NONASSOCIATIVE CYCLIC ALGEBRAS AND THE SEMIASSOCIATIVE BRAUER MONOID

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Abstract. We look at classes of semiassociative algebras, with an emphasis on those that are canonical generalizations of associative (generalized) cyclic algebras. We investigate their behaviour in the semiassociative Brauer monoid defined by Blachar, Haile, Matri, Rein, and Vishne. A possible generalization of this monoid in characteristic p that includes nonassociative differential algebras is briefly considered.

Introduction

Recently, semiassociative algebras and their semiassociative Brauer monoid were introduced by Blachar, Haile, Matri, Rein, and Vishne as a canonical generalization of associative central simple algebras and their Brauer group [4]. Semiassociative algebras A over a field F are F-central and are characterized by having an étale algebra E contained in their nucleus, such that A is cyclic and faithful as an $E \otimes_F E$-module via the action $(e \otimes e')a = eae'$ for all $a \in A$, $e, e' \in E$. This definition makes it possible to use classical Brauer factor sets [12, Chapter 2] when developing the theory, and guarantees that the algebras are forms of skew matrix algebras, which are defined and investigated in depth [4].

Together with the tensor product, equivalence classes of semiassociative algebras over F form a monoid denoted $Br^{sa}(F)$ that contains the classical Brauer group as the unique maximal subgroup. The skew matrix algebras play the role of classical matrices in the Brauer group. In particular, a semiassociative algebra is called split if it is isomorphic to a skew matrix algebra. The authors state that "the key example for semiassociative algebras are skew matrices" [4].

In this paper we will look at another important example of semiassociative algebras; the nonassociative (generalized) cyclic algebras (and their opposite algebras). It is well known that (generalized) cyclic algebras play a prominent role in the structure theory of classical central simple algebras. Now we look at the role nonassociative (generalized) cyclic algebras play in the structure theory of semiassociative simple algebras. These algebras are canonical generalizations of associative cyclic algebras (respectively, of the generalized associative cyclic algebras introduced by Jacobson [12]) over F.

If F has a cyclic Galois field extension of degree n, then there exist nonassociative (generalized) cyclic algebras, and these are semiassociative algebras of degree n that have this

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cyclic field extension as their nucleus (respectively, a central simple algebra in their nucleus with center a separable field extension of F). These semiassociative algebras are not semicentral, and thus in particular not homogeneous. They have infinite order in $Br^{sa}(F)$, even when they are not division algebras. Nonassociative (generalized) cyclic algebras also show that the splitting behaviour of an algebra in $Br^{sa}(F)$ (i.e., whether or not it will be split or split under a field extension) only depends on its nucleus.

Cyclic algebras $(K/F, \sigma, d)$ of degree n that are not associative are division algebras in many cases, e.g. for all prime n. Some of the tensor products we consider yield examples of semiassociative algebras of degree mn which again are division algebras, and are obvious generalizations of the classical generalized cyclic algebras.

The nonassociative (generalized) cyclic algebras we present here already appeared in space-time block coding [27, 26, 19, 21], and in (f, σ, δ)-codes [23]. We now generalize their definition which previously usually employed skew polynomials in $D[t; \sigma]$ over division rings D, and define them in a more general setting, dropping the assumption that D has no zero divisors. Generalized Menichetti algebras [16, 28] can be seen as generalizations of both crossed products and nonassociative cyclic algebras, and make up the second class of semiassociative algebras we present.

We finish by suggesting possible generalizations of the semiassociative Brauer monoid, which allow us to include nonassociative (generalized) differential extensions as classes of algebras in the monoid, if the characteristic of F is prime.

The structure of the paper is as follows: we collect the basic results needed in Section 1. In Section 2, we generalize the definition of nonassociative cyclic algebras $(K/F, \sigma, d)$ to allow for the case that K/F is an étale extension, and the definition of generalized nonassociative cyclic algebras (D, σ, d) to include the case that the algebra D employed in the construction with the skew polynomial $t^m - d \in D[t; \sigma]$ has zero divisors, collecting and generalizing several previously achieved results. In Section 3, we look at the tensor product of a central simple algebra and a nonassociative cyclic algebra. We investigate the behaviour of nonassociative (generalized) cyclic algebras in the Brauer monoid $Br^{sa}(F)$ in Section 4, and briefly look at $Br^{sa}(\mathbb{R})$ and $Br^{sa}(\mathbb{F}_q)$.

When F is a field of prime characteristic p, the definition of $Br^{sa}(F)$ may benefit from a generalization that includes a class of algebras that generalize algebras that are associative differential extensions [12]. All associative central division algebras over a field F of characteristic zero can be constructed using differential polynomials (Amitsur [2], and later [11], [12, Sections 1.5, 1.8, 1.9]). The construction method is an analogue to the well-known crossed product construction, except that instead of their algebraic splitting fields it uses splitting fields K of the algebras, where the field F is algebraically closed in K. For p-algebras over base fields of characteristic $p > 0$ the construction employs differential polynomial rings $D[t; \delta]$ (where D is a division algebra over F), factoring out a two-sided ideal generated by $f \in D[t; \delta]$. This construction was generalized to the nonassociative setting in [24]. We briefly consider these algebras in characteristic p and the challenges to include them in potential generalizations of $Br^{sa}(F)$ in the last two sections.

1. Preliminaries

1.1. **Nonassociative algebras.** Let F be a field. An F-vector space A is an *algebra* over F, if there exists an F-bilinear map $A\times A\to A$, $(x,y)\mapsto x\cdot y$, denoted simply by juxtaposition xy, the *multiplication* of A. An algebra A is called *unital* if there is an element in A, denoted by 1, such that $1x=x1=x$ for all $x\in A$. We only consider unital algebras. The *center* of A is $\mathrm{C}(A)=\{x\in \mathrm{Nuc}(A)\,|\,xy=yx \text{ for all } y\in A\}$ and A is called *(F-)central* if it has center F.

Associativity in A is measured by the *left nucleus* $\mathrm{Nuc}_l(A)=\{x\in A\,|\,[x,A,A]=0\}$, the *middle nucleus* $\mathrm{Nuc}_m(A)=\{x\in A\,|\,[A,x,A]=0\}$ and the *right nucleus* $\mathrm{Nuc}_r(A)=\{x\in A\,|\,[A,A,x]=0\}$ of A, where $[x,y,z]=(xy)z-x(yz)$ is the *associator*. $\mathrm{Nuc}_l(A)$, $\mathrm{Nuc}_m(A)$, and $\mathrm{Nuc}_r(A)$ are associative subalgebras of A, and their intersection $\mathrm{Nuc}(A)=\{x\in A\,|\,[x,A,A]=[A,x,A]=[A,A,x]=0\}$ is the *nucleus* of A. $\mathrm{Nuc}(A)$ is an associative subalgebra of A containing $F1$ and $x(yz)=(xy)z$ whenever one of the elements x,y,z is in $\mathrm{Nuc}(A)$. Multiplication on both sides make A into a bimodule over its nucleus. Moreover, for every subalgebra N of the nucleus the N-bimodule structure of A can be viewed as a left module structure over the ring $N^e=N\otimes_F N^{op}$.

An algebra $A\neq 0$ is called a *division algebra* if for any $a\in A$, $a\neq 0$, the left multiplication with a, $L_a(x)=ax$, and the right multiplication with a, $R_a(x)=xa$, are bijective. If A has finite dimension over F, A is a division algebra if and only if A has no zero divisors [30, pp. 15, 16].

An *étale* algebra over F is a finite direct product of finite separable field extensions of F.

1.2. **Semiassociative algebras.** (cf. [4])

A finite dimensional nonassociative F-central algebra A is called *semiassociative* if its nucleus has an étale F-subalgebra E, such that A is cyclic and faithful over $E\otimes_F E$ via the action $(e\otimes e')a=eae'$ for all $a\in A$, $e,e'\in E$. The dimension of a semiassociative algebra A is a square [4, Corollary 3.4] and the root of the dimension of A is called the *degree* of A. If A is semiassociative of degree n, then any n-dimensional étale subalgebra E of $\mathrm{Nuc}(A)$ is a maximal commutative subalgebra of A [4, Corollary 7.3].

If A is a nonassociative algebra containing an étale subalgebra E in its nucleus, then any two of the following conditions imply the third: A is faithful over $E\otimes E^{\mathrm{op}}$, A is cyclic over $E\otimes E^{\mathrm{op}}$, and $dimA=(dimE)^2$ [4, Remark 3.3].

Every associative central simple algebra of degree n has a maximal étale subalgebra E of dimension n and is semiassociative. We call A *E-semiassociative* if E is an étale F-subalgebra of its nucleus, such that A is cyclic and faithful over $E^e=E\otimes_F E$. The nucleus of a nonassociative algebra may contain more than one étale subalgebra of the same dimension. However, if A is a semiassociative algebra with respect to one étale subalgebra of its nucleus, then it is semiassociative with respect to all étale subalgebras of its nucleus [4, Proposition 3.6]. A scalar extension of a semiassociative algebra is semiassociative [4, Proposition 12.1].

A tensor c_{ijk} of degree n of $n\times n\times n$ scalars in F is called a *skew set c of degree n*. A skew set c is called *reduced* if $c_{iij}=c_{jii}=1$ for all i,j. Let c be such a reduced skew set.

Then the *skew matrix algebra* $M_n(F;c)$ is the F-vector space with basis the matrix units e_{ij} and multiplication given by $e_{ij}e_{kl}=\delta_{jk}c_{ijl}e_{il}$. Note that $M_n(F)=M_n(F;1)$.

