Power-associative algebras that are train algebras

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Abstract

We investigate the structure of power-associative algebras that are train algebras. We first show the existence of idempotents, which are all principal and absolutely primitive. We then study the train equation involving the Peirce decomposition. When the algebra is finite-dimensional, it turns out that the dimensions of the Peirce components are invariant and that the upper bounds for their nil-indexes are reached for some idempotent. Further, locally train algebras are shown to be train algebras. We then get a complete description of the set of idempotents by giving their explicit formulas, including several illustrative examples. Some attention is paid to the Jordan case, where we discuss conditions forcing power-associative train algebras to be Jordan algebras. It is also shown that finitely generated Jordan train algebras are finite-dimensional. For a n^{th}-order Bernstein algebra of period p, we prove that power-associativity necessitates $p = 1$. In this case, there are 2^{n-1} possible train equations, which are explicitly described.

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0 INTRODUCTION

The class of power-associative algebras assumes an important place in the theory of "nearly associative algebras", including Jordan algebras whose origin lies in the algebraic formulation of quantum mechanics. Another kind of non-associative algebras consists of the so-called train algebras, which have been introduced by Etherington [5] in connection with the symbolism of genetics. During the past twenty years, a lot of effort was made to study train algebras from various points of view, particularly for low ranks. However, due to their intrinsic complexity, little is known about train algebras of arbitrary ranks. In [26], Schafer discovered the presence of some Jordan algebras among train algebras, through the gametic and zygotic algebras for simple Mendelian inheritance (see also [12]). Motivated by these results, Ouattara presented subsequently a study on Jordan train algebras in a more general context [21]. Later, Guzzo and Vicente [10] founded the coefficients of the train equation of a power-associative train algebra. Recently, Mallol and Varro [17] used the Peirce decomposition to analyze the train equation of a train algebra that is power-associative or alternative. But as far as we know, power-associative algebras that are train algebras have not been studied systematically.

On the other hand, the existence of idempotents in power-associative algebras, as well as in train algebras, is quite important, since idempotents produce the Peirce decompositions of the algebra [1,8,11,27]. But in addition to their mathematical importance, idempotents also have genetic significance in train algebras [24,33]. Some results in this direction were given in [6,14,33] for ranks ≤ 4.

The main goal of the present paper is to develop a structure theory for power-associative algebras that are train algebras. Our point of departure is the previous paper [20], where two special cases have been examined. This paper is organized as follows. After a section of preliminaries, we first prove in Section 2 the existence of idempotents, which are all principal and absolutely primitive. The train equations of such algebras are revisited in order to illuminate some new aspects. In particular, for finite-dimensional algebras, we give a partial affirmative answer to an open question raised in [17] by establishing that the upper bounds for the nil-indexes of the Peirce components are achieved for some idempotent. We also show that the dimensions of the Peirce components are independent of the idempotent and that every locally train algebra is a train algebra.

In Section 3 we study with two different methods the behavior of the set of idempotents by furnishing their specific expressions and applying to concrete situations. Section 4 is devoted to the Jordan case by providing conditions under which power-associative train algebras become Jordan algebras. It is also proved that Jordan train algebras that are finitely generated are finite-dimensional. In the final section, dedicated to n^{th}-order Bernstein algebras, we establish that any power-associative n^{th}-order Bernstein algebra of period p is necessarily a n^{th}-order Bernstein algebra. Furthermore, 2^{n-1} possible train equations are found for n^{th}-order Bernstein algebras that are power-associative. Various examples are presented throughout the article to serve as motivation and illustration for our results. Some connections of our development with other approaches are also discussed.

1 PRELIMINARIES

In this section we briefly summarize notation, terminology and classical properties for both train algebras and power-associative algebras. We would still recommend [15,33] for train algebras, although there is now the most readable [24]. The reader may opt for [1,27] for references about power-associative algebras. Throughout this paper, unless otherwise mentioned, A is a commutative non-associative algebra of arbitrary dimension over an infinite field K of characteristic $\neq 2, 3, 5$, even if many results hold in any characteristic $\neq 2$. We say that A is a *baric algebra* if there exists a nonzero homomorphism of algebras $\omega : A \to K$, called the *weight function*. Denoting by H *the unit hyperplane* $H = \{x \in A \;/\; \omega(x) = 1\}$, we have $A = Ka \oplus \ker(\omega)$ for each $a \in H$. A baric algebra (A, ω) is called a *train algebra of rank* r if there exist $\gamma_1, \dots, \gamma_{r-1}$ in K such that

$$x^r + \gamma_1\omega(x)x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x = 0, \tag{1.1}$$

for all $x \in A$, where $r \geq 2$ is the smallest integer for which such an equation holds, and $x^1 = x, \dots, x^{k+1} = x^k x$ are the *principal powers* of x. Equation (1.1) is called the *train equation* of A, where we have necessarily $1 + \gamma_1 + \cdots + \gamma_{r-1} = 0$. Then the weight function ω is unique and $\ker(\omega)$ is the set of nilpotent elements. Consider the ordinary polynomial $P(X) = X^r + \gamma_1 X^{r-1} + \cdots + \gamma_{r-1}X$, called the *train polynomial* of A. In a suitable extension of K, $P(X)$ splits into linear factors $P(X) = X(X-1)(X-\lambda_1)\cdots(X-\lambda_{r-2})$, where $\lambda_0 = 1, \lambda_1, \dots, \lambda_{r-2}$ are called the *principal train roots* of A. In an abuse of notation as in [33], we write (1.1) in the form $x(x-\omega(x))(x-\lambda_1\omega(x))\cdots(x-\lambda_{r-2}\,\omega(x)) = 0$, which really means $(L_x - \omega(x)\,\mathtt{id}_A)\,(L_x - \lambda_1\omega(x)\,\mathtt{id}_A)\cdots(L_x - \lambda_{r-2}\,\omega(x)\,\mathtt{id}_A)\,x = 0$ for all $x \in A$, where L_x indicates the multiplication by x and $\mathtt{id}_A$ stands for the identity mapping.

For any polynomial $Q(X) = b_0X^s + \cdots + b_{s-1}X$ in $K[X]$ with no constant term, we define $Q(a) = b_0\,a^s + \cdots + b_{s-1}\,a$ for all $a \in H$. It is known that the set of polynomials $Q(X)$ with no constant term satisfying $Q(a) = 0$ for all $a \in H$ is an ideal in $K[X]$ generated by the train polynomial $P(X)$ (see [8,15]). An element $e \in A$ is an *idempotent* if $e^2 = e \neq 0$. We will denote by $I(A)$ the set of idempotents of A. It is important to note that every idempotent in a train algebra has weight 1. For a finite sequence $\{x_1, \dots, x_n\}$ of elements of an algebra A, we shall write $< x_1, \dots, x_n >$ for the subspace spanned by $x_1, \dots, x_n$.

On the other hand, an algebra A is *power-associative* if every element lies in an associative subalgebra. A is called a *Jordan algebra* if the identity $x^2(yx) = (x^2y)x$ holds in A. It is well known that power-associativity is equivalent to the identity $x^2x^2 = x^4$, and that Jordan algebras are power-associative.

Let A be a power-associative algebra that possesses an idempotent e. Then we have a Peirce decomposition $A = A_1 \oplus A_{1/2} \oplus A_0$, where $A_\lambda = \{x \in A \ / \ ex = \lambda x\}$. The Peirce components A_λ are connected according to the relations

$$A_\lambda A_\lambda \subseteq A_\lambda,\ A_\lambda A_{1/2} \subseteq A_{1/2} \oplus A_{1-\lambda}\ (\lambda = 0,1),\ A_{1/2}A_{1/2} \subseteq A_0 \oplus A_1,\ A_0 A_1 = 0. \tag{1.2}$$

Following the notation of Albert [1], for each $x_1 \in A_1$ we define the maps $S_{1/2}(x_1) : A_{1/2} \to A_{1/2},\ x_{1/2} \mapsto (x_1 x_{1/2})_{1/2}$ and $S_0(x_1) : A_{1/2} \to A_0,\ x_{1/2} \mapsto (x_1 x_{1/2})_0$.
Similarly, each $x_0 \in A_0$ defines the maps $T_{1/2}(x_0) : A_{1/2} \to A_{1/2},\ x_{1/2} \mapsto (x_0 x_{1/2})_{1/2}$ and $T_1(x_0) : A_{1/2} \to A_1,\ x_{1/2} \mapsto (x_0 x_{1/2})_1$. Then we have the following crucial Peirce identities (see details in [1]).

