FACTORING SKEW POLYNOMIALS OVER HAMILTON'S QUATERNION ALGEBRA AND THE COMPLEX NUMBERS

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Abstract. Using nonassociative algebras constructed out of skew-polynomial rings as introduced by Petit, we show that all non-constant polynomials in the skew-polynomial ring $\mathbb{H}[t;\sigma,\delta]$ decompose into a product of linear factors, and that all non-constant polynomials in the skew-polynomial ring $\mathbb{C}[t;\sigma,\delta]$ decompose into a product of linear and quadratic irreducible factors.

Introduction

By the Fundamental Theorem of Algebra, every non-constant complex polynomial splits into a product of linear factors and thus has a root in $\mathbb{C}$. For the ring of left polynomials over Hamilton's quaternion algebra $\mathbb{H}[t]=\mathbb{H}_L[t]$ with elements $f=a_mt^m+\cdots+a_1t+a_0$, $a_i\in\mathbb{H}$, where the variable t commutes with every $z\in\mathbb{H}$, it is again well-known that every non-constant polynomial f splits into a product of linear polynomials, cf. for instance [6], [11, Theorem 5.2], and that every non-constant $f\in\mathbb{H}[t]$ has a root in $\mathbb{H}$ (see [15] for the earliest proof).

In the following, we look at skew-polynomial rings $D[t;\sigma,\delta]$ with σ a ring endomorphism of D and δ a σ-derivation, where D is either the quaternion division algebra over a real closed field F or its quadratic field extension $F(\sqrt{-1})$. We show that each non-constant polynomial $f\in D[t;\sigma,\delta]$ splits into linear, quadratic or, if $D=F(\sqrt{-1})$, quartic irreducible factors (Theorems 3 and 6). As a consequence, we obtain that each non-constant polynomial $f\in\mathbb{H}[t;\sigma,\delta]$ splits into linear factors and that each non-constant polynomial $f\in\mathbb{C}[t;\sigma,\delta]$ splits into linear or quadratic irreducible factors (Corollaries 4 and 7).

For the proofs, we employ nonassociative algebras S_f constructed out of skew-polynomial rings defined by Petit [16], together with the fact that over real-closed fields, division algebras only exist in dimensions 1, 2, 4 or 8. As a corollary, we obtain a new proof for the Fundamental Theorem of Algebra for polynomials in $\mathbb{H}[t]$. We believe that our approach deserves attention, as the proofs are straightforward and only use results from the theory of nonassociative algebras.

As a consequence, we also obtain that a real algebra S_f is a division algebra implies that $f\in\mathbb{C}[t;\sigma,\delta]$ must be irreducible of degree 2. Every real division algebra A of dimension 4

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which is a two-dimensional vector space over $\mathbb{C} \subset \mathrm{Nuc}_l(A) \cap \mathrm{Nuc}_m(A)$ is isomorphic to S_f for some irreducible $f(t) = t^2 - d_1 t - d_0 \in \mathbb{C}[t;\sigma,\delta]$, where σ and δ are suitably defined.

1. Preliminaries

1.1. **Nonassociative algebras.** Let F be a field and let A be a finite-dimensional F-vector space. We call A an *algebra* over F if there exists an F-bilinear map $A \times A \to A$, $(x,y) \mapsto x \cdot y$, denoted simply by juxtaposition xy, the *multiplication* of A. An algebra A is called *unital* if there is an element in A, denoted by 1, such that $1x = x1 = x$ for all $x \in A$. We will only consider unital algebras.

An algebra $A \neq 0$ is called a *division algebra* if for any $a \in A$, $a \neq 0$, the left multiplication with a, $L_a(x) = ax$, and the right multiplication with a, $R_a(x) = xa$, are bijective. A is a division algebra if and only if A has no zero divisors ([17], pp. 15, 16).

