OCTONION ALGEBRAS OBTAINED FROM ASSOCIATIVE ALGEBRAS WITH INVOLUTION

Holger P. Petersson
Fachbereich Mathematik
FernUniversität
D-58084 Hagen
Bundesrepublik Deutschland

Michel L. Racine*
Department of Mathematics
University of Ottawa
Ottawa, Ontario
K1N 6N5
Canada

Abstract: A natural octonion algebra structure on the symmetric elements of trace 0 of central simple associative algebras of degree 3 with involution of the second kind is obtained.

Introduction

Given a central simple associative algebra of degree 3 with involution of the second kind $(B, *)$, several constructions in the literature can be viewed as associating to it an octonion algebra. Our main concern in this paper is to derive an explicit formula for the multiplication of an octonion algebra living on the space of symmetric trace 0 element of $(B, *)$ and to relate it to previous constructions.

The first construction is due independently to Okubo [6] and Faulkner [2] and yields an eight-dimensional (nonunital) symmetric composition algebra on the elements of trace 0 of a central simple associative algebra A of degree 3 provided that the base field has characteristic not 2 or 3 and contains the cube roots of unity. Since, by passing to a principal Albert isotope (see, for example, [1]), every eight-dimensional composition algebra determines a unique octonion algebra having the same norm, the Okubo-Faulkner construction, in a roundabout way, produces an octonion algebra structure on the elements of trace 0 in A.

Next there is a construction due to Haile, Knus, Rost and Tignol [3] attaching to any central simple associative algebra $(B, *)$ of degree 3 with

* The second author's research is supported in part by a grant from NSERC.

involution of the second kind a 3-fold Pfister form which reflects important cohomological invariants of algebraic groups and, in particular, classifies the involutions of the second kind on a fixed algebra B. We refer to [5, Chapter V] for a systematic account of the construction. Since 3-fold Pfister forms are exactly the norm forms of octonion algebras and since an octonion algebra is determined up to isomorphism by its norm form, the Haile-Knus-Rost-Tignol construction again produces an octonion algebra structure, this time determined in a roundabout manner by $(B, *)$.

Our purpose is to make this octonion algebra structure explicit. Rather than working within the narrow confines of $(B, *)$ itself, we prefer to enlarge it to a reduced Albert algebra J by means of the Tits process [7,8], allowing us, via the explicit coordinatization obtained in [11], to identify the pure octonions of the coordinate algebra of J with certain symmetric elements of trace 0 in $(B, *)$. An analogous approach had been adopted previously for the coordinate norm [9,10], leading to an alternate treatment of the 3-fold Pfister form attached to $(B, *)$ (see also [5, Chapter 40] for related results). But while the latter approach works in all characteristics, here we have to exclude characteristics 2 and 3. Finally, we specialize $(B, *)$ to $A + A^{op}$ equipped with the exchange involution, A being a central simple associative algebra of degree 3, and in this way relate our octonion algebra structure to the symmetric composition algebra obtained by Okubo and Faulkner.

Algebras of degree 3

We start by recalling facts concerning algebras of degree 3. Let $(B, *)$ be a central simple associative algebra with involution over a field k. Then B is either simple or the direct sum $A \oplus A^{op}$, where A is simple and A^{op} denotes the opposite algebra, i.e., $a^{op}b := ba$; in this case * is the exchange involution which permutes the summands. Assume that * is of the second kind, i.e., is not the identity on the centre of B. Then K, the centre of B, is either a separable quadratic field extension of k or $K = k \oplus k$. Let $H(B, *) = \{b \in B \mid b^* = b\}$, the symmetric elements of B. If $B = A \oplus A^{op}$ then $H(B, *) = \{(a, a) \mid a \in A\}$ which can be identified with A, at least as a vector space. We will also identify K with $K1 \subseteq B$.