Lemma 1.1 *For all $x_0, y_0 \in A_0,\ x_1, y_1 \in A_1$ and $a_{1/2} \in A_{1/2}$, we have*

(i) $S_{1/2}(x_1 y_1) = S_{1/2}(x_1)S_{1/2}(y_1) + S_{1/2}(y_1)S_{1/2}(x_1)$,
$\frac{1}{2}S_0(x_1 y_1) = S_0(x_1)S_{1/2}(y_1) + S_0(y_1)S_{1/2}(x_1)$;
(ii) $T_{1/2}(x_0 y_0) = T_{1/2}(x_0)T_{1/2}(y_0) + T_{1/2}(y_0)T_{1/2}(x_0)$,
$\frac{1}{2}T_1(x_0 y_0) = T_1(x_0)T_{1/2}(y_0) + T_1(y_0)T_{1/2}(x_0)$;
(iii) $T_{1/2}(x_0)S_{1/2}(y_1) = S_{1/2}(y_1)T_{1/2}(x_0)$;
(iv) $[T_1(x_0)a_{1/2}]y_1 = 2T_1(x_0)S_{1/2}(y_1)a_{1/2}$,
$[S_0(y_1)a_{1/2}]x_0 = 2S_0(y_1)T_{1/2}(x_0)a_{1/2}$;
(v) $x_\lambda(x_{1/2}y_{1/2}) = [x_{1/2}(x_\lambda y_{1/2})_{1/2} + y_{1/2}(x_\lambda x_{1/2})_{1/2}]_\lambda$
$+\frac{1}{2}[x_{1/2}(x_\lambda y_{1/2})_{1-\lambda} + y_{1/2}(x_\lambda x_{1/2})_{1-\lambda}]_\lambda \quad (\lambda = 0,1)$;
(vi) $S_{1/2}(w_1)a_{1/2} = T_{1/2}(w_0)a_{1/2}$, *where* $a_{1/2}^2 = w_1 + w_0$.

Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a power-associative algebra A relative to an idempotent e. The idempotent e is said to be *principal* if there is no idempotent in A_0, and *primitive* if it is the unique idempotent in A_1. The idempotent e is called *absolutely primitive* if each element of A_1 has the form $\alpha e + x$, where $\alpha \in K$ and x is a nilpotent element. In this case, $A_1 = Ke \oplus \overline{A_1}$ where $\overline{A_1}$ is a nil-subalgebra of A. We say that the algebra A is *e-stable* if $A_i A_{1/2} \subseteq A_{1/2}$ $(i = 0,1)$, and A is *stable* if it is stable for every idempotent e. In particular, Jordan algebras are stable.

2 BASIC RESULTS

In this section we deal with the structure of power-associative train algebras involving their Peirce decompositions.

Proposition 2.1 *Let A be a power-associative algebra. If A is a train algebra, then*
(i) *A admits at least an idempotent;*
(ii) *Every idempotent of A is both principal and absolutely primitive.*

Proof. (i) Let $x \in A$ with $\omega(x) \neq 0$. By (1.1), the subalgebra $K[x]$ generated by x coincides with the subspace $< x, x^2, \ldots, x^{r-1} >$. Hence, $K[x]$ is a finite-dimensional power-associative algebra that is not a nil-algebra. It follows from [27, Proposition 3.3] that $K[x]$ has an idempotent.
(ii) Let e be an idempotent of A and let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of A induced by e. Each element $x_0 \in A_0$ satisfies $ex_0 = 0$, so $\omega(x_0) = \omega(ex_0) = 0$ and therefore x_0 is nilpotent. Thus, A_0 is a nil-algebra, so e is principal. Clearly, e is absolutely primitive, because every $x \in A_1$ is expressible in the form $\alpha e + y$, where $\alpha = \omega(x)$ and $y \in \ker(\omega)$ is a nilpotent element. Moreover, it easily seen that $A_1 = Ke \oplus \overline{A_1}$, where $\overline{A_1} = A_1 \cap \ker(\omega)$. ■

The previous result has a converse in the finite-dimensional case, which we state as:

Proposition 2.2 *Let A be a finite-dimensional power-associative algebra.*
If A has an idempotent e which is both principal and absolutely primitive, then
(i) *The linear map $\omega : A = Ke \oplus \overline{A_1} \oplus A_{1/2} \oplus A_0 \to K,\ x = \alpha e + x_1 + x_{1/2} + x_0 \mapsto \alpha$, is the unique weight function of A;*
(ii) *(A, ω) is a train algebra.*

Proof. (i) According to [20, Lemme 0.1] and its proof, the given linear map ω is the unique weight function of A and $\ker(\omega) = \overline{A_1} \oplus A_{1/2} \oplus A_0$ is the nil-radical of A.
(ii) Let m be the nil-index of $\ker(\omega)$. For any $x \in A$, since $x^2 - \omega(x)x \in \ker(\omega)$, we have $(x^2 - \omega(x)x)^m = 0$. This gives, by power-associativity,

$$x^{2m} + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \omega(x)^k x^{2m-k} = 0,$$

and hence (A, ω) is a train algebra. ■

Combining Propositions 2.1 and 2.2 actually shows that, if a finite-dimensional power-associative algebra A admits an idempotent e that is both principal and absolutely primitive, then so is any other idempotent of A.

One of the present authors explored in [20] the cases in which $A_0 = 0$ or $\overline{A_1} = 0$. Here we consider the general situation. Our first main result in this section is the following theorem.

Theorem 2.3 *Let A be a power-associative train algebra. Then*
(i) *The train equation of A is $x^s(x - \omega(x))^t = 0$, for some integers $s, t \geq 1$;*
(ii) *If $A = Ke \oplus \overline{A_1}(e) \oplus A_{1/2}(e) \oplus A_0(e)$ is the Peirce decomposition of A associated to an idempotent e, we have nil-index of $A_0(e) \leq s$ and nil-index of $\overline{A_1}(e) \leq t$;*
(iii) *If A is finite-dimensional, then the above bounds s and t are simultaneously attained for some idempotent e.*

To prove this, we need the following auxiliary lemma, which covers the associative setting:

Lemma 2.4 *Let $A = Ke \oplus \overline{A_1} \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a power-associative train algebra attached to an idempotent e. If $A_{1/2} = 0$, then*
(i) *e is the unique idempotent of A;*
(ii) *The train equation of A is $x^s(x-\omega(x))^t = 0$, where s and t are respectively the nil-indexes of A_0 and $\overline{A_1}$.*

Proof. (i) Let e' be an idempotent of A and decompose $e' = e + x_1 + x_0$, where $x_1 \in \overline{A_1}$ and $x_0 \in A_0$. Then $e'^2 = e'$ implies that $x_1 = -x_1^2$ and $x_0 = x_0^2$. Since x_1 and x_0 are nilpotent, $x_1 = x_0 = 0$, so $e' = e$.
(ii) Recall from [9] that if (A, ω) is a baric algebra with idempotent e such that $\ker(\omega)$ is nil of nil-index $d \geq 2$, and if there is $\lambda \in K$ with $ea = \lambda a$ for all $a \in \ker(\omega)$, then (A, ω) is a train algebra of train polynomial $X^{d+1} + \gamma_1 X^d + \cdots + \gamma_d X$, where

$$\gamma_i = (d, i, \lambda) := (-1)^i \left(\binom{d-1}{i} \lambda^i + \binom{d-1}{i-1} \lambda^{i-1} \right), \text{ for } 1 \leq i \leq d.$$

Applying this fact to the baric subalgebras $B = Ke \oplus A_0$ and $C = Ke \oplus \overline{A_1}$, and observing that $(s, i, 0) = 0$ for $2 \leq i \leq s$, $(s, 1, 0) = -1$, and $(t, i, 1) = (-1)^i \binom{t}{i}$, we infer that B and C are train algebras with respective train polynomials $P(X) = X^{s+1} - X^s$ and $Q(X) = X^{t+1} - \binom{t}{1} X^t + \binom{t}{2} X^{t-1} + \cdots + (-1)^t X = X(X-1)^t$.
On the other hand, since $A_0 \overline{A_1} = 0$, the baric algebra $A = Ke \oplus \overline{A_1} \oplus A_0$ is isomorphic to the join $B \vee C$ of the baric algebras B and C (see [3] for details about the join of baric algebras). It follows from [4] that the train polynomial of A is the least common multiple $X^s(X-1)^t$ of $P(X)$ and $Q(X)$. ■

Proof of Theorem 2.3. Part (i) is obtained as for Proposition 2.2(ii) by writing $(x^2 - \omega(x)x)^r = 0$ for all $x \in A$, where $r = \text{rank } A$. The train polynomial of A, which must divide $(X^2 - X)^r$, takes the form $X^s(X-1)^t$.
(ii) By Lemma 2.4, the subalgebra $B_e = Ke \oplus \overline{A_1}(e) \oplus A_0(e)$ is a train algebra of train equation $x^{s_e}(x - \omega(x))^{t_e} = 0$, where t_e and s_e are respectively the nil-indexes of $\overline{A_1}(e)$ and $A_0(e)$. Now, since the train polynomial of B_e divides that of A, we get $s_e \leq s$ and $t_e \leq t$.
(iii) In view of [15] (see also [8] and [17, Proposition 2]), there exists $a \in A$ with $\omega(a) = 1$ such that $a, a^2, \ldots, a^{r-1}$ are linearly independent, where $r = s + t = \text{rank } A$. Thus, the subalgebra $S := K[a]$ generated by a is a train algebra of rank r and train equation $x^s(x - \omega(x))^t = 0$. Let e be the unique idempotent of S and consider the Peirce decomposition $S = Ke \oplus \overline{S_1}(e) \oplus S_0(e)$. According to Lemma 2.4, the train equation of S is $x^{s_a}(x - \omega(x))^{t_a} = 0$, where t_a and s_a are respectively the nil-indexes of $\overline{S_1}(e)$ and $S_0(e)$. Consequently, $s_a = s$ and $t_a = t$. But the idempotent e produces a Peirce decomposition $A = Ke \oplus \overline{A_1}(e) \oplus A_{1/2}(e) \oplus A_0(e)$ of A, where $\overline{S_1}(e) \subseteq \overline{A_1}(e)$ and $S_0(e) \subseteq A_0(e)$. Therefore $t = t_a \leq$ nil-index of $\overline{A_1}(e)$ and $s = s_a \leq$ nil-index of $A_0(e)$. The proof is finally complete, because the reverse inequalities hold by part (ii). ■

