INFINITE DIMENSIONAL

LIE AND JORDAN ALGEBRAS

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F algebraically closed field of zero characteristic

- Z-graded algebras

V. Kac → Local Lie algebra

If $L = \oplus_{i \in Z} L_i$, is a simple graded Lie algebra of finite growth (i.e. dim $L_i \leq |i|^c + d$) and L is generated by the vector subspace $L_{-1} \oplus L_0 \oplus L_1$, L infinite dimensional and L_{-1} is faithful simple L_0-module, then L is isomorphic to an affine algebra or L is a Cartan algebra.

Kac's Conjecture

O. MATHIEU: Simple graded Lie algebras of polynomial growth :

- (twisted) loop algebra, or
- Cartan type algebra , or
- the Virasoro algebra, **Vir**

- $\mathcal{G}$ simple finite dimensional Lie (resp. Jordan) algebra, $\mathcal{L}(\mathcal{G}) = \mathcal{G} \otimes F[t^{-1}, t]$ its (non twisted)loop algebra.

- If $\mathcal{G}$ is Z/lZ-graded, $\mathcal{G} = \mathcal{G}_0 + \cdots + \mathcal{G}_{l-1}$, then $\sum_{i=j\, mod\, l} \mathcal{G}_i \otimes t^j$ is " twisted loop algebra".

- $W =$ Vir is the algebra of derivations of Laurent polynomials.

It has a basis $\{e_j | j \in Z\}$ with multiplication

$$[e_i, e_j] = (j - i)e_{i+j}$$

- W_n is the algebra of derivations of the polynomial ring $F[t_1, \ldots, t_n]$.

- Cartan algebras are some particular subalgebras of W_n.

Examples of simple graded Jordan algebras of finite growth

- Loop algebras $\mathcal{L}(\mathcal{G})$, where $\mathcal{G}$ is now a Z/lZ-graded simple Jordan algebra.

- $J = F1 + V$ the Jordan algebra of a bilineal form, where V is a Z-graded vector space, $V = \sum_{i \in Z} V_i$, s.t. $\dim V_i \leq |i|^c + d$ and with a nondegenerate symmetric bilineal form defined on V.

Theorem: Let $J = \sum_{i \in Z} J_i$ be a Z-graded simple graded Jordan algebra of finite growth. Suppose J is infinite dimensional. Then J is isomorphic to one of the following Jordan algebras:

(a) a loop algebra

(b) a simple Jordan algebra of a bilineal form over an infinite dimensional vector space V.

Prime Z-graded Jordan algebras

- An algebra J is prime if $IJ \neq (0)$ if I, J are non zero ideals of L.
- A Jordan algebra is nondegenerate if it has not absolute zero divisors.

Change finite growth by growth ≤ 1

C. M. and E. Zelmanov :
If J is a finitely generated Jordan algebra with GK-dimension one, then its McCrimmon radical is nilpotent. Furthermore, if J is nondegenerate then $Z(J) \neq (0)$ and J is a finite module over $Z(J)$. In particular J is PI.

Theorem: Let $J = \sum_{i \in Z}$ be a prime nondegenerate Jordan algebra satisfying that $\dim J_i < d \ \forall i \in Z$. Then

(a) Either J is graded simple or

(b) $\exists s \geq 1$ such that $J_i = 0$ for $i < -s$ (resp. $i > s$). Furthermore, there is a finite dimensional Z/lZ-graded algebra $\mathcal{G}$ and an isomorphism $\phi : J \longrightarrow \mathcal{L}(\mathcal{G})$ s.t. $\phi(J_k) = (\mathcal{L}(\mathcal{G}))_k \ \forall k > m$ (resp. $k < -m$) for some $m \geq 1$. (one sided graded)

Application to superconformal algebras

Definition: A *superconformal algebra* is a Z-graded simple Lie superalgebra, $L = \sum_{i \in Z} L_i$, with $\dim L_i \leq d \; \forall i \in Z$ and containing the Virasoro algebra.