For an F-algebra A, associativity in A is measured by the *associator* $[x,y,z] = (xy)z - x(yz)$. The *left nucleus* of A is defined as $\mathrm{Nuc}_l(A) = \{x \in A \,|\, [x,A,A] = 0\}$, the *middle nucleus* of A is defined as $\mathrm{Nuc}_m(A) = \{x \in A \,|\, [A,x,A] = 0\}$ and the *right nucleus* of A is defined as $\mathrm{Nuc}_r(A) = \{x \in A \,|\, [A,A,x] = 0\}$. Their intersection $\mathrm{Nuc}(A) = \{x \in A \,|\, [x,A,A] = [A,x,A] = [A,A,x] = 0\}$ is the *nucleus* of A. $\mathrm{Nuc}(A)$ is an associative subalgebra of A containing $F1$ and $x(yz) = (xy)z$ whenever one of the elements x,y,z is in $\mathrm{Nuc}(A)$. The *center* of A is $\mathrm{C}(A) = \{x \in A \,|\, x \in \mathrm{Nuc}(A) \text{ and } xy = yx \text{ for all } y \in A\}$.

1.2. **How to obtain nonassociative division algebras from skew-polynomial rings.** In the following, we use the terminology used by Jacobson [10] and Petit [16] and summarize their most important results needed in this paper. Let D be a unital division ring, σ a ring endomorphism of D and δ a *left σ-derivation* of D, i.e. an additive map such that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$$

for all $a, b \in D$, implying $\delta(1) = 0$. The *skew-polynomial ring* $D[t;\sigma,\delta]$ is the set of polynomials

$$a_0 + a_1 t + \cdots + a_n t^n$$

with $a_i \in D$, where addition is defined term-wise and multiplication by

$$ta = \sigma(a)t + \delta(a) \quad (a \in D).$$

That means,

$$at^n bt^m = \sum_{j=0}^{n} a(S_{n,j} b) t^{m+j}$$

$(a, b \in D)$, where the map $S_{n,j}$ is defined recursively via

$$S_{n,j} = \delta(S_{n-1,j}) + \sigma(S_{n-1,j-1}),$$

with $S_{0,0} = id_D$, $S_{1,0} = \delta$, $S_{1,1} = \sigma$ and so $S_{n,j}$ is the sum of all polynomials in σ and δ of degree j in σ and degree $n-j$ in δ [10, p. 2]. If $\delta = 0$, then $S_{n,j} = \sigma^n$. $D[t;\sigma] = D[t;\sigma,0]$ is called a *twisted polynomial ring* and $D[t;\delta] = D[t;id,\delta]$ a *differential polynomial ring*. For the special case that $\sigma = id$ and $\delta = 0$, we obtain the usual ring of left polynomials

$D[t] = D[t; id, 0]$, often also denoted $D_L[t]$ in the literature [8], with its multiplication given by

$$(\sum_{i=1}^{s} a_i t^i)(\sum_{i=1}^{t} b_i t^i) = \sum_{i,j} a_i b_j t^{i+j}.$$

If D has finite dimension over its center and σ is a ring automorphism of D, then $R = D[t;\sigma,\delta]$ is either a twisted polynomial or a differential polynomial ring by a linear change of variables [10, Thm. 1.2.21]. Note also that if σ and δ are F-linear maps then $D[t;\sigma,\delta] \cong D[t]$ by a linear change of variables.

For $f = a_0 + a_1 t + \cdots + a_n t^n$ with $a_n \neq 0$ define $\deg(f) = n$ and $\deg(0) = -\infty$. Then $\deg(fg) = \deg(f) + \deg(g)$. An element $f \in R$ is *irreducible* in R if it is no unit and it has no proper factors, i.e if there do not exist $g, h \in R$ with $\deg(g), \deg(h) < \deg(f)$ such that $f = gh$.