In a recent paper [17] by Mallol and Varro, finite-dimensional power-associative and alternative train algebras that are not necessarily commutative were considered. Let us explain the relationships between our previous results and those of [17]. Let (A, ω) be a noncommutative power-associative train algebra of arbitrary dimension with train equation (1.1). Then the symmetrized algebra A^+ with product $x.y = \frac{1}{2}(xy + yx)$ is a baric algebra with weight function ω. Since powers in A^+ coincide with those in A, then (A^+, ω) is a commutative power-associative train algebra with the same train equation (1.1). It follows from Proposition 2.1 and Theorem 2.3(i) that A possesses an idempotent e and that the train equation of A has the form $x^s(x - \omega(x))^t = 0$. These facts extend the results of [17, Théorèmes 5 and 6] to the infinite-dimensional case. Concerning the Peirce decomposition of A, it is known [27, page 131] that $A = A_1 \oplus A_{1/2} \oplus A_0$, where $A_i = \{x \in A \ / \ ex = xe = ix\}$ for $i = 0, 1$, and $A_{1/2} = \{x \in A \ / \ ex + xe = x\}$. Since $A_1 = Ke \oplus \overline{A_1}$, where $\overline{A_1} = A_1 \cap \ker(\omega)$, it follows that the two inequalities in Theorem 2.3(ii) are still valid in the noncommutative case, generalizing the result of [17, Théorème 7] obtained for finite-dimensional alternative algebras.

Now, if A is alternative, it was conjectured in [17] that t =nil-index of $\overline{A_1}$ and s =nil-index of A_0. Actually, our statement (iii) in Theorem 2.3 provides a partial affirmative answer to this question, even in the power-associative case. Clearly, the natural question to know whether the nil-indexes of $\overline{A_1}(e)$ and $A_0(e)$ are independent of the chosen idempotent e remains open.

Returning to Theorem 2.3, we have the following consequence, containing [10, Theorem 2.1].

Corollary 2.5 *Let A be a train algebra of rank r with train equation* (1.1). *If A is power-associative, then there exists an integer t with $1 \leq t \leq r-1$ such that*

$$\gamma_k = (-1)^k \binom{t}{k} \quad \textit{for} \ \ 1 \leq k \leq t, \quad \textit{and} \ \gamma_k = 0 \quad \textit{for} \ \ t+1 \leq k \leq r.$$

In addition, $x_0^{r-t} = 0$ and $x_1^t = 0$ for all idempotent $e \in A$, $x_0 \in A_0(e)$ and $x_1 \in \overline{A_1}(e)$.

An arbitrary train algebra is said to be of *presentation* (s, t) if its train equation is $x^s(x - \omega(x))^t = 0$. By Corollary 2.5, there are exactly $r-1$ possible presentations (or train equations) for power-associative train algebras of rank r. We note in passing that, contrarily to the case when rank $A \leq 3$ (see [21, Théorème 2.1] or [10, Proposition 2.2]), a train algebra of presentation (s, t) need not be power-associative. In fact, for each $r \geq 4$, it has been exhibited in [10, Example 1] a train algebra of presentation $(r-1, 1)$ which is not power-associative.

Remark 2.6 As pointed out in [17, Théorème 7], there is a duality between power-associative train algebras of presentation (s, t) and those of presentation (t, s). Precisely, one assigns to every baric algebra (A, ω) a new baric algebra $(A^\star, \omega)$ with the same vector space A and multiplication $x \star y = \omega(x)y + \omega(y)x - xy$ (see a more general construction in [16]). Then A is a power-associative train algebra of presentation (s,t) if and only if $A^\star$ is a power-associative train algebra of presentation (t,s). Moreover, $I(A)=I(A^\star)$, and if $A=Ke\oplus\overline{A_1}\oplus A_{1/2}\oplus A_0$ and $A^\star=Ke\oplus\overline{A_1^\star}\oplus A_{1/2}^\star\oplus A_0^\star$ are the Peirce decompositions of A and $A^\star$ with respect to an idempotent e, then $\overline{A_1}=A_0^\star$, $A_{1/2}=A_{1/2}^\star$ and $A_0=\overline{A_1^\star}$. We will use occasionally this duality in order to simplify some proofs in the text.

We now continue to establish some fundamental properties of power-associative train algebras. Our next task is to discuss the invariance of the dimensions of the Peirce components.

Theorem 2.7 *Let $A=A_1(e)\oplus A_{1/2}(e)\oplus A_0(e)$ be the Peirce decomposition of a finite-dimensional power-associative train algebra A. Then the dimensions of $A_1(e)$, $A_{1/2}(e)$ and $A_0(e)$ are independent of the choice of the idempotent e.*

Proof. Since A is a train algebra, it follows from [15, page 110] (see also [8]) that the characteristic polynomial $P(X)=\det(L_a-X\,\mathtt{id}_A)$ of the operator L_a is the same for all $a\in H$. In particular, $P(X)=\det(L_e-X\,\mathtt{id}_A)$ for each idempotent $e\in A$. On the other hand, the direct sum $A=A_1(e)\oplus A_{1/2}(e)\oplus A_0(e)$ says that the only possible eigenvalues of L_e are $1,0$ and $\frac{1}{2}$. Hence $P(X)=(X-1)^{n(1)}(X-\frac{1}{2})^{n(\frac{1}{2})}X^{n(0)}$, where the integers $n(1),n(\frac{1}{2})$ and $n(0)$ are independent of the chosen idempotent e. Finally, since $A=\ker(L_e-\mathtt{id}_A)^{n(1)}\oplus\ker(L_e-\frac{1}{2}\,\mathtt{id}_A)^{n(\frac{1}{2})}\oplus\ker(L_e^{n(0)})$ and $A_k(e)=\ker(L_e-k\,\mathtt{id}_A)\subseteq\ker(L_e-k\,\mathtt{id}_A)^{n(k)}$, we conclude that $A_k(e)=\ker(L_e-k\,\mathtt{id}_A)^{n(k)}$ has dimension $n(k)$, for $k=1,\frac{1}{2},0$. ■

The triplet $(\dim A_1,\dim A_{\frac{1}{2}},\dim A_0)$, whose uniqueness has just been proved, is called the *type* of A.

On the other hand, it is evident that, if (A,ω) is an arbitrary train algebra, then also is every baric subalgebra B of A. We shall proceed to show that the converse is also true in the finite-dimensional power-associative situation. We say that a baric algebra (A,ω) is a *locally train algebra* if the subalgebra $K[x]$ is a train algebra for every $x\in H$.

Theorem 2.8 *Let (A,ω) be a finite-dimensional power-associative baric algebra. If (A,ω) is a locally train algebra, then (A,ω) is a train algebra.*

Proof. By hypothesis and Lemma 2.4, for each element $x\in H$, there exist integers $s_x\geq 1$ and $t_x\geq 1$ such that $y^{s_x}(y-\omega(y))^{t_x}=0$ for all $y\in K[x]$. Moreover, s_x and t_x are the nil-indexes of the Peirce components of $K[x]$. Since A is finite-dimensional, the sets $\mathcal{S}=\{s_x\ /\ x\in H\}$ and $\mathcal{T}=\{t_x\ /\ x\in H\}$ are bounded. Setting $s=\max\mathcal{S}$ and $t=\max\mathcal{T}$, we have $x^s(x-\omega(x))^t=0$ for all $x\in H$. It follows from [16, Proposition 3] (see also [8]) that $x^s(x-\omega(x))^t=0$ for all $x\in A$. ■

It is not known if the previous result remains true when power-associativity is relaxed.

3 ON THE SET OF IDEMPOTENTS

A very interesting topic in the study of train algebras is the existence and the knowledge of idempotents. Guzzo [11] and Gutiérrez [8] assumed the existence of an idempotent to find the Peirce decomposition of a train algebra. We have already proved in Proposition 2.1 the existence of such elements in power-associative train algebras of arbitrary dimensions. In this section we will derive in two distinct ways interesting formulas for the idempotents.

A) The theorem below shows how to compute idempotents in terms of elements of weight 1.

Theorem 3.1 *Let (A, ω) be a power-associative train algebra of train equation $x^s(x - \omega(x))^t = 0$. Let $f(X)$ and $g(X)$ be the unique polynomials satisfying the Bezout identity*

$$f(X)X^s + g(X)(X-1)^t = 1, \qquad \deg f(X) < t, \quad \deg g(X) < s. \tag{3.1}$$

Then the set of all idempotents of A is given by $I(A) = \{e_a := h(a) \ / \ a \in H\}$, where $h(X) = f(X)X^s$.

