbf{a}} \xrightarrow{\cong} \mathbf{V}_{12}$, and the change of coordinates $\phi = \psi_2'^{-1} \circ \psi_1'\colon k_{\mathbf{u}} \times k_{\mathbf{a}} \to k_{\mathbf{u}} \times k_{\mathbf{a}}$ is given by*

$$\phi(\lambda,\beta) = \big(\lambda^{-1},\ \lambda(1-\lambda\beta)\big) \tag{4}$$

*for all* $(\lambda,\beta) \in R^\times \times R$*,* $R \in k$-**alg***, and satisfies*

$$\phi \circ \phi = \mathrm{Id}. \tag{5}$$

*Proof.* (a) Since $\mathbf{V}_i$ is the inverse image of the open subschemes $\mathbf{U}_i$, it is open in $\mathbf{X}$. (Alternatively, $\mathbf{V}_i$ is the inverse image of $(k^2)_{\mathbf{u}}$ under the morphism $z_i\colon \mathbf{X} \to k^2_{\mathbf{a}}$, by 1.3.1). If $R$ is a field, at least one row of $c \in \mathbf{X}(R)$ is non-zero, which proves the covering statement.

(b) It is obvious from (1) that $\psi_1$ takes values in $\mathbf{V}_1$. Conversely, assume that $c = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{V}_1(R)$. The transpose of $c$ is in $\mathbf{X}(R)$ along with $c$, so the span of the rows of $c$ is a direct summand of rank 1 of the dual $(R^2)^*$. Since $(\alpha,\beta)$ is unimodular by 1.3.1, there exists a unique $\lambda \in R$ such that $(\gamma,\delta) = \lambda(\alpha,\beta)$. Now one shows easily, using the fact that $\mathrm{tr}(c) = 1$, that $c \mapsto (\lambda,\beta)$ is the inverse map of $\psi_1$. From (1) it is clear that the columns $v_1$, $v_2$ of $\psi_1(\lambda,\beta)$ are multiples of $\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$, and that $\begin{pmatrix} 1 \\ \lambda \end{pmatrix} = v_1 + \lambda v_2$. This proves $(\pi \circ \psi_1)(\lambda,\beta) = (1\colon\lambda) = \varphi_1(\lambda)$, so (3) commutes. The proof for $\psi_2$ is analogous.

(c) Since $\mathbf{V}_{12} = \pi^{-1}(\mathbf{U}_{12})$, (3) and 1.2.1 imply $\psi_i^{-1}(\mathbf{V}_{12}) = \mathrm{pr}_1^{-1}\big(\varphi_i^{-1}(\mathbf{U}_{12})\big) = k_{\mathbf{u}} \times k_{\mathbf{a}}$. Now (4) follows from (1) and (2). These formulas show also that $\phi^{-1}(\mu,\gamma) = ({\psi'_1}^{-1} \circ \psi'_2)(\mu,\gamma) = \big(\mu^{-1},\ \mu(1-\mu\gamma)\big)$. Thus $\phi^{-1} = \phi$, proving (5).

**1.5. Remarks.** (i) By 1.4.4, the second component of $\phi$ is an affine, but not a linear function of $\beta$, in accordance with the fact that $\mathbf{X}$ is an affine, but not a vector bundle over $\mathbf{P}_1$. The occurrence of the factor $\lambda^2$ at $\beta$ corresponds to the fact that the fibre of $\pi$ over $L$ is isomorphic to the affine space determined by $L^{\otimes 2}$, as remarked in 1.3.

(ii) Formula 1.4.5 is the analogue of the fact that, by 1.2.1, the change of coordinates $\varphi_2^{-1} \circ \varphi_1$ in $k[\mathbf{U}_{12}] \cong k[\mathbf{t},\mathbf{t}^{-1}]$ is inversion $\lambda \mapsto \lambda^{-1}$ which obviously has period two. This will be important later in the proof of Theorem 4.2.

(iii) There is a second projection $\pi'\colon \mathbf{X} \to \mathbf{P}_1$ given by $\pi'(c) = \mathrm{Ker}(c)$. Since an element $c \in \mathbf{X}(R)$ can be identified with the decomposition $R^2 = \mathrm{Im}(c) \oplus \mathrm{Ker}(c)$, it is clear that $(\pi,\pi')$ is an isomorphism of $\mathbf{X}$ onto the open subscheme $\mathbf{W} \subset \mathbf{P}_1 \times \mathbf{P}_1$ given by $\mathbf{W}(R) = \{(L,M) \in \mathbf{P}_1(R)^2 : R^2 = L \oplus M\}$. If $R = K$ is a field, then $(L,M) \in \mathbf{W}(K)$ if and only if $L \neq M$, so $\mathbf{W}(K)$ is the complement of the diagonal in $\mathbf{P}_1(K)^2$.

(iv) There is no section of $\pi\colon \mathbf{X} \to \mathbf{P}_1$. Indeed, assume to the contrary that $\sigma\colon \mathbf{P}_1 \to \mathbf{X}$ satisfies $\pi \circ \sigma = \mathrm{Id}$. Then $\sigma_i = \sigma\big|\mathbf{U}_i\colon \mathbf{U}_i \to \mathbf{V}_i$ are sections of $\pi\big|\mathbf{V}_i$. Identify the affine algebras $k[\mathbf{U}_i]$ with the polynomial ring $k[\mathbf{t}]$ by means of $\varphi_i$. Then $\sigma_i(\varphi_i(\mathbf{t})) = \psi_i(\mathbf{t},\ f_i(\mathbf{t}))$ where the $f_i(\mathbf{t})$ are polynomials in $\mathbf{t}$, and 1.4.4 and 1.2.1 imply

$$f_2(\mathbf{t}^{-1}) = \mathbf{t} \cdot (1 - \mathbf{t} f_1(\mathbf{t}))$$

in the Laurent polynomial ring $k[\mathbf{t},\mathbf{t}^{-1}] \cong k[\mathbf{U}_{12}]$ which is impossible.

