On polynomial identities in associative and Jordan pairs

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Abstract

We prove that a Jordan system satisfies a polynomial identity if and only if it satisfies a homotope polynomial identity. In the obtention of that result, we also prove an analogue for associative pairs with involution of Amitsur's theorem on associative algebras satisfying a polynomial identity with involution.

1 Introducction

Polynomial identities play a basic structural role in nonassociative theory. In contrast with the study of associative rings satisfying polynomial identities, where that condition can be understood as a kind of finiteness condition which allows a strengthen form for the structural results of the

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general theory, its nonassociative counterpart is an unavoidable ingredient in the obtention of general structure theories. In particular, that is the case in Zelmanov's fundamental results [McZ] on Jordan algebras, where the existence of the so called hermitian polynomials leads to the study of algebras that satisfy a particular kind of identities (Clifford identities) prior to the obtention of the general classification theorem. For Jordan pairs and triple systems the situation is entirely analogous to the algebra case. The only difference is that the role played by polynomial identities is now played by homotope polynomial identities (see [DA, DAMc1, DAMc2, M1, M2]), that is, polynomial identities that hold in all homotope algebras of the system. This partially motivated the study of "local PI-theory" in [M1, M2, M3], that is, the sudy of Jordan systems with local algebras satisfying a polynomial identity, and in particular, of Jordan systems satisfying a homotope polynomial identity.

In spite of the effectiveness of the use of homotope polynomials in the structural results, usual identities deserve some attention, both because they are conceptually simpler, and because they seem easier to obtain than homotope identities. In that respect, the study of graded polynomial identities on 3-graded Lie algebras, which will be the subject of a forthcoming paper by the authors, naturally leads through the Kantor-Koecher-Tits construction, to general identities in Jordan pairs. On the other hand, the result of Zelmanov [Z, Theorem 3] asserting that in a PI-Jordan system, the McCrimmon radical and the nil-radical coincide, suggests that some significant results could be reached for general polynomial identities.

The question of whether a theory of general identities of Jordan systems could be developed remained however open, and was the content of a question raised in [M2], namely: does every PI Jordan system satisfy a homotope polynomial identity?

In this paper we answer in the affirmative that conjecture. Since our approach makes use of the structure theory, we need to gain first some information on Jordan pairs of hermitian elements $H(A, *)$ for an associative pair A with involution $*$. Thus, after a section of preliminaries, we devote a section to the study of associative pairs satisfying a $*$-polynomial identity (a polynomial identity with involution). We prove there a pair analogue of Amitsur's celebrated result [Am] on algebras with involution satisfying a $*$-polynomial identity. Our proof is closely patterned after the proof of the algebra result as exposed by Herstein in [H2], which is due to Montgomery (see [H2, page 185]). Apart from its own interest, that result is instrumental in the final section for the proof of the affirmative answer to the foregoing question. Finally, we combine that positive answer with the local PI-theory to obtain Jordan analogues of Kaplansky's and Posner-Rowen's theorems.

In view of the approach followed in the paper, one may ask whether the structural results on PI-Jordan systems could be reached directly without appealing to the local PI-theory. This might well be so, but a careful analysis of the associative GPI-theory uncovers the role played in it by local algebras (see for instance [Ro1, Ro2], and specially the approach followed in [BMM]). On the other hand, looking at the standard embedding of associative systems, the PI-condition seems to be in an intermediate place between a PI and a GPI condition, and this suggests that, one way or another, local PI-algebras should make its appearance in their study.

2 Preliminaries

2.1 We will work with associative and Jordan systems over a unital commutative ring of scalars Φ which will be fixed throughtout. We refer to [L1, Me, McZ] for notation, terminology and basic results. We recall in this section some of those notations and basic results.

2.2 An *associative pair* over Φ is a pair $A = (A^+, A^-)$ of Φ-modules together with Φ-trilinear maps

$$\begin{array}{rccc} \langle\ ,\ ,\ \rangle^\sigma : & A^\sigma \times A^{-\sigma} \times A^\sigma & \to & A^\sigma \\ & (x, y, z) & \mapsto & \langle x, y, z\rangle^\sigma \end{array}$$

such that $\langle\langle x, y, z\rangle^\sigma, u, v\rangle^\sigma = \langle x, \langle y, z, u\rangle^{-\sigma}, v\rangle^\sigma = \langle x, y, \langle z, u, v\rangle^\sigma\rangle^\sigma$ for all x, z, $v \in A^\sigma$, y, $u \in A^{-\sigma}$ and $\sigma = \pm$.

An *involution* in the associative pair A is a pair of linear maps

$$\begin{array}{rccc} * : & A^\sigma & \to & A^\sigma \\ & x & \mapsto & x^* \end{array}$$

such that $(x^*)^* = x$ and $\langle x, y, z\rangle^* = \langle z^*, y^*, x^*\rangle$ for all $x,\ z \in A^\sigma,\ \ y \in A^{-\sigma}$, $\sigma = \pm$.

To alleviate the notation, we will usually denote the products in associative pairs simply by juxtaposition, so that if $a, c \in A^\sigma$, and $b \in A^{-\sigma}$ for $\sigma = \pm$, abc will mean $\langle a, b, c\rangle^\sigma$. We recall that associative pairs have a standard imbedding into an associative algebra where the juxtaposition of any two factors makes sense.

Let A be an associative pair. An *A-module* is a pair of Φ-modules (M^+, M^-) endowed with two bilinear maps

$$\begin{array}{rcl} M^\sigma \times A^\sigma & \to & M^{-\sigma} \\ (m, x) & \mapsto & mx \end{array}$$

for $\sigma = \pm$, that satisfy $((mx)y)z = m(xyz)$ for all $x, z \in A^\sigma$, $y \in A^{-\sigma}$, and $m \in M^\sigma$. It is clear how to define the notion of A-submodule of an A-module, and the notions of *irreducible* and *faithful A-modules.*

2.3 Given any associative pair $A = (A^+, A^-)$ we denote by $\mathcal{U}_A$, the *standard Φ-imbedding* of A. Recall that $\mathcal{U}_A$ is a unital associative algebra with two idempotents $e_1 + e_2 = 1$ such that if we consider the Peirce decomposition of $\mathcal{U}_A$ with respect these idempotents, then $A = ((\mathcal{U}_A)_{12}, (\mathcal{U}_A)_{21})$ with the usual triple product. Every involution of A extends uniquely to an algebra involution of $\mathcal{U}_A$ that satisfies $e_1^* = e_2$ [FT, 3.2].

2.4 The *socle* of $A = (A^+, A^-)$ is $Soc(A) = (Soc(A^+), Soc(A^-))$, where $Soc(A^\sigma)$ is the sum of all minimal right ideals of A^σ. If A has no minimal right ideals, we write $Soc(A) = 0$. An associative pair A has *finite capacity* if it satisfies both the ascending and the descending chain condition on principal inner ideals (see [L1] for definitions). In that case A equals its socle and it contains a maximal idempotent: an idempotent $e = (e^+, e^-)$ whose Peirce 00-space A_{00} vanishes (see [L1]).