A semiassociative algebra A is called *split*, if it is a skew matrix algebra. The split nonassociative quaternion algebra from [33] is the skew matrix algebra of degree 2 mentioned in [4, Example 6.12 (3)]. A field extension K/F *splits* a semiassociative algebra A, if $A_K=A\otimes_F K$ is split. A semiassociative algebra of degree n is split if and only if F^n is a unital subalgebra of the nucleus [4, Proposition 7.2]. Let A be an E-semiassociative algebra with $E=\mathrm{Nuc}(A)$. Then a field extension K/F splits A if and only if it splits E [4, Corollary 7.5]. If K is a field that splits an étale subalgebra in the nucleus of an n-dimensional semiassociative algebra A of degree n, and F is an infinite field, then K splits A [4, Theorem 7.1].

If A is semiassociative of degree n, then any n-dimensional étale subalgebra E of $\mathrm{Nuc}(A)$ is a maximal commutative subalgebra of A [4, Corollary 7.3].

Let $J(\mathrm{Nuc}(A))$ denote the radical of the associative algebra $\mathrm{Nuc}(A)$. For a semiassociative algebra A, the simple components of the semisimple quotient $\sigma(A)=\mathrm{Nuc}(A)/J(\mathrm{Nuc}(A))$ are called the *atoms* of A. A semiassociative algebra over F is called *semicentral*, if all of its atoms are F-central [4, Definition 16.1]. A semiassociative algebra is *homogeneous* if it is semicentral, and the atoms are all Brauer equivalent to each other [4, Definition 17.2].

Two semiassociative algebras A and B over F are called *Brauer equivalent*, if there exist skew matrix algebras $M_n(F;c)$ and $M_m(F,c')$ such that $A\otimes_F M_n(F;c)\cong B\otimes_F M_m(F;c')$. The *semiassociative Brauer monoid* $Br^{sa}(F)$ is the set of equivalence classes with respect to Brauer equivalence, with product $[A]^{sa}[B]^{sa}=[A\otimes_F B]^{sa}$ and unit element $[F]^{sa}$. If A is a homogeneous semiassociative algebra, and D the (associative) underlying division algebra of its atoms, then there is a decomposition $A\cong D\otimes_F M$, where M is a skew matrix algebra and D is the unique member of minimal degree in the class $[D]^{sa}\in Br^{sa}(F)$ [4, Proposition 18.2, Corollary 18.3].

1.3. **Nonassociative algebras obtained from skew polynomial rings.** (for details, cf. [23])

Let S be a unital associative noncommutative ring, $\sigma\in\mathrm{Aut}(S)$, and $\delta:S\to S$ a σ-*derivation*, i.e. an additive map such that $\delta(ab)=\sigma(a)\delta(b)+\delta(a)b$ for all $a,b\in S$. The *skew polynomial ring* $R=S[t;\sigma,\delta]$ is the set of skew polynomials $a_0+a_1t+\cdots+a_nt^n$ with $a_i\in S$, where addition is defined term-wise and multiplication by $ta=\sigma(a)t+\delta(a)$ for all $a\in S$. For $\sigma=id$ and $\delta=0$, this is the ring of left polynomials $S[t]=S[t;id,0]$.

For $f(t)=a_0+a_1t+\cdots+a_nt^n$ with $a_n\neq 0$ define $\deg(f)=n$ and $\deg(0)=-\infty$. Then $\deg(gh)\leq\deg(g)+\deg(h)$ for $f,g\in S[t]$ (with equality if h or g have an invertible leading coefficient, or if S is a division ring). An element $f\in R$ is *irreducible* in R if it is not a unit and it has no proper factors, i.e if there do not exist $g,h\in R$ with $\deg(g),\deg(h)<\deg(f)$ such that $f=gh$. We call $f\in R$ a *(right) semi-invariant* polynomial if for every $a\in D$ there is $b\in D$ such that $f(t)a=bf(t)$. If also $f(t)t=(ct+d)f(t)$ for some $c,d\in D$ then f is called *(right) invariant*. The invariant polynomials are also called *two-sided*, as the ideals they generate are left and right ideals.

Let $f \in R$ have degree m and an invertible leading coefficient. Then for all $g(t) \in R$ of degree $l \geq m$, there exist uniquely determined $r, q \in R$ with $\deg(r) < \deg(f)$, such that $g(t) = q(t)f(t) + r(t)$. This generalizes the right division algorithm in R that is well-known when S is a division ring [12, p. 6]. In that case, $R = S[t;\sigma,\delta]$ is a left principal ideal domain (i.e., every left ideal in R is of the form Rf).

From now on we assume that f is monic. Let $\text{mod}_r f$ denote the remainder of right division by f. Since the remainder is uniquely determined, the skew polynomials of degree less that m canonically represent the elements of the left $S[t;\sigma,\delta]$-module $S[t;\sigma,\delta]/S[t;\sigma,\delta]f$.

Suppose $f \in R = S[t;\sigma,\delta]$ is monic of degree m. The additive group $\{g \in R \,|\, \deg(g) < m\}$, together with the multiplication $g \circ h = gh \text{ mod}_r f$ for all $g, h \in R$ of degree less than m, is a unital nonassociative algebra over $S_0 = \{a \in S \,|\, ah = ha \text{ for all } h \in S_f\}$, denoted by S_f or R/Rf, and called a *Petit algebra*, as the construction for S a division algebra goes back to Petit [17]. S_0 is a commutative subring of S, and if S is a division algebra, it is a subfield of S. S_f is associative if and only if f is two-sided. If S_f is not associative then $S \subset \text{Nuc}_l(S_f)$, $S \subset \text{Nuc}_m(S_f)$ (if S is a division ring, the inclusions become equalities), and the eigenspace of f is the right nucleus: $\text{Nuc}_r(S_f) = \{g \in R \,|\, \deg(g) < m \text{ and } fg \in Rf\}$.

If $f \in S[t;\sigma,\delta]$ is reducible then S_f contains zero divisors: when $f = gh$ then g and h are zero divisors in S_f. If S is a division ring, then S_f has no zero divisors if and only if f is irreducible.

If Rf is a two-sided ideal in R (i.e. f is *two-sided*, also called *invariant*) then S_f is the associative quotient algebra obtained by factoring out the ideal generated by a two-sided $f \in S[t;\sigma,\delta]$.

For all $g \in R$ of degree $l \geq m$, there also exist uniquely determined $r, q \in R$ with $\deg(r) < \deg(f)$, such that $g(t) = f(t)q(t) + r(t)$. Let $\text{mod}_l f$ denote the remainder of left division by f. Then the additive group $\{g \in R \,|\, \deg(g) < m\}$ together with the multiplication $g \diamond h = gh \text{ mod}_l f$ defined for all $g, h \in R$ of degree less than m, is also a unital nonassociative algebra ${}_fS$ over S_0 denoted by R/fR. Moreover, The canonical anti-automorphism

$$\psi : S[t;\sigma,\delta] \to S^{op}[t;\sigma^{-1}, -\delta \circ \sigma^{-1}], \quad \psi(\sum_{k=0}^{n} a_k t^k) = \sum_{k=0}^{n} (\sum_{i=0}^{k} \Delta_{n,i}(a_k)) t^k$$

induces an anti-automorphism between the rings $S_f = S[t;\sigma,\delta]/S[t;\sigma,\delta]f$ and ${}_{\psi(f)}S = S^{op}[t;\sigma^{-1}, -\delta \circ \sigma^{-1}]/\psi(f)S^{op}[t;\sigma^{-1}, -\delta \circ \sigma^{-1}]$. Recall that $\Delta_{n,j}$ is defined recursively via $\Delta_{n,j} = \delta(\Delta_{n-1,j}) + \sigma(\Delta_{n-1,j-1})$, with $\Delta_{0,0} = id_S$, $\Delta_{1,0} = \delta$, $\Delta_{1,1} = \sigma$ and so $\Delta_{n,j}$ is the sum of all polynomials in σ and δ of degree j in σ and degree $n - j$ in δ [12, p. 2]. If $\delta = 0$, then $\Delta_{n,j} = \sigma^n$.

Note that $S_f^{op} = {}_{\psi(f)}S$

2. Nonassociative (generalized) cyclic algebras and (generalized) Menichetti algebras

2.1. **Nonassociative cyclic algebras.** The equivalence class of a homogeneous semiassociative algebra in $Br^{sa}(F)$ is represented by a unique central associative division algebra [4]. However, we will see now that the question whether an algebra is a division algebra or not is much less important in $Br^{sa}(F)$.

Nonassociative cyclic algebras of degree n are canonical generalizations of associative cyclic algebras of degree n and were first introduced over finite fields by Sandler [29]. Indeed, nonassociative quaternion algebras (the case $n = 2$) were the first known example of a nonassociative division algebra [9]. Over arbitrary fields they were investigated by Steele [31, 32] (Steele studied the opposite algebras of the nonassociative algebras we define here, but used our notation).

Definition 1. Let K be an étale algebra of dimension n over F and $\sigma \in \mathrm{Aut}_F(K)$ of order n. Let $f(t) = t^n - d \in K[t;\sigma]$ with $d \in K^\times$. We call the F-central algebra $S_f = K[t;\sigma]/K[t;\sigma]f$ a *nonassociative cyclic algebra* over F and denote it by $(K/F, \sigma, d)$.

This definition of a nonassociative cyclic algebra generalizes the one treated so far in most papers, where K is assumed to be a cyclic Galois field extension of degree n (for an earlier generalization, cf. [23]).

If K/F is a cyclic Galois field extension of degree n with Galois group $\mathrm{Gal}(K/F) = \langle \sigma \rangle$, then $(K/F, \sigma, d)$ is a classical associative cyclic algebra over F of degree n if $d \in F^\times$, and a nonassociative cyclic algebra as defined in [6, 29, 31], if $d \in K \setminus F$. We note that with our more general definition, we now have for instance that $(K/F, \sigma, d) \otimes_F K \cong (K \otimes_F K, \sigma, d)$ with σ denoting the canonical extension $\sigma \otimes id$ of σ to $K \otimes_F K$. For more details and proofs in the case that K is a division algebra, cf. [6, 31].