If B is of degree 3 over its centre then every element satisfies its reduced characteristic polynomial

$$x^3 - T(x)x^2 + S(x)x - N(x), \tag{1}$$

where T is the *reduced trace*, N the *reduced norm* and S a quadratic form, which we refer to as the *quadratic trace*. The forms T, S and N on all of B take values in K but their restrictions to $H(B, *)$ take values in k. The quadratic trace $S(x)$ is given by

$$S(x) := T(x^{\#}), \tag{2}$$

where $^{\#}$ is the *adjoint*, i.e., the numerator of the inversion map,

$$x^{\#} = x^2 - T(x)x + S(x). \tag{3}$$

Recall that for $x \times y := (x+y)^{\#} - x^{\#} - y^{\#}$,

$$1 \times y = T(y) - y. \tag{4}$$

Let

$$T(a,b) := T(ab).$$

From now on assume that $k = H(K,*)$ is of characteristic not 2 or 3. In this case, combining (2) and (3),

$$S(x) = \frac{1}{2}(T(x)^2 - T(x^2)). \tag{5}$$

We wish to obtain more precise information on the behaviour of the quadratic form S on $H(B,*)_0$, the symmetric elements of trace 0. For example, (5) becomes

$$S(x) = -\frac{1}{2}T(x^2), \quad x \in H(B,*)_0, \tag{5'}$$

which linearizes to

$$S(x,y) = -T(x,y), \quad x,y \in H(B,*)_0. \tag{5''}$$

From $x^{\#\#} = N(x)x$ and $S(x^{\#}) = T(x^{\#\#}) = N(x)T(x)$, we deduce

$$S(x^{\#}) = 0, \quad x \in H(B,*)_0. \tag{6}$$

Moreover

$$S(x^2,1) = T(x^2 \times 1) = T(T(x^2) - x^2) = 2T(x^2).$$

So, by (5′),

$$S(x^2,1) = -4S(x). \tag{7}$$

$$\begin{aligned} S(x^{\#}) &= S(x^2 + S(x)) \\ &= S(x^2) + S(x)S(x^2,1) + S(x)^2 S(1) \\ &= S(x^2) - 4S(x)^2 + 3S(x)^2 \end{aligned}$$

and, by (6),

$$S(x^2) = S(x)^2, \quad x \in H(B,*)_0. \tag{8}$$

Since the characteristic is not 2, there exists an element $\theta \in K$ such that $K = k[\theta]$, $\theta^* = -\theta$ and $\theta^2 = \rho \in k$. Recall that a *frame* of B is a set of pairwise orthogonal primitive idempotents whose sum is the unit element of B. We will need the existence of certain symmetric elements of B.

Lemma 1. *If $(B, *)$ is a central simple associative algebra of degree 3 with involution of the second kind, then there exists a $d \in H(B, *)$ satisfying*

$$T(d) = 0 \quad \text{and} \quad S(d) \neq 0 \neq N(d). \tag{9}$$

Proof. If B is a division algebra or $B = A + A^{op}$, with A a division algebra, then the norm form is anisotropic on $H(B, *)$. Since $H(B, *) = k + H(B, *)_0$ is an orthogonal splitting with respect to the trace bilinear form $T(x, y)$ and, by $(5'')$, $S(x, y) = -T(x, y)$ on $H(B, *)_0$, $S(x)$ is nonsingular on $H(B, *)_0$ and we may choose a $d \in H(B, *)_0$ with $S(d) \neq 0 \neq N(d)$. On the other hand, if $H(B, *)$ contains a frame $\{e_1, e_2, e_3\}$, one checks that $d = e_1 + e_2 - 2e_3$ satisfies $T(d) = 0$, $S(d) = -3$ and $N(d) = -2$. So in all cases there exists a $d \in H(B, *)$ satisfying $T(d) = 0$ and $S(d) \neq 0 \neq N(d)$. □

Remark. If the base field is infinite, such d's are Zariski dense.