2.5 Any element $a \in A^{-\sigma}$, determines a *homotope algebra* $(A^\sigma)^{(a)}$, an associative algebra over the Φ-module A^σ with multiplication $x \cdot_a y = xay$ for any $x,\ y \in A^\sigma$. The set $Ker_A\, a = Ker\, a = \{x \in A^\sigma \mid axa = 0\}$ is an ideal of $A^{\sigma(a)}$, and the quotient $A^\sigma_a = (A^\sigma)^{(a)}/Ker\, a$ is an associative algebra called the *local algebra of A at a.*

2.6 Let $X = (X^+, X^-)$ be a pair of nonempty sets. We denote by $FAP(X)$ the free associative pair over Φ on X, and call its elements (*pair*) *polynomials*. Note that $FAP(X)$ is the subpair of the pair $(FA(X^+ \cup X^-), FA(X^+ \cup X^-))$ obtained by doubling the free associative algebra $FA(X^+ \cup X^-)$, generated by (X^+, X^-). Clearly, every nonzero pair polynomial has odd degree. By the universal property of $FAP(X)$, any polynomial $f(x_1^+, \dots, x_n^+, x_1^-, \dots, x_n^-)$ can be evaluated in an associative pair A on fixed values $x_i^\sigma = a_i^\sigma \in A^\sigma$ for the indeterminates $x^\sigma \in X^\sigma$. An associative polynomial $f_\sigma \in FAP(X)^\sigma$ is a *polynomial identity* of an associative pair A, and A is then said to be a *PI-associative pair*, if f_σ is monic in the sense that some leading monomial in f has coefficient 1, and all the evaluations of f_σ in A vanish. A polynomial $p_\sigma \in FAP(X)^\sigma$ of degree $m = 2d + 1$, and involving the variables $x_1^\sigma, \dots, x_{d+1}^\sigma$ and $x_1^{-\sigma}, \dots, x_d^{-\sigma}$ is *multilinear* if each monomial appearing in p_σ (note that $FA(X)^\sigma$ is a free Φ-module over the set of monomials) its degree in each variable $x^{\pm\sigma} \in X$ is exactly 1. As for algebras, it is easy to see that if an associative pair A satisfies a polynomial identity of degree m, then it satisfies a multilinear identity of degree m.

2.7 Let $X = (X^+, X^-)$ and $Z = (Z^+, Z^-)$ be two pairs of sets and assume that there are bijections $* : X^\sigma \to Z^\sigma$ (whose inverse we also denote by $*$). Then the associative pair $FAP(X, Z) = FAP(X^+ \cup Z^+, X^- \cup Z^-)$ can be endowed with an involution extending $*$ in the obvious way. Its elements are called $*$-*polynomials*. We have a notion of degree for any $*$-polynomial $p_\sigma(x^+, x^-, (x^+)^*, (x^-)^*) = p_\sigma(x_1^+, \dots, x_n^+, (x_1^+)^*, \dots, (x_n^+)^*, x_1^-, \dots, x_n^-, (x_1^-)^*, \dots, (x_n^-)^*) \in FAP(X, Z)^\sigma = FAP(X, X^*)^\sigma$. An associative pair $A = (A^+, A^-)$ with involution $*$ satisfies $p_\sigma(x^+, x^-, (x^+)^*, (x^-)^*)$ as before if that polynomial vanishes under every substitution $x_i^\sigma \in A^\sigma$ (see [H2, p. 185]). An associative pair is a $*$-PI-associative pair, if it satisfies a monic $*$-polynomial. A $*$-polynomial $p_\sigma(x_1^+, \dots, x_n^+, (x_1^+)^*, \dots, (x_n^+)^*, x_1^-, \dots, x_n^-, (x_1^-)^*, \dots, (x_n^-)^*) \in FAP(X, X^*)$ is *multilinear* if the polynomial $p_\sigma(x_1^+, \dots, x_n^+, z_1^+, \dots, z_n^+, x_1^-, \dots, x_n^-, z_1^-, \dots, z_n^-) \in FAP(X \cup Z)$ is multilinear.

2.8 Lemma. *Let $A = (A^+, A^-)$ be an associative pair endowed with an involution $*$, and satisfying a $*$-polynomial identity $p_\sigma(x^+, x^-, (x^+)^*, (x^-)^*) = m_\sigma(x^+, x^-, (x^+)^*, (x^-)^*) + \cdots$, of degree $2d + 1$, where $0 \neq \alpha \in \Phi$ and*

$m_\sigma(x^+, x^-, (x^+)^*, (x^-)^*)$ is a monomial of degree $2d+1$. Then A satisfies $\widetilde{p}_\sigma(x^+, x^-, (x^+)^*, (x^-)^*) = x_1^\sigma x_1^{-\sigma} \dots x_d^\sigma x_d^{-\sigma} x_{d+1}^\sigma + q_\sigma(x^+, x^-, (x^+)^*, (x^-)^*)$, where each monomial of $q_\sigma(x^+, x^-, (x^+)^*, (x^-)^*)$ is of degree $2d+1$, involves each x_i^σ or $(x_i^\sigma)^*$ for every $\sigma \in \{+, -\}$, but not both, and where $x_1^\sigma x_1^{-\sigma} \dots x_d^\sigma$ does not occur in $q_\sigma(x^+, x^-, (x^+)^*, (x^-)^*)$. Therefore, A satisfies a multilinear identity of degree $2d+1$.

Proof. See [H2, Lemma 5.1.1]. □

2.9 A Jordan algebra has products $U_x y$ and x^2, quadratic in x and linear in y, whose linearizations are $U_{x,z}y = V_{x,y}x = \{x, y, z\} = U_{x+z}y - U_x y - U_z y$ and $x \circ y = V_x y = (x+y)^2 - x^2 - y^2$.

A Jordan pair $V = (V^\sigma, V^{-\sigma})$ has products $Q_x y$ for $x \in V^\sigma$ and $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z}y = D_{x,y}z = \{x, y, z\} = Q_{x+z}y - Q_x y - Q_z y$.

A Jordan triple system T has product $P_x y$, whose linearizations are $P_{x,z}y = L_{x,y}z = \{x, y, z\} = P_{x+z}y - P_x y - P_z y$.

Any Jordan pair $V = (V^+, V^-)$ gives rise to a *polarized Jordan triple system* $T(V) = V^+ \oplus V^-$ with product $P_{x^+ \oplus x^-} y^+ \oplus y^- = P_{x^+} y^- \oplus P_{x^-} y^+$. Conversely doubling a Jordan triple system T produces a *double Jordan pair* $V(T) = (T, T)$ with $Q_x y = P_x y$ for any x, $y \in T$.

We denote by $\Gamma(J)$ the centroid of a Jordan system J (see [L1, Mc] for definitions). If J is a strongly prime Jordan system, then $\Gamma(J)$ is a domain acting faithfully on J, and we can form the central closure $\Gamma(J)^{-1}J$ as the quotient module (or pair of modules if J is a Jordan pair) of J, which is a Jordan system of the same type as J over the field of fractions $\Gamma(J)^{-1}\Gamma(J)$ of J.

We refer to [M2] for the related notion of *extended centroid* of a Jordan system J, which we denote $\mathcal{C}(J)$, and the attached scalar extension (for a nondegenerate J): its *extended central closure* $\mathcal{C}(J)J$.

2.10 Jordan systems can be obtained from associative systems by symmetrization. Every associative algebra A gives rise to a Jordan algebra $A^{(+)}$, by taking $U_x y = xyx$ and $x^2 = xx$ for x, $y \in A$. Similarly, every associative pair $A = (A^+, A^-)$ produces a Jordan pair $A^{(+)}$ is obtained from an associative pair $A = (A^+, A^-)$ simply by defining $Q_x y = xyx$ for $x^\sigma \in A^\sigma$, $y^{-\sigma} \in A^{-\sigma}$, $\sigma = \pm$.

A Jordan system (algebra, pair or triple system) is *special* if it is isomorphic to a Jordan subsystem of $A^{(+)}$ for some associative system A.

2.11 Associative systems with involution give rise to important examples of special Jordan systems. Given any associative algebra A with involution $*$, the set $H(A,*) = \{a \in A \mid a^* = a\} \subset A^{(+)}$ of symmetric elements of A is a *hermitian Jordan algebra*. More generally, we can consider *ample hermitian subspaces* $H_0(A,*) \subseteq H(A,*)$ of symmetric elements containing all traces $\{a\} = a + a^*$ and norms aa^* of the elements of A and such that $aH_0(A,*)a^* \subset H_0(A,*)$ for all $a \in A$. Recall that if $\frac{1}{2} \in \Phi$, the only ample subspace is $H_0(A,*) = H(A,*)$ [McZ].