If K is a cyclic Galois field extension of F, then $\mathrm{Nuc}((K/F, \sigma, d)) = K$. The proof that $K \subset \mathrm{Nuc}_r((K/F, \sigma, d))$ in the general case that K is étale is analogous to the case where K is a field and implies that $K \subset \mathrm{Nuc}((K/F, \sigma, d))$ holds in the general setup, too.

The easiest example of a nonassociative cyclic algebra is a nonassociative quaternion division algebra $(K/F, \sigma, d)$, where K/F is a quadratic field extension, and $d \in K \setminus F$. This is, up to isomorphism, also the only simple K-semiassociative division algebra of degree 2 that is not associative [33]. This algebra and the simple skew matrix algebra that represents it when it splits over the field extension K are presented in [33].

Theorem 1. *Let K/F be a cyclic Galois field extension of degree n with Galois group $\mathrm{Gal}(K/F) = \langle \sigma \rangle$ and $d \in K \setminus F$. Let $H = \{\tau \in G \,|\, \tau(d) = d\}$. Then $H = \langle \sigma^s \rangle$ for some integer s such that $n = sr$ and*

$$\mathrm{Nuc}_r((K/F, \sigma, d)) = (K/E, \sigma^s, d)$$

is a cyclic algebra of degree r over $E = \mathrm{Fix}(\sigma^s)$, where $[E : F] = |H|$. In particular, if n is prime then $\mathrm{Nuc}_r((K/F, \sigma, d)) = K$.

Proof. By [32, Proposition 3.2.3], $\mathrm{Nuc}_r((K/F, \sigma, d)) = K \oplus Kt^s \oplus \cdots \oplus Kt^{(r-1)s}$. By [31, Theorem 5.1], the linear subspace $K \oplus Kt^s \oplus \cdots \oplus Kt^{(r-1)s}$ is the cyclic subalgebra $(K/E, \sigma^s, d)$ of degree r over $E = \mathrm{Fix}(\sigma^s)$, where $[E : F] = |H|$ and $d \in E$. □

Corollary 2. *Let K/F be a cyclic Galois field extension of degree n with Galois group $\mathrm{Gal}(K/F) = \langle \sigma \rangle$ and $d \in K \setminus F$. Let $H = \{\tau \in G \,|\, \tau(d) = d\} = \langle \sigma^s \rangle$ for some integer s with $1 < s < n$. Then $(K/F, \sigma, d)^{op}$ is not isomorphic to a nonassociative cyclic algebra.*

Proof. If $1 < s < n$ then $\text{Nuc}_r((K/F,\sigma,d)) = (K/F,\sigma^s,d)$ and E is a proper intermediate field of K/F. Thus $\text{Nuc}_l((K/F,\sigma,d)^{op}) = (K/F,\sigma^s,d)$ is unequal to $K = \text{Nuc}_l((K/F,\sigma,d'))$ for any nonassociative cyclic algebra $(K/F,\sigma,d')$. (Note that the fact that the middle nucleus of a nonassociative cyclic algebra is K, so if A is a nonassociative cyclic algebra isomorphic to $(K/F,\sigma,d)^{op}$, A must involve the same field extension K/F.) □

Nonassociative cyclic algebras (and analogously, their opposite algebras) are important examples of semiassociative (division) algebras that are not semicentral, thus in particular not homogeneous:

Proposition 3. *(i) Every nonassociative cyclic algebra $(K/F,\sigma,d)$ over F is K-semiassociative of degree n and thus semiassociative.*
(ii) $(K/F,\sigma,d)$ is split if and only if $K = F^n$.
(iii) $(K/F,\sigma,d)\otimes_F K$ splits.
(iv) [31] *Let K/F be a cyclic field extension of degree n. Then $(K/F,\sigma,d)$ is a division algebra for all $d \in K\setminus F$, such that 1, $d,\dots,d^{n-1}$ are linearly independent over F. If K/F has prime degree then $(K/F,\sigma,d)$ is a division algebra for all $d\in K\setminus F$.*
(v) Let K/F be a cyclic field extension, then for all $d \in K\setminus F$, $(K/F,\sigma,d)$ is not semicentral.

The proof of (i), (ii), (iii) is straightforward employing results from [4] listed in Section 1.2. For $n = 2$, (i) was already pointed out in [4]. (v) is clear because $K = \text{Nuc}((K/F,\sigma,d))$.

Remark 4. Let K be an étale algebra of dimension n over F and $\sigma \in \text{Aut}_F(K)$ of order n. Let $f(t) = t^n \in K[t;\sigma]$. Then $S_f = K[t;\sigma]/K[t;\sigma]f$ is an associative algebra over F which is semiassociative but not simple; the semisimple quotient is K. Abusing notation we denote it by $(K/F,\sigma,0)$ (cf. [4, Remark 3.8] for $n = 2$).

2.2. **Nonassociative generalized cyclic algebras.** (for details on the case that B is a division algebra, cf. [6])

Let B be a central simple algebra over F (i.e., F-central) of degree n, and $\sigma \in \text{Aut}(B)$ such that $\sigma|_F$ has finite order m and for $F_0 = \text{Fix}(\sigma)\cap F$ assume that F/F_0 is a cyclic Galois field extension of degree m with $\text{Gal}(F/F_0) = \langle \sigma|_F\rangle$. (This last assumption is automatically satisfied, if B is a division algebra.)

Definition 2. Let $f(t) = t^m - d \in B[t;\sigma]$, $d \in B^\times$. We call the Petit algebra

$$(B,\sigma,d) = B[t;\sigma]/B[t;\sigma]f$$

a *nonassociative generalized cyclic algebra* over F_0.

This definition generalizes the definition of both a nonassociative and an associative generalized cyclic algebra in [6] (see [12, p. 19] for the associative case), which also assumed that B is a division algebra.

The algebra (B,σ,d) has dimension m^2n^2 over F_0 and is F_0-central. If $d \in F_0^\times$ and B is a division algebra, then (B,σ,d) is a classical associative generalized cyclic algebra over F_0 of degree mn. Indeed, (B,σ,d) is associative if and only if $d \in F_0$.

In particular, if $B=F$, F/F_0 is a cyclic Galois extension of degree m with Galois group generated by σ and $f(t)=t^m-d\in F[t;\sigma]$, we obtain the nonassociative cyclic algebra $(F/F_0,\sigma,d)$.

Lemma 5. *(i)* [7] $B\subset \mathrm{Nuc}_l((B,\sigma,d))=\mathrm{Nuc}_m((B,\sigma,d))$ *with equality when B is a division algebra.*
(ii) [7] *Suppose that $d\in F\setminus F_0$. Then $B\subset \mathrm{Nuc}_r((B,\sigma,d))$, i.e. $B\subset \mathrm{Nuc}((B,\sigma,d))$ with equalities when B is a division algebra.*
(ii) [4, Remark 3.3] *Let K be a maximal étale subalgebra of B of dimension n. Then (B,σ,d) has the étale algebra K/F_0 of dimension mn in its nucleus. (B,σ,d) is a faithful K^e-module, thus cyclic as a K^e-module.*

If $B=D$ is a division algebra over F, then (D,σ,d) is a division algebra over F_0 if and only if $f(t)=t^m-d\in D[t;\sigma]$ is irreducible [17, (7)]. We know that $f(t)=t^2-d\in D[t;\sigma]$ is irreducible if and only if $\sigma(z)z\neq d$ for all $z\in D$, $f(t)=t^3-d\in D[t;\sigma]$ is irreducible if and only if $d\neq\sigma^2(z)\sigma(z)z$ for all $z\in D$, and $f(t)=t^4-d\in D[t;\sigma]$ is irreducible if and only if $\sigma^2(y)\sigma(y)y+\sigma^2(x)y+\sigma^2(y)\sigma(x)\neq 0$ or $\sigma^2(x)x+\sigma^2(y)\sigma(y)x\neq d$ for all $x,y\in D$ (cf. [17, 25], and [5, Theorem 3.19], [7]). More generally, if F_0 contains a primitive mth root of unity and m is prime then $f(t)=t^m-d\in D[t;\sigma]$ is irreducible if and only if $d\neq\sigma^{m-1}(z)\cdots\sigma(z)z$ for all $z\in D$ ([5, Theorem 3.11], see also [25, Theorem 6]), which generalizes the equivalent condition in the associative setup.

Lemma 6. *Suppose that B contains the maximal étale subalgebra K of dimension n. Then for all $d\in F^\times$, (B,σ,d) is a K-semiassociative algebra over F_0 of degree mn. (B,σ,d) is not semicentral.*

Proof. If $d\in F_0$ then (B,σ,d) is an associative central simple algebra over F_0 and trivially semiassociative. For all $d\in C(B)=F$, $d\notin F_0$ we have $B\subset\mathrm{Nuc}((B,\sigma,d))$ and the étale algebra K/F_0 of degree mn lies in $\mathrm{Nuc}((B,\sigma,d))$. The rest is a straightforward calculation as well. In particular, since $B\subset\mathrm{Nuc}((B,\sigma,d))$ is an F-central simple algebra, (B,σ,d) is not semicentral. □

2.3. **Menichetti algebras.** ([16, 28])

Let K/F be a Galois field extension of F of degree m with $\mathrm{Gal}(K/F)=\{\tau_0,\dots,\tau_{m-1}\}$. Let $k_i\in K^\times$, $i\in 0,\dots,m-1$, and let

$$c_{i,j}=k_0^{-1}k_1^{-1}\cdots k_{j-1}^{-1}k_ik_{i+1}\cdots k_{i+j-1}$$

for all $i,j\in\mathbb{Z}_m$. Let $z_0,\dots,z^{m-1}$ be an F-basis of K^m and define a multiplication on $(K/F,k_0,\dots,k_{m-1})=K^m$ via

$$(az_i)\cdot(bu_j)=\tau_j(a)b(u_i\cdot u_j),\quad z_i\cdot z_0=z_0\cdot z_i=z_i \text{ for all } i\in\mathbb{Z}_m,$$

and

$$z_i\cdot z_j=c_{ji}z_{i+j} \text{ for all } i\in\mathbb{Z}_m\setminus\{0\}$$

for all $a,b\in K$. Then $(K/F,k_0,\dots,k_{m-1})$ is a nonassociative unital algebra over F of dimension m^2, called a *Menichetti algebra*, as the construction generalizes the one in [16].