Let $\mathfrak{U}$ (resp. $\mathfrak{V}$) be the open covering of $\mathbf{P}_1$ (resp. $\mathbf{X}$) given by $\mathbf{U}_1$ and $\mathbf{U}_2$ (resp. $\mathbf{V}_1$ and $\mathbf{V}_2$). Our aim is to determine the subgroups $\mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1)$ and $\mathrm{Pic}_{\mathfrak{V}}(\mathbf{X})$ of the respective Picard groups consisting of all (isomorphism classes of) line bundles whose restriction to the $\mathbf{U}_i$ (resp. $\mathbf{V}_i$) is trivial. We begin by constructing the standard examples of such bundles.

## 2. The line bundles E and L

**2.1.** The tautological bundle $\mathbf{E}$ over $\mathbf{P}_1$ is the line bundle whose fibre over a point $L \in \mathbf{P}_1(R)$ is the $R$-module $L$ itself, whence the name "tautological". More formally, it is the $k$-functor

$$\mathbf{E}(R) = \{(L, x) : L \in \mathbf{P}_1(R),\ x \in L\} \quad (R \in k\text{-}\mathbf{alg}),$$

with projection $\mathrm{pr}_1\colon \mathbf{E} \to \mathbf{P}_1$. The sheaf $\mathscr{E}$ of sections of $\mathbf{E}$ is the sheaf usually denoted $\mathscr{O}_{\mathbf{P}_1}(-1)$. Now let $\mathbf{L} = \pi^*(\mathbf{E})$ be the inverse image of $\mathbf{E}$ on $\mathbf{X}$ under $\pi$, that is, the fibre product $\mathbf{L} = \mathbf{X} \times_{\mathbf{P}_1} \mathbf{E}$. Thus, for every $R \in k\text{-}\mathbf{alg}$, $\mathbf{L}(R)$ is the set of all pairs $\big(c, (L, x)\big)$ where $c \in \mathbf{X}(R)$, $(L, x) \in \mathbf{E}(R)$ and $\pi(c) = L$. Since $L$ is already determined by $c$, we can and will identify $\mathbf{L}$ with the functor

$$\mathbf{L}(R) = \{(c, x) : c \in \mathbf{X}(R),\ x \in \mathrm{Im}(c)\} \quad (R \in k\text{-}\mathbf{alg}).$$

Then the following diagram is commutative and Cartesian:

$$\begin{array}{ccc}
\mathbf{L} & \xrightarrow{\ \eta\ } & \mathbf{E} \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
\mathbf{X} & \xrightarrow[\ \pi\ ]{} & \mathbf{P}_1
\end{array}$$

where $\eta(c, x) = (\pi(c), x)$. Denote by $\mathscr{L}$ the sheaf of sections of $\mathbf{L}$.

Now let $A = k[\mathbf{X}]$ be the affine algebra of $\mathbf{X}$, thus $A = k[\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}]$, subject to the relations $\boldsymbol{\alpha}\boldsymbol{\delta} = \boldsymbol{\beta}\boldsymbol{\gamma}$ and $\boldsymbol{\alpha} + \boldsymbol{\delta} = 1$. Let

$$\mathbf{e} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\gamma} & \boldsymbol{\delta} \end{pmatrix} \in \mathbf{X}(A)$$

be the "generic" element of $\mathbf{X}$, corresponding to the identity map under the identification of $\mathbf{X}(R)$ with $\mathrm{Hom}_{k\text{-}\mathbf{alg}}(A, R)$, for all $R \in k\text{-}\mathbf{alg}$. Any $c \in \mathbf{X}(R)$ determines an invertible $R$-module $L = \mathrm{Im}(c) \subset R^2$. In particular, $\mathrm{Im}(\mathbf{e}) \subset A^2$ is an invertible $A$-module; this is the module denoted $L_e$ in [**4**, Sect. 7], and it is related to $\mathscr{L}$ as follows.

**2.2. Lemma.** $\mathrm{Im}(\mathbf{e})$ *is canonically isomorphic to the $A$-module $\mathscr{L}(\mathbf{X})$ of global sections of* $\mathbf{L}$.

*Proof.* An element $s \in \mathscr{L}(\mathbf{X})$, i.e., a section $s\colon \mathbf{X} \to \mathbf{L}$ of $\mathrm{pr}_1\colon \mathbf{L} \to \mathbf{X}$, is of the form $s(c) = \big(c, v(c)\big)$ where $v(c) \in \mathrm{Im}(c)$, for all $c \in \mathbf{X}(R)$, $R \in k\text{-}\mathbf{alg}$. In particular, $v(\mathbf{e}) \in \mathrm{Im}(\mathbf{e})$, so we obtain a map $\mathscr{L}(\mathbf{X}) \to \mathrm{Im}(\mathbf{e})$ sending $s$ to $v(\mathbf{e})$. Conversely, let $w \in \mathrm{Im}(\mathbf{e})$ and define a section $s\colon \mathbf{X} \to \mathbf{L}$ as follows. For $R \in k\text{-}\mathbf{alg}$ and $c \in \mathbf{X}(R)$, let $\varrho_c\colon A \to R$ be the $k$-algebra homomorphism corresponding to $c$. Then $s(c) := (c, \varrho_c(w)) \in \mathbf{L}(R)$ defines a section $s\colon \mathbf{X} \to \mathbf{L}$. One sees immediately that the constructions are inverse to each other.

