Values for the levels and sublevels of algebras obtained by the Cayley-Dickson process

Cristina Flaut
"Ovidius" University of Constanta, Romania
e-mail: cflaut@univ-ovidius.ro;cristina_flaut@yahoo.com

Abstract. In this paper we will prove that any number $n \in \mathbb{N} - \{0\}$ could be realised as level of an algebra obtained by the Cayley-Dickson process over a suitable field.

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0. Introduction

A. Pfister, in [Pf; 65], showed that if a field has a finite level this level is a power of 2 and any power of 2 could be realised as the level of a field. For the quaternion and octonion division algebras, was found 2^k and $2^k + 1$ for all $k \in \mathbb{N} - \{0\}$ like values for the level of these algebras(see [Lew; 87], [La,Ma; 01], [Pu; 05]) and 6 and 7 for the level of the octonion division algebras(see [O' Sh; 10]). These values, 6 and 7, remained the only known exact values for the level of quaternion and octonion division algebras, other than 2^k or $2^k + 1$, $k \in \mathbb{N} - \{0\}$. It is still unknown what exact numbers can be realized as levels and sublevels of quaternion and octonion division algebras. The level problem for the integral domains was solved in [Da, La, Pe; 80] when Z.D. Dai, T. Y. Lam, C. K. Peng proved that any positive integer could be realised as the level of an integral domain.

In this paper, we will show that in the case of level for algebras obtained by the Cayley-Dickson process the situation is the same like for the integral domains, proving that for any positive integer n, there is an algebra A, obtained by the Cayley-Dickson process with the norm form anisotropic over a suitable field, which has level n.

1. Preliminaries

In this paper, we assume that K is a field, $charK \neq 2$ and all the quadratic forms are nondegenerate. For the basic terminology of quadratic and symmetric bilinear spaces, the reader is referred to [Sch; 85] .

Let φ be a $n-$dimensional quadratic irreducible form over K, $n \in N, n > 1$, which is not isometric to the hyperbolic plane. We may consider φ as a homogeneous polynomial of degree 2, $\varphi(X) = \varphi(X_1, ...X_n) = \sum a_{ij} X_i X_j$, $a_{ij} \in K^*$. The *functions field of* φ, denoted $K(\varphi)$, is the quotient field of the integral domain $K[X_1, ..., X_n] / (\varphi(X_1, ..., X_n))$.

A subset P of K is called an *ordering* of K if $P + P \subset P, P \cdot P \subset P, -1 \notin P$, $\{x \in K \ / x \text{ is a sum of squares in } K\} \subset P, P \cup -P = K, P \cap -P = 0$.

A field K with an ordering is called an *ordered* field. For $x, y \in K$, K an ordered field, we define $x > y$ if $(x-y) \in P$.

A *quadratic semi-ordering* (or *q-ordering*) of a field K is a subset P with the following properties: $P + P \subset P, K^2 \cdot P \subset P, 1 \in P, P \cup -P = K, P \cap -P = 0$.

Obviously, every ordering is a q-ordering [Sch; 85]. Let P_0 be a q-preordering, i.e. $P_0 + P_0 \subset P_0, K^2 \cdot P_0 \subset P_0, P_0 \cap -P_0 = 0$. Then there is a q-ordering P such that $P_0 \subset P$ or $-P_0 \subset P$. ([Sch; 85], p.133)

If $\varphi \cong < a_1, ..., a_n >$ is a quadratic form over a formally real field K and P is an ordering on K, *the signature* of φ at P is

$$sgn\left(\varphi\right) = \left|\left\{i \mid \ a_i >_P 0\right\}\right| - \left|\left\{\left\{i \mid \ a_i <_P 0\right\}\right\}\right|.$$

The quadratic form q is *indefinite* at ordering P if $\dim \varphi > |sgn \varphi|$.

The *Witt index* of a quadratic form φ, denoted by $i_W\left(\varphi\right)$, is the dimension of a maximal totally isotropic subform of φ. Indeed, if $\varphi \cong \varphi_{an} \perp \varphi_h$, with φ_{an} anisotropic and φ_h hiperbolic, the Witt index of φ is $\frac{1}{2}\dim \varphi_h$. The *first Witt index* of a quadratic form φ is the Witt index of φ over its function field and is denoted by $i_1\left(\varphi\right)$. The *essential dimension* of φ is $\dim_{es}\left(\varphi\right) = \dim\left(\varphi\right) - i_1\left(\varphi\right) + 1$.