Define

$$M(x_0,\dots,x_{m-1})=\begin{bmatrix} x_0 & c_{m-1,1}\tau_1(x_{m-1}) & \dots & c_{1,m-1}\tau_{m-1}(x_1)\\ x_1 & \tau_1(x_0) & \dots & c_{2,m-1}\tau_{m-1}(x_2)\\ x_2 & c_{1,1}\tau_1(x_1) & \tau_2(x_0) & c_{3,m-1}\tau_{m-1}(x_3)\\ \dots & \dots & \dots & \dots\\ x_{m-2} & c_{m-3,1}\tau_1(x_{m-3}) & \dots & c_{m-1,m-1}\tau_{m-1}(x_{m-1})\\ x_{m-1} & c_{m-2,1}\tau_1(x_{m-2}) & \dots & \tau_{m-1}(x_0)\end{bmatrix},$$

identify $x_0z_0+\cdots+x_{m-1}z_{m-1}$ with $(x_0,\dots,x_{m-1})$, $x_i\in K$, then

$$(x_0,\dots,x_{m-1})\cdot(y_0,\dots,y_{m-1})=M(x_0,\dots,x_{m-1})(y_0,\dots,y_{m-1})^t.$$

It is easy to see that $K\subset \mathrm{Nuc}((K/F,k_0,\dots,k_{m-1}))$ and that $(K/F,k_0,\dots,k_{m-1})$ is a semiassociative algebra over F that generalizes nonassociative cyclic algebras [28].

2.4. **Generalized Menichetti algebras.** ([28])

Let D be a central simple algebra over F of degree n. Let $\sigma\in \mathrm{Aut}(D)$ such that $\sigma|_F$ has finite order m, and put $F_0=\mathrm{Fix}(\sigma)\cap F$. Assume that F/F_0 is a cyclic Galois field extension of degree m with $\mathrm{Gal}(F/F_0)=\langle\sigma|_F\rangle$. (This is automatically satisfied, if D is a division algebra.) Let $k_i\in F^\times$, $i\in 0,\dots,m-1$, and

$$c_{i,j}=k_0^{-1}k_1^{-1}\cdots k_{j-1}^{-1}k_ik_{i+1}\cdots k_{i+j-1}$$

for all $i,j\in\mathbb{Z}_m$. Let $z_0,\dots,z^{m-1}$ be an F_0-basis of D^m and define a multiplication on D^m via

$$(az_i)\cdot(bz_j)=\sigma^j(a)b(z_i\cdot z_j),$$
$$z_i\cdot z_0=z_0\cdot z_i=z_i \text{ for all } i\in\mathbb{Z}_m,$$
$$z_i\cdot z_j=c_{ji}z_{i+j} \text{ for all } i\in\mathbb{Z}_m\setminus\{0\}$$

for all $a,b\in D$. This yields a nonassociative unital algebra over F_0 of dimension n^2m^2 that we denote by $(D,\sigma,k_0,\dots,k_{m-1})$ and call a *generalized Menichetti algebra of degree mn*. Its multiplication is given by the matrix

$$M(x_0,\dots,x_{m-1})=\begin{bmatrix} x_0 & c_{m-1,1}\sigma(x_{m-1}) & \dots & c_{1,m-1}\sigma^{m-1}(x_1)\\ x_1 & \sigma(x_0) & \dots & c_{2,m-1}\sigma^{m-1}(x_2)\\ x_2 & c_{1,1}\sigma(x_1) & \sigma^2(x_0) & c_{3,m-1}\sigma^{m-1}(x_3)\\ \dots & \dots & \dots & \dots\\ x_{m-2} & c_{m-3,1}\sigma(x_{m-3}) & \dots & c_{m-1,m-1}\sigma^{m-1}(x_{m-1})\\ x_{m-1} & c_{m-2,1}\sigma(x_{m-2}) & \dots & \sigma^{m-1}(x_0)\end{bmatrix}$$

for all $x_i\in D$. Let E be a maximal étale subalgebra of D of dimension n. Then $(D,\sigma,k_0,\dots,k_{m-1})$ has the étale algebra E/F_0 of dimension mn in its nucleus. $(D,\sigma,k_0,\dots,k_{m-1})$ is a faithful E^e-module, thus is cyclic as a E^e-module [4, Remark 3.3] and is a semiassociative algebra [4].

Let now K/F be a cyclic field extension of degree m with $\mathrm{Gal}(K/F)=\langle\sigma\rangle$. Let D_0 be a central simple algebra over F of degree n, and put $D=D_0\otimes_F K$. Let $\widetilde{\sigma}$ be the unique extension of σ to D such that $\widetilde{\sigma}|_{D_0}=id_{D_0}$. Then

$$A=D_0\otimes_F(K/F,k_0,\dots,k_{m-1})\cong(D_0\otimes_F K,\widetilde{\sigma},k_0,\dots,k_{m-1})$$

is a nonassociative generalized Menichetti algebra over F of degree mn. Since $\text{Nuc}(A) = D_0 \otimes \text{Nuc}((K/F, k_0, \dots, k_{m-1}))$, we have $D \subset \text{Nuc}(A)$. A is a semiassociative algebra.

3. The tensor product of an associative algebra and a nonassociative cyclic algebra

The tensor product of two semiassociative algebras is again a semiassociative algebra. The special case of tensoring an associative central division algebra and a nonassociative cyclic division algebra appeared already in the context of constructing space time block codes [27, 18, 20, 15, 25]. The resulting algebra is a generalized nonassociative cyclic algebra (cf. [12, p. 36] for the associative setup, where D is assumed to be a division algebra, but the proof goes through verbatim if not):

Let E/F be a cyclic field extension of degree m with $\text{Gal}(E/F) = \langle \tau \rangle$. Let D_0 be a central simple algebra over F of degree n, and put $D = D_0 \otimes_F E$. Let $D_1 = (E/F, \tau, d)$ be a nonassociative cyclic algebra over F of degree m (i.e. $c \in F^\times$ and $d \in E^\times$), and let $\widetilde{\tau}$ be the unique extension of τ to D such that $\widetilde{\tau}|_{D_0} = id_{D_0}$. Then

$$A = D_0 \otimes_F (E/F, \tau, d) \cong (D_0 \otimes_F E)[t, \widetilde{\tau}]/(D_0 \otimes_F E)[t, \widetilde{\tau}](t^m - d) = (D, \widetilde{\tau}, d)$$

is a nonassociative generalized cyclic algebra over F of degree mn. Now $(E/F, \tau, d)$ is associative if and only if $d \in F^\times$, so assume that $d \in E \setminus F$. Then $\text{Nuc}((D, \widetilde{\tau}, d)) = D_0 \otimes_F E = D$ is a normal algebra over F and $D_0 \otimes_F (E/F, \tau, d)$ is a semiassociative algebra over F of degree mn that is not semicentral.

For any maximal étale subalgebra L in D_0, $K = L \otimes_F E \subset \text{Nuc}(A)$ is a maximal étale subalgebra of A of degree mn over F.

Proposition 7. *Let $H = \{\gamma \in \text{Gal}(E/F) \,|\, \gamma(d) = d\} = \langle \tau^s \rangle$ for some integer s such that $m = sr$.*
(i) Let $M = \text{Fix}(\tau^s)$. Then

$$\text{Nuc}_r((D, \widetilde{\tau}, d)) = D_0 \otimes_F (K/M, \sigma^s, d) \cong D \oplus Dt^s \oplus \dots \oplus Dt^{s(r-1)}$$

with $(K/M, \sigma^s, d)$ a cyclic algebra of degree r over M, where $[M : F] = |H|$. In particular, if m is prime then $\text{Nuc}_r((D, \widetilde{\tau}, d)) = D$.
(ii) If $1 < s < m$, then $(D, \widetilde{\tau}, d)^{op}$ is not a generalized nonassociative cyclic algebra.

Proof. (i) We have $\text{Nuc}_r((D, \widetilde{\tau}, d)) = D_0 \otimes_F \text{Nuc}_r((E/F, \tau, d))$, where $\text{Nuc}_r((E/F, \tau, d))$ is the cyclic subalgebra $(E/M, \tau^s, d)$ of degree r over $M = \text{Fix}(\tau^s)$. Here, $\text{Nuc}_r((K/F, \tau, d)) = (K/M, \sigma^s, d)$ is a cyclic algebra of degree r over $M = \text{Fix}(\tau^s)$, and $[M : F] = |H|$. In particular, if m is prime then $\text{Nuc}_r((K/F, \tau, d)) = D_0 \otimes_F E = D$.
(ii) The proof is straightforward, we just compare the left nuclei. □

Remark 8. (i) Let K be an étale algebra of dimension n over F. Note that the associative algebra $D_0 \otimes_F (K/F, \sigma, 0)$ also is a semiassociative algebra over F_0. It is not simple, the semisimple quotient is D. Abusing notation we denote it by $(D_0 \otimes_F K, \sigma, 0)$ (cf. [4, Remark 3.8] for $n = 2$).
(ii) The nucleus of $(D, \widetilde{\tau}, d) \otimes_F (D, \widetilde{\tau}, d)^{op}$ is $M_{n^2m^2}(K)$, so this algebra is not split.