**2.3.** There are sections $s_i \in \mathscr{E}(\mathbf{U}_i)$ given by

$$s_1(\varphi_1(\lambda)) = \Big((1\colon \lambda), \begin{pmatrix} 1 \\ \lambda \end{pmatrix}\Big), \quad s_2(\varphi_2(\mu)) = \Big((\mu\colon 1), \begin{pmatrix} \mu \\ 1 \end{pmatrix}\Big) \quad (\lambda, \mu \in R,\ R \in k\text{-}\mathbf{alg}).$$

These sections "vanish nowhere", i.e., they form bases for the $k[\mathbf{U}_i]$-modules $\mathscr{E}(\mathbf{U}_i)$ of sections of $\mathbf{E}$ over $\mathbf{U}_i$, so $\mathscr{E}$ represents an element of $\mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1)$. The sections $s_i$ are related on $\mathbf{U}_{12}$ by

$$s_2(\varphi_2(\lambda^{-1})) = s_1(\varphi_1(\lambda) \cdot \lambda^{-1} \quad (\lambda \in R^\times,\ R \in k\text{-}\mathbf{alg}), \tag{1}$$

since $\varphi_1(\lambda) = \varphi_2(\mu)$ if and only if $\lambda\mu = 1$ by 1.2.1, and $\begin{pmatrix} \mu \\ 1 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ \mu^{-1} \end{pmatrix} = \mu \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$. On the other hand, it is well-known (and follows easily from (1)) that zero is the only section of $\mathbf{E}$ over all of $\mathbf{P}_1$.

The sections $s_i$ may be lifted to nowhere vanishing sections $\tilde{s}_i \in \mathscr{L}(\mathbf{V}_i)$ by

$$\begin{aligned} \tilde{s}_1(c) &= \big(c, \begin{pmatrix} 1 \\ \lambda \end{pmatrix}\big) \quad \text{for } c = \psi_1(\lambda, \beta) \in \mathbf{V}_1(R), \\ \tilde{s}_2(c) &= \big(c, \begin{pmatrix} \mu \\ 1 \end{pmatrix}\big) \quad \text{for } c = \psi_2(\mu, \gamma) \in \mathbf{V}_2(R), \end{aligned} .$$

Hence $\mathbf{L}$ represents an element of $\mathrm{Pic}_{\mathfrak{Y}}(\mathbf{X})$. The sections $\tilde{s}_i$ are related on $\mathbf{V}_{12}$ in the same way as before:

$$\tilde{s}_2(c) = \tilde{s}_1(c) \cdot \lambda^{-1} \tag{2}$$

for $c = \psi_1(\lambda, \beta) = \psi_2(\mu, \gamma) \in \mathbf{V}_{12}(R)$ since $\mu = \lambda^{-1}$ by 1.4.4.

## 3. Auxiliary results on Laurent polynomials over rings

**3.1.** Recall the constant $k$-group scheme $\mathbf{Z}$ defined by the integers: $\mathbf{Z}(R)$ is the set of all locally constant maps $\mathfrak{d}\colon \mathrm{Spec}(R) \to \mathbb{Z}$ with the obvious (additive) group structure. The elements of $\mathbf{Z}(R)$ are in bijection with families $\varepsilon = (\varepsilon_n)_{n\in\mathbb{Z}}$ of orthogonal idempotents of $R$ with $\varepsilon_n \neq 0$ for only finitely many $n$, and $\sum \varepsilon_n = 1$, by means of the relations

$$\mathfrak{d}(\mathfrak{p}) = n \quad \iff \quad \varepsilon_n(\mathfrak{p}) = 1_{\boldsymbol{\kappa}(\mathfrak{p})}, \tag{1}$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$, $R \in k$-**alg**. Here we use the notation $r(\mathfrak{p})$ for the canonical image of an element $r \in R$ in the quotient field $\boldsymbol{\kappa}(\mathfrak{p})$ of $R/\mathfrak{p}$. Then the group law in $\mathbf{Z}(R)$ is described (multiplicatively) by

$$(\varepsilon \cdot \varepsilon')_n = \sum_{l+m=n} \varepsilon_l \varepsilon'_m, \tag{2}$$

so the inverse of $\varepsilon$ is $\varepsilon^{-1} = (\varepsilon_{-n})_{n\in\mathbb{Z}}$, and the unit element of $\mathbf{Z}(R)$, i.e., the constant map $0\colon S \to \mathbb{Z}$, corresponds to the family $\varepsilon_n = \begin{Bmatrix} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{Bmatrix}$. Let $R[\mathbf{t}, \mathbf{t}^{-1}]$ be the Laurent polynomial ring in one variable $\mathbf{t}$ over $R$. Then (2) implies that there is a group monomorphism

$$\mathbf{Z}(R) \to R[\mathbf{t}, \mathbf{t}^{-1}]^\times, \quad \mathfrak{d} \mapsto \mathbf{t}^{\mathfrak{d}} := \sum_{n\in\mathbb{Z}} \varepsilon_n \mathbf{t}^n.$$

**3.2. Lemma.** *Let $R$ be a commutative ring and* $\mathbf{t}$ *an indeterminate. Denote by* $\mathrm{Nil}(R)$ *the nil radical of $R$.*

(a) *A polynomial $f(\mathbf{t}) = \sum_{i\geqslant 0} r_i \mathbf{t}^i$ is a unit in $R[\mathbf{t}]$ if and only if $r_0 \in R^\times$ and $r_i \in \mathrm{Nil}(R)$ for all $i > 0$.*

(b) *A Laurent polynomial $g \in R[\mathbf{t}, \mathbf{t}^{-1}]$ is a unit in $R[\mathbf{t}, \mathbf{t}^{-1}]$ if and only if there exists an element $\mathfrak{d} \in \mathbf{Z}(R)$, a unit $u \in R^\times$ and a nilpotent $h \in R[\mathbf{t}, \mathbf{t}^{-1}]$ such that*

$$g = u\,\mathbf{t}^{\mathfrak{d}} + h. \tag{1}$$

*The element $\mathfrak{d}$ is uniquely determined by $g$, called the* degree *of $g$, and the map*

$$\deg\colon R[\mathbf{t}, \mathbf{t}^{-1}]^\times \to \mathbf{Z}(R), \quad \deg(u\mathbf{t}^{\mathfrak{d}} + h) := \mathfrak{d},$$

*is a group homomorphism.*

Note, however, that $u$ and $h$ are not uniquely determined by $g$.