Conjecture. (V. Kac, van de Leur, 1989) $W(1,n)+$ Cartan type sub-superalgebras $+$ Cheng-Kac $CK(6)$.

$$W(1,n) = Der F[t^{-1}, t, \xi_1, \ldots, \xi_n].$$

- (V. Kac + C.M. + E. Zelmanov)

Confirmation of the above conjecture in the "Jordan case"

Kantor-Koecher-Tits Construction

J Jordan algebra $\rightarrow$ $K(J) = J^- + [J^-, J^+] + J^+$ Lie algebra

$Fe + Fh + Ff \subseteq L$, $adh : L \to L$ diagonalizable

$$L = L_{-2} + L_0 + L_2$$

$J = L_{-2}$ is a Jordan algebra: $x_{-2} \cdot y_{-2} = [[x_{-2}, f], y_{-2}]$

L can be recovered (up to central extensions) from J

If $L = L_{\bar{0}} + L_{\bar{1}}$ is a Z-graded Lie superalgebra with the dimensions dimL_i uniformly bounded.

- $L_0 = L_{0\bar{0}} + L_{0\bar{1}}$ is a finite dimensional Lie superalgebra.

- If L_0 is not solvable, then $L_{0\bar{0}}$ contains a copy of $sl_2(F) = Fe + Fh + Ff$, $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

- The adjoint operator $ad(h) : L \to L$ is diagonalizable and has finitely many eigenvalues.

- L has a finite grading that is compatible with the initial Z-grading.

Jordan case if only -2, 0, 2 appear.

Theorem: Let $J = \sum_{i \in Z}$ be a Z-graded simple graded Jordan algebra with $\dim J_i \leq d$. Then J is isomorphic to one of the following:

i) a loop superalgebra $\mathcal{L}(\mathcal{G})$,

ii) $J = F1 + V$ a Jordan superalgebra of a nondegenerate bilineal form on $V = V_{\bar{0}} + V_{\bar{1}}$,

iii) A Kantor double $J = A + Ax$ with $A = \sum_{i \in Z} G(V)_{\bar{i}} \otimes t^i$ a commutative associative superalgebra with a Jordan bracket.

iv) A superalgebra of Cartan type

v) A Cheng Kac superalgebra corresponding to $CK(6)$.

If A is the even part of J, I the biggest $\mathcal{D}$-invariant ideal of A with I_0 nilpotent, $\mathcal{D}$ the linear span of derivations $R(x)^2 : J \to J$, $x \in J_{\bar{1}}$, then

1) A/I loop algebra, $I \neq (0)$ nilpotent,

2) A/I is *one sided graded*,

3) $I \neq (0)$ and A/I simple finite dimensional of a bilinear form,

4) $A = A^1 \oplus A^2$ with A^1, A^2 simple finite dimensional or loop algebras or one sided graded or infinite dimensional of a bilinear form,

5) A is a loop algebra

6) A is simple finite dimensional

7) A/I simple infinite dimensional of a bilinear form, $I \neq (0)$,

8) A simple infinite dimensional of a bilinear form.

In the general case

What is the structure of the even part of a conformal algebra?

Z- graded prime nondegenerate Lie algebras of growth one

- L is nondegenerate if $a \in L$ and $[[L,a],a] = (0)$ implies $a = 0$.

- L has a finite grading if

$$L = \sum_{i \in Z} L_{(i)},$$

with $[L_{(i)}, L_{(j)}] \subseteq L_{(i+j)}$, where $\{i | L_{(i)} \neq (0)\}$ is finite.

The grading is nontrivial if

$$\sum_{i \neq 0} L_{(i)} \neq (0)$$

$\longleftrightarrow$ Jordan algebras and their generalizations.

If $\mathcal{G}$ is simple Z/lZ-graded, then $\mathcal{L}(\mathcal{G})$ has a Z-grading and a Z/lZ-grading that are compatible.