If $A = (A^+, A^-)$ is an associative pair with an involution $*$, then $H(A,*) = \big(H(A^+,*), H(A^-,*)\big) \subset A^{(+)}$ where $H(A^\sigma,*) = \{a \in A^\sigma \mid a^* = a\}$ is a *hermitian Jordan pair*. An *ample hermitian subpair* is a subpair $H_0(A,*) = \big(H_0(A^+,*), H_0(A^-,*)\big) \subseteq H(A,*)$ that contains all traces $\{a\} = a + a^*$ of elements $a \in A^\sigma$ and satisfies $aH_0(A^{-\sigma},*)a^* \subseteq H_0(A^\sigma,*)$ for all $a \in A^\sigma$, $\sigma = \pm$.

2.12 Given a Jordan pair $V = (V^+, V^-)$ and $a \in V^{-\sigma}$ the Φ-module V^σ becomes a Jordan algebra denoted $(V^\sigma)^{(a)}$ and called the *a-homotope of* V by defining $U_x^{(a)} y = Q_x Q_a y$ and $x^{(2,a)} = Q_x a$, for any $x,\ y \in V^\sigma$. The set $Ker_V\, a = Ker\, a = \{x \in V^\sigma \mid Q_a x = Q_a Q_x a = 0\}$ is an ideal of $(V^\sigma)^{(a)}$ and the quotient $V_a^\sigma = (V^\sigma)^{(a)}/Ker\, a$ is a Jordan algebra, called the *local algebra of* V *at* a. If V is nondegenerate, then $Ker\, a = \{x \in V^\sigma \mid Q_a x = 0\}$.

2.13 The *socle* $Soc(V) = \big(Soc(V^+), Soc(V^-)\big)$ of a nondegenerate Jordan pair $V = (V^+, V^-)$ is the sum of all minimal inner ideals of V. The socle is a direct sum of simple ideals [L2], therefore $Soc(V)$ is simple if the Jordan pair is strongly prime. As for associative pairs, a Jordan pair V has finite capacity if it satisfies both the ascending and the descending chain condition on inner ideals. Again in this case, V equals its socle, and contains a maximal idempotent e, an idempotent whose Peirce 0-space $V_0(e)$ vanishes (see [L3]).

2.14 We refer to [Ro1, Ro2, M1, M2] for the basic notions on PI-theory for associative and Jordan systems.

A polynomial $f = f(x_1, \ldots, x_n) \in FJ(X)$, the free Jordan algebra on a set X, is called *essential* if its image in the free special Jordan algebra $FSJ(X)$ under the natural homomorphism has the same degree as f, and has a monic leading term as an associative polynomial (note that $FSJ(X)$ is isomorphically embedded in the symmetrized Jordan algebra $FA(X)^{(+)}$ of the free associative algebra $FA(X)$) . A Jordan *PI-algebra* is a Jordan algebra which satisfies some essential $f(x_1, \ldots, x_n)$.

This definition extends to Jordan pairs by considering the free Jordan pair $FJP(X^+, X^-)$ on the sets of generators (X^+, X^-) (see [N]). Here one considers the free special Jordan pair $FSJP(X^+, X^-)$, which embeds isomorphically into the Jordan pair $FAP(X^+, X^-)$, and the natural homomorphism $\tau : FJP(X^+, X^-) \to FSJP(X^+, X^-)$ extending the identity on X^σ, $\sigma = \pm$, and defines an essential polynomial as a nonzero polynomial $f \in FJP(X^+, X^-)$ such that $\tau(f)$ has the same degree as f, and has a monic coefficient as an element of $FAP(X^+, X^-)$. A *PI-Jordan pair* is then a Jordan pair satisfying some essential polynomial. This definition extends in the obvious way to Jordan triple systems.

If $f(x_1, \ldots, x_n)$ is a polynomial in the free Jordan algebra $FJ[X]$ on a countable set of generators X and z is an element of the free Jordan triple system $FJT(X)$, the polynomial $f(z; x_1, \ldots, x_n) = f^{(z)}(x_1, \ldots, x_n)$ is the image of f under the only homomorphism $FJ(X) \to FJT(X)^{(z)}$ extending the identity on X [DA, DAMc1].

A Jordan triple system T satisfies a *homotope polynomial identity* (homotope-PI or HPI, for short) if there exists an essential polynomial $f(x_1, \ldots, x_n)$ in $FJ[X]$ such that $f(y; x_1, \ldots, x_n)$ with $y \in X$ different from x_i vanishes under all substitutions of elements $y,\ x_i \in T$.

This definition extends to Jordan pairs V by considering polarized triple systems. Indeed, since for every $a^+ \oplus a^- \in T(V)$ the homotope $T(V)^{(a^+ \oplus a^-)}$ is isomorphic to the product $(V^+)^{(a^-)} \times (V^-)^{(a^+)}$, a polynomial $f(x_1, \ldots, x_n) \in FJ[X]$ is an identity of all homotopes of $T(V)$ if and only if it is an identity of all homotopes of V. Note that a homotope polynomial identity on a

Jordan pair is a *pair* of usual identities: If V satisfies $f(y; x_1, \dots, x_n)$ for an admissible Jordan polynomial f, then it satisfies $g_{-\sigma} = f(y^{-\sigma}; x_1^{\sigma}, \dots, x_n^{\sigma})$ for $\sigma = \pm$ (see however [GM]).

2.15 The fact that a Jordan system J satisfies a homotope-PI means that all homotopes, and consequently all local algebras, satisfy a given identity. Nonetheless, sometimes we are also be interested in the existence of some $a^{\sigma} \in V^{\sigma}$ such that the local algebra $V_{a^{\sigma}}^{-\sigma}$ is PI. Such elements are called *PI-elements.* If V is a Jordan pair we write $PI(V) = (PI(V^+), PI(V^-))$ for the set of PI-elements of V. It was proved in [M1, 5.4] that $PI(J)$ is an ideal in every nondegenerate Jordan system J.

3 Polynomial identities in associative pairs.

The purpose of this section is to relate the existence of polynomial and homotope polynomial identities on associative pairs endowed with involution. From now on we will assume those polynomial identities are as in 2.8.

We begin with primitive pairs.

3.1 Let $A = (A^+, A^-)$ be a primitive associative pair. By the Density Theorem [CGM, Theorem 1], there is a division Φ-algebra Δ and two nonzero Δ-vector spaces M^+, M^- such that A is isomorphic to a dense subpair of $\mathcal{H} = (Hom_{\Delta}(M^-, M^+), Hom_{\Delta}(M^+, M^-))$. Besides the standard imbedding $\mathcal{U}_A$ of A is a primitive associative algebra and $M = M^- \oplus M^+$ is a faithful irreducible right $\mathcal{U}_A$-module such that Δ is isomorphic to $End\ (M_{\mathcal{U}_A})$.

3.2 Lemma. *Let $A = (A^+, A^-)$ be an associative pair, and let (M^+, M^-) be a faithful irreducible A-module. Then either A has no minimal right ideals or given two finite dimensional vector spaces $W^{\sigma} \subseteq M^{\sigma}$, $\sigma = \pm$, and $v^{\sigma} \in M^{\sigma}$ a vector not contained in W^{σ}, there exists $a^{-\sigma} \in A^{-\sigma}$ such that $W^{\sigma} a^{-\sigma} = 0$ and $v^{\sigma} a^{-\sigma} \notin W^{-\sigma}$.*

Proof. Suppose that A does not have minimal right ideals. Then A has no nonzero elements of finite rank, and therefore, $M^{\sigma} a^{-\sigma}$ is an infinite dimensional vector space for all nonzero $a^{-\sigma} \in A^{-\sigma}$. Since $W^{-\sigma}$ is finite dimensional, this implies that there exists $a^{-\sigma} \in A^{-\sigma}$ such that $W^{\sigma}a^{-\sigma} = 0$ and $v^{\sigma}a^{-\sigma} \notin W^{-\sigma}$ by [CGM, Theorem 1]. $\square$

3.3 Lemma. *Let $A = (A^+, A^-)$ be a primitive associative pair with a polarized involution $*$ and (M^+, M^-) be a faithful irreducible A-module. Then*

(i) either A has minimal right ideals,

(ii) or given any two finite dimensional vector subspaces $W^{\sigma} \subset M^{\sigma}$, $\sigma = \pm$, and any vector $v^{-\sigma} \in M^{-\sigma}$ not contained in $W^{-\sigma}$, there exists an element $r^{\sigma} \in A^{\sigma}$ that satisfies the followings conditions:

$$\begin{array}{llll} (a) & W^{-\sigma}r^{\sigma} = 0, & (c) & v^{-\sigma}(r^{\sigma})^* = 0, \\ (b) & W^{-\sigma}(r^{\sigma})^* = 0, & (d) & v^{-\sigma}r^{\sigma} \notin W^{\sigma}. \end{array}$$

Proof. Suppose that A has no minimal right ideals. Then, by [CGM, Theorem 2], A has no nonzero elements of finite rank and $M^{-\sigma}x^{\sigma}$ is an infinite dimensional Δ-subspace of M^{σ} for all $0 \neq x^{\sigma} \in A^{\sigma}$.