The *sublevel* of the algebra A, denoted by $\underline{s}(A)$, is the least integer n such that 0 is a sum of $n+1$ nonzero squares of elements in A. The *level* of the algebra A, denoted by $s\left(A\right)$, is the least integer n such that -1 is a sum of n squares in A. If these numbers do not exist, then the level and sublevel are infinite. Obviously, $\underline{s}(A) \leq s\left(A\right)$.

In the following, we recall shortly the *Cayley-Dickson process* and the properties of the obtained algebras.

Let A be a finite dimensional unitary algebra over a field K, with a *scalar involution* $\overline{} : A \rightarrow A, a \rightarrow \overline{a}$, i.e. a liniar map satisfying the following relations: $\overline{ab} = \overline{b}\overline{a}$, $\overline{\overline{a}} = a$, and $a + \overline{a}$, $a\overline{a} \in K \cdot 1$ for all $a, b \in A$. The element $\overline{a}$ is called the *conjugate* of the element a, the linear form $t : A \rightarrow K$, $t\left(a\right) = a + \overline{a}$ and the quadratic form $n : A \rightarrow K$, $n\left(a\right) = a\overline{a}$ are called the *trace* and the *norm* of the element a. It results that an algebra A with a scalar involution is quadratic.

Let $\gamma \in K$ be a fixed non-zero element. We define the following algebra multiplication on the vector space $A \oplus A$.

$$\left(a_1, a_2\right)\left(b_1, b_2\right) = \left(a_1 b_1 + \gamma \overline{b_2} a_2, a_2 \overline{b_1} + b_2 a_1\right). \tag{1.1.}$$

We obtain an algebra structure over $A \oplus A$, denoted by (A, γ) and called the *algebra obtained from A by the Cayley-Dickson process.* We have $\dim\left(A, \gamma\right) = 2\dim A$.

Let $x \in \left(A, \gamma\right), x = \left(a_1, a_2\right)$. The map $\overline{} : \left(A, \gamma\right) \rightarrow \left(A, \gamma\right)$, $x \rightarrow \bar{x} = \left(\overline{a}_1, -a_2\right)$, is a scalar involution of the algebra (A, γ), extending the involution $\overline{}$ of the algebra A, therefore the algebra (A, γ) is quadratic, with $t\left(x\right) = t(a_1)$ and $n\left(x\right) = n\left(a_1\right) - \gamma n(a_2)$ the *trace*, respectively, the *norm* of the element $x \in \left(A, \gamma\right)$.

If we take $A=K$ and apply this process t times, $t\geq 1$, we obtain an algebra over K, $A_t=K\{\alpha_1,...,\alpha_t\}$. By induction, in this algebra we find a basis $\{1,f_2,...,f_q\}, q=2^t$, satisfying the properties: $f_i^2=\alpha_i 1,\ \alpha_i\in K,\alpha_i\neq 0,\ i=2,...,q$ and $f_if_j=-f_jf_i=\beta_{ij}f_k,\ \beta_{ij}\in K,\ \beta_{ij}\neq 0, i\neq j, i,j=2,...q,\ \beta_{ij}$ and f_k being uniquely determined by f_i and f_j. We denote $(A_t)_0=\{x\in A_t \mid t(x)=0\}$.

If $x\in A_t, x=x_1 1+\sum\limits_{i=2}^{q}x_if_i$, the quadratic form $T_C: A_t\rightarrow K$,

$T_C=<1,\alpha_1,\alpha_2,-\alpha_1\alpha_2,\alpha_3,...,(-1)^t(\prod\limits_{i=1}^{t}\alpha_i)>=<1,\beta_2,...,\beta_q>$ is called *the trace form* and the quadratic form $T_P=T_C\mid_{(A_t)_0}: (A_t)_0\rightarrow K$,

$T_P=<\alpha_1,\alpha_2,-\alpha_1\alpha_2,\alpha_3,...,(-1)^t(\prod\limits_{i=1}^{t}\alpha_i)>=<\beta_2,...,\beta_q>$ is called *the pure trace form* of the algebra A_t. We remark that $T_C=<1>\perp T_P$ (the orthogonal sum of two quadratic forms) and the norm $n=n_C=<1>\perp -T_P$, therefore $n_C=<1,-\alpha_1,-\alpha_2,\alpha_1\alpha_2,\alpha_3,...,(-1)^{t+1}(\prod\limits_{i=1}^{t}\alpha_i)>=<1,-\beta_2,...,-\beta_q>$.