*Proof.* (a) is evident if $R$ is a field. In general, consider $r \in R$ and $\mathfrak{p} \in S := \mathrm{Spec}(R)$. Then $r \in R^\times \iff r(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in S$, and $r \in \mathrm{Nil}(R) \iff r(\mathfrak{p}) = 0$, for all $\mathfrak{p} \in S$. This proves (a).

(b) Clearly an element as in (1) is a unit in $R[\mathbf{t},\mathbf{t}^{-1}]$. Conversely, let $g \in R[\mathbf{t},\mathbf{t}^{-1}]^\times$, and consider again first the case where $R$ is a field. We leave it to the reader to show that $g = a_n\mathbf{t}^n$ is a non-zero monomial.

Now let $R$ be arbitrary, write $g = \sum_{n\in\mathbb{Z}} r_n\mathbf{t}^n$ where $r_n \in R$, and let $\mathfrak{p} \in S$. By applying the above to $g\otimes\boldsymbol{\kappa}(\mathfrak{p})$, we see that there exists a unique index $n =: \mathfrak{d}(\mathfrak{p}) \in \mathbb{Z}$ such that $r_n(\mathfrak{p}) \neq 0$. The map $\mathfrak{d}\colon S \to \mathbb{Z}$ thus defined is locally constant. Indeed, if $\mathfrak{d}(\mathfrak{p}_0) = n$ then $r_n(\mathfrak{p}_0) \neq 0$ and hence $r_n(\mathfrak{p}) \neq 0$ for all $\mathfrak{p}$ in the basic open neighbourhood $U$ of $\mathfrak{p}_0$ in $S$ defined by $r_n$. Since $r_j(\mathfrak{p}) = 0$ for all other $j \neq n$, the function $\mathfrak{d}$ is constant equal to $n$ on $U$. This proves $\mathfrak{d} \in \mathbf{Z}(R)$.

Let $(\varepsilon_n)_{n\in\mathbb{Z}}$ be the family of idempotents corresponding to $\mathfrak{d}$. Then $\big(r_n(1-\varepsilon_n)\big)(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in S$. Indeed, if $\mathfrak{d}(\mathfrak{p}) = n$ then $(1-\varepsilon_n)(\mathfrak{p}) = 0$ by 3.1.1, while if $\mathfrak{d}(\mathfrak{p}) \neq n$, then $r_n(\mathfrak{p}) = 0$ by definition of $\mathfrak{d}$. Hence $c_n := r_n(1-\varepsilon_n) \in \operatorname{Nil}(R)$. Moreover, $u := \sum_{n\in\mathbb{Z}} r_n\varepsilon_n \in R^\times$ because, for all $\mathfrak{p} \in S$, by definition of $\mathfrak{d}$,

$$u(\mathfrak{p}) = \sum_{n\in\mathbb{Z}} r_n(\mathfrak{p})\varepsilon_n(\mathfrak{p}) = r_{\mathfrak{d}(\mathfrak{p})}(\mathfrak{p}) \neq 0.$$

Now $u\varepsilon_n = r_n\varepsilon_n$ by orthogonality of the $\varepsilon_n$, and hence

$$g = \sum_{n\in\mathbb{Z}} r_n\varepsilon_n\mathbf{t}^n + \sum_{n\in\mathbb{Z}} c_n\mathbf{t}^n = u\mathbf{t}^{\mathfrak{d}} + h$$

where $h = \sum c_n\mathbf{t}^n$ is nilpotent, being a finite sum of the nilpotent monomials $c_n\mathbf{t}^n$. This proves (1). Uniqueness of $\mathfrak{d} = \deg(g)$ is clear since $g \otimes 1_{\boldsymbol{\kappa}(\mathfrak{p})} = u(\mathfrak{p})\cdot\mathbf{t}^{\mathfrak{d}(\mathfrak{p})}$. Finally, suppose that $g' = u'\,\mathbf{t}^{\mathfrak{d}} + h'$ is a second element of $R[\mathbf{t},\mathbf{t}^{-1}]^\times$. Then

$$gg' = (u\mathbf{t}^{\mathfrak{d}} + h)(u'\mathbf{t}^{\mathfrak{d}'} + h') \equiv uu'\mathbf{t}^{\mathfrak{d}+\mathfrak{d}'} \;\big(\operatorname{mod} \operatorname{Nil}(R[\mathbf{t},\mathbf{t}^{-1}]\big)$$

since $\mathfrak{d} \mapsto \mathbf{t}^{\mathfrak{d}}$ is a group homomorphism, showing deg is a homomorphism.

**3.3. Lemma.** *There is an exact sequence*

$$1 \longrightarrow R^\times \xrightarrow{\ \Delta\ } R[\mathbf{t}]^\times \times R[\mathbf{t}]^\times \xrightarrow{\ \partial\ } R[\mathbf{t},\mathbf{t}^{-1}]^\times \xrightarrow{\ \deg\ } \mathbf{Z}(R) \longrightarrow 0$$

*where $\Delta(r) = (r,r)$ is the diagonal map, $\partial(f_1(\mathbf{t}), f_2(\mathbf{t})) = f_1(\mathbf{t})\cdot f_2(\mathbf{t}^{-1})^{-1}$ and* deg *is as in Lemma* 3.2(b)*.*

*Proof.* Clearly $\partial(f_1,f_2) = 1$ if and only if $f_1(\mathbf{t}) = f_2(\mathbf{t}^{-1})$ if and only if $f_1 = f_2 = r \in R^\times$. Next, $\operatorname{Im}(\partial) \subset \operatorname{Ker}(\deg)$ because $\deg(f_1(\mathbf{t})) = 0 = \deg(f_2(\mathbf{t}^{-1}))$ for $f_i \in R[\mathbf{t}]^\times$ and deg is a homomorphism. Also, deg is surjective since the map $\mathfrak{d} \mapsto \mathbf{t}^{\mathfrak{d}}$ is even a section of deg. Thus it remains to prove the inclusion $\operatorname{Ker}(\deg) \subset \operatorname{Im}(\partial)$.