$R = D[t;\sigma,\delta]$ is a left principal ideal domain and there is a left-division algorithm in R [10, p. 3]: for all $g, f \in R$ there exist unique $r, q \in R$, $g \neq 0$ and $\deg(r) < \deg(f)$, such that

$$g = qf + r.$$

If σ is a ring automorphism then $R = D[t;\sigma,\delta]$ is a left and right principal ideal domain (a PID) [10, p. 6] and there is also a right-division algorithm in R [10, p. 3 and Prop. 1.1.14]. (We point out that the terminology used by Petit in [16] is different from ours; there what we and Jacobson call left is called a right-division algorithm and vice versa.)

If σ is a ring automorphism, two non-zero elements $f, g \in R$ are called *similar* ($f \sim g$) if and only if there exist $h, q, u \in R$ such that

$$1 = hf + qg \text{ and } u'f = gu$$

for some $u' \in R$ if and only if $R/Rf = R/Rg$ [10, p. 11]. If σ is a ring automorphism, $R = D[t;\sigma,\delta]$ is a PID, hence any element $f \in R$, $f \neq 0$ which is not a unit in R, can be written as $f = p_1 \cdots p_s$ with irreducible $p_i \in R$. If $f = p_1 \cdots p_s = p'_1 \cdots p'_t$, where the p_i and the p'_i are irreducible, then $s = t$ and there exists a permutation $\pi \in S_s$ such that $p_i \sim p'_{\pi(i)}$ for all i. This is the Fundamental Theorem of Arithmetic in a PID [10, Theorem 1.2.9]. Obviously, $f \sim g$ implies that $\deg(f) = \deg(g)$.

Definition 1. (cf. [16, (7)]) Let D be a unital associative division algebra and $f \in D[t;\sigma,\delta]$ of degree m, σ a ring endomorphism of D. Let $\mathrm{mod}_l f$ denote the remainder of left division by f. Then the vector space

$$R_m = \{g \in D[t;\sigma,\delta] \,|\, \deg(g) < m\}$$

together with the multiplication

$$g \circ h = gh \;\mathrm{mod}_l f$$

becomes a nonassociative algebra $S_f = (R_m, \circ)$ over $F_0 = \{a \in D \,|\, ah = ha \text{ for all } h \in S_f\}$.

Note that the multiplication is well-defined and that F_0 is a subfield of D [16, (7)].

Remark 1. In [10], only the associative algebras

$$E(f) = \{g \in R \,|\, f \text{ left divides } fg\}$$

for $f \in D[t;\sigma,\delta]$, σ an automorphism, were investigated. $E(f)$ is division if f is irreducible.

Theorem 2. *(cf.* [16, (2), p. 13-03, (5), (7), (9), (14)]*) Let $f \in R = D[t;\sigma,\delta]$.*
(i) If S_f is not associative then

$$\mathrm{Nuc}_l(S_f) = \mathrm{Nuc}_m(S_f) = D$$

and

$$\mathrm{Nuc}_r(S_f) = \{g \in R \,|\, fg \in Rf\}.$$

(ii) The powers of t are associative if and only if $t^m t = t t^m$ if and only if $t \in \mathrm{Nuc}_r(S_f)$ if and only if $ft \in Rf$.
(iii) If f is irreducible then $\mathrm{Nuc}_r(S_f)$ is an associative division algebra.
(iv) Let $f \in R$ be irreducible and S_f a finite-dimensional F_0-vector space or a finite-dimensional right $\mathrm{Nuc}_r(S_f)$-module. Then S_f is a division algebra.
S_f is associative if and only if f is a two-sided element.
(v) Let $f = t^m - \sum_{i=0}^{m-1} d_i t^i \in R = D[t;\sigma]$. Then $f(t)$ is a two-sided element of S_f if and only if $\sigma^m(z) d_i = d_i \sigma^i(z)$ for all $z \in D$, $0 \leq i < m$ and $\sigma(d_i) = d_i$ for all i, $0 \leq i < m$.