We wish to recall a few facts concerning Albert algebras [7, 8]. Every Albert algebra J, central over the field k, can be obtained from a central simple associative algebra $(B, *)$, of degree 3 with involution of the second kind, using the Tits process. For $u \in H(B, *)$ and $\mu \in K$ invertible such that $N(u) = \mu\mu^*$, $J = J(B, *, u, \mu) := H(B, *) \oplus B$, as a vector space. For (a, b), $(r, s) \in J(B, *, u, \mu)$, one defines an adjoint $\#$, a norm form N and a trace form T on J which extend those on $H(B, *)$,

$$(a, b)^{\#} := (a^{\#} - bub^*,\ \mu^* b^{*\#} u^{-1} - ab), \tag{10}$$
$$N((a, b)) := N(a) + \mu N(b) + \mu^* N(b^*) - T(a, bub^*), \tag{11}$$
$$T((a, b)) := T(a). \tag{12}$$

Moreover

$$\begin{aligned} S((a, b)) &= T((a, b)^{\#}) = T(a^{\#} - bub^*) \\ &= S(a) - T(bub^*), \end{aligned} \tag{13}$$
$$(a, b) \times (r, s) = (a \times r - bus^* - sub^*,\ \mu^*(b^* \times s^*)u^{-1} - as - rb). \tag{10'}$$

Since, for $J(B, *, u, \mu)$ to be a division algebra, it is necessary that $\mu \notin N(B)$ [7, Th. 5.2], $J(B, *, 1, 1)$ is not a division algebra. So it is reduced and contains a frame. We will construct an explicit one. Let

$$e_1 = \frac{1}{3}(1, 1) \quad \text{and} \quad f = 1 - e_1 = \frac{1}{3}(2, -1). \tag{14}$$

Since $T(e_1) = 1$ and ${e_1}^{\#} = 0$, e_1 is a primitive idempotent of J. One also checks that

$$f^{\#} = e_1. \tag{15}$$

For d as in Lemma 1, let

$$c = \frac{1}{2}S(d)^{-1}d^2 + \frac{1}{3}. \tag{16}$$

By (5′), $T(c) = 0$ and $c \in H(B,*)_0$. We will also need

$$\begin{aligned} S(c) &= \frac{1}{4}S(d)^{-2}S(d^2) + \frac{1}{6}S(d)^{-1}S(d^2,1) + \frac{1}{3} \\ &= \frac{1}{4} - \frac{2}{3}S(d)^{-1}S(d) + \frac{1}{3} && \text{by (7)} \\ &= -\frac{1}{12}. \end{aligned}$$

So, for c as in (16), we have

$$T(c) = 0 \quad \text{and} \quad S(c) = -\frac{1}{12}. \tag{17}$$

For arbitrary $(a,b) \in J$, we have

$$e_1 \times (a,b) = \frac{1}{3}(T(a) - a - b - b^*,\ T(b^*) - a - b - b^*), \tag{18}$$

$$(2,-1) \times (a,b) = (2T(a) - 2a + b^* + b,\ -T(b^*) + b^* - 2b + a), \tag{19}$$

$$(c,c) \times (a,b) = (a \times c - cb^* - bc,\ c \times b^* - cb - ac), \tag{20}$$

$$(c,c)^{\#} = (c^{\#} - c^2, c^{\#} - c^2) = -(\frac{1}{12}, \frac{1}{12}). \tag{21}$$

We recall [11, equations (28), (29)] that, for a primitive idempotent e and an arbitrary element of J,

$$T(x) = 0 \quad \text{and} \quad e \times x = 0 \iff x \in J_{\frac{1}{2}}(e), \tag{22}$$

$$T(x) = 0 \quad \text{and} \quad e \times x = -x \quad \Rightarrow \quad x \in J_0(e). \tag{23}$$

Lemma 2. *If $(B,*)$ is a central simple associative algebras of degree 3 with involution of the second kind, let d be chosen as in Lemma 1 and c as in equation (16). Then*

$$e_1 = \frac{1}{3}(1,1), \quad e_2 = (\frac{1}{3} + c,\ -\frac{1}{6} + c) \quad \text{and} \quad e_3 = (\frac{1}{3} - c,\ -\frac{1}{6} - c) \tag{24}$$

form a frame of $J(B,*,1,1)$.