In general, algebras A_t of dimension 2^t obtained by the Cayley-Dickson process are not division algebras for all $t\geq 1$. But, there are at least two example of fields on which if we apply the Cayley-Dickson process, the obtained algebras A_t are division algebras for all $t\geq 1$. These fields are power-series field $K\{X_1,X_2,...,X_t\}$ and the rational functions field $K(X_1,X_2,...,X_t)$, where $X_1,X_2,...,X_t$ are t algebraically independent indeterminates over the field K. First construction was given by R. B. Brown in [Br; 67] in which he built, for every t, a division algebra A_t of dimension 2^t over the power-series field $K\{X_1,X_2,...,X_t\}$. The second construction are done by C. Flaut in [Fl; 11(2)] over the rational functions field $K(X_1,X_2,...,X_t)$.

In the folllowing, we briefly present this construction. For every t we construct a division algebra A_t over a field F_t. Let $X_1,X_2,...,X_t$ be t algebraically independent indeterminates over the field K and $F_t=K(X_1,X_2,...,X_t)$ be the rational functions field. For $i=1,...,t$, we construct the algebra A_i over the rational functions field $K(X_1,X_2,...,X_i)$ by setting $\alpha_j=X_j$ for $j=1,2,...,i$. Let $A_0=K$. By induction over i, assuming that A_{i-1} is a division algebra over the field $F_{i-1}=K(X_1,X_2,...,X_{i-1})$, then the algebra A_i is a division algebra over the field $F_i=K(X_1,X_2,...,X_i)$.

Indeed, let $A_{F_i}^{i-1}=F_i\otimes_{F_{i-1}}A_{i-1}$. For $\alpha_i=X_i$ we apply the Cayley-Dickson process to algebra $A_{F_i}^{i-1}$. The obtained algebra, denoted A_i, is an algebra over the field F_i and has dimension 2^i.

Proposition 1.1. [Fl; 11(1)] *Let A be an algebra obtained by the Cayley-Dickson process. The following statements are true:*

a) If $n\in\mathbb{N}-\{0\}$, such that $n=2^k-1$, for $k>1$, then $s(A)\leq n$ if and only if $<1>\perp n\times T_P$ is isotropic.

b) If -1 is a square in K, then $\underline{s}(A)=s(A)=1$.

c) If $-1\notin K^{*2}$, then $s(A)=1$ if and only if T_C is isotropic.

Remark 1.2. Using the above proposition, if the algebra A is an algebra obtained by the Cayley-Dickson process, of dimension greater than 2 and if n_C is isotropic, then $s(A) = \underline{s}(A) = 1$. Indeed, if -1 is a square in K, the statement results from above. If $-1 \notin K^{*2}$, since $n_C =< 1 >\perp -T_P$ and n_C is a Pfister form, we obtain that $-T_P$ is isotropic, therefore T_C is isotropic and, from above proposition, we have that $s(A) = \underline{s}(A) = 1$.

Proposition 1.3. [Fl; 11(1)] *Let A be an algebra obtained by the Cayley-Dickson process. With above notations, we have:*

i) If $s(A) \leq n$ then -1 is represented by the quadratic form $n \times T_C$.

ii) -1 is a sum of n squares of pure elements in A if and only if the quadratic form $n \times T_P$ represents -1.

iii) For $n \in \mathbb{N} - \{0\}$, if the quadratic form $< 1 >\perp n \times T_P$ is isotropic over K, then $s(A) \leq n$.