2. Factorization Theorems for skewed polynomials if F is a real closed field

Let F be a *real closed field*, that is a formally real field such that every polynomial of odd degree with coefficients in F has at least one root in F, and for every element $a \in F$ there is $b \in F$ such that $a = b^2$ or $a = -b^2$. Equivalently, F is a real closed field if it is not algebraically closed but if the field extension $F(\sqrt{-1})$ is algebraically closed. Then every division algebra over F has dimension 1, 2, 4 or 8. Moreover, up to isomorphism, there are exactly three associative division algebras over F, one each of dimension 1, 2, and 4. The first is a classical and well-known result for algebras over $\mathbb{R}$ (cf. [3] and [12]) and was generalized to any closed fields using model theory in [4], the second is also well-known for $\mathbb{R}$ [5], for real closed fields see [13] or [4].

Let D be a finite-dimensional unital associative division algebra over F (hence either the quadratic field extension $F(\sqrt{-1})$, or the quaternion division algebra) and σ an injective ring homomorphism of D. By [10, Thm. 1.1.21], after a linear change of variables, $D[t;\sigma,\delta]$ is either a twisted polynomial ring or a differential polynomial ring.

Using the results quoted above, we are immediately able to prove:

Theorem 3. *(Factorization Theorem I)*
(i) Let D be the quaternion division algebra over a real closed field. Then every polynomial $f \in D[t;\sigma,\delta]$ of degree $m > 2$ decomposes into the product of linear or quadratic irreducible polynomials. No quadratic irreducible polynomial is a two-sided element in R.
(ii) If σ is a ring automorphism and $f = p_1 \cdots p_s = p'_1 \cdots p'_t$ are two such decompositions, then $s = t$ and there is a permutation $\pi \in S_s$ such that

$$p'_{\pi(i)} \sim p_i$$

for all i, $1 \leq i \leq s$.
(iii) Every irreducible monic quadratic polynomial $f = t^2 - d_1 t - d_0 \in D[t;\sigma]$ satisfies $\sigma^2(z) d_i \neq d_i \sigma^i(z)$ for some $z \in D$, $i \in \{0,1\}$, or $\sigma(d_i) \neq d_i$ for some $i \in \{0,1\}$.

Proof. (i) Suppose that $f(t) \in R = D[t;\sigma,\delta]$ has degree $m > 2$ and is irreducible. Then S_f is a unital nonassociative division algebra of dimension $4m > 8$ over F_0 and F_0, being a subfield of D, is either F or $F(\sqrt{-1})$. The case that $F_0 = F(\sqrt{-1})$ is not possible, as there are no division algebras over $F(\sqrt{-1})$ of dimension > 1. Thus $F_0 = F$. However, there are no division algebras over F of dimension > 8, again a contradiction. Thus $f(t)$ must be reducible and so $f = gh$ for suitable $f, g \in R$ with $\deg(f), \deg(h) < m$. By applying the same argument to g and h and iterating it, we conclude that f is a product of linear and quadratic irreducible polynomials. If f is an irreducible quadratic polynomial, then S_f has dimension 8 and is a nonassociative division algebra. Theorem 2 (iv) implies that f cannot be a two-sided element.
(ii) By the Fundamental Theorem of Arithmetic [10, Theorem 1.2.9], this decomposition is unique up to a permutation of the factors and similarity.
(iii) The assertion follows from Theorem 2 (v). □

For the next result, recall that if σ is a ring automorphism then $t-a$ and $t-b$ are called (σ,δ)-*conjugate* if and only if there is some $u \in D^\times$ such that $a = \sigma(u)bu^{-1} + \delta(u)u^{-1}$.

Let $\mathbb{H}$ be Hamilton's quaternion division algebra over $\mathbb{R}$.