Fix $\sigma \in \{+, -\}$. Then, given a finite dimensional Δ-space $W^{-\sigma} \subset M^{-\sigma}$ and $v^{-\sigma} \in M^{-\sigma}$ not in $W^{-\sigma}$, by [CGM, Theorem 2], there exists a nonzero element $a^{\sigma} \in A^{\sigma}$ such that $W^{-\sigma}a^{\sigma} = 0$ and $v^{-\sigma}a^{\sigma} = 0$. Moreover, since W^{σ} is finite dimensional over Δ and $0 \neq M^{-\sigma}(a^{\sigma})^*$ is an infinite dimensional vector Δ-subspace of M^{σ}, we can take $0 \neq u^{\sigma} \in M^{-\sigma}(a^{\sigma})^*$ such that $u^{\sigma} \notin W^{\sigma}$.

Consider now $B^{\sigma} = \{y^{\sigma} \in A^{\sigma} \mid W^{-\sigma}y^{\sigma} = 0\}$, which is a right ideal of A, and satisfies $v^{-\sigma}B^{\sigma} \neq 0$ by 3.2. It follows from the equalities $v^{-\sigma}B^{\sigma} = M^{\sigma}$ and $((v^{-\sigma}B^{\sigma})A^{-\sigma})A^{\sigma} \subseteq v^{-\sigma}(B^{\sigma}A^{-\sigma}A^{\sigma}) \subseteq v^{-\sigma}B^{\sigma}$, that $M = (v^{-\sigma}B^{\sigma}, (v^{-\sigma}B^{\sigma})A^{-\sigma})$. Hence there exists $b^{\sigma} \in B^{\sigma}$ such that $0 \neq v^{-\sigma}b^{\sigma}$ and therefore we have $M^{-\sigma} = (v^{-\sigma}b^{\sigma})A^{-\sigma}$.

Finally since $u^{\sigma} \in M^{-\sigma}(a^{\sigma})^*$, we can write $u^{\sigma} = ((v^{-\sigma}b^{\sigma})x^{-\sigma})(a^{\sigma})^*$ for some $x^{-\sigma} \in A^{-\sigma}$ and it is easily seen that $r^{\sigma} = b^{\sigma}x^{-\sigma}(a^{\sigma})^*$ satisfies the required properties. $\square$

3.4 Proposition. *Let $A = (A^+, A^-)$ be a primitive associative pair.*

a) If A is PI, then A has nonzero socle *b) If A is endowed with an involution $*$, and A is $*$-PI, then A has nonzero socle.*

Proof. Let us first prove b). Suppose that A has no minimal right ideals. Then, with the notation of 3.1, and applying 3.3 to $W^{\pm} = 0$ and an arbitrary nonzero vector $v^{-\sigma} \in M^{-\sigma}$, we obtain $a_1^{\sigma} \in A^{\sigma}$ such that

$$\begin{aligned} W^{-\sigma} a_1^{\sigma} &= 0, & v^{-\sigma}(a_1^{\sigma})^* &= 0,\\ W^{-\sigma}(a_1^{\sigma})^* &= 0, & v^{-\sigma} a_1^{\sigma} &\notin W^{\sigma}. \end{aligned}$$

Since $v^{-\sigma} a_1^{\sigma} \neq 0$, we can apply 3.3 again with $W^{-\sigma} = \Delta v^{-\sigma}$, $W^{\sigma} = 0$, and $v^{-\sigma} a_1^{\sigma} \neq 0$, to obtain an element $a_1^{-\sigma} \in A^{-\sigma}$ such that

$$\begin{aligned} W^{\sigma} a_1^{-\sigma} &= 0, & (v^{-\sigma} a_1^{\sigma})(a_1^{-\sigma})^* &= 0,\\ W^{\sigma}(a_1^{-\sigma})^* &= 0, & (v^{-\sigma} a_1^{\sigma}) a_1^{-\sigma} &\notin W^{-\sigma} = \Delta v^{-\sigma}. \end{aligned}$$

We claim that repeated application of 3.3 produces sequences $a_1^+, \cdots, a_m^+ \in A^+$ and $a_1^-, \ldots, a_m^- \in A^-$, for any $m \in \mathbb{N}$, that satisfy the following conditions:

(1) (1.a) $((((\ldots(v^{-\sigma} a_1^{\sigma}) a_1^{-\sigma}) \ldots) a_i^{\sigma}) a_i^{-\sigma}) a_j^{\sigma} = 0$ for all $i + 1 < j \leq m$,
(1.b) $(((\ldots(v^{-\sigma} a_1^{\sigma}) a_1^{-\sigma}) \ldots) a_i^{\sigma}) a_j^{-\sigma} = 0$ for all $i < j \leq m$,

(2) (2.a) $((((\ldots(v^{-\sigma} a_1^{\sigma}) a_1^{-\sigma}) \ldots) a_i^{\sigma}) a_i^{-\sigma})(a_j^{\sigma})^* = 0$ for all $i + 1 \leq j \leq m$,
(2.b) $(((\ldots(v^{-\sigma} a_1^{\sigma}) a_1^{-\sigma}) \ldots) a_i^{\sigma})(a_j^{-\sigma})^* = 0$ for all $i \leq j \leq m$,

(3) The sets

(3.a) $\{v^{-\sigma}, (v^{-\sigma} a_1^{\sigma}) a_1^{-\sigma}, \ldots, (((\ldots(v^{-\sigma} a_1^{\sigma}) a_1^{-\sigma}) \ldots) a_m^{\sigma}) a_m^{-\sigma}\} \subseteq M^{-\sigma}$,
(3.b) $\{v^{-\sigma} a_1^{\sigma}, ((v^{-\sigma} a_1^{\sigma}) a_1^{-\sigma}) a_2^{\sigma}, \ldots, ((\ldots(v^{-\sigma} a_1^{\sigma}) \ldots) a_{m-1}^{-\sigma}) a_m^{\sigma}\} \subseteq M^{\sigma}$.

are linearly independent over Δ

Indeed, take $a_1^{\sigma}, a_1^{-\sigma}, \ldots, a_k^{\sigma}, a_k^{-\sigma}$ satisfying (1)-(3). Then, since both $U^{-\sigma} = \Delta v^{-\sigma} + \Delta((v^{-\sigma} a_1^{\sigma}) a_1^{-\sigma}) + \cdots + \Delta(((v^{-\sigma} a_1^{\sigma}) \ldots) a_{k-1}^{-\sigma}) \subseteq M^{-\sigma}$ and $U^{\sigma} = \Delta(v^{-\sigma} a_1^{\sigma}) + \cdots + \Delta((((v^{-\sigma} a_1^{\sigma}) \ldots) a_{k-1}^{-\sigma}) a_k^{\sigma}) \subseteq M^{\sigma}$ are finite dimensional over Δ and $v_k^{-\sigma} = (((v^{-\sigma} a_1^{\sigma}) \ldots) a_k^{\sigma}) a_k^{-\sigma} \notin U^{-\sigma}$ by 3.3, there exists $a_{k+1}^{\sigma} \in A^{\sigma}$ such that

$$\begin{aligned} U^{-\sigma} a_{k+1}^{\sigma} &= 0, & v_k^{-\sigma}(a_{k+1}^{\sigma})^* &= 0,\\ U^{-\sigma}(a_{k+1}^{\sigma})^* &= 0, & v_k^{-\sigma} a_{k+1}^{\sigma} &\notin U^{\sigma}. \end{aligned}$$