Proposition 1.4. [Hoff; 08, Lemma 2.5.] *Let φ be a quadratic form over a formally real field K_0, $\dim \varphi \geq 2$, and let P be an ordering on K_0. Then P extends to $K_0(\varphi)$ if and only if φ is indefinite at P. In this situation, if ψ is another form over K_0, then $\dim \left(\psi_{K_0(\varphi)}\right)_{an} \geq |sgn_P(\psi)|$.*

Proposition 1.5. [Hoff; 08, Theorem 2.2.] *Let φ and ψ be anisotropic forms over an arbitary field K. If $\varphi_{K(\psi)}$ is isotropic, then $\dim \varphi - i_1(\varphi) \geq \dim \psi - i_1(\psi)$ and equality holds if and only if $\psi_{F(\varphi)}$ is isotropic.*

2. Main results

Let A_t be a division algebra over a field $K = K_0(X_1, ..., X_t)$ obtained by the Cayley-Dickson process, of dimension $q = 2^t$, where K_0 is a formally real field, $X_1, ..., X_t$ algebraically independent indeterminates over the field K_0, T_C and T_P are its trace and pure trace forms. Let

$$\varphi_n =< 1 > \perp n \times T_P, \psi_m =< 1 > \perp m \times T_C,$$

$$A_t(n) = A_t \otimes_K K(< 1 > \perp n \times T_P), n \in \mathbb{N} - \{0\}. \qquad (2.1.)$$

Remark 2.1. i)We denote $K_n = K(< 1 > \perp n \times T_P)$, and let $n_C^{A_t}$ be the norm form of the algebra A_t. Since A_t is a division algebra, it results that $n_C^{A_t}$ is anisotropic over K and $n_C^{A_t(n)}$ is anisotropic over K_n. Indeed, let $x \in A_t(n)$, $x = y \otimes 1, y \in A_t$. If $0 = n_C^{A_t(n)}(x) = x\overline{x} = (y \otimes 1)(\overline{y} \otimes 1) = y\overline{y} \otimes 1 = n_C^A(y)(1 \otimes 1)$, it results that $n_C^{A_t} = 0$ or $1 \otimes 1 = 0$, false. The algebra $A_t(n)$ has dimension 2^t and is not necessarily division algebra, but, using Remark 1.2., this algebra has level greater than 1.

ii) From Proposition 1.3. i) and iii), if ψ_m is anisotropic and if φ_n is isotropic over K_n, then $s(A_t(n)) \in [m + 1, n]$.

Proposition 2.2. $i_1(< 1 > \perp n \times T_P) = 1$ *for all* $n \in \mathbb{N} - \{0\}$, *where T_P is the pure trace form for the algebra A_t.*

Proof. Consider P an arbitrary ordering over K such that $\beta_2, ..., \beta_q <_P 0$. We remark that such an ordering always exists. Indeed, since φ_n is anisotropic over K, from Springer's Theorem, we have that $P_0 = \{a \mid a = 0$ or a is represented by $\varphi_n\}$ is a $q-$preordering, therefore there is a $q-$ordering P containing

P_0 or $-P_0$. We have $|sgn\varphi_n| = |sgn\,(<1> \perp n \times T_P)| = (2^t-1)\,n-1 < (2^t-1)n+1 = \dim\varphi_n$. It results that φ_n is indefinite at P over K, then P extends to K_n. Therefore $(2^t-1)\,n-1 \geq \dim((\varphi_n)_{K_n})_{an}$, since we have only one positive coefficient with respect to P. From Proposition 1.4., it results $(2^t-1)\,n-1 \geq \dim((\varphi_n)_{K_n})_{an} \geq |sgn\varphi_n| = (2^t-1)\,n-1$, so $\dim((\varphi_n)_{K_n})_{an} = (2^t-1)\,n-1 = \dim\varphi_n -2$ and thus $i_1(\varphi_n) = \frac{1}{2}2 = 1.\square$

Theorem 2.3. *With the above notations, we have* $s\,(A_t\,(n)) \in [n-[\frac{n}{2^t}],n]$.