Virasoro acts naturally on $\mathcal{L}(\mathcal{G})$ and the semidirect sum $L \simeq \mathcal{L}(\mathcal{G}) > \triangleleft Vir$ is prime nondegenerate.

THEOREM I

Let $L = \sum_{i \in Z} L_i$ be a prime nondegenerate Z-graded Lie algebra containing the Virasoro algebra and having the dimensions $dim L_i$ uniformly bounded. Let's assume that L has a nontrivial finite grading that is compatible with the Z-grading. Then $L \simeq \mathcal{L}(\mathcal{G}) \rtimes Vir$ for some simple finite dimensional Lie algebra $\mathcal{G}$.

Strongly PI Case

Let $f(x_1, \ldots, x_n)$ be a nonzero element of the free associative algebra.

-A satisfies the polynomial identity

$$f(x_1, \ldots, x_n) = 0$$

if $f(a_1, \ldots, a_n) = 0$ for arbitrary elements $a_1, \ldots, a_n \in A$.

- An algebra that satisfies some polynomial identity is said to be a PI-algebra.

- If A is an algebra, its multiplication algebra $M(A)$ is the subalgebra of $End_F(A)$ generated by the right multiplications

$$R(a) : x \to xa,$$

and the left multiplications

$$L(a) : x \to ax \quad a \in A.$$

- An algebra A is *strongly PI* if its multiplication algebra $M(A)$ is PI.

- An element a in a Lie algebra L over a field F is said to be of rank 1 if $[[L, a], a] \subseteq Fa$.

- An ideal of the free Lie (resp. associative) algebra is called a T-ideal if it is invariant with respect to substitutions.

- The ideal that consists of all the identities that an arbitrary algebra L satisfies is a T-ideal.

Lemma 1

Let $A = \sum_{i \in Z} A_i$ be a graded algebra whose centroid $\Gamma = \sum_{i \in Z} \Gamma_i$ contains an invertible homogeneous element $\gamma \in \Gamma_i$ of degree $i \neq 0$. Then $A \simeq \mathcal{L}(\mathcal{G})$ is a (twisted) loop algebra.

Lemma 2 Let $\Gamma = \sum \Gamma_i$ be a (commutative and associative) Z-graded domain over an algebraically closed field F with the dimensions $dim_F \Gamma_i$ uniformly bounded.

Then, $\Gamma \simeq F[t^{-m}, t^m]$ or

$\sum_{i \geq k} F t^{mi} \subseteq \Gamma \subseteq F[t^m]$ or

$\sum_{i \geq k} F t^{-mi} \subseteq \Gamma \subseteq F[t^{-m}]$, where $m \geq 1$, $k \geq 1$.

Let $L = \sum_{i \in Z} L_i$ be a Z-graded Lie algebra that is *strongly PI*, prime and non-degenerate.

Let's consider $d = max_{i \in Z} dim L_i$.

Γ the centroid of L (the centralizer of the multiplication algebra $M(L)$ in $End_F(L)$, Γ_h the set of homogeneous elements of Γ.

Lemma 3

(1) $\Gamma \neq (0)$ is an integral domain and the fractions ring $(\Gamma \setminus \{0\})^{-1} L$ is a simple finite dimensional Lie algebra over the field $K = (\Gamma \setminus \{0\})\Gamma$.

(2) The algebra $\tilde{L} = (\Gamma_h \setminus \{0\})^{-1} L$ is a graded simple algebra and $\dim_F \tilde{L}_i \leq d$, for an arbitrary $i \in Z$.

(3) Either L is isomorphic to a loop algebra or there exists a graded embedding $\varphi : \Gamma \to F[t^{-m}, t^m]$ s.t.

$\sum_{i \geq k} Ft^{im} \subseteq \varphi(\Gamma) \subseteq F[t^m]$ or

$\sum_{i \geq k} Ft^{-im} \subseteq \varphi(\Gamma) \subseteq F[t^{-m}]$.