Continuing in this way, with $\overline{U^\sigma} = U^\sigma$, $\overline{U^{-\sigma}} = U^{-\sigma} + \Delta(((v^{-\sigma}a_1^\sigma)\ldots)a_k^{-\sigma}) = U^{-\sigma} + \Delta v_k^{-\sigma}$, and $v_{k+1}^\sigma = v_k^{-\sigma}a_{k+1}^\sigma = ((((v_k^{-\sigma}a_1^\sigma)a_1^{-\sigma})\ldots)a_k^{-\sigma})a_{k+1}^\sigma$, we obtain $a_{k+1}^{-\sigma} \in A^{-\sigma}$ such that $a_1^\sigma, a_1^{-\sigma}, \ldots, a_k^\sigma, a_k^{-\sigma}, a_{k+1}^\sigma, a_{k+1}^{-\sigma}$ satisfies (1)-(3), thus proving the claim.

We can assume that A satisfies a multilinear polynomial identity p_σ of degree $2d+1$ of the form 2.8. We claim that the above inductive process gives rise to a sequence of elements in A on which that polynomial identity does not vanish. Indeed, if we build the above sequence for $m = d+1$, and we evaluate p_σ in $x_1^\sigma = a_1^\sigma$, $x_1^{-\sigma} = a_1^{-\sigma}, \ldots, x_d^{-\sigma} = a_d^{-\sigma}, x_{d+1}^\sigma = a_{d+1}^\sigma$, we obtain

$$\begin{aligned} 0 &= v^{-\sigma}p_\sigma(a_1^\sigma, \ldots, a_{d+1}^\sigma, a_1^{-\sigma}, \ldots, a_d^{-\sigma}, (a_1^\sigma)^*, \ldots, (a_d^{-\sigma})^*) = \\ &= (((v^{-\sigma}a_1^\sigma)a_1^{-\sigma})\ldots)a_{d+1}^\sigma + v^{-\sigma}q_\sigma(a_1^\sigma, \ldots, a_1^{-\sigma}, \ldots). \end{aligned}$$

So it suffices to check that $v^{-\sigma}q_\sigma(a_1^\sigma, \ldots, a_1^{-\sigma}, \ldots) = 0$, since this would imply $(((v^{-\sigma}a_1^\sigma)a_1^{-\sigma})\ldots)a_d^\sigma = 0$, contrary to (1)-(3).

To do this, first note that any monomial in $q_\sigma(a_1^\sigma, \ldots, a_1^{-\sigma}, \ldots)$ which does not begin with a_1^σ annihilates $v^{-\sigma}$. Indeed, by (1.a), $v^{-\sigma}a_i^\sigma = 0$ for all $i > 1$ and, by (2.a), $v^{-\sigma}(a_i^\sigma)^* = 0$ for all $i \geq 1$. Thus, only those monomials in $q_\sigma(a_1^\sigma, \ldots, a_1^{-\sigma}, \ldots)$ beginning by a_1^σ give a nonzero contribution to $v^{-\sigma}q_\sigma(a_1^\sigma, \ldots, a_1^{-\sigma}, \ldots)$. Similarly, by (1.b) and (2.b), only those monomials which continue with $a_1^{-\sigma}$ give a nonzero contribution to $v^{-\sigma}q_\sigma(a_1^\sigma, \ldots, a_1^{-\sigma}, \ldots)$. Therefore

$$v^{-\sigma}q_\sigma(a_1^\sigma, \ldots, a_1^{-\sigma}, \ldots) = ((v^{-\sigma}a_1^\sigma)a_1^{-\sigma})q_{\sigma,1}(a_2^\sigma, \ldots, a_2^{-\sigma}, \ldots),$$

where $q_{\sigma,1}$ denotes a sum of monomials involving either x_i^τ or $(x_i^\tau)^*$ for all $\tau \in \{+,-\}$, but not both, for all $2 \leq i \leq d$.

Now, since the monomial $x_1^\sigma x_1^{-\sigma} \ldots x_d^\sigma$ does not occur in q_σ, repeated application of (1) and (2) above yields $v^{-\sigma}q_\sigma(a_1^\sigma, \ldots, a_1^{-\sigma}, \ldots) = 0$. Hence $(((v^{-\sigma}a_1^\sigma)a_1^{-\sigma})\ldots)a_d^\sigma = 0$, contrary to (1)-(3).

That proves b), and a) is proved in the same way by using 3.2 instead of 3.3.

$\square$

3.5 Remark. Let A be an associative algebra, and consider the associative pair (A, A). If (A, A) satisfies a polynomial identity $p(x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^-)$ of degree d, then A satisfies the algebra identity $p(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of degree d, and therefore it is a PI algebra. Similarly, if A possesses an involution $*$, then $*$ induces an involution on (A, A), and if (A, A) satisfies a $*$-polynomial identity $p(x^+, x^-, (x^+)^*, (x^-)^*)$, then A satisfies an algebra $*$-identity $p(x, y, x^*, y^*)$ of the same degree.

3.6 Theorem. *Let $A = (A^+, A^-)$ be a primitive associative pair, let (M^+, M^-) be a faithful irreducible right A-module, and set $\Delta = End_{\mathcal{U}_A}(M^+ \oplus M^-)$.*

a) If A satisfies a polynomial identity of degree $2d+1$, then for some $\sigma = +$ or $-$, the dimension of M^σ over Δ is at most d.

b) If A has an involution $*$, and it satisfies a $*$-polynomial identity of degree m, then M^+ and M^- are at most m-dimensional over Δ.

In both cases A is simple of finite capacity.

Proof. a) By 3.4, A has nonzero socle, and it is prime since it is primitive. By [CGM, 2.8], $Soc(A)$ is a simple associative pair [CFGM, Theorem 1] and, by [CGM, Theorem 2], we have

$$\begin{aligned} Soc(A) &= (\mathcal{F}_\Delta(M^-, M^+), \mathcal{F}_\Delta(M^+, M^-)) \\ &\qquad \lhd A \subseteq (Hom_\Delta(M^-, M^+), Hom_\Delta(M^+, M^-)). \end{aligned}$$

Then for all $n \in \mathbb{Z}^+$ with $n \leq \dim_\Delta M^+$ and $\leq \dim_\Delta M^-$, there is a subpair of matrices $\mathcal{M}_n(\Delta) = (M_n(\Delta), M_n(\Delta)) \subseteq Soc(A)$. This is the associative pair of the algebra $M_n(\Delta)$, and since it is a subpair of A, it satisfies a polynomial identity of degree $2d+1$, hence $M_n(\Delta)$ satisfies an algebra identity of degree $2d+1$ by 3.5. Then $2d+1 \geq 2n$, and we get $n \leq d$, hence $\dim_\Delta M^\sigma \leq d$ for one of the $\sigma = \pm$.

b) Since A has nonzero socle by 3.4, A is strongly prime [CGM, 2.8], and therefore, by [FT, 3.14], Δ has an involution $^-$, and there is a mapping $g : M^+ \times M^- \to \Delta$ such that (M^+, M^-, g) form a pair of skew dual vector spaces the associative division Φ-algebra with involution $(\Delta, -)$ (see [FT, 3.11]), and $Soc(A) = (\mathcal{F}_\Delta(M^-, M^+), \mathcal{F}_\Delta(M^+, M^-))$. Moreover, by [FT, 3.20], for any $n \leq dim_\Delta(X)$, $Soc(A)$ contains a $*$-subpair S $*$-isomorphic to a full pair of matrices with involution $\big((M_n(\Delta), M_n(\Delta)), *\big)$, and either $* = \sharp$ and then $B^* = \overline{B}^t$ or Δ is a field, $-$ is the identity, $* = -\sharp$, and $B^* = -B^t$ for all $B \in M_n(\Delta)$, where t is the transpose involution.