In the following, let L/F be a cyclic Galois field extension of degree n with $\mathrm{Gal}(L/F) = \langle\sigma\rangle$. Let $K = L \otimes_F E$, then σ and τ canonically extend to K.

Let now $D_0 = (L/F, \sigma, c)$ be an associative cyclic algebra over F and $D_1 = (E/F, \tau, d)$ be a nonassociative cyclic algebra over F, i.e. $c \in F^\times$. Then $D = (L/F, \sigma, c) \otimes_F E$ is a central simple algebra over E of degree n and K/E is a maximal étale subalgebra of D of degree n (i.e., we also have $D \cong (K/E, \sigma, c)$ using our new generalized definition of a cyclic algebra employing Petit algebras). Then

$$A = (L/F, \sigma, c) \otimes_F (E/F, \tau, d)$$

is a semiassociative algebra over F of degree mn, and if $d \in E \setminus F$ then $\mathrm{Nuc}(A) = D_0 \otimes_F E = D$ and $K = L \otimes_F E \subset \mathrm{Nuc}(A)$ is a maximal commutative subalgebra of A that is an étale algebra of degree mn over F (if $d \in F^\times$ then $(E/F, \tau, d)$ is associative).

In this case $\widetilde{\tau}$ is the unique L-linear automorphism of D such that $\widetilde{\tau}|_K = \tau$, i.e. for $x = x_0 + x_1 t + x_2 t^2 + \cdots + x_{n-1} t^{n-1} \in D$ ($x_i \in K$, $1 \leq i \leq n$), define

$$\widetilde{\tau}(x) = \tau(x_0) + \tau(x_1)t + \tau(x_2)t^2 + \cdots + \tau(x_{n-1})t^{n-1}.$$

So here $\widetilde{\tau}|_E$ has order m and $\mathrm{Fix}(\widetilde{\tau}) = F$.

Corollary 9. *The algebra $(L/F, \sigma, c) \otimes_F (E/F, \tau, d) \cong (D, \widetilde{\tau}, d)$ is a generalized nonassociative cyclic algebra over F, which is associative if $d \in F^\times$, and has nucleus D if $d \in E \setminus F$. It is K-semiassociative of degree mn and, if it is not associative, it is not semicentral.*

Remark 10. The proof of this result is also a straightforward generalization of the proof [25, Theorem 11] which assumed that D is a division algebra, and that L and E are linearly disjoint over F, so that K is a field: the proof that $(L/F, \sigma, c) \otimes_F (E/F, \tau, d) \cong (D, \widetilde{\tau}, d) D[t; \widetilde{\tau}]/D[t; \widetilde{\tau}]f$ where $f(t) = t^m - d$ goes through verbatim in our more general setting, as the whole theory does not depend on these two assumptions (it was originally developed for space-time block codes which are built from division algebras). Since we look at the opposite cyclic algebras than the one employed throughout [25, Theorem 11], $\widetilde{\tau}^{-1}$ in [25, Theorem 11] in our setup becomes $\widetilde{\tau}$. If $d \in E \setminus F$ the algebra has as the nucleus the central simple algebra $D = (L/F, \sigma, c) \otimes_F E$, and K/F is a maximal étale subalgebra of the nucleus of degree mn.

Let $(E/F, \tau, d)$ be a cyclic associative division algebra of prime degree m. Suppose that B_0 is a central associative algebra over F such that $B = B_0 \otimes_F E$ is a division algebra. By a classical result by Jacobson, the tensor product $B_0 \otimes_F (E/F, \tau, d)$ is a division algebra if and only if $d \neq \widetilde{\tau}^{m-1}(z) \cdots \widetilde{\tau}(z) z$ for all $z \in B$ ([12, Theorem 1.9.8], see also [1, Theorem 12, Ch. XI]).

This result can be generalized to the tensor product of a cyclic and a nonassociative cyclic algebra, if the base field contains a suitable root of unity [25]. We now put the main results from [25] into the context of semiassociative algebras, adjusting them where needed (some of the algebras studied in [25, Section 3] are the opposite algebras of ours).

The generalization of Jacobson's condition is a necessary condition for $d \in E^\times$ in our general nonassociative case as well:

Proposition 11. [25, Proposition 20] *Let $D_0=(L/F,\sigma,c)$ be an associative cyclic algebra of degree n over F, such that $D=D_0\otimes_F E$ is a division algebra. If $D_0\otimes_F(E/F,\tau,d)$ is a division algebra then $d\neq z\widetilde{\tau}(z)\cdots\widetilde{\tau}^{m-1}(z)$ for all $z\in D$.*

From now on, let L and E be linearly disjoint over F, then $K=L\otimes_F E=L\cdot E$ is the composite of L and E over F, with Galois group $\mathrm{Gal}(K/F)=\langle\sigma\rangle\times\langle\tau\rangle$, and K/E is a maximal separable subfield of $D=(L/F,\sigma,c)\otimes_F E$ of degree n.

Theorem 12. *(*[25, Theorems 14, 15, 16]*) Suppose that $D=(L/F,\sigma,c)\otimes_F E$ is a division algebra, m is prime and in case $m\neq 2,3$, additionally that F contains a primitive mth root of unity.*
(i) $(L/F,\sigma,c)\otimes_F(E/F,\tau,d)$ is a semiassociative division algebra if and only if $d\neq z\widetilde{\tau}(z)\cdots\widetilde{\tau}^{m-1}(z)$ for all $z\in D$, if and only if $f(t)=t^m-d\in D[t;\widetilde{\tau}]$ is irreducible.
(ii) If $\tau(d^n)\neq d^n$ then $(L/F,\sigma,c)\otimes_F(E/F,\tau,d)$ is a K-semiassociative division algebra of degree mn with nucleus D.
(iii) If $d\in E$ such that $d^n\notin N_{D/F}(D^\times)$, then $(L/F,\sigma,c)\otimes_F(E/F,\tau,d)$ is a K-semiassociative division algebra of degree mn with nucleus D. In particular, for all $d\in E\setminus F$ with $d^n\notin F$, $(L/F,\sigma,c)\otimes_F(E/F,\tau,d)$ is a K-semiassociative division algebra of degree mn.

Theorem 13. [25, Theorem 17] *Let F be of characteristic not 2. Let $(a,c)_F$ be a quaternion algebra over F which is a division algebra over $E=F(\sqrt{b})$, and $(F(\sqrt{b})/F,\tau,d)$ a nonassociative quaternion algebra over F. Then $(a,c)_F\otimes_F(F(\sqrt{b})/F,\tau,d)$ is a semiassociative division algebra over F of degree 4 with nucleus $(a,c)_{F(\sqrt{b})}$.*

More generally, let B be a central simple algebra over F of degree n, and $\sigma\in\mathrm{Aut}(B)$ such that $\sigma|_F$ has finite order m, $F_0=\mathrm{Fix}(\sigma)\cap F$ and F/F_0 is a cyclic Galois field extension of degree m with $\mathrm{Gal}(F/F_0)=\langle\sigma|_F\rangle$ and $d\in F^\times$. Let D_0 be a central simple algebra over F_0 of degree s, and let $\widetilde{\sigma}$ be the unique extension of σ to $D_0\otimes_{F_0}B$ such that $\widetilde{\sigma}|_{D_0}=id_{D_0}$ Then $\widetilde{\sigma}$ has order m over F_0, and

$$D_0\otimes_{F_0}(B,\sigma,d)\cong(D_0\otimes B,\widetilde{\sigma},d)$$

with $(D_0\otimes_{F_0}B,\widetilde{\sigma},d)=(D_0\otimes_{F_0}B)[t;\widetilde{\sigma}]/(D_0\otimes_{F_0}B)[t;\widetilde{\sigma}](t^m-d)$. We get a generalized nonassociative cyclic algebra of degree mns with $D_0\otimes_{F_0}B$ contained in its nucleus.

4. The semiassociative Brauer monoid

4.1. The classes in $Br^{sa}(F)$ that contain the homogeneous semiassociative algebras are determined by the Brauer group and are of the kind $[B]^{sa}$ with B an associative central simple algebra over F. In particular, if D is an associative F-central division algebra, then $[D]^{sa}$ is the unique element of minimal degree in the class $[D]^{sa}\in Br^{sa}(F)$ which contains the homogeneous semiassociative algebras of the kind $D\otimes_F M$, where M is a skew matrix algebra [4, Example 14.5, Corollary 18.3]. Moreover, if F is a field with nontrivial Brauer group, then $Br^{sa}(F)$ has elements $[A]^{sa}$ of infinite order [4, Corollary 20.4]. From the proof of [4, Corollary 20.4], it is clear that these elements are constructed by finding semiassociative algebras A, such that $\sigma(A)=F\oplus B$, where B is a central division algebra over F of index p, so the similarity class $[A]^{sa}$ contains elements that are all semicentral (although the algebras are not explicitly constructed there).

We now collect some observations on elements in the semiassociative Brauer monoid $Br^{sa}(F)$. In particular, $Br^{sa}(F)$ can be nontrivial even if the classical Brauer group is trivial, as we can easily conclude from our previous results:

Proposition 14. *(i) Let F be a field that has a cyclic Galois field extension K/F of degree n, $\mathrm{Gal}(K/F) = \langle\sigma\rangle$. Then $[F]^{sa} \neq [(K/F,\sigma,d)]^{sa}$ for all $d \in K \setminus F$ and so $Br^{sa}(F)$ is nontrivial. Moreover, $[(K/F,\sigma,d)]^{sa}$ has infinite order in $Br^{sa}(F)$, i.e. the powers of these element are distinct.*
(ii) Let F be a field that has a Galois field extension K/F of degree n, and $(K/F,k_0,\dots,k_{m-1})$ be any Menichetti algebra that is not associative. Then $[F]^{sa} \neq [(K/F,k_0,\dots,k_{m-1})]^{sa}$ and so $Br^{sa}(F)$ is nontrivial, and $[(K/F,k_0,\dots,k_{m-1})]^{sa}$ has infinite order in $Br^{sa}(F)$.

