INVERTIBLE AND NILPOTENT ELEMENTS IN THE GROUP ALGEBRA OF A UNIQUE PRODUCT GROUP

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Abstract. We describe the nilpotent and invertible elements in group algebras $k[G]$ for k a commutative associative unital ring and G a unique product group, for example an ordered group.

Introduction. A fundamental problem in the theory of group algebras is to determine their units = invertible elements. The reader can find a short introduction to this question in [3, §6] and a much more substantial one in [6, Ch. 13] and [7, Ch. II and VI]. Since $1 - a$ is invertible for any nilpotent element a, a closely related problem is that of describing all nilpotent elements.

In this short note we give a description of the nilpotent and invertible elements in group algebras $k[G]$ where k is an arbitrary commutative associative unital ring and G is a unique product group, e.g. an ordered group (Cor. **5.**). Our result is well-known in case k is an integral domain: If G is a unique product group, 0 is the only nilpotent element and all units are trivial. So the main point here is the generality of k.

Our approach uses a little bit of algebraic geometry and might possibly also be of interest to solve other problems related to group algebras. It is inspired by a recent result of Ottmar Loos in [4], where he determines the invertible elements in a Laurent polynomial ring $k[t^{\pm 1}]$. In Th. **3** we describe the nilpotent and invertible elements in $k[G]$ under the assumption that for all k-algebras K which are fields the group algebra $K[G]$ is a domain or, respectively, has only the trivial units. The case of group algebras $k[G]$ for G a unique product group is then an immediate corollary.

A different characterization of the units in $k[G]$ for G a right-ordered and thus unique product group is proven in [5].

1. Notation. Throughout we use the following notation: k is a commutative associative unital ring, $\mathrm{Spec}(k)$ is the prime spectrum of k equipped with the Zariski topology, $\kappa(\mathfrak{p})$ is the quotient field of $k/\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(k)$, $x(\mathfrak{p})$ is the canonical image of $x \in k$ in $\kappa(\mathfrak{p})$ and k-**alg** is the category of associative commutative and unital k-algebras. The invertible elements of an associative unital k-algebra A are denoted $A^\times$.

Let G be a group, written multiplicatively and let $A = k[G]$ be the group algebra of G over k. Thus A is a free k-module with k-basis $(u_g : g \in G)$ in bijection with

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G by $g \mapsto u_g$ and the multiplication of the k-algebra A is determined by the rule $u_g u_h = u_{gh}$ for $g, h \in G$. It is immediate that any element xu_g, $x \in k^\times$ is invertible. These are the so-called *trivial units* of A.

We endow G with the discrete topology. Recall the definition of the constant group scheme $\mathbf{G}$ associated to G [2, II, §1, no. 2.12]: For $R \in k$-**alg**, $\mathbf{G}(R)$ is the set of continuous (= locally constant) maps $\mathfrak{d} : \mathrm{Spec}(R) \to G$ with the group structure inherited from G. In particular, this applies to $R = k$.

Since $\mathrm{Spec}(k)$ is quasi-compact [1, II, §4.3, Prop. 12], it follows from [1, II, §4.3, Prop. 15] that there exists a bijection $\mathcal{I} : \mathbf{G}(k) \to \mathcal{E}$ from $\mathbf{G}(k)$ to the set $\mathcal{E}$ of all families $\varepsilon = (\varepsilon_g)_{g \in G}$ of orthogonal idempotents in k with $\varepsilon_g \neq 0$ for only finitely many $g \in G$ and $\sum_{g \in G} \varepsilon_g = 1_k$. The bijection $\mathfrak{d} \mapsto \mathcal{I}(\mathfrak{d}) = (\varepsilon_g)_{g \in G}$ is given by the relations

$$\mathfrak{d}(\mathfrak{p}) = g \iff \varepsilon_g \notin \mathfrak{p} \iff \mathfrak{p} \in (\mathrm{Spec}(k))_{\varepsilon_g} \iff \varepsilon_g(\mathfrak{p}) = 1_{\kappa(\mathfrak{p})} \tag{1}$$

where, for $x \in k$, $(\mathrm{Spec}(k))_x$ denotes the basic open subset of all $\mathfrak{p} \in \mathrm{Spec}(k)$ with $x \notin \mathfrak{p}$. We will usually view $\mathcal{I}$ as an identification. The product of $\varepsilon = (\varepsilon_g)$ and $\varepsilon' = (\varepsilon'_g)$ in the group $\mathbf{G}(k)$ is then given by the formula

$$(\varepsilon \cdot \varepsilon')_x = \textstyle\sum_{gh=x} \varepsilon_g \varepsilon'_h, \quad (x \in G) \tag{2}$$