By Lemma 3.2(b), an invertible Laurent polynomial of degree zero has the form $g(\mathbf{t}) = u\cdot 1 + h(\mathbf{t})$ where $u \in R^\times$ and $h(\mathbf{t}) = \sum_{i\geqslant -n} c_i\,\mathbf{t}^i$ for some $n \in \mathbb{N}$, with $c_i \in \operatorname{Nil}(R)$. Hence $G(\mathbf{t}) := \mathbf{t}^n g(\mathbf{t}) \in R[\mathbf{t}]$. Denote the canonical maps $R \to \bar{R} = R/\operatorname{Nil}(R)$ and $R[\mathbf{t}] \to \bar{R}[\mathbf{t}]$ by a bar. Then $\bar{G}(\mathbf{t}) = \mathbf{t}^n\bar{u} = \bar{P}(\mathbf{t})\cdot\bar{Q}(\mathbf{t})$ where $\bar{P}(\mathbf{t}) = \mathbf{t}^n$ is monic and $\bar{Q}(\mathbf{t}) = \bar{u} \in \bar{R}^\times$. Clearly $\bar{P}$ and $\bar{Q}$ are strongly relatively prime in $\bar{R}[\mathbf{t}]$, so by Hensel's Lemma [**1**, III, §4.3, Theorem 1], applied to the discretely topologized ring $A = R$ and the ideal $\mathfrak{m} = \operatorname{Nil}(R)$, the polynomials $\bar{P}$, $\bar{Q}$ lift uniquely to polynomials $P, Q \in R[\mathbf{t}]$, $P$ monic, such that $G = P\cdot Q$. Write $P(\mathbf{t}) = \mathbf{t}^m + a_1\mathbf{t}^{m-1} + \cdots + a_m$ and $Q(\mathbf{t}) = b_0 + b_1\mathbf{t} + \cdots$. Then $\bar{P}(\mathbf{t}) = \mathbf{t}^n$ and $\bar{Q} = \bar{u}$ shows $m = n$, $b_0 \in R^\times$ and $a_i, b_i \in \operatorname{Nil}(R)$ for $i > 0$. By Lemma 3.2(a), the polynomial $F(\mathbf{t}) := 1 + a_1\mathbf{t} + \cdots + a_n\mathbf{t}^n = \mathbf{t}^n P(\mathbf{t}^{-1})$ belongs to $R[\mathbf{t}]^\times$. Now put $f_1(\mathbf{t}) := Q(\mathbf{t})$ and $f_2(\mathbf{t}) := F(\mathbf{t})^{-1}$. Then

$$f_1(\mathbf{t})f_2(\mathbf{t}^{-1})^{-1} = Q(\mathbf{t})F(\mathbf{t}^{-1}) = Q(\mathbf{t})\mathbf{t}^{-n}P(\mathbf{t}) = \mathbf{t}^{-n}G(\mathbf{t}) = g(\mathbf{t}),$$

as desired.

## 4. Determination of $\mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1)$ and $\mathrm{Pic}_{\mathfrak{V}}(\mathbf{X})$.

**4.1. Theorem.** *There is a natural isomorphism* $\Phi\colon \mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1) \xrightarrow{\cong} \mathbf{Z}(k)$ *mapping the tautological bundle to* $-1 \in \mathbb{Z} \subset \mathbf{Z}(k)$ *as follows.*

*Identify* $k[\mathbf{U}_i]$ *with the polynomial ring* $k[\mathbf{t}]$ *by means of the isomorphisms* $\varphi_i$ *of* 1.2 *and identify* $k[\mathbf{U}_{12}]$ *with the Laurent polynomial ring* $k[\mathbf{t},\mathbf{t}^{-1}]$ *by means of the open embedding* $\mathbf{U}_{12} \subset \mathbf{U}_1$. *Let* $\mathscr{M}$ *be a representative of an element* $[\mathscr{M}] \in \mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1)$, *and let* $s_i \in \mathscr{M}(\mathbf{U}_i)$ *be sections trivializing* $\mathscr{M}$ *over* $\mathbf{U}_i$, *so that* $s_2 = s_1 \cdot g_{12}$ *on* $\mathbf{U}_{12}$ *where* $g_{12} \in k[\mathbf{t},\mathbf{t}^{-1}]^\times$. *Then the element* $\deg(g_{12}) \in \mathbf{Z}(k)$ *depends only on the isomorphism class of* $\mathscr{M}$, *and* $[\mathscr{M}] \mapsto \deg(g_{12})$ *is the desired isomorphism.*

*Proof.* By standard facts, computing $\mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1)$ amounts to computing the Čech cohomology group $H^1 = H^1(\mathfrak{U}, \mathscr{F})$ of the sheaf $\mathscr{F} = \mathscr{O}_{\mathbf{P}_1}^\times$ with respect to the covering $\mathfrak{U}$. Recall that $H^1 = Z^1/B^1$ where $Z^1 = Z^1(\mathfrak{U}, \mathscr{F})$ is the group of Čech 1-cocycles $(g_{ij}) \in \mathscr{F}(\mathbf{U}_i \cap \mathbf{U}_j)$ and $B^1 = \partial^0(C^0)$ is the group of coboundaries.