Corollary 4. *(Fundamental Theorem)*
(i) Every polynomial $f(t) \in \mathbb{H}[t;\sigma,\delta]$ of degree m decomposes into the product of m linear polynomials. In particular, $f(t)$ has a root.
(ii) If σ is a ring automorphism of $\mathbb{H}$ and $f = p_1 \cdots p_m = p'_1 \cdots p'_m$, then there is a permutation $\pi \in S_s$ such that $p'_{\pi(i)}$ and p_i are (σ,δ)-conjugate for all i, $1 \leq i \leq s$.
(iii) If σ is a ring automorphism of $\mathbb{H}$ then each root of a monic polynomial $f(t) \in \mathbb{H}[t;\sigma,\delta]$ in $\mathbb{H}$ is the (σ,δ)-conjugate to some a_i, where the $t - a_i$ are the linear factors of f.

Proof. (i) It remains to show that here all quadratic polynomials are reducible: Suppose f is an irreducible quadratic polynomial. Then the unital F-algebra S_f has dimension 8 and since it is nonassociative we have $\mathrm{Nuc}_l(S_f) = \mathbb{H}$ by Theorem 2 (i). Since for any unital real division algebra A, $\mathrm{Nuc}_l(A) = \mathbb{R}$ by [2], this yields a contradiction. Thus $f(t)$ is reducible and the product of two linear polynomials. In particular, $f(t)$ has a root in $\mathbb{H}$: for $f(t) = g(t)(at-b)$, $f(a^{-1}b) = g(a^{-1}b)(aa^{-1}b - b) = 0$.
(ii) is clear and (iii) is [13, (16.4)]: Each root of $f(t) = (t-a_1)\cdots(t-a_m)$ in $\mathbb{H}$ is a (σ,δ)-conjugate to some a_i. □

Factorization Theorems for polynomials in $\mathbb{H}[t]$ and related results were proved for instance in [6], [7], [9] and [11, Theorem 5.2]. The Fundamental Theorem of Algebra for polynomials in $\mathbb{H}[t]$ was proved by [15]. We obtain it as a special case:

Corollary 5. *(i) Every non-constant polynomial $f(t) \in \mathbb{H}[t]$ of degree m decomposes into the product of m linear polynomials. Moreover, if $f = p_1 \cdots p_m = p'_1 \cdots p'_m$, then there is a permutation $\pi \in S_s$ and $z_i \in \mathbb{H}$ such that*

$$p'_{\pi(i)} = z_i p_i z_i^{-1}$$

for all i, $1 \leq i \leq m$.
(ii) If the monic polynomial $f(t) \in \mathbb{H}[t]$ has the m linear factors $t - a_i$ then there is a zero of f in $\mathbb{H}$ in each of the congruence classes $[a_i]$, $1 \leq i \leq m$.

Proof. (i) This follows from Corollary 4 and the fact that conjugation in $\mathbb{H}[t]$ is the usual conjugacy [10, p. 15].
(ii) This is proved in [18]. □

Theorem 6. *(Factorization Theorem II)*
(i) Every polynomial $f(t) \in F(\sqrt{-1})[t;\sigma,\delta]$ of degree $m > 3$ decomposes into the product of linear, quadratic or quartic irreducible polynomials. No quartic irreducible polynomial is a two-sided element in $F(\sqrt{-1})[t;\sigma,\delta]$.
(ii) If σ is a ring automorphism of $F(\sqrt{-1})$ and $f = p_1 \cdots p_s = p'_1 \cdots p'_t$ are two such decompositions, then $s = t$ and there is a permutation $\pi \in S_s$ such that

$$p'_{\pi(i)} \sim p_i$$

for all i, $1 \leq i \leq s$.
(iii) If σ is a ring automorphism of $F(\sqrt{-1})$ then every irreducible monic quartic polynomial $f = t^4 - d_3t^3 - d_2t^2 - d_1t - d_0 \in F(\sqrt{-1})[t;\sigma]$ satisfies $\sigma^4(z)d_i \neq d_i\sigma^i(z)$ for some $z \in D$, $i \in \{0,1,2,3\}$, or $\sigma(d_i) \neq d_i$ for some $i \in \{0,1,2,3\}$.