Indeed, a locally constant function $\mathfrak{d} : \mathrm{Spec}(k) \to k$ gives rise to a partition of $\mathrm{Spec}(k)$ by basic open sets $\mathrm{Spec}(k\varepsilon_g) = \mathrm{Spec}(k)_{\varepsilon_g}$, where $\varepsilon = (\varepsilon_g)$ is the complete orthogonal system corresponding to $\mathfrak{d}$ and where $\mathrm{Spec}(k)_x$ is canonically identified with a subset of $\mathrm{Spec}(k)$. Given two locally constant functions $\mathfrak{d}$ and $\mathfrak{d}'$ with corresponding orthogonal systems $\varepsilon = \mathcal{I}(\mathfrak{d})$ and $\varepsilon' = \mathcal{I}(\mathfrak{d}')$ we get a partition of $\mathrm{Spec}(k)$ by open sets

$$\big(\mathrm{Spec}(k)\big)_{\varepsilon_g} \cap \big(\mathrm{Spec}(k)\big)_{\varepsilon'_h} = \big(\mathrm{Spec}(k)\big)_{\varepsilon_g \varepsilon'_h} = \mathrm{Spec}(k\varepsilon_g\varepsilon'_h)$$

on which the function $\mathfrak{d}\mathfrak{d}'$ has the value gh. Hence $\mathfrak{d}\mathfrak{d}'$ has value $x \in G$ precisely on

$$\textstyle\bigcup_{gh=x} \mathrm{Spec}(k\varepsilon_g\varepsilon'_h) = \mathrm{Spec}\,\big(k(\sum_{gh=x} \varepsilon_g\varepsilon'_h)\big).$$

In terms of the ε's, the unit element of $\mathbf{G}(k)$ is the family $\varepsilon^{(0)} = (\varepsilon_g^{(0)})$ with

$$\varepsilon_g^{(0)} = \begin{cases} 1_k, & g = 1_G, \\ 0, & g \neq 1_G. \end{cases}$$

and the inverse of $\varepsilon = (\varepsilon_g)_{g \in G}$ is $\varepsilon^{-1} = (\varepsilon_g^{-1})_{g \in G}$ with $\varepsilon_g^{-1} = \varepsilon_{g^{-1}}$.

Let now $A = k[G]$ be the group algebra of G. We then have a group monomorphism

$$\mathbf{G}(k) \to k[G]^\times, \quad \mathfrak{d} \mapsto u_{\mathfrak{d}} := \textstyle\sum_{g \in G} \varepsilon_g u_g, \text{ for } \varepsilon = \mathcal{I}(\mathfrak{d}).$$

Indeed, it follows from (2) that $u_{\mathfrak{d}} u_{\mathfrak{d}'} = u_{\mathfrak{d}\mathfrak{d}'}$ for all $\mathfrak{d}, \mathfrak{d}' \in \mathbf{G}(k)$.

We recall that a *nil ideal* of an associative algebra A is an ideal consisting of nilpotent elements. By definition [3, 10.26], the *upper nil radical* of an associative algebra A is the sum $\mathrm{Nil}^*(A)$ of all nil ideals of A, equivalently, $\mathrm{Nil}^*(A)$ is the biggest nil ideal of A. If A is also commutative, $\mathrm{Nil}^*(A) = \{a \in A : a \text{ nilpotent}\} = \mathrm{Nil}(A)$, the *nil radical* of A.

2. Theorem. (a) *Assume $K[G]$ is a domain for every field $K \in k$-**alg**. Then the upper nil radical of $k[G]$ is*

$$\mathrm{Nil}^*(k[G]) = \big\{ \textstyle\sum_{g\in G} n_g u_g : n_g \in \mathrm{Nil}(k) \textit{ for every } g \in G \big\} \cong \big(\mathrm{Nil}(k)\big)[G]. \qquad (3)$$

It coincides with the set of nilpotent elements of $k[G]$.