Proof. Consider the train polynomial $P(X) = X^s(X-1)^t$ and write $f(X) = \sum_{i=0}^{t-1} b_i X^i$. It is clear from (3.1) that $f(1) = 1$. Hence $\omega(e_a) = \sum_{i=0}^{t-1} b_i\, \omega(a^{s+i}) = \sum_{i=0}^{t-1} b_i = f(1) = 1$, and therefore $e_a \neq 0$. On the other hand, (3.1) implies that $h(X)^2 - h(X) = h(X)[h(X) - 1] = f(X)X^s\left[-g(X)(X-1)^t\right] = -P(X)f(X)g(X) \equiv 0 \mod P(X)$. This shows that $e_a^2 = h(a)^2 = h(a) = e_a$. Conversely, each idempotent e arises in this fashion, since $h(e) = \sum_{i=0}^{t-1} b_i e^{s+i} = \left(\sum_{i=0}^{t-1} b_i\right) e = e$. ■

The foregoing theorem shows that to obtain the full determination of the idempotents, we only need to compute the polynomial $f(X)$. This is the subject of the next lemma.

Lemma 3.2 *The polynomial $f(X)$ in (3.1) is given by*

$$f(X) = \sum_{p=0}^{t-1} \binom{p+s-1}{p} (1-X)^p. \tag{3.2}$$

Proof. The case $t = 1$ is immediate, since $\deg f(X) < 1$ implies that $f(X) = f(1) = 1$, which equals the right side of (3.2). Let $t \geq 2$ and set

$$T(X) := Xf'(X) + sf(X). \tag{3.3}$$

A derivation of (3.1) gives

$$X^{s-1}T(X) = -(X-1)^{t-1}\left[tg(X) + g'(X)(X-1)\right].$$

Hence, $(X-1)^{t-1}$ divides $X^{s-1}T(X)$, and so divides also $T(X)$. Since $\deg T(X) < t$, we have

$$T(X) = \alpha(X-1)^{t-1}, \tag{3.4}$$

for some $\alpha \in K$. Now, making use of (3.9), it follows that

$$T^{(p+1)}(X) = Xf^{(p+1)}(X) + (s+p)f^{(p)}(X), \tag{3.5}$$

for every $p \geq 0$. Taking $X = 1$ in (3.5) yields

$$f^{(p+1)}(1) = -(s+p)f^{(p)}(1), \quad \text{for all } p \in \{0, 1, \dots, t-2\}, \tag{3.6}$$

because in this case we have $T^{(p)}(1) = 0$ in virtue of (3.4). An easy induction on $p \geq 0$ shows, with the aid of (3.6), that

$$f^{(p)}(1) = (-1)^p \frac{(s+p-1)!}{(s-1)!}, \quad \text{for all } p \in \{0, 1, \dots, t-1\}. \tag{3.7}$$

Finally, by insertion of (3.7) into Taylor's formula, we get the required relation (3.2). ■

Remark 3.3 Let $\widetilde{A}$ be the algebra obtained from A by the usual unitization process. Then, for every $a \in H$, the element $e'_a = g(a)(a-1)^t$ is an idempotent in $\widetilde{A}$, which is orthogonal to $e_a = f(a)a^s$. Notice that it is also possible to evaluate the polynomial $g(X)$. For this, replacing X by $1-X$ in (3.1), where we write $f_{s,t}(X)$ and $g_{s,t}(X)$ instead of $f(X)$ and $g(X)$, gets

$$(-1)^t g_{s,t}(1-X)X^t + (-1)^s f_{s,t}(1-X)(X-1)^s = 1.$$

Comparison of this identity with $f_{t,s}(X)X^t + g_{t,s}(X)(X-1)^s = 1$ shows that $f_{t,s}(X) = (-1)^t g_{s,t}(1-X)$, implying that $g_{s,t}(X) = (-1)^t f_{t,s}(1-X)$. It follows from Lemma 3.2 that

$$g_{s,t}(X) = (-1)^t \sum_{p=0}^{s-1} \binom{p+t-1}{p} X^p. \tag{3.8}$$

Remark 3.4 Using Theorem 3.1, an alternative proof of Theorem 2.3(iii) can be furnished. Indeed, pick $a \in H$ such that $a, a^2, \cdots, a^{s+t-1}$ are linearly independent, so that $X^s(X-1)^t$ is the minimal polynomial of a, and decompose $S = K[a] = Ke \oplus \overline{S_1}(e) \oplus S_0(e)$, as in the proof of Theorem 2.3(iii). Since e is the unique idempotent of S, we have $e = e_a = f(a)a^s$ by Theorem 3.1. Now, the element $x_0 := a(a-1)^t \in S$ satisfies $ex_0 = f(a)a\,[a^s(a-1)^t] = 0$, and therefore $x_0 \in S_0(e)$. Further, as $X^s(X-1)^t$ does not divide $X^{s-1}(X-1)^{t(s-1)}$, we have $x_0^{s-1} = a^{s-1}(a-1)^{t(s-1)} \neq 0$, so x_0 is nilpotent of index s. It follows from this and Theorem 2.3(ii) that s is the nil-index of the subspace $S_0(e)$. We may show analogously that the element $x_1 := a^s(a-1)$ belongs to $\overline{S_1}(e)$ and $x_1^{t-1} \neq 0$, so that $\overline{S_1}(e)$ has nil-index t.
By the way we point out that $S_0(e_a) =< a(a-1)^t, a^2(a-1)^t, \dots, a^{s-1}(a-1)^t >$ and $\overline{S_1}(e_a) =< a^s(a-1), a^s(a-1)^2, \dots, a^s(a-1)^{t-1} >$.

Applying Theorem 3.1 together with Lemma 3.2, we obtain the following consequences.

Corollary 3.5 *Let (A,ω) be a power-associative train algebra of rank n. If the presentation of A is either $(n-1,1),(n-2,2),(n-3,3)$, $(n-4,4)$ or $(n-5,5)$, then $I(A)=\{e_a \ / \ a\in H\}$, where we have respectively*

$$\begin{aligned}
e_a &= a^{n-1},\\
e_a &= (n-1)a^{n-2}-(n-2)a^{n-1},\\
e_a &= \tfrac{(n-1)(n-2)}{2}\ a^{n-3}-(n-1)(n-3)\ a^{n-2}+\tfrac{(n-2)(n-3)}{2}\ a^{n-1},\\
e_a &= \tfrac{(n-1)(n-2)(n-3)}{6}\ a^{n-4}-\tfrac{(n-1)(n-2)(n-4)}{2}\ a^{n-3}\\
&\quad+\tfrac{(n-2)(n-3)(n-4)}{2}\ a^{n-2}-\tfrac{(n-1)(n-3)(n-4)}{6}\ a^{n-1}\\
e_a &= \tfrac{(n-1)(n-2)(n-3)(n-4)}{24}\ a^{n-5}-\tfrac{(n-1)(n-2)(n-3)(n-5)}{6}\ a^{n-4}+\tfrac{(n-1)(n-3)(n-4)(n-5)}{4}\ a^{n-3}\\
&\quad-\tfrac{(n-1)(n-3)(n-4)(n-5)}{6}\ a^{n-2}+\tfrac{(n-2)(n-3)(n-4)(n-5)}{24}\ a^{n-1}.
\end{aligned}$$

Proof. It suffices to calculate the polynomial $f(X)$ with the aid of Lemma 3.2 in each case. The verification is straightforward and is left to the reader. $\blacksquare$.

We note that the particular cases $(n-1,1),(n-2,2)$ and $(n-3,3)$ were already accomplished by Giovanni Reyes in his Ph.D. thesis [25], using other techniques.

The general expression of the idempotents in presentation (t,s) can be immediately deduced with a slight modification from that in presentation (s,t). Indeed, let (A,ω) be a power-associative train algebra of presentation (t,s). In view of Remark 2.6, the attached baric algebra $(A^\star,\omega)$ is a power-associative train algebra of presentation (s,t) and $I(A)=I(A^\star)$. Let $f_{s,t}(X)=\sum\limits_{i=0}^{t-1} b_iX^i$ and $h_{s,t}(X)=f_{s,t}(X)X^s$. By Theorem 3.1, $I(A^\star)=\{e_a:=h^\star_{s,t}(a)\ /\ a\in H\}$, where $h^\star_{s,t}(a)=\sum\limits_{i=0}^{t-1} b_ia^{\star(i+s)}$ and the $a^{\star(i+s)}$ are the powers of a in $A^\star$.
On the other hand, according to [16, page 6], we have $a^{\star k}=\sum\limits_{i=1}^{k}(-1)^k\binom{k}{i}a^i$, which can be formally written as $a^{\star k}=1-(1-a)^k$. Therefore,

$$\begin{aligned}
e_a=h^\star_{s,t}(a)&=\sum_{i=0}^{t-1} b_i\left[1-(1-a)^{i+s}\right]=\sum_{i=0}^{t-1} b_i-\sum_{i=0}^{t-1} b_i(1-a)^{i+s}\\
&=1-\sum_{i=0}^{t-1} b_i(1-a)^{i+s}. \qquad (3.9)
\end{aligned}$$

For instance, taking into account Corollary 3.5, we obtain directly from (3.9) the following corollary.