Lemma 4

Let $L = \sum_{i \in Z} L_i$ be a prime Lie algebra, that is nondegenerate and strongly PI, with $\dim L_i \leq d$.

Let us assume that $Vir = \sum_{i \in Z} Vir_i$ can be embedded in $Der(L)$ as a graded algebra. Then L is isomorphic to a (non-twisted) loop algebra.

Lemma 5

Let L be a prime nondegenerate Lie algebra and let I be a nonzero ideal of L. Then I is a prime nondegenerate algebra.

Lemma 6

Let $L = \sum_{i \in Z}^{n} L_i$ be a Z-graded, prime non- degenerated Lie algebra containing the Virasoro algebra and with the dimensions $\dim L_i$ uniformly bounded. Let us assume that L contains a nonzero graded ideal I that is strongly PI. Then L is isomorphic to the semidirect sum of a loop algebra $\mathcal{L}(\mathcal{G})$ (fore some simple finite-dimensional lie algebra $\mathcal{G}$) and the Virasoro algebra.

Lie-Jordan Connections

A Jordan pair $P = (P^-, P^+)$ is a pair of vector spaces with two trilinear operations :

$$\{\ ,\ ,\ \} : P^- \times P^+ \times P^- \to P^-,$$

$$\{\ ,\ ,\ \} : P^+ \times P^- \times P^+ \to P^+$$

satisfying:

(P.1) $\{x^\sigma, y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, z^{-\sigma}\}, x^\sigma\}$,

(P.2) $\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, y^{-\sigma}, u^\sigma\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, y^{-\sigma}\}, u^\sigma\}$,

(P.3) $\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, z^{-\sigma}, \{x^\sigma, y^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}, y^{-\sigma}\}, x^\sigma\}$,

for every $x^\sigma, u^\sigma \in P^\sigma$, $y^{-\sigma}, z^{-\sigma} \in P^{-\sigma}$, $\sigma = \pm$.

If $L = \sum_{i=-n}^{n} L_{(i)}$ has a finite grading, then the pair $(L_{(-n)}, L_{(n)})$ with

$$\{x^\sigma, y^{-\sigma}, z^\sigma\} = [[x^\sigma, y^{-\sigma}], z^\sigma],$$

$\sigma = \pm$ is a Jordan pair.

Lemma 7

Let L be a Lie algebra with a finite grading

$$L = \sum_{k=-n}^{n} L_{(k)}, \; L_{(0)} = \sum_{k=1}^{n} [L_{(-k)}, L_{(k)}]$$

and $L_{(n)} \neq (0)$.

If L is prime and nondegenerate, then:

(1) Every nonzero ideal of L has nonzero intersection with $L_{(n)}$,

(2) The Jordan pair $V = (L_{(-n)}, L_{(n)})$ is prime and nondegenerate.

Lemma 8

Let $L = \sum_{k=-n}^{n} L_{(k)}$ be a Lie algebra with a finite grading. Let's assume that the Jordan pair $V = (L_{(-n)}, L_{(n)})$ is prime and nondegenerate and an arbitrary nonzero ideal of L has nonzero intersection with V. Then L is prime and nondegenerate.

THEOREM II

Let $V = (V^-, V^+) = \sum_{i\in Z} V_i$ be a Z-graded Jordan pair that is prime and non-degenerated, having the dimensions $\dim V_i$ uniformly bounded. Then either V is isomorphic to a (twisted) loop pair $\mathcal{L}(W)$, with W a simple finite-dimensional Jordan pair or V can be embedded in $\mathcal{L}(W)$.

Furthermore $\sum_{i\geq k} \mathcal{L}(W)_i \subseteq V \subseteq \mathcal{L}(W)$ or $\sum_{i\geq k} \mathcal{L}(W)_{-i} \subseteq V \subseteq \mathcal{L}(W)$.