Since $\mathfrak{U}$ has only two elements, we have a group isomorphism $Z^1 \cong \mathscr{F}(\mathbf{U}_{12})$ sending $(g_{ij})$ to $g_{12}$. Note that this isomorphism is not unique; $(g_{ij}) \mapsto g_{21} = g_{12}^{-1}$ would have been just as good. We identify the group $C^0$ of 0-cochains with $\mathscr{F}(\mathbf{U}_1) \times \mathscr{F}(\mathbf{U}_2)$. Then the coboundary operator $\partial^0\colon C^0 \to C^1$ is given by

$$\partial^0(g_1, g_2) = \varrho_1(g_1) \cdot \varrho_2(g_2)^{-1}, \tag{1}$$

where $g_i \in \mathscr{F}(\mathbf{U}_i)$ and $\varrho_i\colon \mathscr{F}(\mathbf{U}_i) \to \mathscr{F}(\mathbf{U}_{12})$ are the restriction homomorphisms.

Now consider the isomorphisms $\varphi_i\colon k_{\mathbf{a}} \to \mathbf{U}_i$ and $\varphi_i'\colon k_{\mathbf{u}} \to \mathbf{U}_{12}$ of 1.2. After identifying the affine algebras of $k_{\mathbf{a}}$ and $k_{\mathbf{u}}$ with $k[\mathbf{t}]$ and $k[\mathbf{t},\mathbf{t}^{-1}]$, we have induced isomorphisms $\varphi_i^*\colon \mathscr{F}(\mathbf{U}_i) \to k[\mathbf{t}]^\times$ and $\varphi_i'^*\colon \mathscr{F}(\mathbf{U}_{12}) \to k[\mathbf{t},\mathbf{t}^{-1}]^\times$. Under these isomorphisms, the coboundary operator $\partial^0$ corresponds to the map $\partial'\colon k[\mathbf{t}]^\times \times k[\mathbf{t}]^\times \to k[\mathbf{t},\mathbf{t}^{-1}]^\times$ given by

$$\partial'(f_1, f_2) = f_1 \cdot \phi^*(f_2)^{-1} \tag{2}$$

where $\phi = \varphi_2'^{-1} \circ \varphi_1'$ is the change of coordinates map. Details are left to the reader. By 1.2.1, $\phi$ is inversion on $k_{\mathbf{u}}$, so $\phi^*$ is the automorphism $\mathbf{t} \mapsto \mathbf{t}^{-1}$ of $k[\mathbf{t},\mathbf{t}^{-1}]$. It follows that $\partial' = \partial$, the map considered in Lemma 3.3. Hence the diagram

$$\begin{array}{ccccccc}
C^0 & \xrightarrow{\partial^0} & Z^1 & \xrightarrow{\text{can}} & H^1 & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \| & & \\
\mathscr{F}(\mathbf{U}_1) \times \mathscr{F}(\mathbf{U}_2) & \xrightarrow{\partial^0} & \mathscr{F}(\mathbf{U}_{12}) & \longrightarrow & H^1 & \longrightarrow & 0 \\
{\scriptstyle \varphi_1^* \times \varphi_2^*}\downarrow\cong & & {\scriptstyle \varphi_1'^*}\downarrow\cong & & \vdots\cong & & \\
k[\mathbf{t}]^\times \times k[\mathbf{t}]^\times & \xrightarrow[\partial]{} & k[\mathbf{t},\mathbf{t}^{-1}]^\times & \xrightarrow[\deg]{} & \mathbf{Z}(k) & \longrightarrow & 0
\end{array}$$

is commutative and has exact rows, so there is a unique isomorphism $H^1 \to \mathbf{Z}(k)$ making the diagram commutative. Explicitly, it is given by the procedure described in the statement of the theorem. Finally, 2.3.1 implies that the tautological bundle is mapped to $-1 \in \mathbf{Z}(k)$.

**4.2. Theorem.** *There is a natural isomorphism* $\Psi\colon \mathrm{Pic}_{\mathfrak{V}}(\mathbf{X}) \xrightarrow{\cong} \mathbf{Z}(k)$ *making the diagram*

$$\begin{array}{ccc}
\mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1) & \xrightarrow{\pi^*} & \mathrm{Pic}_{\mathfrak{V}}(\mathbf{X}) \\
& {\scriptstyle \Phi}\searrow \quad \swarrow{\scriptstyle \Psi} & \\
& \mathbf{Z}(k) &
\end{array} \tag{1}$$

*commutative. Hence* $\pi^*$ *is an isomorphism, and the bundle* $\mathbf{L} = \pi^*(\mathbf{E})$ *of* 2.1 *is mapped to* $-1$ *under* $\pi^*$.

*Proof.* We proceed as in the proof of 4.1. Let $\mathscr{G}$ be the sheaf $\mathscr{O}_{\mathbf{X}}^{\times}$ and identify $C^0(\mathfrak{V}) \cong \mathscr{G}(\mathbf{V}_1) \times \mathscr{G}(\mathbf{V}_2)$ and $Z^1(\mathfrak{V}) \cong \mathscr{G}(\mathbf{V}_{12})$. Then the coboundary operator $\partial^0 \colon C^0(\mathfrak{V}) \to Z^1(\mathfrak{V})$ is given by 4.1.1. Again as before, we consider the isomorphisms $\psi_i \colon k_{\mathbf{a}}^2 \to \mathbf{V}_i$ and $\psi_i' \colon k_{\mathbf{u}} \times k_{\mathbf{a}} \to \mathbf{V}_{12}$ of 1.4. After identifying the affine algebra of $k_{\mathbf{a}}^2$ with the polynomial ring $k[\mathbf{t}, \mathbf{y}]$ in two variables and the affine algebra of $k_{\mathbf{u}} \times k_{\mathbf{a}}$ with $k[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{y}]$, we have induced isomorphisms $\psi_i^* \colon \mathscr{G}(\mathbf{V}_i) \to k[\mathbf{t}, \mathbf{y}]^{\times}$ and $\psi_i'^* \colon \mathscr{G}(\mathbf{V}_{12}) \to k[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{y}]^{\times}$. Let $\phi$ be the change of coordinates 1.4.4. Then again $\partial^0$ corresponds to the map $\partial'$ of 4.1.2.