Proof. From Proposition 2.2., we have that $\dim\varphi_n - i_1(\varphi_n) = (2^t-1)n+1-i_1(\varphi_n) = (2^t-1)n$. For the quadratic form ψ_m, relation $\dim\psi_m - i_1(\psi_m) = 2^t n+1-i_1(\psi_m)$ holds. The form φ_n and ψ_m are anisotropic over $K = K_0(X_1,...,X_t)$, by Springer's Theorem. Supposing that ψ_m are isotropic over K_n, from Proposition 1.5., we have $\dim\psi_m - i_1(\psi_m) \geq \dim\varphi_n - i_1(\varphi_n)$. It results $2^t m+1-i_1(\psi_m) \geq (2^t-1)n$. Therefore, if

$$2^t m+1-i_1(\psi_m) < (2^t-1)n, \tag{2.2.}$$

we have ψ_m is anisotropic over K_n, so, from Remark 2.1. i), it results $s\,(A_t\,(n)) \in [m+1,n]$. We note that $i_1(\psi_m) \geq 1$ and, from relation (2.2.), the highest value of m such that ψ_m is isotropic over K_n is $n-[\frac{n}{2^t}]-1$. Indeed, relation (2.2.) implies $2^t m < (2^t-1)n$, therefore $m < n-\frac{n}{2^t}$ and we obtain $m \leq n-[\frac{n}{2^t}]-1.\square$

Theorem 2.4. *With the above notations, we have* $\underline{s}(A_t\,(n)) \in [n-[\frac{n+2^t-1}{2^t}],n], n \in \mathbb{N}-\{0\}$.

Proof. Using Proposition 1.3. i), if the quadratic form $\phi_m = (m+1)\times T_C$ is anisotropic, then $\underline{s}(A_t\,(n)) \geq m+1$ and if φ_n is isotropic, then $\underline{s}(A_t\,(n)) \leq n$. Using the same arguments like in the proof of Theorem 2.3., if

$$2^t\,(m+1)-i_1(\phi_m) < (2^t-1)n, \tag{2.3.}$$

we have ϕ_m is anisotropic over K_n, therefore $\underline{s}(A_t\,(n)) \in [m+1,n]$. We note that $i_1(\phi_m) \geq 1$ and, from relation (2.3.), the highest value of m such that ϕ_m is isotropic over K_n is $n-[\frac{n+2^t-1}{2^t}]-1$. Indeed, relation (2.3.) implies $2^t\,(m+1)-1 < (2^t-1)n$, therefore $m < n\frac{2^t-1}{2^t}+\frac{1}{2^t}-1 = n-\frac{n+2^t-1}{2^t}$ and we obtain $m \leq n-[\frac{n+2^t-1}{2^t}]-1.\square$

Theorem 2.5. *With the above notation, for each* $n \in \mathbb{N}-\{0\}$ *there is an algebra* $A_t\,(n)$ *such that* $s\,(A_t\,(n)) = n$ *and* $\underline{s}(A_t\,(n)) \in \{n-1,n\}$.

Proof. Let $n \in \mathbb{N}-\{0\}$ and m be the least positive integer such that $n \leq 2^m$. For $n = 2^m$, there are quaternion $(A_2\,(n))$ and octonion $(A_3\,(n))$ division algebras of level $n = 2^m$, (see [La,Ma; 01] and [Pu; 05]). We suppose that $n < 2^m$. For $t = m$, let $A_t\,(n)$ be algebra of dimension $q = 2^t$ given by the relation (2.1.). From Theorem 2.3., this algebra has level $s\,(A_t\,(n)) \in [n-[\frac{n}{2^t}],n]$ and sublevel $\underline{s}(A_t\,(n)) \in [n-[\frac{n+2^t-1}{2^t}],n], n \in \mathbb{N}-\{0\}$. Since $n < 2^t$, it results that $[\frac{n}{2^t}] = 0$ and $[\frac{n+2^t-1}{2^t}] = 1$, therefore $s\,(A_t\,(n)) = n$ and $\underline{s}(A_t\,(n)) \in \{n-1,n\}.\square$

Remark. 2.6. Theorem 2.5. gives a positive partial answer to the question that any number $n \in \mathbb{N}-\{0\}$ can be realised as a level of composition algebras.

The answer becomes positive if we replace "composition algebras" with "algebras obtained by the Cayley-Dickson process", therefore, we can say that any number $n \in \mathbb{N} - \{0\}$ can be realised as a level of an algebra obtained by the Cayley-Dickson process over a suitable field.

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