Proof of Theorem I from Theorem II

If $L = \sum_{i\in Z} L_i = \sum_{k=-n}^{k=n} L_{(k)}$ is our Lie algebra and $V = (L_{(-n)}, L_{(n)}$ the associated Jordan pair, then V can be embedded into $\mathcal{L}(W) = \sum_{i=q \; mod \, l} W_i \otimes t^q$ the (twisted) loop pair associated to W a simple, f.d. Jordan pair graded by Z/lZ, $W = \sum_{i=0}^{l-1} W_i$.

From x element of first order in $\mathcal{G}$, the Lie algebra associated to W we find an element $a \in V$ such the ideal $\mathrm{id}_L(a)$ is strongly PI $\longrightarrow$ Apply Lemma 6

Proof of Theorem II

- $K(V)$ is strongly PI

- $\mathcal{K}(V) = \mathcal{K}(V)_{-1} + \mathcal{K}(V)_0 + \mathcal{K}(V)_1$ is a Z-graded Lie algebra,

$$\mathcal{K}(V)_0 = [\mathcal{K}(V)_{-1}, \mathcal{K}(V)_1],$$

$(\mathcal{K}(V)_{-1}, \mathcal{K}(V)_1) = V$ and $\mathcal{K}(V)_0$ does not contain nonzero ideals of $\mathcal{K}(V)$.

- Every nonzero ideal of $\mathcal{K}(V)$ has nonzero intersection with V^+.

- $\mathcal{K}(V)$ is prime.

- $L = \mathcal{K}(V)$ prime and strongly PI $\Rightarrow$ the centroid Γ de L is nonzero and $(\Gamma \setminus \{0\})^{-1}L$ f. d. over $(\Gamma \setminus \{0\})^{-1}\Gamma$.

- $\Gamma \simeq$ centroid of V.

- Γ is a commutative graded domain, $\Gamma = \sum_{i \in Z} \Gamma_i$ with $\dim \Gamma_i \leq 1$.

- If $\Gamma = \Gamma_0 \Rightarrow \Gamma = F$ and $\dim_F V < \infty$.

- If there exist $i, j \geq 1$ with $\Gamma_i \neq (0) \neq \Gamma_{-j}$, then V is a (twisted) loop Jordan pair.

- If there are only negative components in Γ (resp. positive) then V is embedded in a loop pair.

f.g. Case

E. Zelmanov :V is either strongly PI or V is special

Lema 11

If V is a finitely generated special Jordan pair and A is an associative algebra such that $(V^-, V^+) \subseteq (A^-, A^+)$ and $A = A^- + (A^- A^+ + A^+ A^-) + A^+$, then $GK - dim(V) = GK - dim(A)$.

By the result of Small, Stafford and Warfield Jr,$GK - dim(A) = 1 \Rightarrow A$ is PI.

V is strongly PI (previous case)

General Case

Lema 12

Let $V = \sum_{i \in Z} V_i$ a Z-graded Jordan pair with the dimensions $\dim V_i$ uniformly bounded. Then the locally nilpotent radical $Loc(V)$ coincides with the McCrimmon radical $M(V)$.

- Let V be a Jordan pair as in Theor. 2 and $\tilde{V}$ a f.g. graded subpair of V:

- The (nondeg.) pair $\tilde{V}/M(\tilde{V}))$ is approximated by f. g.prime Jordan pairs

- By the f.g. case each of those pairs is $\mathcal{L}(U)$ or can be embedded in a loop pair $\mathcal{L}(U)$, with U simple finite - dimensional.

- $\dim U \leq N(d)$, where $d = max dim V_i$.

- T ideal of the free Jordan pair that consists of elements that are identically zero in all Jordan pairs of dimension $\leq N(d)$.

- $T(\tilde{V}) \subseteq \mathrm{Loc}(\tilde{V}) \Rightarrow T(V) \subseteq Loc(V)$.

$\mathrm{Loc}(V) = M(V) = (0) \Rightarrow T(V) = (0)$.

- The pair V is strongly PI.