Put $R = k[\mathbf{y}]$, so that $k[\mathbf{t}, \mathbf{y}] = R[\mathbf{t}]$ and $k[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{y}] = R[\mathbf{t}, \mathbf{t}^{-1}]$. We wish to apply Lemma 3.3. However, the automorphism $\phi^*$ of the $k$-algebra $R[\mathbf{t}, \mathbf{t}^{-1}]$ is no longer just given by $\mathbf{t} \mapsto \mathbf{t}^{-1}$ but also involves the variable $\mathbf{y}$, so $\partial'$ is not equal to the map $\partial$ of Lemma 3.3. Hence the following detour is required.

From 1.4.4, we see that $\phi$ can be factored in the form $\phi = \iota \circ \vartheta$ where $\iota(\lambda, \beta) = (\lambda^{-1}, \beta)$ and $\vartheta(\lambda, \beta) = \big(\lambda, \lambda(1 - \lambda\beta)\big)$. Putting $I = \iota^*$ and $\Theta = \vartheta^*$, this shows

$$\phi^* = \Theta \circ I \tag{2}$$

where $\Theta$ and $I$ are the automorphisms of $k[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{y}]$ given by the formulas

$$\Theta(\mathbf{t}) = \mathbf{t}, \quad \Theta(\mathbf{y}) = \mathbf{t}(1 - \mathbf{t}\mathbf{y}), \tag{3}$$
$$I(\mathbf{t}) = \mathbf{t}^{-1}, \quad I(\mathbf{y}) = \mathbf{y}. \tag{4}$$

By 1.4.5, $\phi^*$ squares to the identity and obviously $I^2 = \mathrm{Id}$. Hence (2) implies

$$I = \Theta \circ I \circ \Theta. \tag{5}$$

Next observe (cf. [**3**, 0.12.2]) that an idempotent $\varepsilon$ of the polynomial ring $R = k[\mathbf{y}]$ belongs to $k$. Hence the natural homomorphism $k \to R$ induces an isomorphism

$$\mathbf{Z}(k) \xrightarrow{\cong} \mathbf{Z}(R). \tag{6}$$

Using the description of the units of $R = k[\mathbf{y}]$ in Lemma 3.2(a), part (b) of that lemma shows that $g \in R[\mathbf{t}, \mathbf{t}^{-1}]^{\times}$ if and only if $g = \mu\, \mathbf{t}^{\mathfrak{d}} + h$ where $\mu \in k^{\times}$, $\mathfrak{d} = \deg(g) \in \mathbf{Z}(k)$ and $h \in R[\mathbf{t}, \mathbf{t}^{-1}]$ is nilpotent. From this and the formulas for $\Theta$ and $I$ we see

$$\deg(\Theta(g)) = \deg(g), \quad \deg(I(g)) = -\deg(g) \quad (g \in R[\mathbf{t}, \mathbf{t}^{-1}]^{\times}). \tag{7}$$

With the notations introduced above, the map $\partial$ of Lemma 3.3 is expressed by

$$\partial(f_1, f_2) = f_1 \cdot I(f_2)^{-1}, \tag{8}$$

while by 4.1.2 and (2),

$$\partial'(f_1, f_2) = f_1 \cdot \Theta(I(f_2))^{-1}, \tag{9}$$

for $f_i \in R[\mathbf{t}]^{\times}$. We claim that

$$\mathrm{Im}(\partial') = \mathrm{Im}(\partial) = \mathrm{Ker}(\deg). \tag{10}$$

Indeed, the second equality follows from Lemma 3.3. As deg vanishes on $R[\mathbf{t}]^{\times}$, it follows from (7) and (9) that $\mathrm{Im}(\partial') \subset \mathrm{Ker}(\deg) = \mathrm{Im}(\partial)$. To prove $\mathrm{Im}(\partial) \subset \mathrm{Im}(\partial')$, it suffices by (8) to have $I(f) \in \mathrm{Im}(\partial')$, for all $f \in R[\mathbf{t}]^{\times}$. The automorphism $\Theta$ of $R[\mathbf{t}, \mathbf{t}^{-1}]$ induces an endomorphism (but not an automorphism) of the subring $R[\mathbf{t}]$. This is evident from (3). Hence $\Theta(f) \in R[\mathbf{t}]^{\times}$, and by (5), $I(f) = \Theta(I(\Theta(f))) = \partial'(1, \Theta(f)^{-1}) \in \mathrm{Im}(\partial')$. Now (10) and Lemma 3.3 together with (6) yields the desired isomorphism

$$\Psi \colon \mathrm{Pic}_{\mathfrak{V}}(\mathbf{X}) \cong H^1(\mathfrak{V}, \mathscr{G}) \cong R[\mathbf{t}, \mathbf{t}^{-1}]^{\times} / \mathrm{Im}(\partial') \xrightarrow{\deg} \mathbf{Z}(k).$$