Proof. (i) Suppose that $f(t) \in R = F(\sqrt{-1})[t;\sigma,\delta]$ has degree $m > 4$ or degree 3 and is irreducible. Then S_f is a unital nonassociative division algebra over F_0 and F_0, being a subfield of $F(\sqrt{-1})$, is either F or $F(\sqrt{-1})$. The case that $F_0 = F(\sqrt{-1})$ is not possible, as there are no division algebras over $F(\sqrt{-1})$ of dimension > 1. Thus $F_0 = F$. However, there are no division algebras over F of dimension $2m > 8$ or 6, again a contradiction. Thus $f(t)$ must be reducible and so $f = gh$ for suitable $f, g \in R$ with $\deg(f), \deg(h) < m$. By applying the same argument to g and h and iterating it, we conclude that f is a product of linear, quadratic and quartic irreducible polynomials. If f is an irreducible quartic polynomial, then S_f has dimension 8 over F and is a nonassociative division algebra. Theorem 2 (iv) implies that f cannot be a two-sided element.
(ii) By the Fundamental Theorem of Arithmetic for a PID, this decomposition is unique up to a permutation of the factors and similarity.
(iii) The assertion follows from Theorem 2 (v). □

Corollary 7. *Every polynomial $f(t) \in R = \mathbb{C}[t;\sigma,\delta]$ of degree ≥ 1 decomposes into the product of linear and quadratic irreducible polynomials. Moreover, if σ is a ring automorphism and $f = p_1 \cdots p_s = p'_1 \cdots p'_t$, then $s = t$ and there is a permutation $\pi \in S_s$ such that*

$$p'_{\pi(i)} \sim p_i$$

for all i, $1 \leq i \leq s$.

Proof. It remains to show that all quartic polynomials are reducible: Suppose f is an irreducible quartic polynomial. Then the unital F-algebra S_f has dimension 8 and since it is nonassociative we have $\mathrm{Nuc}_l(S_f) = \mathbb{C}$ by Theorem 2 (i). Since for any unital real division algebra A, $\mathrm{Nuc}_l(A) = \mathbb{R}$ by [2], this yields a contradiction. Thus $f(t)$ is reducible. □

Remark 8. (i) If it is possible to prove (most likely by using model theory) that also for a real closed field F, any unital division algebra has left nucleus F, then there are no irreducible quadratic polynomials in $D[t;\sigma,\delta]$ and no irreducible quartic polynomials in $F(\sqrt{-1})[t;\sigma,\delta]$, either.
(ii) It is easy to see that for any $a\in\mathbb{R}\setminus\mathbb{Q}$, there is an id-derivation δ on $\mathbb{R}$ such that $\delta(a)\neq 0$, that means δ is not $\mathbb{R}$-linear. This yields an id-derivation δ on $\mathbb{H}$, $\delta(x_0+x_1i+x_2j+x_3k)=\delta(x_0)+\delta(x_1)i+\delta(x_2)j+\delta(x_3)k$, and an id-derivation δ on $\mathbb{C}$ via $\delta(x_0+x_1i)=\delta(x_0)+\delta(x_1)i$, which are both not $\mathbb{R}$-linear. Along the same line of thought, we can extend any ring isomorphism $\sigma:\mathbb{R}\to\mathbb{R}$ to a ring isomorphism on $\mathbb{C}$ and on $\mathbb{H}$.

3. A remark on four-dimensional real division algebras

If σ is a ring isomorphism, we also have a right division algorithm and can use it to define a second algebra construction (cf. [16]): Let $f\in D[t;\sigma]$ be of degree m and let $\mathrm{mod}_r f$ denote the remainder of right division by f. Then the vector space $V=\{g\in D[t;\sigma]\,|\,\deg(g)<m\}$ together with the multiplication

$$g\circ h=gh\ \mathrm{mod}_r f$$

becomes a nonassociative algebra ${}_fS=(V,\circ)$, which, however, turns out to be anti-isomorphic to a suitable algebra S'_g.