Suppose first that $B^* = \overline{B}^t$. Then, $\big((M_n(\Delta), M_n(\Delta)), *\big)$ is the pair with involution obtained from the algebra $M_n(\Delta)$ endowed with the involution $*$. Since A satisfies a $*$-polynomial identity of degree m, so does S, hence $(M_n(\Delta), *)$ satisfies an algebra $*$-polynomial identity of degree m by 3.5. Then, by a theorem of Amitsur [Am], $M_n(\Delta)$ satisfies the standard identity S_{2m} by [Ro2, 1.4.1], hence $2n \leq 2m$, and we get $n \leq m$, which yields $\dim_\Delta M^+ = \dim_\Delta M^- \leq m$.

Suppose finally that $B^* = -B^t$. Since by [FT, 3.20], Δ is now a field we write $\Delta = F$. If $dim_F M^\sigma \geq 2n$ for some (hence both) σ, by [FT, 3.20], then $Soc(A)$, hence A, contains a $*$-subpair S $*$-isomorphic to a full pair of matrices with involution $\big((M_{2n}(F), M_{2n}(F)), *\big)$, such that, for any $B = (C_{ij}) \in M_{2n}(F)$, $B^* = (C_{ij})^* = (C_{ji}^*)$, where $C \in M_2(F)$ and

$$C^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

for all α, β, γ, $\delta \in F$.

Again, S, hence $\big((M_{2n}(F), M_{2n}(F)), *\big)$, inherits the $*$-identity of A, and since $\big((M_{2n}(F), M_{2n}(F)), *\big)$ is the $*$-pair obtained from the algebra $M_{2n}(F)$ endowed with some involution $*$, by 3.5, it satisfies an algebra $*$-identity of degree m, hence by [Am], it satisfies the standard identity S_{2m}. Thus we get $4n \leq 2m$ by [Ro2, 1.4.3], hence $2n \leq m - 1$ since m is odd. This implies $\dim_F M^\sigma \leq m$, $\sigma = \pm$. $\square$

3.7 Remark. We point out that, as a consequence of the above proof, if A is a primitive associative pair with involution $*$, and it satisfies a $*$-polynomial identity of degree m, then there exists a division algebra Δ and a positive integer n such that A is isomorphic to a pair $\big(M_n(\Delta), M_n(\Delta)\big)$, and $M_n(\Delta)$ satisfies the standard identity S_{2m}.

3.8 Lemma. *Let $A = (A^+, A^-)$ be an associative pair with an involution $*$ that satisfies a polynomial identity $p_\sigma(x, x^*)$ of degree m. If P is a primitive ideal of A , then A/P is simple, has finite capacity and all its local algebras satisfy the standard identity S_{2m}.*

Proof. If $P = P^*$, the pair A/P is primitive with an involution induced by that of A, and clearly it satisfies the polynomial identity $p_\sigma(x, x^*)$ of degree m. Thus, in this case we can assume that A is primitive. By 3.7, A is then a simple pair of finite capacity of the form $\big(M_n(\Delta), M_n(\Delta)\big)$ for a division algebra Δ, and the matrix algebra $M_n(\Delta)$ satisfies the standard identity S_{2m}. By a well known scalar extension argument (see [H1]), $M_n(\Delta)$ embeds in a matrix algebra $M_k(F)$ over a field F for some $k \leq m$ which also satisfies the identity S_{2m}. If now $a \in A^\sigma$, then the local algebra $A_a^{-\sigma} = M_n(\Delta)_a$ is (isomorphic to) a subalgebra of $M_k(F)_a$, and it is easy to see that $M_k(F)_a \cong M_r(F)$, where $r \leq k \leq m$ is that rank of a. Therefore the local algebra $A_a^{-\sigma}$ satisfies the standard identity S_{2m} by Amitsur-Levitzki's theorem [Ro2, 1.4.1]

We assume next that $P \not\supseteq P^*$. (Note that this case includes the possibility of A being $*$-primitive but not primitive, since then A has a primitive ideal P such that $P \cap P^* = 0$. See [Ro2, 7.3.5].)

Factoring out the ideal $P \cap P^*$ we can assume that $P \cap P^* = 0$. Set $I = P + P^* = P \oplus P^*$. Since I is a $*$-ideal of A, it inherits the $*$-polynomial identity of A. Now, if $(A, *)$ satisfies the $*$-polynomial $p(x^+, x^-, (x^+)^*, (x^-)^*)$ of degree m, then P^* satisfies the polynomial $q = p(x^+, x^-, y^+, y^-)$, which we can assume to be multilinear. Moreover $P^* \cong I/P$ is an ideal of the primitive pair A/P, hence it is itself a primitive pair (since any faithful irreducible A/P-module is easily seen to be a faithful irreducible I/P-module). Therefore, P^* is a pair of finite capacity by 3.6, and by [CGM, Proposition 1] has the form $\big(Hom_\Delta(X, Y), Hom_\Delta(Y, X)\big)$ for a division algebra Δ, and a pair of Δ-vector spaces X, Y, with one of the dimensions $\dim_\Delta X$ or $\dim_\Delta Y$ finite. Note now that P^* is an associative pair over the center $Z = Z(\Delta)$ of Δ, and let F be a maximal subfield of Δ. Consider now the scalar extension $P_F^* = P^* \otimes_Z F$. Then, (Y, X) becomes an irreducible P_F^*-module, and $F = End_{\mathcal{U}_{P_F^*}}(Y \oplus X) = F$. Now P_F^* still satisfies the multilinear identity q

of degree m, hence by 3.6, if we set $m = 2d+1$ (recall that m is odd), we have that either $\dim_F X \leq d$ or $\dim_F Y \leq d$, and P_F^* is isomorphic to the pair $B = \big(Hom_F(X,Y), Hom_F(Y,X)\big)$. Now, if $b^\sigma \in B^\sigma$, it is easy to see that the local algebra $B_{b^\sigma}^{-\sigma}$ is isomorphic to a matrix algebra $M_n(F)$ where n is the rank of the linear transformation b^σ, which is at most the minimum of $\dim_F X$ and $\dim_F Y$, and hence $r \leq d < m$. Therefore, every local algebra of B satisfies the standard identity S_{2m}, and so does every local algebra of the subpair $I/P = P^*$, of B.

Next we claim that $I/P = A/P$. Indeed, since I/P has finite capacity by [L3, Theorem 3(v)], there is an idempotent $e = (e^+, e^-)$ of I/P such that $(I/P)_{00}(e) = 0$. Then $I/P = (I/P)_{11}(e) \oplus (I/P)_{10}(e) \oplus (I/P)_{01}(e)$. But, since $(I/P)_{ij}(e) = (A/P)_{ij}(e)$ for all $i,\ j \in \{0,1\}$ such that $(i,j) \neq (0,0)$, we have $A/P = I/P \oplus (A/P)_{00}(e)$. Take now $z^\sigma \in (A/P)_{00}^\sigma(e)$. By the Peirce relations (see [L1, pp. 94-95]),

$$\langle z^\sigma, L^{-\sigma}, z^\sigma\rangle \subset \sum_{(i,j)\neq(0,0)} \langle (A/P)_{00}^\sigma(e), (A/P)_{ij}^{-\sigma}(e), (A/P)_{00}^\sigma(e)\rangle = 0,$$

which gives $z^\sigma \in Ann_{(A/P)^J}((I/P)^{-\sigma}) = 0$ [CFGM, Lemma 2]. Hence $(A/P)_{00}^\sigma(e) = 0$ and then $A/P = I/P$, hence A/P has finite capacity and all its local algebras satisfy the standard identity S_{2m}. □