Corollary 3.6 *If the presentation of A is either $(1,n-1),(2,n-2),(3,n-3),(4,n-4)$ or*

$(5, n-5)$, *then* $I(A) = \{e_a \;/\; a \in H\}$, *where we have respectively*

$$\begin{aligned}
e_a &= 1-(1-a)^{n-1}, \\
e_a &= 1-(n-1)(1-a)^{n-2}+(n-2)(1-a)^{n-1}, \\
e_a &= 1-\tfrac{(n-1)(n-2)}{2}\,(1-a)^{n-3}+(n-1)(n-3)\,(1-a)^{n-2}-\tfrac{(n-2)(n-3)}{2}\,(1-a)^{n-1}, \\
e_a &= 1-\tfrac{(n-1)(n-2)(n-3)}{6}\,(1-a)^{n-4}+\tfrac{(n-1)(n-2)(n-4)}{2}\,(1-a)^{n-3} \\
&\quad -\tfrac{(n-1)(n-3)(n-4)}{2}\,(1-a)^{n-2}+\tfrac{(n-2)(n-3)(n-4)}{6}\,(1-a)^{n-1} \\
e_a &= 1-\tfrac{(n-1)(n-2)(n-3)(n-4)}{24}\,(1-a)^{n-5}+\tfrac{(n-1)(n-2)(n-3)(n-5)}{6}\,(1-a)^{n-4} \\
&\quad -\tfrac{(n-1)(n-3)(n-4)(n-5)}{4}\,(1-a)^{n-3}+\tfrac{(n-1)(n-3)(n-4)(n-5)}{6}\,(1-a)^{n-2} \\
&\quad -\tfrac{(n-2)(n-3)(n-4)(n-5)}{24}\,(1-a)^{n-1}.
\end{aligned}$$

As illustration, we offer in the following table the list of all train equations of ranks ≤ 7 with their manifolds of idempotents. The results are quickly obtained from Theorem 3.1 by direct application of the extended Euclidean algorithm in a symbolic computation software as MAPLE or MATHEMATICA.

Rank	(s,t)	Train-equation	Idempotent e_a
2	$(1,1)$	$x^2-\omega(x)x=0$	a
3	$(2,1)$	$x^3-\omega(x)x^2=0$	a^2
	$(1,2)$	$x^3-2\omega(x)x^2+\omega(x)^2x=0$	$2a-a^2$
4	$(3,1)$	$x^4-\omega(x)x^3=0$	a^3
	$(2,2)$	$x^4-2\omega(x)x^3+\omega(x)^2x^2=0$	$3a^2-2a^3$
	$(1,3)$	$x^4-3\omega(x)x^3+3\omega(x)^2x^2-\omega(x)^3x=0$	$3a-3a^2+a^3$
5	$(4,1)$	$x^5-\omega(x)x^4=0$	a^4
	$(3,2)$	$x^5-2\omega(x)x^4+\omega(x)^2x^3=0$	$4a^3-3a^4$
	$(2,3)$	$x^5-3\omega(x)x^4+3\omega(x)^2x^3-\omega(x)^3x^2=0$	$6a^2-8a^3+3a^4$
	$(1,4)$	$x^5-4\omega(x)x^4+6\omega(x)^2x^3-4\omega(x)^3x^2+\omega(x)^4x=0$	$4a-6a^2+4a^3-a^4$

Rank	(s,t)	Train-equation	Idempotent e_a
6	$(5,1)$	$x^5 - \omega(x)x^4 = 0$	a^5
	$(4,2)$	$x^5 - 2\omega(x)x^4 + \omega(x)^2x^3 = 0$	$5a^4 - 4a^5$
	$(3,3)$	$x^5 - 3\omega(x)x^4 + 3\omega(x)^2x^3 - \omega(x)^3x^2 = 0$	$10a^3 - 15a^4 + 6a^5$
	$(2,4)$	$x^5 - 4\omega(x)x^4 + 6\omega(x)^2x^3 - 4\omega(x)^3x^2 + \omega(x)^4x = 0$	$10a^2 - 20a^3 + 15a^4 - 4a^5$
	$(1,5)$	$x^5 - 4\omega(x)x^4 + 6\omega(x)^2x^3 - 4\omega(x)^3x^2 + \omega(x)^4x = 0$	$5a - 10a^2 + 10a^3 - 5a^4 + a^5$
7	$(6,1)$	$x^5 - \omega(x)x^4 = 0$	a^6
	$(5,2)$	$x^5 - 2\omega(x)x^4 + \omega(x)^2x^3 = 0$	$6a^5 - 5a^6$
	$(4,3)$	$x^5 - 3\omega(x)x^4 + 3\omega(x)^2x^3 - \omega(x)^3x^2 = 0$	$15a^4 - 24a^5 + 10a^6$
	$(3,4)$	$x^5 - 4\omega(x)x^4 + 6\omega(x)^2x^3 - 4\omega(x)^3x^2 + \omega(x)^4x = 0$	$20a^3 - 45a^4 + 36a^5 - 10a^6$
	$(2,5)$	$x^5 - 4\omega(x)x^4 + 6\omega(x)^2x^3 - 4\omega(x)^3x^2 + \omega(x)^4x = 0$	$15a^2 - 40a^3 + 45a^4 - 24a^5 + 5a^6$
	$(1,6)$	$x^5 - 4\omega(x)x^4 + 6\omega(x)^2x^3 - 4\omega(x)^3x^2 + \omega(x)^4x = 0$	$6a - 15a^2 + 20a^3 - 15a^4 + 6a^5 - a^6$

B) The next objective is to presenting an alternative characterization of the set of idempotents. A natural question consists on finding all the idempotents starting with a fixed one. To this end, we assume in the following that the algebra A is e-stable. Before starting discussion, we require some preparation. The first key ingredient is the next lemma, which is just a reformulation of Lemma 1.1 for e-stable algebras.

Lemma 3.7 *Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a power-associative algebra induced by an idempotent e. If A is e-stable, then for all $x_i, y_i \in A_i$ $(i = 0,1)$ and $a_{1/2} \in A_{1/2}$, we have*
(a) $(x_1y_1)a_{1/2} = x_1(y_1a_{1/2}) + y_1(x_1a_{1/2})$;
(b) $(x_0y_0)a_{1/2} = x_0(y_0a_{1/2}) + y_0(x_0a_{1/2})$;
(c) $x_0(y_1a_{1/2}) = y_1(x_0a_{1/2})$;
(d) $x_0(x_{1/2}y_{1/2}) = [x_{1/2}(y_{1/2}x_0)]_0 + [y_{1/2}(x_{1/2}x_0)]_0$;
$x_1(x_{1/2}y_{1/2}) = [x_{1/2}(y_{1/2}x_1)]_1 + [y_{1/2}(x_{1/2}x_1)]_1$;
(e) $(a_{1/2}^2)_1a_{1/2} = (a_{1/2}^2)_0a_{1/2} = \frac{1}{2}a_{1/2}^3$.

Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of an e-stable power-associative algebra produced by e. For all $\lambda \in \{0,1\}$ and $x_\lambda \in A_\lambda$, we consider the map $S_{x_\lambda} : A_{1/2} \to$

$A_{1/2}, x_{1/2} \mapsto x_\lambda x_{1/2}$. By parts (a) and (b) of Lemma 3.7, we have $S_{x_\lambda}S_{y_\lambda} + S_{y_\lambda}S_{x_\lambda} = S_{x_\lambda y_\lambda}$. Thus, one easily obtain as in [20, Lemme 1.4] the following

Lemma 3.8 *For all $k \geq 2, \lambda \in \{0,1\}$ and $x_\lambda \in A_\lambda$, we have:*
(i) $S_{x_\lambda{}^k} = 2^{k-1}S_{x_\lambda}^k$;
(ii) $S_{x_\lambda{}^k} = 2S_{x_\lambda}S_{x_\lambda{}^{k-1}}$;
(iii) $S_{x_\lambda}S_{x_\lambda{}^{k-1}} = S_{x_\lambda{}^{k-1}}S_{x_\lambda}$.

The next lemma is also necessary for our intended applications.

Lemma 3.9 *For all $k \geq 1$ and $x_{1/2} \in A_{1/2}$, we have*

$$x_{1/2}(x_{1/2}^2)_0^k \overset{\text{(iv)}}{=} x_{1/2}(x_{1/2}^2)_1^k \overset{\text{(v)}}{=} \frac{1}{2}x_{1/2}^{2k+1}.$$

Proof. We carry out an induction to show (iv). For $k = 1$ the result is just Lemma 3.7(e). Let $k \geq 2$. With the aid of Lemmas 3.7 and 3.8, together with the induction hypothesis (I.H.), we have:

$$\begin{aligned} x_{1/2}(x_{1/2}^2)_0^{k+1} &\overset{(ii)}{=} 2(x_{1/2}^2)_0[x_{1/2}(x_{1/2}^2)_0^k] \overset{(I.H.)}{=} 2(x_{1/2}^2)_0[x_{1/2}(x_{1/2}^2)_1^k] \\ &\overset{(c)}{=} 2(x_{1/2}^2)_1^k[x_{1/2}(x_{1/2}^2)_0] \overset{(e)}{=} 2(x_{1/2}^2)_1^k[x_{1/2}(x_{1/2}^2)_1] \\ &\overset{(iii)}{=} 2(x_{1/2}^2)_1[x_{1/2}(x_{1/2}^2)_1^k] \overset{(ii)}{=} x_{1/2}(x_{1/2}^2)_1^{k+1}. \end{aligned}$$

which proves (iv). Now (v) follows from (iv), because
$x_{1/2}^{2k+1} = x_{1/2}(x_{1/2}^2)^k = x_{1/2}[(x_{1/2}^2)_0^k + (x_{1/2}^2)_1^k] \overset{\text{(iv)}}{=} 2x_{1/2}(x_{1/2}^2)_1^k$. ■

Combining Lemma 3.9 and Theorem 2.3(ii) allows us to state:

Corollary 3.10 *Let A be an e-stable power-associative train algebra of presentation (s,t). Then $x_{1/2}^{2r+1} = 0$ for all $x_{1/2} \in A_{1/2}$, where $r = \min(s,t)$.*

It follows from the preceding corollary that, if $s = 1$ or $t = 1$, that is $A_0 = 0$ or $\overline{A_1} = 0$, then $x_{1/2}^3 = 0$ for all $x_{1/2} \in A_{1/2}$.