Proof. A semiassociative algebra over F of degree kn is split if and only if F^{kn} is contained in its nucleus as a unital subalgebra.
(i) Now $A = (K/F,\sigma,d) \otimes_F \cdots \otimes_F (K/F,\sigma,d)$ (k-times) has degree kn and nucleus $K \otimes_F K \otimes_F \cdots \otimes_F K$ (k-times). If K/F is a cyclic field extension of degree n with Galois group G then $K \otimes_F K \otimes_F \cdots \otimes_F K \cong \prod_{G^{k-1}} K$, where the index set G^{k-1} is the $(k-1)$-fold product of G. So clearly the étale algebra F^{nk-n} is a unital subalgebra of the nucleus of A, but F^{kn} is not.
(ii) $(K/F,k_0,\dots,k_{m-1}) \otimes_F \cdots \otimes_F (K/F,k_0,\dots,k_{m-1})$ has degree kn and nucleus $K \otimes_F K \otimes_F \cdots \otimes_F K$ (k-times), so the assertion follows as in (i). □

Since $(K/F,\sigma,d)$ is not semicentral for all $d \in K \setminus F$, it does not lie in the similarity class of any F-central simple algebra B in $Br^{sa}(F)$, and if n is prime (or if $1, d, \dots, d^{n-1}$ are linearly independent over F), then $(K/F,\sigma,d)$ is always a division algebra, thus is a division algebra of smallest degree in $[(K/F,\sigma,d)]^{sa}$.

Lemma 15. *Let K/F be a field extension of degree m and D be a central simple algebra over F of degree n. Then*

$$[D]^{sa}[(K/F,\sigma,d)]^{sa} = [(D \otimes_F K, \widetilde{\sigma}, d)]^{sa}$$

for all $d \in K$. In particular, for all $d \in K \setminus F$ we have

$$[(K/F,\sigma,d)]^{sa} = [(M_n(K), \widetilde{\sigma}, d)]^{sa} \neq [F]^{sa}$$

and

$$[D]^{sa}[(K/F,\sigma,d)]^{sa} = [(M_n(K), \widetilde{\sigma}, d)]^{sa} = [(K/F,\sigma,d)]^{sa}$$

if K is a splitting field of D.
Moreover, for a generalized nonassociative cyclic algebra (B,σ,d) over F, we have

$$[D]^{sa}[(B,\sigma,d)]^{sa} = [(D \otimes_F B, \widetilde{\sigma}, d)]^{sa},$$

where $\widetilde{\sigma}$ is the unique extension of σ to $D \otimes_F B$ such that $\widetilde{\sigma}|_D = id_D$.
For a Menichetti algebra $(K/F,k_0,\dots,k_{m-1})$ over F, we have analogously

$$[D]^{sa}[(K/F,k_0,\dots,k_{m-1})]^{sa} = [(D \otimes_F K, \widetilde{\sigma}, k_0,\dots,k_{m-1})]^{sa}$$

and for a generalized Menichetti algebra $(B,\sigma,k_0,\dots,k_{m-1})$ *over* F*, we have analogously*

$$[D]^{sa}[(B,\sigma,k_0,\dots,k_{m-1})]^{sa}=[(D\otimes_F B,\widetilde{\sigma},k_0,\dots,k_{m-1})]^{sa}$$

for all $k_i\in F$*, where* $\widetilde{\sigma}$ *is the unique extension of* σ *to* $D\otimes_F B$ *such that* $\widetilde{\sigma}|_D=id_D$*.*

Proof. Since $D\otimes_F(K/F,\sigma,d)\cong(D\otimes_F K,\widetilde{\sigma},d)$ is a nonassociative generalized cyclic algebra over F of degree nm, we obtain

$$[D]^{sa}[(K/F,\sigma,d)]^{sa}=[(D\otimes_F K,\widetilde{\sigma},d)]^{sa}=[M_2(F)]^{sa}[(K/F,\sigma,d)]^{sa}=[(K/F,\sigma,d)]^{sa}$$

for all $d\in K\setminus F$. In particular, $M_n(F)\otimes_F(K/F,\sigma,d)\cong(M_n(K),\widetilde{\sigma},d)$ has nucleus $M_n(K)$. The maximal étale F-algebra in its nucleus is K^n. This yields the assertion that $[F]^{sa}\neq[(M_n(K),\widetilde{\sigma},d)]^{sa}=[M_n(F)]^{sa}[(K/F,\sigma,d)]^{sa}=[(K/F,\sigma,d)]^{sa}$. The rest is clear. □

Theorem 16. *Let* A *and* A' *be two semiasssociative algebras over* F*.*
(i) Let $\mathrm{Nuc}(A)=K$ *and* $\mathrm{Nuc}(A')=L$ *be two field extensions of* F*. If* $[A]^{sa}=[A']^{sa}\in Br^{sa}(F)$ *then* $K\cong L$*.*
(ii) Let A *and* A' *have a simple nucleus* N*, respectively* N'*, where* N *is an* E*-central simple algebra and* N' *is an* E'*-central simple algebra, with* E *and* E' *some separable field extensions of* F*. If* $[A]^{sa}=[A']^{sa}\in Br^{sa}(F)$ *then* $E\cong E'$ *and* $[N]=[N']\in Br(E)$*.*

Proof. Since $A\sim A'$ we have $A\otimes_F M_n(F;c)\cong A'\otimes_F M_s(F;c')$ for suitable skewed matrix algebras $M_n(F;c)$, $M_s(F;c')$. From $\sigma(A\otimes_F M_n(F;c))\cong\sigma(A'\otimes_F M_s(F;c'))$ it follows that $\sigma(A)\otimes_F\sigma(M_n(F;c))\cong\sigma(A')\otimes_F\sigma(M_s(F;c'))$ by [4, Proposition 13.5]. Now $\sigma(M_n(F;c))$ and $\sigma(M_s(F;c'))$ are sums of matrix algebras over F whose degrees sum up to n, respectively to s: $\sigma(M_n(F;c))\cong M_{n_1}(F)\oplus\cdots\oplus M_{n_r}(F)$, respectively $\sigma(M_s(F;c'))\cong M_{s_1}(F)\oplus\cdots\oplus M_{s_j}(F)$.
(i) Since K and L are fields we have $\mathrm{Nuc}(A)=K=\sigma(K)$ and $\mathrm{Nuc}(A')=L=\sigma(L)$. We obtain $M_{n_1}(K)\oplus\cdots\oplus M_{n_r}(K)\cong M_{s_1}(L)\oplus\cdots\oplus M_{s_j}(L)$. These decompositions are unique up to permutations of summands, so $r=j$ and $K\cong L$.
(ii) Here, $J(\mathrm{Nuc}(A))=J(\mathrm{Nuc}(A'))=0$ and so $N=\mathrm{Nuc}(A)=\sigma(A)$ and $N'=\mathrm{Nuc}(A')=\sigma(A')$ and the above argument yields $M_{n_1}(N)\oplus\cdots\oplus M_{n_r}(N)\cong M_{s_1}(N')\oplus\cdots\oplus M_{s_j}(N')$. These decompositions are unique up to permutations of summands, so $r=j$ and $M_{n_1}(N)\cong M_{n_t}(N')$ for some t, where N is an E-central simple algebra and N' is an E'-central simple algebra, with E and E' some separable field extensions of F. This implies that $E\cong E'$ as both algebras must have the same center. Moreover, then $[N]=[N']\in Br(E)$.

□

In particular, if K/F and L/F are two cyclic field extensions and $[(K/F,\sigma,d)]^{sa}=[(L/F,\tau,d')]^{sa}$ then $K=L$. It is an open and seemingly non-trivial problem, if two non-isomorphic cyclic algebras $(K/F,\sigma,d)$ and $(K/F,\sigma,d')$ which are both not associative, can lie in the same similarity class in $Br^{sa}(F)$.

Corollary 17. *(i) Let* K/F *and* L/F *be two cyclic field extensions and* $(K/F,\sigma,d)$*,* $(L/F,\sigma',d')$ *be two nonassociative cyclic algebras. If* K *and* L *are not isomorphic then* $[(K/F,\sigma,d)]^{sa}\neq[(L/F,\sigma',d')]^{sa}$ *in* $Br^{sa}(F)$*.*
(ii) Let $\mathrm{Nuc}(A)=K$ *be a field extension of degree* n *and* $\mathrm{Nuc}(A')=D$ *an* F*-central algebra of degree* $m\geq 2$*. Then* $[A]^{sa}\neq[A']^{sa}$ *in* $Br^{sa}(F)$*.*

(iii) Let D_0, D_0' be two central simple algebras, and $A = D_0 \otimes_F (E/F, \tau, d) \cong (D, \widetilde{\tau}, d)$ and $B = D_0' \otimes_F (E'/F, \tau', d') \cong (D', \widetilde{\tau'}, d')$ with E/F, E'/F two separable field extensions and $d \in E \setminus F$, $d' \in E' \setminus F$. If $[A]^{sa} = [A']^{sa} \in Br^{sa}(F)$ then $E \cong E'$ and $[D_0 \otimes_F E] = [D_0' \otimes_F E] \in Br(E)$.