Proof. Since $e_2 = \frac{1}{2}f + (c,c)$ and $e_3 = \frac{1}{2}f - (c,c)$, we have $T(e_i) = 1$, $e_1 + e_2 + e_3 = 1$ and, by (19) and (21),

$$\begin{aligned} e_2^{\#} &= \frac{1}{4}f^{\#} + \frac{1}{2}f \times (c,c) + (c,c)^{\#} \\ &= \frac{1}{12}(1,1) - (\frac{1}{12}, \frac{1}{12}) = (0,0). \end{aligned}$$

Similarly $e_3^\# = (0,0)$ and we have a system of primitive idempotents. Now $T((c,c)) = T(c) = 0$ and, by (18), $e_1 \times (c,c) = -(c,c)$. So, by (23), $(c,c) \in J_0(e_1)$. Since $f \in J_0(e_1)$, by (24), e_2 and e_3 are also elements of $J_0(e_1)$. Thus $e_1 \perp e_2$, e_3 and since e_2 is a primitive idempotent of $JU_f = J_0(e_1)$, $f - e_2 = e_3 \perp e_2$ and we have a frame of J. □

An Octonion Structure on $H(B,*)_0$

Let $\mathcal{C}$ be an octonion algebra over k with norm n and trace t. Since char $k \neq 2$, $\mathcal{C} = k + \mathcal{C}_0$, $\mathcal{C}_0 = \{a \in \mathcal{C} \mid t(a) = 0\}$, and the multiplication of $\mathcal{C}$ is determined by the product of *pure octonions*, i.e., elements of $\mathcal{C}_0$.

Theorem. *Let $(B,*)$ be a central simple associative algebra of degree 3 with involution of the second kind over a field k of characteristic not 2 or 3. Write $K = k(\theta)$, ($\theta^* = -\theta$, $\theta^2 = \rho$, $\rho \in k$) for the centre of B and let d be a fixed element of $H(B,*)_0$ with $S(d) \neq 0 \neq N(d)$. For arbitrary elements a, $b \in H(B,*)_0 \cap d^{2\perp}$, $d^{2\perp}$ the orthogonal complement of d^2 with respect to T, the product*

$$\begin{aligned} a \cdot b := {} & 18\rho T(a,b)S(d)^2 w \\ & +3\theta(-3[a,\ b] \circ d^2 - 3ad^2b + 3bd^2a + 2T(b,d)[a,\ d] \\ & \quad - 2T(a,\ d)[b,\ d] - 4S(d)[a,\ b] + 2T([a,\ b],d)d + T([a,\ b],d^2)), \end{aligned} \tag{25}$$

where $[a,b] = ab - ba$, $a \circ b = ab + ba$ and w is a formal unit element, endows $H(B,*)_0 \cap d^{2\perp}$ with a pure octonion structure with unit element w and norm form

$$n(a) := 36\rho S(d)^2 S(a).$$

Proof. We will prove this by embedding $H(B,*)_0 \cap d^{2\perp}$ in the reduced Albert algebra $J = J(B,*,1,1)$ and then using the explicit coordinatization obtained in [11]. Recall that this is done by choosing a frame, fixing a Peirce space, say J_{23}, choosing elements $u \in J_{12}$ and $v \in J_{31}$ with $S(u) \neq 0 \neq S(v)$ and defining a product on J_{23} as

$$a \cdot b := (v \times a) \times (u \times b), \quad a, b \in J_{23}. \tag{26}$$

This endows J_{23} with an octonion algebra structure whose norm and unit element are

$$n(a) : = -S(u)S(v)S(a), \quad a \in J_{23} \tag{27}$$

$$1 : = (S(u)S(v))^{-1} u \times v. \tag{28}$$

Let $J = ke_1 + ke_2 + ke_3 + J_{12} + J_{23} + J_{31}$ be the Peirce decomposition of J with respect to the frame e_1, e_2, e_3 as in Lemma 2. By (18) and (12),