Remark 3.11 Since $x_{1/2}^{2r+1} = 0$, we have $(x_{1/2}^2)^{r+1} = 0$. Hence $(x_{1/2}^2)_0^{r+1} = (x_{1/2}^2)_1^{r+1} = 0$, and so $(x_{1/2}^2)_0^k = (x_{1/2}^2)_1^k = 0$ for all $k \geq r+1$.

Having these preparations at hand, we come now to the following principal result.

Theorem 3.12 *Let $A = Ke \oplus \overline{A_1} \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of an e-stable power-associative train algebra of presentation (s,t). Then the set of idempotents of A is given by*

$$I(A) = \left\{ e + x_{1/2} + \sum_{p=1}^{r} \mu_p \left[(x_{1/2}^2)_0^p - (x_{1/2}^2)_1^p \right] \ / \ x_{1/2} \in A_{1/2} \right\}, \tag{3.10}$$

where $r = \min(s,t)$ *and the scalars* $\mu_1, \dots, \mu_r$ *depending only on* r *satisfy the following combinatorial equation*

$$\mu_1 = 1 \quad \textit{and} \quad \mu_p = \sum_{p/2 \le k \le p-1} \mu_k \, (-1)^{p-k+1} \binom{k}{p-k} \qquad (p \ge 2). \tag{3.11}$$

The proof of the theorem depends upon the following technical lemma, which seems to be of independent interest:

Lemma 3.13 *Let* a *be an element in a power-associative algebra* A *with* $a^r = 0$.
(i) *If* $v = a - a^2$, *then* $a = \sum_{p=1}^{r-1} \mu_p v^p$, *where the sequence* $\{\mu_p\}$ *is given by* (3.11).
(ii) *Conversely, if* $a = \sum_{p=1}^{r-1} \mu_p v^p$ *and* $v \in A$ *with* $v^r = 0$, *then* $v = a - a^2$.

Proof. (i) Without loss of generality, we may assume that $a^{r-1} \neq 0$. As $v^r = 0$ and $v^{r-1} \neq 0$, the family $\{v, v^2, \dots, v^{r-1}\}$ is a basis of $K[a]$. Hence, there are $\mu_1, \dots, \mu_{r-1}$ in K such that:

$$a = \sum_{k=1}^{r-1} \mu_k v^k = \sum_{k=1}^{r-1} \mu_k \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^{k+i} \right).$$

This implies that $a = \sum_{p=1}^{r-1} \lambda_p \, a^p$, where

$$\lambda_1 = \mu_1 = 1 \quad \text{and} \quad \lambda_p = \sum_{1 \le k \le r-1, \; 0 \le i \le k, \; k+i=p} \mu_k (-1)^i \binom{k}{i} \qquad (2 \le p \le r-1).$$

But the conditions $0 \le i \le k$ and $k + i = p$ entail $k \le p \le 2k$. It follows that

$$\begin{aligned} \lambda_p &= \sum_{p/2 \le k \le p} \mu_k \, (-1)^{p-k} \binom{k}{p-k} \\ &= \mu_p + \sum_{p/2 \le k \le p-1} \mu_k \, (-1)^{p-k} \binom{k}{p-k} \qquad (2 \le p \le r-1). \end{aligned} \tag{3.12}$$

On the other hand, we have obviously $a = \sum_{p=1}^{r-1} \lambda_p \, a^p$, with $\lambda_1 = 1$ and $\lambda_p = 0$ for $2 \le p \le r-1$. Comparing this with (3.12), we conclude the desired relation (3.11).
(ii) By hypothesis and part (i), we have $a = \sum_{p=1}^{r-1} \mu_p v^p = \sum_{p=1}^{r-1} \mu_p w^p$, where $w = a - a^2$. Then $a^{r-1} = v^{r-1} = w^{r-1}$. Now, $a^{r-2} = v^{r-2} + (r-2)\, \mu_2 \, v^{r-1} = w^{r-2} + (r-2)\, \mu_2 \, w^{r-1}$ yields $v^{r-2} = w^{r-2}$. Continuing in this way with the powers a^k $(1 \le k \le r-1)$, we get finally $v = w$. ■

Putting $l = p - k$, (3.11) becomes

$$\mu_1 = 1 \quad \text{and} \quad \mu_p = \sum_{1 \le l \le p/2} (-1)^{l+1} \binom{p-l}{l} \mu_{p-l} \quad (p \ge 2).$$

The first values of the sequence $\{\mu_p\}_p$ up to $p = 10$ are available in the next table.

p	1	2	3	4	5	6	7	8	9	10
μ_p	1	1	2	5	14	42	132	429	1430	4862

We emphasize in passing the following consequence of the lemma: Let a and b two elements in a power-associative algebra A. If $a^r = b^r = 0$ and $a - a^2 = b - b^2$, then $a = b$.

Proof of Theorem 3.12. Let $f = e + x_{1/2} + x_0 + x_1$ be an idempotent of A, necessarily of weight 1. Then $f^2 = f$ is equivalent to

$$(x_{1/2}^2)_0 + x_0^2 = x_0, \quad (x_{1/2}^2)_1 + x_1^2 + x_1 = 0, \quad x_{1/2}x_0 + x_{1/2}x_1 = 0. \tag{3.13}$$

Since $x_0^s = 0$ and $x_1^t = 0$, we apply Lemma 3.13(i) to $a = x_0$, $v = (x_{1/2}^2)_0$ on the one hand, and to $a = -x_1$, $v = (x_{1/2}^2)_1$ on the other hand, to derive

$$x_0 = \sum_{p=1}^{s-1} \mu_p (x_{1/2}^2)_0^p \quad \text{and} \quad x_1 = -\sum_{p=1}^{t-1} \mu_p (x_{1/2}^2)_1^p.$$

As $(x_{1/2}^2)_0^s = 0$ and $(x_{1/2}^2)_1^t = 0$, we have also

$$x_0 = \sum_{p=1}^{s} \mu_p (x_{1/2}^2)_0^p \quad \text{and} \quad x_1 = -\sum_{p=1}^{t} \mu_p (x_{1/2}^2)_1^p. \tag{3.14}$$

Now, by Remark 3.11 and (3.14), we infer that

$$x_0 = \sum_{p=1}^{r} \mu_p (x_{1/2}^2)_0^p \quad \text{and} \quad x_1 = -\sum_{p=1}^{r} \mu_p (x_{1/2}^2)_1^p, \tag{3.15}$$

so f takes the form in (3.10).

Conversely, every element $f = e + x_{1/2} + x_0 + x_1$ satisfying (3.15) is an idempotent. Indeed, $x_{1/2}x_0 + x_{1/2}x_1 = \sum_{p=1}^{r} \mu_p [x_{1/2}(x_{1/2}^2)_0^p - x_{1/2}(x_{1/2}^2)_1^p] = 0$, thanks to Lemma 3.9. Further, since $(x_{1/2}^2)_0^{r+1} = (x_{1/2}^2)_1^{r+1} = 0$ by Remark 3.12, it follows from (3.15) that $x_0^{r+1} = x_1^{r+1} = 0$. Consequently, we obtain from (3.15) and Lemma 3.13(ii) that $(x_{1/2}^2)_0 = x_0 - x_0^2$ and $(x_{1/2}^2)_1 = -x_1 - x_1^2$. Thus (3.13) is satisfied, that is f is an idempotent, which ends the proof of the theorem. ∎

Theorem 3.12 shows that the set of idempotents of A is parameterized by the subspace $A_{1/2}$. This confirms that $\dim A_{1/2}$ must be independent of the idempotent e, as was already mentioned in Theorem 2.7.

Specializing Theorem 3.12 to the cases of presentations $(s, 1)$ and $(1, t)$, we obtain the earlier results [20, Propositions 1.5 and 4.2]:

Corollary 3.14 *Under the hypotheses of Theorem* 3.12*, we have:*
(i) *If* $A = Ke \oplus A_{1/2} \oplus A_0$ *is of presentation* $(s, 1)$*, then* $I(A) = \{e + x_{1/2} + x_{1/2}^2 \ / \ x_{1/2} \in A_{1/2}\}$.
(i) *If* $A = Ke \oplus \bar{A}_1 \oplus A_{1/2}$ *is of presentation* $(1, t)$*, then* $I(A) = \{e + x_{1/2} - x_{1/2}^2 \ / \ x_{1/2} \in A_{1/2}\}$.

4 JORDAN ALGEBRAS

In this section we concentrate our efforts on Jordan train algebras. In particular, we provide some conditions guaranteeing power-associative train algebras to be Jordan algebras. Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a Jordan algebra induced by the idempotent e. It is known [1,23] that A is stable and satisfies the identity

$$[(x_\lambda x_{1/2}) y_{1/2}]_{1-\lambda} = [(x_\lambda y_{1/2}) x_{1/2}]_{1-\lambda}, \tag{4.1}$$

for all $x_\lambda \in A_\lambda$, $x_{1/2}, y_{1/2} \in A_{1/2}$ and $\lambda = 0, 1$.