Proof. (i) is clear.
(ii) The analogous argument as in Theorem 16 (i) and (ii) shows that if $[A]^{sa} = [A']^{sa} \in Br^{sa}$, then $M_{n_1}(K) \oplus \cdots \oplus M_{n_r}(K) \cong M_{s_1}(D) \oplus \cdots \oplus M_{s_j}(D)$ implies $M_{n_1}(K) \cong M_{n_t b}(D_0)$ for some b, t and some F-central division algebra D_0, a contradiction.
(iii) Since $d \in E \setminus F$ and $d' \in E' \setminus F$, we have $\text{Nuc}((D, \widetilde{\tau}, d)) = D_0 \otimes_F E = D$, and $\text{Nuc}((D', \widetilde{\tau'}, d')) = D_0' \otimes_F E' = D'$, therefore $[(D, \widetilde{\tau}, d)]^{sa} = [(D', \widetilde{\tau'}, d')]^{sa}$ implies $[D_0 \otimes_F E] = [D_0' \otimes_F E'] \in Br(E)$ by Theorem 16 (iii). □

Proposition 18. *Let $(K_i/F, \sigma_i, d_i)$ be nonassociative cyclic algebras of degree n_i which are all not associative (i.e., $d_i \in K \setminus F$), $i = 1, \dots, r$, and let*

$$A = (K_1/F, \sigma_1, d_1) \otimes_F \cdots \otimes_F (K_r/F, \sigma_r, d_r)$$

be their tensor product (which is a semiassociative algebra of degree $n_1 \cdots n_r$).
(i) The nucleus of A is the étale algebra $E = K_1 \otimes_F \cdots \otimes_F K_r$.
(ii) A is split if and only if E is a split étale algebra.
(iii) If $K_1, \dots, K_r$ are linearly disjoint field extensions over F (e.g. all of different prime degrees) then A is a semiassociative algebra of degree $n_1 \cdots n_r$ with nucleus the field extension E/F of degree $n_1 \cdots n_r$. In particular, A is not semicentral.
(iv) If A is a division algebra then $K_1, \dots, K_r$ are linearly disjoint field extensions over F and E/F is a field extension of degree $n_1 \cdots d_r$. In particular, A is not semicentral.

The proof is trivial, employing previously mentioned results from [4].

Mirrowing the classical setup, the semiassociative Brauer monoid of an algebraically closed field is trivial, $Br_{sa}(\mathbb{C}) = 1$ [4, Example 14.5], and any semiassociative algebra over $\mathbb{C}$ splits.

4.2. $Br^{sa}(\mathbb{R})$**.** It is well-known that $Br(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\}$ is a cyclic group of order 2; and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$. Therefore the two classes in $Br^{sa}(\mathbb{R})$ that contain the homogeneous semiassociative algebras are $[\mathbb{R}]^{sa}$ and $[\mathbb{H}]^{sa}$.

Up to isomorphism, every nonassociative simple algebra of dimension 4 with $\mathbb{C}$ as its nucleus is a nonassociative quaternion algebra [33] (ote that $(\mathbb{C}/\mathbb{R}, ^-, 0)$ is semiassociative, even associative, but not simple). For every $a \in \mathbb{C} \setminus \mathbb{R}$, the nonassociative quaternion algebra $(\mathbb{C}/\mathbb{R}, ^-, a)$ is a semiassociatve division algebra over $\mathbb{R}$ of degree two that is not semicentral, and $[(\mathbb{C}/\mathbb{R}, ^-, a)]^{sa}$ has infinite order in $Br^{sa}(\mathbb{R})$. The class $[(\mathbb{C}/\mathbb{R}, ^-, a)]^{sa}$ thus contains algebras that are not semicentral and $(\mathbb{C}/\mathbb{R}, ^-, a)$ is a division algebra of smallest degree in $[(\mathbb{C}/\mathbb{R}, ^-, a)]^{sa}$. We know that for $a, b \in \mathbb{C} \setminus \mathbb{R}$, we have $(\mathbb{C}/\mathbb{R}, ^-, a) \cong (\mathbb{C}/\mathbb{R}, ^-, b)$ if and only if there is $x \in \mathbb{R}$ such that either $a = x^2 b$ or $\bar{a} = x^2 b$ [33]. It is not clear, however, if two nonisomorphic quaternion division algebras can lie in the same similarity class in $Br^{sa}(\mathbb{R})$.

Furthermore, for all $d \in \mathbb{C} \setminus \mathbb{R}$ we have

$$[\mathbb{H}]^{sa}[(\mathbb{C}/\mathbb{R}, ^-, d)]^{sa} = [(M_2(\mathbb{C}), \widetilde{^-}, d)]^{sa} \neq [\mathbb{R}]^{sa},$$

$$[M_n(\mathbb{R})]^{sa}[(\mathbb{C}/\mathbb{R}, ^-, d)]^{sa} = [M_n(\mathbb{C}), \widetilde{^-}, d)]^{sa},$$

so that

$$[\mathbb{H}]^{sa}[(\mathbb{C}/\mathbb{R}, ^-, d)]^{sa} = [(M_2(\mathbb{C}), \widetilde{^-}, d)]^{sa} = [M_2(\mathbb{R})]^{sa}[(\mathbb{C}/\mathbb{R}, ^-, d)]^{sa} = [(\mathbb{C}/\mathbb{R}, ^-, d)]^{sa}.$$

4.3. $Br^{sa}(\mathbb{F}_q)$**.** The Brauer group $Br(\mathbb{F}_q)$ is trivial. Therefore the only class in $Br^{sa}(\mathbb{F}_q)$ that contains homogeneous semiassociative algebras is the trivial class $[\mathbb{F}_q]^{sa} = [M_n(\mathbb{F}_q; c)]^{sa}$ with $M_n(\mathbb{F}_q; c)$ a skew matrix algebra over F. The semiassociative Brauer monoid $Br^{sa}(\mathbb{F}_q)$ is not trivial:

For each finite field extension $K/\mathbb{F}_q$ of degree n, there exist simple nonassociative cyclic algebras $(K/\mathbb{F}_q, \sigma, a)$ of degree n, with nucleus K if $a \in K \setminus \mathbb{F}_q$. Two such algebras $(K/\mathbb{F}_q, \sigma, a)$ and $(K'/\mathbb{F}_q, \sigma', a')$ will automatically be nonisomorphic for two nonisomorphic field extensions K and K', and are not semisimple.

Here, $[M_n(\mathbb{F}_q)]^{sa}[(K/\mathbb{F}_q, \sigma, d)]^{sa} = [(M_n(K), \widetilde{\sigma}, d)]^{sa}$, i.e. $[(K/\mathbb{F}_q, \sigma, d)]^{sa} = [(M_n(K), \widetilde{\sigma}, d)]^{sa}$.

There also exist large classes of semifields, e.g. Menichetti algebras, to name just one, that are all semiassociative.

5. Algebras that are not semiassociative

Semiassociative algebras over F may be defined in terms of simple subalgebras of the nucleus whose center is separable over F [4, Section 5]. This excludes nonassociative algebras that have nuclei that are simple subalgebras but whose centers are (purely) inseparable over F, so may create restrictions when $char(F) = p$ is prime. The definition of semiassociative algebras in particular also excludes algebras that have a purely inseparable field extension as their nucleus. This avoids problems when tensoring these algebras, as the tensor product of two field extensions that are not both separable may

An example of such algebras are nonassociative algebras of square dimension that are a canonical generalization of cyclic p-algebras, and of Amitsur's differential algebras ([2, 3, 11], [12, Sections 1.5, 1.8, 1.9]). Their nucleus is a purely inseparable field extension of F:

5.1. Nonassociative differential extensions of a field. Let K be a field of characteristic p together with a algebraic derivation $\delta : K \to K$ of K of degree p with minimum polynomial $g(t) = t^p - t \in F[t]$, where $F = \mathrm{Const}(\delta) = \{a \in K \mid \delta(a) = 0\}$. Put $R = K[t; \delta]$. Then K/F is a purely inseparable extension of exponent one, and $[K : F] = p$. Let $f(t) = t^p - t - d \in K[t; \delta]$, then the nonassociative F-central algebra

$$(K, \delta, d) = K[t; \delta]/K[t; \delta]f$$

has dimension p^2 and is called a *(nonassociative) differential extension* of K. (K, δ, d) is associative if and only if $d \in F$, and is a division algebra if and only if $f \in K[t; \delta]$ is irreducible. If it is not associative, then (K, δ, d) has nucleus K [24].

If $f \in F[t]$ then (K, δ, d) is an associative central simple algebra over F, and K is a maximal subfield of (K, δ, d) of dimension p [12, p. 23]. If $f \in F[t]$ is irreducible, then (K, δ, d) contains the cyclic separable field extension $F[t]/(t^p - t - d)$ of degree p, so can be seen as a canonical generalization of a cyclic p-algebra.

However, for $d \in K \setminus F$, (K,δ,d) is not semiassociative as its nucleus is the purely inseparable field extension K/F of degree p.

The simple K-algebra $(K,\delta,d)\otimes_F K$ contains the simple truncated polynomial algebra $K\otimes_F K$ in its nucleus, and is called a *split differential extension.*

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A semiassociative algebra of degree n is split if and only if F^n is a unital subalgebra of the nucleus . If K is a field that splits an étale subalgebra in the nucleus of an n-dimensional semiassociative algebra A of degree n, and F is an infinite field, then K splits A [4, Theorem 7.1].

5.2. Nonassociative differential extensions of a division algebra. There are classes of algebras over F that have a central simple algebra D over a field C as their nucleus, but the field extension C/F is purely inseparable of degree p, so the center C of D is purely inseparable over F [24]:

Let C be a field of characteristic p and D be a central simple algebra over C of degree n ($D=C$ is allowed and brings us back to the setup of the previous section). Let δ be a derivation of D, such that $\delta|_C$ is algebraic with minimum polynomial $g(t)=t^p-t$, and let $F=\text{Const}(\delta)$. Assume that $g(\delta)=id_{d_0}$ is an inner derivation of D and that there exists $d_0\in F$ so that $\delta(d_0)=0$ (this is always possible if D is a division algebra [12, Lemma 1.5.3]). The center of $R=D[t;\delta]$ is $F[z]$ with $z=g(t)-d_0$. For all $a\in C$, define $V(a)=V_g(a)=V_p(a)-a$. Then $V: C\to F$ is a homomorphism of the additive groups C and F [13].