$H(B,*)_0$ embeds in $\{(a,a) \mid a \in H(B,*)_0\}$ an 8-dimensional subspace of $J_0(e_1)$. By (19) and (20), for $a \in H(B,*)_0$,

$$f \times (a,a) = (0,0),$$
$$(c,c) \times (a,a) = -T(a,c)(1,1).$$

So, by (22), $\{(a,a) \mid a \in H(B,*)_0, T(a,c) = 0\}$ is a 7-dimensional subspace of J_{23}. For $a \in H(B,*)_0$, $T(a,c) = \frac{1}{2}S(d)^{-1}T(a,d^2)$ and $T(a,c) = 0$ if and only if $T(a,d^2) = 0$. By (18), $e_1 \times (0,\theta) = -(0,\ \theta)$ and $(0,\ \theta) \in J_0(e_1)$. By (19) and (20),

$$\frac{1}{6}(2,-1) \times (0,\theta) = (0,0),$$
$$(c,c) \times (0,\theta) = (\theta c - c\theta,\ -\theta 1 \times c - \theta c) = (0,0)$$

and $(0,\theta) \in J_{23}$. So

$$J_{23} = k(0,\theta) \oplus \{(a,a) \mid a \in H(B,*)_0, T(a,c) = 0\}. \tag{29}$$

Moreover $S((0,\theta)) = T(\theta^2) = 3\rho \neq 0$.

To define an octonion structure on J_{23} we need not determine J_{12} and J_{31} but we only need to find an element of each of these spaces whose quadratic trace is nonzero. For this we will need

$$\begin{aligned} cd &= \frac{1}{2}S(d)^{-1}d^3 + \frac{1}{3}d = \frac{1}{2}S(d)^{-1}(-S(d)d + N(d)) + \frac{1}{3}d \\ &= \frac{1}{2}S(d)^{-1}N(d) - \frac{1}{6}d \end{aligned} \tag{30}$$

and

$$S(c,d) = -T(c,d) = -T(cd) = -\frac{3}{2}S(d)^{-1}N(d). \tag{31}$$

Let $u = (0,\theta d)$. Using (19), (20), (30) and (31), we have $f \times u = -u$, $(c,c) \times u = (0, -\theta(c \times d + cd)) = (0, -\theta(3cd + S(c,d)) = \frac{1}{2}u$. So $e_3 \times u = -u$, $e_2 \times u = 0$ and $u \in J_{12}$. Moreover $S(u) = T(u^\#) = T(\theta^2 d^2) = -2\rho S(d) \neq 0$. Let $v = (2d, -d)$. Again using (19), (20), (30) and (31), we have $f \times v = -v$, $(c,c) \times v = (6cd + 2S(c,d), -3cd - S(c,d)) = -\frac{1}{2}v$. Hence $e_2 \times v = -v$, $e_3 \times v = 0$ and $v \in J_{31}$. Also $S((2d,\ -d)) = T(4d^\# - d^2) = 4S(d) + 2S(d) = 6S(d)$ and $u \times v = 2S(d)(0,\theta)$.

If $x,\ y \in J_{23}$,

$$x \cdot y := ((2d,-d) \times x) \times ((0,\theta d) \times y)$$

defines an octonion stucture on J_{23} with norm form

$$n(x) := 12\rho S(d)^2 S(x),$$

and unit element $(S(u)S(v))^{-1}u \times v = (-2\rho S(d)6S(d))^{-1}2S(d)(0,\theta) = -\frac{1}{6}\rho^{-1}S(d)^{-1}(0,\theta)$. By (13) and (5′), for $a \in H(B,*)_0$, $S((a,a)) = 3S(a)$. So for $a \in H(B,*)_0 \cap d^{2\perp}$,

$$n(a) = 36\rho S(d)^2 S(a). \tag{27′}$$

Using (10′), $(0,\theta 1) \times (a,a) = (0,0)$ and the decomposition in (29) is orthogonal with respect to S and hence with respect to n. Since the octonion unit element is a scalar multiple of $(0,\theta 1)$, the pure octonions, i.e., the octonions of trace 0, correspond to $H(B,*)_0 \cap d^{2\perp}$.