In [1], Albert constructed an example of a commutative power-associative algebra that is not a Jordan algebra, because it is not stable. The next example shows that e-stable power-associative train algebras need not satisfy (4.1).

Example 4.1 Let $A = < e, u_1, u_2, u_3, u_4, v, w >$ be the commutative algebra with multiplication table given by $e^2 = e$, $eu_i = \frac{1}{2} u_i$ $(i = 1, \ldots, 4)$, $ew = w$, $u_1 v = u_1 w = u_3$, $u_2 v = u_2 w = u_4$, $u_2 u_3 = -u_1 u_4 = v + w$, other products being zero. Then A is equipped with the weight function ω such that $\omega(e) = 1$, $\omega(u_i) = \omega(v) = \omega(w) = 0$. By straightforward calculation, one may check that A is a power-associative train algebra of presentation $(2, 2)$, with $\overline{A_1}(e) = < w >$, $A_{1/2}(e) = < u_1, u_2, u_3, u_4 >$ and $A_0(e) = < v >$. Moreover, A is e-stable. Since $[(vu_1)u_2]_1 = w$ and $[(vu_2)u_1]_1 = -w$, A does not satisfy (4.1), and so it is no longer a Jordan algebra.

Next, we recall the result below, stated in [20, Théorème 1.3].

Proposition 4.2 *Let* $A = A_1 \oplus A_{1/2} \oplus A_0$ *be a power-associative train algebra of presentation* $(s, 1)$ *or* $(1, t)$*. Then* A *is a Jordan algebra if and only if the subalgebras* A_0 *and* A_1 *are Jordan algebras.*

It is well known that any commutative nil-algebra of nil-index ≤ 3 is a Jordan algebra. Hence we deduce from Proposition 4.2 and Theorem 2.3(ii):

Corollary 4.3 *Every power-associative train algebra of presentation* $(s, 1)$ *or* $(1, t)$ *with* $1 \leq s, t \leq 3$ *is a Jordan algebra*

A special case of this corollary is:

Corollary 4.4 *Every power-associative train algebra of rank* ≤ 3 *is a Jordan algebra.*

It is worth noticing that the simplest case of rank 2 corresponds to the well-known gametic algebras for simple Mendelian inheritance [33], which are special Jordan algebras [26]. For rank 3, the presentation is either $(2,1)$ or $(1,2)$, so the train equation is either $x^3-\omega(x)x^2=0$ or $x^3-2\omega(x)x^2+\omega(x)^2x=0$, recovering the results of [10, Proposition 2.2] and [21, Théorème 2.1]. It should be pointed out that the train equation $x^3-\omega(x)x^2=0$ characterizes also the class of Bernstein-Jordan algebras [19,31].

For rank 4, Corollary 4.3 permits us to get immediately the results of [2,13] concerning respectively the presentations $(3,1)$ and $(1,3)$. Namely,

Corollary 4.5 *Let A be a power-associative train algebra of rank 4 with train equation $x^4-\omega(x)x^3=0$ (resp. $x^4-3\omega(x)x^3+3\omega(x)^2x^2-\omega(x)^3x=0$). Then A is a Jordan algebra.*

Proof. Apply Corollary 4.3 after observing that the presentation is either $(3,1)$ or $(1,3)$. ■

The remainder class of power-associative train algebras of rank 4 have presentation $(2,2)$ and train equation $x^4-2\omega(x)x^3+\omega(x)^2x^2=0$. We emphasize that such algebras are no longer Jordan algebras [21, Exemple 2.2], so that 3 is the best rank in Corollary 4.4.

At present, we give a criterion for some power-associative train algebras of rank 5 to be Jordan.

Proposition 4.6 *Let A be a power-associative train algebra of rank 5 with train equation $x^5-\omega(x)x^4=0$ (resp. $x^5-4\omega(x)x^4+6\omega(x)^2x^3-4\omega(x)^3x^2+\omega(x)^4x=0$). Then A is a Jordan algebra if and only if the identity $(x^2y)x=0$ holds in A_0 (resp. in $\overline{A_1}$).*

Proof. Here the presentation is $(4,1)$ (resp. $(1,4)$), so A_0 (resp. $\overline{A_1}$) has nil-index ≤ 4. Linearizing the identity $(x^2)^2=0$ implies $x^2(yx)=0$. The result follows then from Proposition 4.2. ■

In concluding this section, we shall develop a result about finitely generated algebras. It is known [28, Corollary 1] that every finitely generated Jordan Bernstein algebra is finite-dimensional. This result was extended in [36, Theorem 6.7] to n^{th}-order Bernstein algebras. Our contribution in this spirit is to establish an analogous version for train algebras. To this aim, the characteristic of K is assumed to be zero in the following theorem.

Theorem 4.7 *Every finitely generated Jordan train algebra is finite-dimensional.*

Proof. We start by showing that $N=\ker(\omega)$ is finitely generated as an algebra. Let $A=Ke\oplus\overline{A_1}\oplus A_{1/2}\oplus A_0$ be the Peirce decomposition of A with respect to an idempotent e. Choose a system of generators $a_1,\ldots,a_n$ of A, and decompose each a_i into $a_i=\alpha_ie+b_i+c_i+d_i$, with $\alpha_i\in K$, $b_i\in\overline{A_1}$, $c_i\in A_{1/2}$ and $d_i\in A_0$. Let $x=f(a_1,...,a_n)\in N$, where f is a non-associative polynomial. Then $x=f(\alpha_1,\ldots,\alpha_n)e+g(e,b_1,\ldots,b_n,c_1,\ldots,c_n,d_1,\ldots,d_n)$ for some non-associative polynomial g such that $y=g(e,b_1,\ldots,b_n,c_1,\ldots,c_n,d_1,\ldots,d_n)\in N$.

Since $\omega(x) = 0,\ \omega(e) = 1$ and $\omega(y) = 0$, then $f(\alpha_1, \ldots, \alpha_n) = 0$ and $x = y$. By the inclusion $(A_{1/2}^2 \subseteq \overline{A_1} + A_0$, we may write $c_i c_j = b'_{ij} + d'_{ij}$, where $b'_{ij} \in \overline{A_1}$ and $d'_{ij} \in A_0$. Keeping in mind that $A_0\overline{A_1} = 0$, $\overline{A_1}A_{1/2} \subseteq A_{1/2}$, and $A_0A_{1/2} \subseteq A_{1/2}$, it is not difficult to see that

$$x = h\left(b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots, d_n,\ \{b'_{ij}\}_{1\leq i,j\leq n},\ \{d'_{ij}\}_{1\leq i,j\leq n}\right),$$

where h is a non-associative polynomial. This proves that N is finitely generated as an algebra.
Next, since N is a nil-algebra of bounded index over a field of characteristic zero, it is solvable [34]. But finitely generated solvable Jordan algebras are nilpotent [35, Theorem 2, page 90]. Hence, N is nilpotent. Finally, to end the proof, it suffices to apply the standard fact that each finitely generated nilpotent algebra is finite-dimensional. ■.

5 BERNSTEIN ALGEBRAS OF ARBITRARY ORDER

In this last section we will touch on some aspects of power-associative Bernstein algebras of order n. Considerably more can be said in this context.

We begin by recalling that the *plenary powers* of an element x in an algebra A are defined by $x^{[1]} = x$ and $x^{[k+1]} = x^{[k]}x^{[k]},\ k \geq 1$. A baric algebra (A, ω) is said to be a n^{th}*-order Bernstein algebra of period* p, or simply a $B(n,p)$*-algebra*, if the identity

$$x^{[n+p+1]} = \omega(x)^{2^n(2^p-1)}x^{[n+1]} \tag{5.1}$$

holds in A, where $n \geq 0,\ p \geq 1$ and the ordered pair (p, n) is minimal for the lexicographic order [30]. $B(n,1)$-algebras satisfy $x^{[n+2]} = \omega(x)^{2^n}x^{[n+1]}$ and are called n^{th}*-order Bernstein algebras* [18]. In particular, $B(1,1)$-algebras are just the well-known Bernstein algebras [19,33].

First of all, we start by stating the following useful lemma.

Lemma 5.1 *Let (A, ω) be a train algebra of rank n and train equation $x^n - \omega(x)^{n-p}x^p = 0$ with $1 \leq p \leq n-1$. If A is power-associative, then $n = p+1$.*

Proof. By Theorem 2.3, the train polynomial of A is $P(X) = X^s(X-1)^t$ for some $s, t \geq 1$. Since $P(X)$ coincides with $Q(X) = X^n - X^p = X^p(X^{n-p} - 1)$, it follows that $s = p,\ t = 1$ and $n - p = 1$. ■

Let us record a couple of consequences of the lemma.

Proposition 5.2 *Let (A, ω) be a train algebra with train equation $x^n - \omega(x)^{n-1}x = 0$, $n \geq 2$ (resp. $x^n - \omega(x)^{n-2}x^2 = 0$, $n \geq 3$). Then the following assertions are equivalent:*
(i) *A is power-associative ;*

(ii) *A is a Jordan algebra ;*

(iii) *A is a Bernstein algebra of order* 0 (*resp. of order* 1)*;*

(iv) $n = 2$ (*resp.* $n = 3$).