Lemma 9. *Let K/F be a separable quadratic field extension with non-trivial automorphism σ and $d\in K\setminus F$. Then the nonassociative quaternion division algebra* $\mathrm{Cay}(K,d)$ *is isomorphic to the algebra S_f with $f(t)=t^2-d\in K[t;\sigma]$.*

The proof is straightforward.

For $F=\mathbb{R}$, we obtain the following characterization of four-dimensional real division algebras which are two-dimensional vector spaces over $\mathbb{C}$ and have $\mathbb{C}\subset\mathrm{Nuc}_l(A)\cap\mathrm{Nuc}_m(A)$ due to [16, (1), p. 13-08] and our above results:

Corollary 10. *(i) If S_f is a real division algebra then $R=\mathbb{C}[t;\sigma,\delta]$, $f\in R$ is irreducible of degree 2, and if S_f is not associative then* $\mathrm{Nuc}_l(A)=\mathrm{Nuc}_m(A)=\mathbb{C}$ *and* $\mathrm{Nuc}_m(A)\in\{\mathbb{R},\mathbb{C}\}$.
(ii) Every real division algebra A with multiplication $\star$ of dimension 4 which is a two-dimensional vector space over $\mathbb{C}\subset\mathrm{Nuc}_l(A)\cap\mathrm{Nuc}_m(A)$ is isomorphic to S_f for a suitable irreducible $f\in\mathbb{C}[t;\sigma,\delta]$, $f=t^2-d_1t-d_0$, where σ and δ are defined via

$$t\star a=\sigma(a)\star t+\delta(a)$$

for all $a\in D$.
(iii) Every real division algebra A of dimension 4 which is a two-dimensional vector space over $\mathbb{C}\subset\mathrm{Nuc}_m(A)\cap\mathrm{Nuc}_r(A)$ is isomorphic to $(S_f)^{op}$ for a suitable irreducible $f\in\mathbb{C}[t;\sigma,\delta]$, $f=t^2-d_1t-d_0$, where σ and δ are defined via

$$t\star a=\sigma(a)\star t+\delta(a)$$

for all $a\in D$ and where $\star$ is the multiplication in S_f. In particular, if σ is a ring isomorphism then $A\cong{}_{H(f)}S'$ with $H(f)=\in R'=\mathbb{C}[t;\sigma^{-1},-\delta\circ\sigma^{-1}]$ and $H(f)=t^2-\sigma(d_1)t-d_0$.

By [16, (17)], $f(t) = t^2 - d_1 t - d_0 \in \mathbb{C}[t;\sigma]$ is irreducible if and only if $\sigma(z)z \neq d_1 z + d_0$ for all $z \in D$.

Example 11. For $R = \mathbb{C}[t;^-]$, $^-$ the complex conjugation, we obtain Hamilton's quaternions by choosing $f(t) = t^2 + 1$, and a nonassociative quaternion division algebra $\mathrm{Cay}(\mathbb{C}, d_0)$ by choosing $f(t) = t^2 - d_0$, for any $d_0 \in \mathbb{C} \setminus \mathbb{R}$. The nonassociative quaternion algebras are up to isomorphism the only unital algebras with complex nucleus [W]. Moreover, $\mathrm{Cay}(\mathbb{C}, d_0)^{op} \cong \mathrm{Cay}(\mathbb{C}, d_0)$.

By [16, (23)], $f(t) = t^2 - d_1 t - d_0 \in \mathbb{C}[t;^-]$ with $d_0, d_1 \in \mathbb{R}^\times$ is irreducible if and only if it has no real roots.

See also [1] for related results on four-dimensional real division algebras with $\mathbb{C} \subset \mathrm{Nuc}_l(A) \cap \mathrm{Nuc}_m(A)$.

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