For $a,\ b \in H(B,*)_0 \cap d^{2\perp}$,

$$\begin{aligned}(2d,\ -d) \times (a,\ a) &= (2a \times d + da + ad,\ -a \times d - 2da + ad)\\ &= (3a \circ d - 2T(a,\ d),\ T(a,\ d) - 3da).\end{aligned}$$

while

$$\begin{aligned}(0,\ \theta d) \times (b,\ b) &= (-\theta db + b\theta d,\ -\theta b \times d - b\theta d)\\ &= (\theta[b,\ d],\ \theta(T(b,d) - 2bd - db))\end{aligned}$$

If we cross the two expressions above, we get

$$\begin{aligned}&((2d,\ -d) \times (a,\ a)) \times ((0,\ \theta d) \times (b,\ b))\\ &\quad = \theta((3a \circ d - 2T(a,d)) \times [b,\ d] + (T(a,d) - 3da)(T(b,d) - 2db - bd)\\ &\qquad - (T(b,d) - 2bd - db)(T(a,d) - 3ad),\\ &\qquad - (T(a,d) - 3ad) \times (T(b,d) - 2db - bd)\\ &\qquad - (3a \circ d - 2T(a,d))(T(b,d) - 2bd - db) - [b,\ d](T(a,d) - 3da)).\end{aligned} \tag{32}$$

Expanding the first component of (32), we get

$$\begin{aligned}&3\theta((a \circ d) \circ [b,\ d] + d[a,\ b]d + 2[dad,\ b]\\ &\qquad\qquad + T(b,d)[a,\ d] - T(a,d)[b,\ d] - T(a \circ d, [b,d]))\\ &\quad = 3\theta([a,\ dbd] - [b,\ dad] + 2d[a,\ b]d - ad^2b + bd^2a\\ &\qquad\qquad + T(b,d)[a,\ d] - T(a,d)[b,\ d] - T([a,\ b], d^2)),\end{aligned}$$

since the trace is invariant under cyclic permutations. In a Jordan algebra of degree 3,

$$xzx = zU_x = T(z,x)x - x^{\#} \times z.$$

For an element z of trace 0,

$$\begin{aligned}dzd &= T(z,d)d - d^{\#} \circ z + S(d)z - S(z,d^{\#}),\\ &= -z \circ d^2 - S(d)z + T(z,d)d + T(z,d^2).\end{aligned} \tag{33}$$

Using this, the above expression can be transformed into

$$3\theta(-3[a,\ b]\circ d^2 - 3ad^2b + 3bd^2a + 2T(b,d)[a,\ d] \\ - 2T(a,\ d)[b,\ d] - 4S(d)[a,\ b] + 2T([a,\ b],d)d + T([a,\ b],d^2)). \quad (32')$$

This is the pure octonion component of the product. We could obtain the appropriate multiple of the unit element by considering the second component of (32) but we choose to use the norm form. In an octonion algebra $x = \frac{1}{2}t(x)1 + x_0 = \frac{1}{2}n(x,1)1 + x_0$, where x_0 is of trace 0. By (27′), $n(ab,1) = n(a,\bar{b}) = -n(a,b) = -36\rho S(d)^2 S(a,b)$ and the coefficient of the unit element in the product of a, b is $-18\rho S(d)^2 S(a,b)$. □

Corollary. *Up to isomorphism, the octonion algebra structure on $H(B,*)$ obtained above is independent of the choice of d. Moreover if K is split then so is the corresponding octonion algebra.*

Proof. Choosing a different d may yield a different frame on $J(B,*,1,1)$. However the coordinate algebras relative to different frames are isomorphic by the Albert-Jacobson Theorem [5, chapter IX, Theorem 1].