Proof. According to Lemma 5.1, (i) implies obviously (iv). On the other hand, it is known that baric algebras satisfying $x^2 - \omega(x)x = 0$ (resp. $x^3 - \omega(x)x^2 = 0$) are Jordan and Bernstein algebras of order 0 (resp. of order 1), so (iv) implies (ii) and (iii). To complete the proof, it only remains to show that (iii) implies (iv). If A is Bernstein of order 0, then clearly $n = 2$. Assume A is Bernstein of order 1. Since, by hypothesis, A is also a train algebra, it follows from [22] or [32] that A satisfies an equation of the form $(x^3 - \omega(x)x^2)(x - \frac{1}{2}\omega(x))^t = 0$, where $t \geq 0$. Hence $P(X) = X^n - X^2$ divides $(X^3 - X^2)(X - \frac{1}{2})^t$, and therefore $n = 3$. ■

Proposition 5.3 *Let* (A, ω) *be a train algebra with train equation* $x^n - \omega(x)^{n-3}x^3 = 0$, $n \geq 4$. *The following assertions are equivalent:*

(i) *A is power-associative;*

(ii) *A is a Jordan algebra ;*

(iii) *A satisfies* $x^{[3]} - \omega(x)x^3 = 0$.

Besides, if one of these conditions is satisfied, then A is a second-order Bernstein algebra.

Proof. In virtue of Lemma 5.1, (i) entails $n = 4$, which yields (iii). Conversely, it is known from [2, Theorem 2.2] that any baric algebra satisfying (iii) is a Jordan algebra that is also a second-order Bernstein algebra. ■

We now give a result about $B(n,p)$-algebras.

Proposition 5.4 *Let* (A, ω) *be a power-associative* $B(n,p)$*-algebra. Then A satisfies* $x^{2^n+1} - \omega(x)x^{2^n} = 0$ *and A is a* n^{th}*-order Bernstein algebra.*

Proof. By power-associativity, (5.1) becomes $x^{2^{n+p}} - \omega(x)^{2^n(2^p-1)}x^{2^n} = 0$, so A is a train algebra. According to Theorem 2.3, the train polynomial of A has the form $P(X) = X^s(X-1)^t$. Since $P(X)$ must divide $Q(X) = X^{2^{n+p}} - X^{2^n} = X^{2^n}(X^{2^n(2^p-1)} - 1)$, we see that $s \leq 2^n$ and $t = 1$. Then A satisfies $x^{s+1} - \omega(x)x^s = 0$, and so also $x^{2^n+1} - \omega(x)x^{2^n} = 0$. An easy induction shows that $x^{2^n+k} = \omega(x)^k x^{2^n}$ for all $k \geq 1$. Putting $k = 2^n$ yields $x^{2^{n+1}} = \omega(x)^{2^n}x^{2^n}$, that is $x^{[n+2]} = \omega(x)^{2^n}x^{[n+1]}$. Comparison of this with (5.1) gives $p = 1$, that is A is a n^{th}-order Bernstein algebra. ■

In [7, Proposition 4.5], the authors indicate that a power-associative train algebra of rank $2^n + 1$ and train equation $x^{2^n+1} - \omega(x)x^{2^n} = 0$ is necessarily a n^{th}-order Bernstein algebra. On the other hand, it was shown in [20, Théorème 3.7] that any power-associative n^{th}-order Bernstein algebra satisfies $x^{2^n+1} - \omega(x)x^{2^n} = 0$, so it is a train algebra of rank $\leq 2^n + 1$. The theorem which follows is of special interest for the matter we are developing in this section. It explores all the possible train equations for a power-associative n^{th}-order Bernstein algebra.

Theorem 5.5 *Let (A, ω) be a power-associative baric algebra. The following statements holds:*
(i) *If A is a n^{th}-order Bernstein algebra, then A is a train algebra of rank $m+1$ and train equation $x^{m+1} - \omega(x)x^m = 0$, for some integer m with $2^{n-1} < m \leq 2^n$.*
(ii) *Conversely, if A is a train algebra of rank $m+1$ and train equation $x^{m+1} - \omega(x)x^m = 0$, then A is a n^{th}-order Bernstein algebra, where n is the unique integer with $2^{n-1} < m \leq 2^n$.*

Proof. (i) Assume that A is a n^{th}-order Bernstein algebra. Then by [20, Théorème 3.7], the identity $x^{2^n+1} - \omega(x)x^{2^n} = 0$ holds in A. Hence A is a train algebra whose train polynomial $P(X)$ divides $Q(X) = X^{2^n+1} - X^{2^n} = X^{2^n}(X-1)$. Thus, $P(X) = X^m(X-1)$ for some $m \leq 2^n$, so the train equation of A is $x^{m+1} - \omega(x)x^m = 0$. We claim that $2^{n-1} < m$. Indeed, suppose instead that $m \leq 2^{n-1}$. Multiplying $x^{m+1} - \omega(x)x^m = 0$ by $x^{2^{n-1}-m}$ gets $x^{2^{n-1}+1} = \omega(x)x^{2^{n-1}}$. From this, we deduce as in the proof of Proposition 5.4 that $x^{[n+1]} = \omega(x)^{2^{n-1}}x^{[n]}$, so A is a Bernstein algebra of order $< n$, a contradiction.
(ii) Conversely, suppose A satisfies the train equation $x^{m+1} - \omega(x)x^m = 0$, and let n be the unique integer with $2^{n-1} < m \leq 2^n$. We multiply $x^{m+1} - \omega(x)x^m = 0$ by $x^{2^\ell - m}$ to obtain $x^{2^\ell+1} - \omega(x)x^{2^\ell} = 0$. Therefore we infer as above that $x^{[\ell+2]} = \omega(x)^{2^\ell}x^{[\ell+1]}$, which means that A is a Bernstein algebra of order $\ell \leq n$. It follows from part (i) that $2^{\ell-1} < m \leq 2^\ell$. Finally, since by hypothesis $2^{n-1} < m$, we have necessarily $\ell = n$, completing the proof. ■

As a consequence of the last result, we have exactly 2^{n-1} possible train equations for a power-associative n^{th}-order Bernstein. For instance, the train equations for $n \leq 3$ are:
- $n = 1:\ \ x^3 - \omega(x)x^2 = 0;$
- $n = 2:\ \ x^4 - \omega(x)x^3 = 0,\ \ x^5 - \omega(x)x^4 = 0;$
- $n = 3:\ \ x^6 - \omega(x)x^5 = 0,\ \ x^7 - \omega(x)x^6 = 0,\ \ x^8 - \omega(x)x^7 = 0,\ \ x^9 - \omega(x)x^8 = 0.$

Note that the particular case of power-associative second-order Bernstein algebras and their corresponding train equations $x^4 - \omega(x)x^3 = 0$ and $x^5 - \omega(x)x^4 = 0$ has been discussed in [7].

Given arbitrary integers $n \geq 1$ and $m \geq 2$ with $2^{n-1} < m \leq 2^n$, we will construct below an associative n^{th}-order Bernstein algebra that is a train algebra of rank $m+1$.

Example 5.6 Let $A_m =< e, v_1, v_2, \ldots, v_{m-1} >$ be the associative algebra with multiplication table $e^2 = e$ and $v_i v_j = v_{i+j}$ whenever $i+j \leq m-1$, other products being zero. Then A_m is endowed with the weight function ω given by $\omega(e) = 1$ and $\omega(v_i) = 0$. Select an element $x = \alpha e + \sum_{i=1}^{m-1} \alpha_i v_i \in A_m$. It is routine to check that

$$x^p = \alpha^p e + \sum_{i=p}^{m-1} \beta_i^{(p)} v_i,$$

for some scalars $\beta_i^{(p)}$. In particular, $x^{m-1} = \alpha^{m-1}e + \beta_{m-1}^{(m-1)} v_{m-1}$, $x^m = \alpha^m e$ and $x^{m+1} = \alpha^{m+1}e$, so $x^{m+1} - \omega(x)x^m = 0$. On the other hand, $x^{[n]} = x^{2^{n-1}} = \alpha^{2^{n-1}}e + \sum_{i=2^{n-1}}^{m-1} \beta_i^{(2^{n-1})} v_i$.

As $m-1 < 2^n \leq i+j$ whenever $2^{n-1} \leq i, j \leq m-1$, it follows that $x^{[n+1]} = \left(x^{[n]}\right)^2 = \alpha^{2^n} e$ and $x^{[n+2]} = \alpha^{2^{n+1}} e$, implying $x^{[n+2]} - \omega(x)^{2^n} x^{[n+1]} = 0$. Consequently, A_m is both a n^{th}-order Bernstein algebra and a train algebra of rank $m+1$.

We close our article by making the following observations. Let (A, ω) be a power-associative n^{th}-order Bernstein algebra. We have seen in Theorem 5.5 that (A, ω) is a train algebra of presentation $(m, 1)$, where $2^{n-1} < m \leq 2^n$. Let $A = Ke \oplus A_{1/2}(e) \oplus A_0(e)$ be the Peirce decomposition of A attached to an idempotent $e \in A$. Then, by Theorem 2.3(ii), we have $x_0^m = 0$ for all $x_0 \in A_0(e)$. Moreover, Theorem 2.7 contains the result by Towers and Bowman [29, Corollary 5.2] asserting that the dimensions of $A_{1/2}(e)$ and $A_0(e)$ are invariant. We note finally that Corollary 3.14(i) was also obtained in [29, Proposition 4.1].

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