(b) *Suppose that $K[G]$ has only trivial units whenever $K \in k$-**alg** is a field. Then an element $a \in k[G]$ is invertible if and only if there exists $\mathfrak{d} \in \mathbf{G}(k)$, a unit $v \in k^\times$ and an element $n \in \mathrm{Nil}^*(k[G])$ such that*

$$a = v\, u_{\mathfrak{d}} + n; \qquad (v \in k^\times,\ \mathfrak{d} \in \mathbf{G}(k),\ n \in \mathrm{Nil}^*(k[G])). \qquad (4)$$

The element $\mathfrak{d}$ is uniquely determined by a, called the degree *of a and the map*

$$\deg : k[G]^\times \to \mathbf{G}(k), \quad \deg(v\, u_{\mathfrak{d}} + n) = \mathfrak{d}$$

is a group homomorphism.

Proof. (a) We abbreviate $A = k[G]$. It is easily seen that $\mathfrak{N} := \big\{ \sum_{g\in G} n_g u_g : n_g \in \mathrm{Nil}(k) \text{ for every } g \in G \big\}$ is an ideal of A consisting of nilpotent elements. Indeed, as an element of A, an $n \in \mathfrak{N}$ has only finitely many non-zero components, say $n_1 u_{g_1}, \dots, n_p u_{g_p}$. Hence there exists $q \in \mathbb{N}$ such that $n_i^q = 0$ for all $1 \le i \le p$. Then n^{pq} is a sum of terms $n_1^{r_1} \cdots n_p^{r_p} u_g$ where g is an appropriate product of pq factors taken from the $g_1, \dots, g_r$ and where at least one $r_i \ge q$. Thus $n^{pq} = 0$. Hence $\mathfrak{N} \subset \mathrm{Nil}^*(A)$ (observe that this holds in general).

To finish the proof of (a), it is now sufficient to show $n \in \mathfrak{N}$ for every nilpotent element n of A. We write $n = \sum_{g\in G} n_g u_g$ with $n_g \in k$ and let $\mathfrak{p} \in \mathrm{Spec}(k)$. The element $n(\mathfrak{p}) \in A \otimes_k \kappa(\mathfrak{p}) \cong \big(\kappa(\mathfrak{p})\big)[G]$ is then nilpotent too. But since by assumption $\kappa(\mathfrak{p})[G]$ is a domain, it follows that $n(\mathfrak{p}) = 0$, i.e., $n_g(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(k)$ and all $g \in G$. Thus, every n_g is nilpotent and $n \in \mathfrak{N}$.

(b) We will first show that any element of the form (4) is invertible. This is clear for vu_ε, so that it suffices to prove invertibility of $(vu_\varepsilon)^{-1} a = 1 + v^{-1} u_\delta n$ for $\delta = \varepsilon^{-1}$. But this is clear since $v^{-1} u_\delta n \in \mathfrak{N}$ is nilpotent.

Conversely, suppose that $a \in k[G]$ is invertible. If k is a field, a has the form $a = vu_g$ for some $v \in k^\times$ by assumption, which is a special case of (4).

Let now k be arbitrary. We write $a = \sum_{g\in G} a_g u_g$ with $a_g \in k$. Let $\mathfrak{p} \in \mathrm{Spec}(k)$. Then there exits a unique $g \in G$ such that $a(\mathfrak{p}) = a_g(\mathfrak{p}) u_g \neq 0$. This gives rise to a map $\mathfrak{d} : \mathrm{Spec}(k) \to G$ which, we claim, is locally constant. Indeed, if $\mathfrak{d}(\mathfrak{p}_0) = g$ then $a_g(\mathfrak{p}) \neq 0$ and hence $a_g(\mathfrak{p}) \neq 0$ for all $\mathfrak{p}$ in the basic open neighborhood $U = (\mathrm{Spec}(k))_x$, $x = a_g$. Since then $a_h(\mathfrak{p}) = 0$ for all $h \neq g$ and $\mathfrak{p} \in U$, we see that $\mathfrak{d}$ is constant equal to g on U. Thus $\mathfrak{d} \in \mathbf{G}(k)$.