If K is split then $B = A \oplus A^{op}$ and $J(B,*,1,1)$ is a first Tits construction [7, Theorem 3.5] which must be split since it is reduced [4, chapter IX, Theorem 20]. □

Remarks. 1. The converse of the last statement of the Corollary is false since a second Tits construction can also be a first Tits construction [8, Theorem 5.2].
2. As pointed out by Faulkner [2], the octonion algebra arising from his construction is split. Since, in our case, it corresponds to K split, so is the octonion algebra of our theorem. □
3. The isometry class of the norm form obtained above corresponds to the mod 2 invariant $f_3(J(B,*,1,1))$ of $J(B,*,1,1)$ [5, section 40; 9].

Let us finally recall the construction of Haile, Knus, Rost and Tignol [3, 5 section 19. B]. Denote by $<< \alpha_1, \alpha_2, \ldots, \alpha_n >>$ the n-fold Pfister form

$$< 1, -\alpha_1 > \otimes < 1, -\alpha_2 > \otimes \cdots \otimes < 1, -\alpha_n > .$$

On $H(B,*)$ as above, the quadratic form

$$Q_*(x) := T(x^2) \simeq < 1,1,1 > \perp < 2 > \cdot << \rho >> \cdot q_*,$$

where q_* is a 3-dimensional quadratic form of determinant 1. Since q_* is of determinant 1, the form $\pi_* = << \rho >> \cdot q_* \perp << \rho >>$ is a 3-fold Pfister form which determines the involution * of B up to conjugacy [5, Theorem 19.6] and yields a cohomological invariant $f_3(B,*)$ [5, Theorem 30.21].

Proposition. *Let* $(B, *)$ *be a central simple associative algebra of degree 3 over* K *with involution of the second kind,* $K = k(\theta)$*,* $\theta^* = -\theta$*,* $\theta^2 = \rho$*,* k *a field of characteristic not 2 or 3 and* n *the unique octonion norm form corresponding to* $(B, *)$*. The 3-fold Pfister forms* π_* *and* n *are isometric and hence the corresponding octonion algebras are isomorphic.*

Proof. By [5, Theorem 40.2, (2)], the invariant $f_3(J(B, *, 1, 1))$ is the f_3-invariant of the involution $*$ of B. □

Remark. Since the f_3-invariant of the involution $*$ of B is an invariant of $(B, *)$ this provides another proof of the Corollary.

References

1. Elduque, A. and Pérez, J.M., *Composition algebras with associative bilinear forms*, Comm. Algebra **24** (1996), 1091–1116.
2. Faulkner, J., *Finding octonion algebras in associative algebras*, Proc. Amer. Math. Soc. **104** (1988), 1027–1030.
3. Haile, D., Knus, M.-A., Rost, M. and Tignol, J.-P., *Algebras of odd degree with involution, trace forms and dihedral extensions*, Israel J. Math. **184** (1996), 299–340.
4. Jacobson, N., Structure and Representations of Jordan Algebras, AMS Colloquium Publications Vol. **39**, 1968, Providence R. I.
5. Knus, M.-A., Merkurjev, A., Rost, M. and Tignol, J.-P., *The Book of Involutions*, American Mathematical Society, Providence R. I., 1998, American Mathematical Society Colloquium Publications Vol. **44**.
6. Okubo, S., *Pseudo-quaternion and pseudo-octonion algebras*, Hadronic J. **1** (1978), 1250–1278.
7. Petersson, H.P., and Racine, M.L., *Jordan algebras of degree 3 and the Tits process*, J. Algebra, **98** (1986), 211–243.
8. Petersson, H.P., and Racine, M.L., *Classification of algebras arising from the Tits process*, J. Algebra, **98** (1986), 244–279.
9. Petersson, H.P., and Racine, M.L., *On the invariants mod 2 of Albert algebras*, *Journal of Algebra* **174** (1995) 1049–1072.
10. Petersson, H.P., Racine, M.L., *Reduced models of Albert algebras*, Math. Zeitschrift **223** (1996) 367–385.
11. Racine, M.L., *A note on quadratic Jordan algebras of degree 3*, Trans. Amer. Math. Soc., **164** (1972), 93–103.