3.9 Theorem. *Let A be an associative pair with an involution $*$. If A has a $*$-polynomial identity of degree m, then there exists a positive integer k such that every local algebra of A satisfies the polynomial identity S_{2m}^k. Moreover, if A is semiprime, then every local algebra satisfies the standard identity S_{2m}.*

Proof. We can assume that A satisfies a multilinear $*$-identity p of degree m as in 2.8. We consider first the case of a semiprime A. To deal with it, we will embed A into a semiprimitive associative pair with the same identities as A by means of its Martindale-McCrimmon embedding [M1, 5.2]: Let $\tilde{A}$ be the pair $(\mathrm{Seq}(A[t_1])[t_2]$, where t_1 and t_2 are polynomial variables, and $\mathrm{Seq}(B) = \prod_1^\infty B$ for an associative pair B. We set $\mathrm{E}(A) = \tilde{A}/\mathrm{Jac}(\tilde{A})$, where Jac denotes the Jacobson radical. We denote by τ the composition $A \subseteq \tilde{A} \to \mathrm{E}(A)$. Clearly, the involution $*$ of A extends to $\mathrm{E}(A)$ making τ a $*$-homomorphism of associative pairs. This is the associative version of the construction of the Martindale-McCrimmon embedding, defined in [MMc] for Jordan algebras, and extended to general Jordan systems in [M1]. In fact, using [L1, 7.9] we easily get $\mathrm{E}(A)^{(+)} = \mathrm{E}(A^{(+)})$, hence by [M1, Lemma 5.3], τ is injective. Moreover, $\mathrm{E}(A)$ satisfies the $*$-identity p, hence by 3.8, for every primitive ideal P of $\mathrm{E}(A)$, all local algebras of $\mathrm{E}(A)/P$ satisfy the standard identity S_{2m}. Therefore, the semipritivity of $\mathrm{E}(A)$ implies that all its local algebras satisfy the identity S_{2m}, and the same goes for A.

Finally, the assertion for a general A follows by Amitsur's argument [Ro2, 1.6.38]. $\square$

As an immediate application of this result we consider associative pairs with involution for which either the set of skew symmetric elements or the set or symmetric elements satisfies a polynomial identity. More generally, we can consider traces: $t(r) = r + r^*$, and the set of all traces $T(A, *)$ of the triple system with involution $(A, *)$.

3.10 Corollary. *If either the set of all symmetric elements (or, more generally, the set of all traces) or the set of all skew elements of an associative pair A with involution satisfies a polynomial identity, then all local algebras of A satisfy an identity S_{2m}^k where S_{2m} is the standard identity.*

4 Polynomial identities in Jordan pairs.

In this section we provide an affirmative answer to following conjecture which was raised in [M2, 6.4].

4.1 Conjecture. *Every PI-Jordan system satifies a homotope-PI.*

We recall here that this result has already been proved for Jordan algebras in [M1, 2.7(ii)]. Thus we will focus on Jordan pairs and triple systems, but we first recall the definition of a family of Jordan polynomials which will play for us to some extent the role of the associative standard identity

4.2 Following [M1, 2.2], we denote by F_m the family of essential polynomials in the free Jordan algebra $FJ[x,y,z]$

$$F_m(x,y,z) = \sum_{\sigma \in S_{m+1}} (-1)^\sigma V_{x^{\sigma(1)},y} \dots V_{x^{\sigma(m+1)},y}\ z.$$

We also write $G_m(x,y,z) = F_m(x,y,z)^3$.

Before studying Jordan pairs, we return to algebras to obtain a sharper version of [M1, 6.4(a)] for semiprime algebras:

4.3 Lemma. *Let A be a semiprime associative algebra. If A satisfies the standard identity S_{2n}, then $A^{(+)}$ satisfies all the identities F_m for all $m \geq n^2$.*

Proof. By Posner-Rowen's theorem, the central localization $B = Z(A)^{-1}A$ of A is a simple algebra finite dimensional over its center, and therefore a matrix algebra over a division ring D, which still satisfies S_{2n}. On the other hand, after a suitable scalar extension of B (for instance by a maximal subfield F of D), we obtain a matrix algebra $M_k(F)$ which still satisfies S_{2n}. Then $k \leq n$ by [Ro2, 1.4.3], hence $M_k(F)$ has dimension at most n^2 over F, and $M_k(F)^{(+)}$ satisfies F_m for $m \geq n^2$ by [M1, 2.3], hence $A^{(+)}$ also satisfies that identity. $\square$

4.4 Recall that a Jordan pair V is said to strictly satisfy a homotope polynomial identity $f(y;x_1,\dots,x_n)$ if every scalar extension $V \otimes \Omega$ for a commutative associative Φ-ring $\Omega \supseteq \Phi$, still satisfies the identity $f(y;x_1,\dots,x_n)$. Since the strict validity of an identity amounts to the validity of all its partial linearizations, it is easy to see that V strictly satisfies $f^{(y)}$, if and only if the polynomial pair $V[t] = V \otimes \Phi[t]$ satisfies $f^{(y)}$.

4.5 Lemma. *Let V be a strongly prime Jordan pair and suppose that a nonzero ideal I of V strictly satisfies some homotope polynomial identity $f(y;x_1,\dots,x_n) = f^{(y)}(x_1,\dots,x_n)$ for a homogeneous admissible Jordan polynomial f, then V satisfies $f^{(y)}$.*

Proof. Consider the extended central closure $\tilde{V} = \mathcal{C}(V)V$ of V, and the ideal $\tilde{I} = \mathcal{C}(V)I$ of $\tilde{V}$ generated as a $\mathcal{C}(V)$ module by I. Take now $x^\sigma \in I^\sigma$ and $y^{-\sigma} \in I^{-\sigma}$ and set $z^\sigma = Q_{x^\sigma} y^{-\sigma}$. Then, for all $a_1^{-\sigma},\dots,a_n^{-\sigma} \in V^{-\sigma}$ we have $Q_{y^{-\sigma}} f(y^{-\sigma}; Q_{x^\sigma} a_1^{-\sigma},\dots,Q_{x^\sigma} a_n^{-\sigma}) = 0$ (since $Q_{x^\sigma} a_i^{-\sigma} \in I^\sigma$ for all i). Thus we have $0 = Q_{x^\sigma} Q_{y^{-\sigma}} f(y^{-\sigma}; Q_{x^\sigma} a_1^{-\sigma}, \ldots, Q_{x^\sigma} a_n^{-\sigma}) = Q_{x^\sigma} Q_{y^{-\sigma}} Q_{x^\sigma} f(Q_{x^\sigma} y^{-\sigma}; a_1^{-\sigma}, \ldots, a_n^{-\sigma}) = Q_{z^\sigma} f(z^\sigma; a_1^{-\sigma}, \ldots, a_n^{-\sigma})$ by [GM, 0.19], hence the local algebra $V_{z^\sigma}^{-\sigma}$ is PI, and thus $Q_{I^\sigma} I^{-\sigma} \subseteq PI(V)$. Since $Q_{I^\sigma} I^{-\sigma} \neq 0$ by [M2, 1.3] and the primality of V, this implies $PI(V) \neq 0$. Therefore, by [M2, 5.1], $\tilde{V}$ is strongly prime with nonzero socle $Soc(\tilde{V}) = PI(\mathcal{C}(V)V)$, which is a simple ideal by 2.13. Thus we get $\tilde{I} = Soc(\tilde{V}) \subseteq \tilde{I}$, and therefore $Soc(\tilde{V})$ satisfies the homotope polynomial identity $f^{(y)}$, hence it has finite capacity by [M1, 4.10]. This implies that $\tilde{I} = Soc(\tilde{V}) = \tilde{V}$ using a complete idempotent as in 3.8, and therefore $\tilde{V}$, hence V, satisfies $f^{(y)}$. $\square$

4.6 Proposition. *Let V be a nondegenerate Jordan pair. If V satisfies a polynomial identity of degree m, then all its local algebras satisfy the identities $F_k(x, y, z)$ for all $k \geq m^2$, hence V satisfies the homotope polynomial identity $G_k(t; x, y, z)$ for all $k \geq m^2$.*

Proof. Since any nondegenerate Jordan pair is a subdirect product of strongly prime Jordan pairs, it clearly suffices to consider the case of a strongly prime V. In that case, by [ACMM, 4.3], either there exits a scalar extension $\tilde{V}$ of V which is Clifford, bi-Cayley or Albert, or V consists of hermitian elements: V has a nonzero ideal $I = H_0(A, *)$ which is an ample subpair of a $*$-prime associative pair A with involution $*$, and $V \subseteq H(Q(A), *)$, where $Q(A)$ is the Martindale pair of symmetric quotients of A.