It remains to show that (1) is commutative. The map $\pi^*\colon \mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1) \to \mathrm{Pic}_{\mathfrak{V}}(\mathbf{X})$ is induced by the maps $\pi_i^*\colon \mathscr{F}(\mathbf{U}_i) \to \mathscr{G}(\mathbf{V}_i)$ and $\pi_{12}^*\colon \mathscr{F}(\mathbf{U}_{12}) \to \mathscr{G}(\mathbf{V}_{12})$, where $\pi_i$ and $\pi_{12}$ are the restrictions of the projection $\pi\colon \mathbf{X} \to \mathbf{P}_1$. After the identifications of these rings with polynomial resp. Laurent polynomial rings as above, these are just the natural injections $k[\mathbf{t}]^\times \to R[\mathbf{t}]^\times$ and $k[\mathbf{t}, \mathbf{t}^{-1}]^\times \to R[\mathbf{t}, \mathbf{t}^{-1}]^\times$ induced from $k \to R$. From (3), (4) and (9) one sees easily that the diagram

$$\begin{array}{ccccccc}
k[\mathbf{t}]^\times \times k[\mathbf{t}]^\times & \xrightarrow{\partial} & k[\mathbf{t}, \mathbf{t}^{-1}]^\times & \xrightarrow{\deg} & \mathbf{Z}(k) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \\
R[\mathbf{t}]^\times \times R[\mathbf{t}]^\times & \xrightarrow[\partial']{} & R[\mathbf{t}, \mathbf{t}^{-1}]^\times & \xrightarrow[\deg]{} & \mathbf{Z}(k) & \longrightarrow & 0
\end{array}$$

is commutative with exact rows. This implies commutativity of (1) and completes the proof.

**4.3. Corollary.** *$\mathscr{E}$ has infinite order in* $\mathrm{Pic}(\mathbf{P}_1)$ *and $\mathscr{L}$ has infinite order in* $\mathrm{Pic}(\mathbf{X})$.

**4.4. Corollary.** *If $k$ is a factorial ring then* $\mathrm{Pic}(\mathbf{P}_1) \cong \mathbb{Z} \cong \mathrm{Pic}(\mathbf{X})$*, generated by $\mathscr{E}$ and $\mathscr{L}$, respectively.*

*Proof.* The Picard group of an integral domain is canonically embedded into the ideal class group, and the latter is trivial for a factorial domain [**1**, VII, §1.2, Remarks after Prop. 4, and §3, Def. 1]. Also, $k[\mathbf{t}]$ is factorial along with $k$. Hence every line bundle on $\mathbf{P}_1$ is trivialized by $\mathfrak{U}$, i.e., $\mathrm{Pic}(\mathbf{P}_1) = \mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1)$. Moreover, $\mathbf{Z}(k) \cong \mathbb{Z}$ since $k$ has no non-trivial idempotents. Now the first isomorphism follows from Theorem 4.1, and the proof of the second one is analogous.

**4.5. Remarks.** (i) The isomorphisms $\Phi$ and $\Psi$ of 4.1 and 4.2 are easily seen to be compatible with base change. Hence, the sub-functors $\mathbf{Pic}_{\mathfrak{U}}(\mathbf{P}_1)$ and $\mathbf{Pic}_{\mathfrak{V}}(\mathbf{X})$ of the Picard functors $\mathbf{Pic}(\mathbf{P}_1)$ and $\mathbf{Pic}(\mathbf{X})$ defined by

$$\mathbf{Pic}_{\mathfrak{U}}(\mathbf{P}_1)(R) = \mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1 \otimes R), \quad \mathbf{Pic}_{\mathfrak{V}}(\mathbf{X})(R) = \mathrm{Pic}_{\mathfrak{V}}(\mathbf{X} \otimes R)$$

are actually isomorphic to $\mathbf{Z}$.

(ii) The canonical projection $p\colon \mathbf{P}_1 \to \mathbf{S} = \mathbf{Spec}(k)$ induces a homomorphism $p^*\colon \mathrm{Pic}(k) \cong \mathrm{Pic}(\mathbf{S}) \to \mathrm{Pic}(\mathbf{P}_1)$. This is an isomorphism onto a direct summand because $p$ has sections (the elements of $\mathbf{P}_1(k)$ are in bijection with the sections of $p$). We claim that $p^*(\mathrm{Pic}(k)) \cap \mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1) = 0$. Indeed, let $i_1\colon \mathbf{U}_1 \to \mathbf{P}_1$ be the inclusion and $p_1 = p\big|\mathbf{U}_1$. Then $p_1 = p \circ i_1$ and hence $p_1^* = i_1^* \circ p^*$. Since $\mathbf{U}_1(k) \neq \emptyset$ as well, $p_1^*\colon \mathbf{Pic}(k) \to \mathbf{Pic}(\mathbf{U}_1)$ is injective, so $i_1^*\colon p^*(\mathbf{Pic}(k)) \to \mathbf{Pic}(\mathbf{U}_1)$ is injective. Hence for an element $p^*([L]) = [\mathscr{M}] \in p^*(\mathbf{Pic}(k)) \cap \mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1)$ we have $i_1^*([\mathscr{M}]) = 0$ (since the restriction of $\mathscr{M}$ to $\mathbf{U}_1$ is trivial) $= p_1^*([L])$ and therefore $[L] = 0$ in $\mathbf{Pic}(k)$. Question: Is

$$p^*(\mathbf{Pic}(k)) \oplus \mathrm{Pic}_{\mathfrak{U}}(\mathbf{P}_1) = \mathrm{Pic}(\mathbf{P}_1)?$$

Analogous statements hold and questions can be asked for $\mathrm{Pic}(\mathbf{X})$.

**Acknowledgement** I am grateful to H. P. Petersson who carefully read an earlier draft of this note and suggested numerous improvements.

### References


[**1**] N. Bourbaki, *Commutative Algebra*, Hermann, Paris, 1972, Translated from the French.

[**2**] M. Demazure and P. Gabriel, *Groupes algébriques*, vol. 1, Masson, Paris, 1970.

[**3**] O. Loos, *Generically algebraic Jordan algebras over commutative rings*, J. Algebra **297** (2006), 474–529.

[**4**] H. P. Petersson, *Idempotent 2-by-2 matrices*, preprint, 2007.