For all $f(t)=t^p-t-d\in D[t;\delta]$, the nonassociative unital F-algebra defined as

$$(D,\delta,d)=S_f=D[t;\delta]/D[t;\delta]f(t)$$

has dimension p^2n^2 over F and is called a *nonassociative generalized differential algebra.* For $d\in F$, $(D,\delta,d)=D[t;\delta]/D[t;\delta]f(t)$ is a central simple algebra over F (cf. [12, p. 23] if D is a division algebra, Amitsur's associative differential extensions of division rings D were generalized to simple rings D already in [14]).

Indeed, (D,δ,d) is an associative algebra if and only if $d\in F$ [24, Theorem 20].

For $d\in C\setminus F$ we have $D=\text{Nuc}((D,\delta,d))$ (this follows from the corrected version of [24, Lemma 19], which implies that $D\subset \text{Nuc}((D,\delta,d))$). Thus every maximal étale subalgebra N of D/C also lies in the nucleus and has dimension pn as algebra over F. As an algebra over F, N is the product of finite dimensional field extensions that are each of the type N_i/F, where we have a tower of field extensions $F\subset C\subset N_i$, such that N_i/C is separable of degree n and C/F purely inseparable of exponent one. This means we can write every N_i as a tensor product $N_i=S_i\otimes_F C$, where S_i is the maximal separable subfield of N_i/F [12, p. 32], and obtain that $N=N_1\times\cdots\times N_r=(S_1\otimes_F C)\times\cdots\times(S_r\otimes_F C)=(S_1\times\cdots\times S_r)\otimes_F C$ is an étale algebra $S_1\times\cdots\times S_r$ over F tensored over F with the purely inseparable field extension C/F of exponent one.

When D is a division algebra then (D,δ,d) is a division algebra if and only if f is irreducible, if and only if $d\neq V_p(z)-z$ for all $z\in D$, if and only if $d\neq(t-z)^p-t^p-z$ for all $z\in D$.

If $d \in C \setminus F$, then the differential algebra $(C, \delta|_C, d)$ is a subalgebra of (D, δ, d) of dimension p^2.

If $f(t) = t^p - t - d \in F[t]$ is irreducible, then (D, δ, d) contains the cyclic field extension $F[t]/(t^p - t - d)$ of dimension p over F as a subfield.

Let F be a field of characteristic p and D be a central simple algebra over F of degree n. Let K/F be a purely inseparable extension of exponent one such that $[K : F] = p$. Let δ be a derivation on K such that $F = \mathrm{Const}(\delta)$, such that δ is an algebraic derivation of degree p with minimum polynomial $g(t) = t^p - t \in F[t]$ of degree p. Let δ be the extension of δ to D_K such that $\delta|_D = 0$. Then $(K, \delta, d) \otimes_F D \cong (D_K, \delta, d)$ is an algebra of dimension n^2p^2 over F.

In particular, if $D_K = D \otimes_F K$ is a division algebra and (K, δ, d) is a division algebra over F, then $(K, \delta, d) \otimes_F D \cong (D_K, \delta, d)$ is a division algebra if and only if $f(t) = t^p - t - d$ is irreducible in $D_K[t; \delta]$, if and only if $d \neq V_p(z) - z$ for all $z \in D_K$, if and only if $d \neq (t - z)^p - t^p - z$ for all $z \in D_K$ [24].

Remark 19. [24, Theorem 23] Let F have characteristic 3, and δ have minimum polynomial $g(t) = t^3 - ct \in F[t]$. Then for $f(t) = t^3 - ct - d \in C[t; \delta]$, (D, δ, d) is a unital algebra over F of dimension 9, and a division algebra if and only if $V_3(z) - cz \neq d$ and $V_3(z) - zc - d + \delta(c) \neq 0$ for all $z \in D$. (D, δ, d) is associative if and only if $d \in F$.

6. Outlook

While there are good reasons to use the existing definition of $Br^{sa}(F)$ (it is the broadest possible one if we want to use Brauer factor sets), we believe it makes sense to discuss (i) a possible refinement of the semiassociative Brauer monoid to include only simple semiassociative algebras in any characteristic, and (ii) a possible generalization of $Br^{sa}(F)$ that allows up to include nonassociative differential algebras, if the base field F is not perfect and has characteristic p:

(i) If we only consider the simple semiassociative algebras we exclude pathological cases like the associative algebras $(K/F, \sigma, 0)$. The simple semiassociative algebras form a submonoid of $Br^{sa}(F)$ that still contains $Br(F)$ as unique maximal subgroup.

(ii) Suppose we want include generalized differential extensions in the definition of the Brauer monoid. Let A be an F-central nonassociative algebra over F of dimension l^2, $\mathrm{char}(F) = p$. We call A a *generalized semisassociative algebra*, if its nucleus contains a tensor product $N = N_1 \otimes_F \cdots \otimes_F N_s$ of finite field extensions N_i/F such that $dim_F N = l$, with N_i either separable or purely inseparable of exponent one and N_i/F primitive of the kind $N_i = F[x]$ for $x_i^p = a \in F$. If all N_i/F are separable then N is an étale algebra over F and we additionally require that A is cyclic and faithful as $N \otimes N$-module, so that A is semiassociative.

The root of the dimension of A is again called the *degree* of A.

Two generalized semiassociative algebras A and B over F are called *Brauer equivalent*, if there exist skew matrix algebras $M_n(F; c)$ and $M_m(F; c')$ such that $A \otimes_F M_n(F; c) \cong B \otimes_F M_m(F; c')$. This is an equivalence relation, as [4, Remark 6.9] still holds. We denote the equivalence class of a generalized semiassociative algebra A by $[A]^{gsa}$ and the monoid of equivalence classes by $Br^{gsa}(F)$.

Note that every finite purely inseparable field extension of exponent one is a tensor product of primitive extensions $F[x_1]\otimes\cdots\otimes F[x_r]$, where $x_i^p-a_i=0$. Also note that if N is purely inseparable of exponent one, then $N\otimes N$ is a truncated polynomial algebra that is isomorphic to $F[G]$ for a finite abelian p-group G [8]. This means that our N is always the tensor product of an étale algebra over F (the "separable part") and another algebra (the "inseparable part"). This inseparable part is either a purely inseparable field extension of F of exponent one, an F-algebras $F[G]$, or a tensor product of an F-algebra $F[G]$ and a purely inseparable field extension of F of exponent one.

Let A be a generalized semiassociative algebra of degree n with $N\subset \text{Nuc}(A)$, $N=N_1\otimes_F\cdots\otimes_F N_s$ of dimension n over F, with finite field extensions N_i/F where N_i either separable or purely inseparable and primitive of exponent one. Then A is called *split*, if $N\cong E\otimes_F F[G]$, where E/F is a split étale algebra, and $F[G]$ is a simple truncated polynomial algebra (G an abelian p-group). We allow here that $E=F$ or $F[G]=F$. A finite-dimensional field extension E/F *splits* A, if $N\otimes_F E\cong S\otimes_E E[G]$ for a suitable abelian p group G, and an étale algebra S over E. We also note that if $N=N_1\otimes_F N_2$ is the tensor product of a separable and a purely inseparable extension of exponent one, then N is a finite field extension, as these are linearly disjoint over F.

Let A be a generalized semiassociative algebra with $K=\text{Nuc}(A)$ a purely inseparable field extension of exponent one. Then a finite-dimensional field extension E/F splits A, if $K\otimes_F E$ is a simple truncated polynomial algebra, which is the case if and only if $F\subset K\subset E$ is an intermediate field. In particular, K splits (K,δ,d). For the nonassociative generalized differential algebra (D,δ,d) we know that if K is a purely inseparable splitting field of the C-central simple algebra D, then $(D,\delta,d)\otimes_F K\cong (M_n(K),\delta,d)$ is a generalized semiassociative algebra over K whose nucleus contains the K-algebra $K\otimes_F K\cong F[G]$, but the algebra is not split, as the dimension of $F[G]$ is too small.

Moreover, for every central simple algebra D over F, we have $D\otimes_F(K,\delta,d)\cong (D_K,\delta,d)$, so if K is a purely inseparable splitting field of D, then $D\otimes_F(K,\delta,d)\cong (M_n(K),\delta,d)$, which is, however, not a split algebra over F.

Let D be a p-algebra of degree p^s with maximal separable splitting field E and purely inseparable simple splitting field L of degree $p^f\leq p^e$ or degree p^s (so D is cyclic). Then $D\otimes_F(K,\delta,d)\cong (D\otimes_F K,\delta,d)$ contains the field extension $E\otimes_F K\subset D\otimes_F K$ of degree $p^sp=p^{e+1}$ and the algebra $L\otimes_F K\subset D\otimes_F K$ in its nucleus. Here, $L\otimes_F K$ is either a finite purely inseparable field extension of exponent one, or - if $L=K$ - an algebra $F(G)$. In the later case, we get $D\otimes_F(K,\delta,d)\cong (M_{p^s}(K),\delta,d)$.

Alternatively, we could define $Br^{gsa}(F)$ as the submonoid of the above generalized one that is generated by $Br^{sa}(F)$ and the algebras (D,δ,d) and (K,δ,d), and call the resulting algebras *generalized semiassociative algebras*.

In either case, we obtain that

$$[(K,\delta,d)]^{gsa}[D]^{gsa}=[(D_K,\delta,d)]^{gsa},$$

in particular $[(K,\delta,d)]^{gsa}=[(K,\delta,d)]^{gsa}[M_n(F)]^{gsa}=[(M_n(K),\delta,d)]^{gsa}$. Furthermore, if D is a p-algebra of degree p^s and K is a finite-dimensional purely inseparable splitting field of D then

$$[(K,\delta,d)]^{gsa}[D]^{gsa}=[(M_{p^s}(K),\delta,d)]^{gsa}.$$

It would be interesting to explore other relations.

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