We consider first the hermitian case. Note that we can assume that V satisfies a multilinear polynomial identity p of degree m. Consider the polynomial extension $I[t] = I \otimes_\Phi \Phi[t]$. It is straightforward that $I[t] = H_0(A[t], *)$ is an ample subpair of the $*$-prime associative pair $A[t]$, and it still satisfies the polynomial identity p. By 3.9, every local algebra of the semiprime pair $A[t]$ satisfies the standard identity S_{2m}, and since these are again semiprime, every local algebra of $A[t]^{(+)}$ satisfies all the identities F_m for all $m \geq n^2$ by 4.3, and so does every local algebra of $I[t]$. Thus I strictly satisfies all the identities $F_m(t; x, y, z)$ for all $m \geq n^2$ by 4.4. Thus V satisfies all the identities $F_m^{(t)}$ for all $m \geq n^2$ by 4.5.

Now if V is bi-Cayley or Albert type, then all its local algebras are at most 27-dimensional over their centroids, and if V is of Clifford type, then its local algebras are generically algebraic of degree 2 over their centroids.

Therefore they all satisfy $F_k(x, y, z)$ for $k \geq 27$. Since $m \geq 3$, all local algebras of V satisfy F_k for $k \geq m^2$. $\square$

4.7 Theorem. *Let J be a Jordan system. If J satisfies a polynomial identity of degree m, then J satisfies the homotope polynomial identity $F_k^l(t; x, y, z) = F_k(t; x, y, z)^{(l,t)}$ for all $k \geq m^2$ and some $l \geq 3$.*

Proof. For Jordan pairs, this follows from 4.6 by Amitsur's argument [Ro2, 1.6.38]. The result for Jordan triple systems follows from that result by using the double pair $V(J)$ attached to a Jordan triple system J. $\square$

As a consequence of that theorem we can improve the Jordan analogues of Kaplansky's and Posner-Rowen theorems obtained in [M1, M2, M3].

4.8 Theorem. *Let V be a Jordan pair, and suppose that V satisfies a polynomial identity, then:*

a) If V is primitive, then it is simple of finite capacity.

b) If V is strongly prime, then its central closure $\Gamma(V)^{-1}V$ is simple of finite capacity.

c) The McCrimmon radical and the properly nilpotent radical of V coincide.

Proof. Since by 4.7 every PI Jordan system satisfies a homotope polynomial identity, the result immediatelly follows from [M1, 4.10(ii),6.3(b)] and [M3, 4.3] $\square$

4.9 Remark. Part c) of 4.8 was first proved by Zelmanov in [Z, Theorem 3]. This raises the question of whether the present proof is independent of that result, which was used in [Z] to prove its prime dichotomy theorem (every strongly prime Jordan system is either i-special or an Albert form), which in turn is needed in the proof of 4.6. However a careful reading of [Z] reveals that the only result that is needed is [M1, 6.3(b)], which does not require the classification theorem of strongly prime Jordan systems.

We can also obtain as a corollary the extension to arbitrary Jordan systems of a result that was proved in [M3] for nondegenerate systems:

4.10 Corollary. *If a Jordan system J satisfies a homotope polynomial identity, then it strictly satisfies a homotope polynomial identity of the form* $F_k^l(t;x,y,z) = F_k(t;x,y,z)^{(l,t)}$.

Proof. Since J satisfies a homotope polynomial identity, it is PI, and the result follows from 4.7. □

References

[Am] S. A. Amitsur, Identities in rings with involution, *Israel J. Math* **1** (1969), 63-68.

[ACMM] J. A. Anquela, T. Cortés, K. McCrimmon, F. Montaner, Strong Primeness of Hermitian Jordan Systems, *J. Algebra* **198** (1999), 311-326.

[BMM] K. I. Beidar, W. S. Martindale, A. V. Mikhalev, *Rings with Generalized Identities*, Marcel Dekker, Inc. New York, 1996.

[CFGM] A. Castellón Serrano, A. Fernández López, A. García Martín, C. Martín González, Strongly prime Alternative Pairs with minimal Inner Ideals, *Manuscripta Math.* **90** (1996), 479-487.

[CGM] J. A. Cuenca Mira, A. García Martín, C. Martín González, Jacobson density for associative pairs and its applications, *Comm. Algebra* **17** (10) (1989), 2595-2610.

[DA] A. D'Amour, Quadratic Jordan Systems of Hermitian Type, *J. Algebra* **149** (1992), 197-233.

[DAMc1] A. D'Amour, K. McCrimmon, The local algebras of Jordan Systems, *J. Algebra* **177** (1995), 199-239.

[DAMc2] A. D'Amour, K. McCrimmon, The Structure of Quadratic Jordan Systems of Clifford Type, *J. Algebra* **234** (2000), 31-89.

[FT] A. Fernández López, M. I. Tocón, Strongly prime Jordan pairs with nonzero socle, *Manuscripta Math.* **111** (2003), 321-340.

[GM] E. García Rus, F. Montaner, Maximal Modular Inner Ideals in Jordan Systems, *Comm. Algebra* **31** (2) (2003), 697-749.

[H1] I. N. Herstein, *Noncommutative Rings*, The Carus Mathematical Monographs, Vol. 15, The Mathematical Association of America, Washington, DC, 1968.

[H2] I. N. Herstein, *Rings with Involution*, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, 1976.

[L1] O. Loos, *Jordan Pairs*, Lecture Notes in Mathematics, Vol. 460, Springer, Berlin/Heidelberg/New York, 1975.

[L2] O. Loos, On the socle of a Jordan pair, *Collect. Math.* **40** (1989), 109-125.

[L3] O. Loos, Finiteness conditions in Jordan pairs, *Math. Z.* **206** (1991), 577-587.

[MMc] W. S. Martindale III, K. McCrimmon, Imbedding nondegenerate Jordan algebras in semiprimitive algebras, *Proc. Amer. Math. Soc.* **103** (4) (1988), 1031-1036.

[Mc] K. McCrimmon, Jordan Centroids, *Comm. Algebra* **27** (2) (1999), 933-954.

[McZ] K. McCrimmon, E. Zelmanov, The Structure of strongly prime Quadratic Jordan Algebras, *Adv. Math.* **69** (2) (1988), 133-222.

[Me] K. Meyberg, *Lectures on Jordan Algebras and Triple Systems.* Lecture Notes, University of Virginia, Charlottesville, 1972.

[M1] F. Montaner, Local PI-theory of Jordan Systems, *J. Algebra* **216** (1999), 302-327.

[M2] F. Montaner, Local PI-theory of Jordan Systems II, *J. Algebra* **241** (2001), 473-514.

[M3] F. Montaner, Homotope polynomial identities in prime Jordan systems, *J. Pure Appl. Algebra* **208** (200), 107-116.

[N] E. Neher, Polynomial identities and nonidentities in split Jordan pairs, *J. Algebra* **211** (1999), 206-224.

[Ro1] L. H. Rowen, The theory of generalized polynomial identities, in *Ring Theory, Proceedings of the Ohio University Conference* (S. K. Kain and K. E. Eldridge, Eds.), Dekker, New York, 1977.

[Ro2] L. H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, New York, 1980.

[Z] E. I. Zelmanov, Prime Jordan triple systems, *Siberian Math. J.* **24** (1983), 509-520.