Let $\varepsilon = (\varepsilon_g)_{g\in G}$ be the family corresponding to $\mathfrak{d}$. Then $\big(a_g(1-\varepsilon_g)\big)(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(k)$. Indeed, if $\mathfrak{d}(\mathfrak{p}) = g$ then $(1-\varepsilon_g)(\mathfrak{p}) = 1_{\kappa(\mathfrak{p})} - 1_{\kappa(\mathfrak{p})} = 0$ by (1), while if $\mathfrak{d}(\mathfrak{p}) \neq g$ then $a_g(\mathfrak{p}) = 0$ by definition of $\mathfrak{d}$. Hence $n_g = a_g(1-\varepsilon_g) \in k$ is nilpotent. Also $v = \sum_{g\in G} a_g \varepsilon_g \in k^\times$ since, for any $\mathfrak{p} \in \mathrm{Spec}(k)$, $v(\mathfrak{p}) = \sum_{g\in G} a_g(\mathfrak{p}) \varepsilon_g(\mathfrak{p}) = a_{\mathfrak{d}(\mathfrak{p})}(\mathfrak{p}) \neq 0$. Thus,

$$a = \textstyle\sum_{g\in G} a_g \varepsilon_g u_g \ + \ \sum_{g\in G} a_g(1-\varepsilon_g) u_g = \big(\sum_{g\in G} a_g\big) \big(\sum_{h\in G} \varepsilon_h u_h\big) + n$$

as required in (4).

Uniqueness of $\mathfrak{d}$, i.e. of ε, is clear from the construction above. For the proof of the last claim, let $a' = v' u_{\varepsilon'} + n'$ be another invertible element of A. Then

$aa' = vv'u_{\varepsilon\varepsilon'} + b$ where $b = vu_\varepsilon n' + v'u_{\varepsilon'}n' + nn' \in \mathfrak{N}$. Thus aa' has degree $\varepsilon\varepsilon' = \mathcal{I}(\mathfrak{d}\mathfrak{d}')$ proving that deg is a homomorphism.

3. Example. Suppose k is reduced (= semiprime) and that $K[G]$ has only trivial units for any field $K \in k$-**alg**. Then Th. **2** says that for any unit $a \in k[G]$ there exists a decomposition of k into a finite direct sum of ideals I such that a decomposes into trivial units in each $I[G]$.

4. Unique product groups. It is well-known that the assumptions in (a) and (b) of Th. **2** are fulfilled for ordered groups, see for example [3, Th. 6.29]. However, they are also fulfilled for the much more general class of so-called unique product groups.

Recall [6] that a group G is called a *unique product group*, abbreviated u.p. group, if, given any two finite non-empty subsets A, B of G, there is an element of AB that can be uniquely written in the form ab with $a \in A$ and $b \in B$. It follows immediately [6, 13, Lemma 1.9(i)] that if R is an integral domain and G is a u.p. group then $R[G]$ is a domain. Furthermore, it is also known [6, Appendix, Th. 15] that a u.p. group G is the same as a *two unique products group*: if $A, B \subset G$ are finite non-empty subsets, not both singletons, there are at least two elements in AB which are uniquely represented. It then follows [6, 13, Lemma 1.9(ii)] that any unit in $R[G]$, R an integral domain, is trivial. To summarize:

5. Corollary. *If G is a u.p. group, e.g. an ordered group, the assumptions in* (a) *and* (b) *are fulfilled. Hence* (3) *and* (4) *describe the nilpotent and invertible elements of the group algebra $k[G]$.*

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References

[1] Nicolas Bourbaki, *Éléments de mathématique. Fascicule XXVII. Algèbre commutative. Chapitre 1: Modules plats. Chapitre 2: Localisation*, Actualités Scientifiques et Industrielles, No. 1290, Herman, Paris, 1961.

[2] Michel Demazure and Pierre Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970, Avec un appendice ıt Corps de classes local par Michiel Hazewinkel.

[3] T. Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 1991.

[4] Ottmar Loos, *Remarks on Holger P. Petersson's paper "Idempotent 2-by-2 matrices"*, Mathematikberichte, Universität Hagen **78** (2007), see also Jordan Theory Preprint Archive, paper 243.

[5] M. M. Parmenter, *Units and isomorphism in group rings*, Quaestiones Math. **8** (1985), no. 1, 9–14.

[6] Donald S. Passman, *The algebraic structure of group rings*, Robert E. Krieger Publishing Co. Inc., Melbourne, FL, 1985, Reprint of the 1977 original.

[7] Sudarshan K. Sehgal, *Topics in group rings*, Monographs and Textbooks in Pure and Applied Math., vol. 50, Marcel Dekker Inc., New York, 1978.

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