A CONSTRUCTION METHOD FOR SOME REAL DIVISION ALGEBRAS WITH $su(3)$ AS DERIVATION ALGEBRA

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Abstract. We obtain a new family of eight-dimensional unital division algebras over a field F out of a separable quadratic field extension S of F, a three-dimensional anisotropic hermitian form h over S of determinant one and a scalar $c \in S^\times$ not contained in F. These algebras are not third-power associative.

Over $\mathbb{R}$, this yields a family of unital division algebras with derivation algebra isomorphic to $su(3)$ and automorphism group isomorphic to $SU(3)$. The algebra is the direct sum of two one-dimensional modules and a six-dimensional irreducible $su(3)$-module. Mutually non-isomorphic families of Albert isotopes of these algebras with the same properties are considered as well.

Introduction

In the early 1980s, real division algebras were roughly classified by Benkart and Osborn according to the isomorphism type of their derivation algebra [B-O1]. In the special case that the Lie algebra of derivations of an eight-dimensional real division algebra A is isomorphic to $su(3)$, they showed that A must be either an eight-dimensional irreducible $su(3)$-module or the direct sum of two one-dimensional modules and a six-dimensional irreducible $su(3)$-module [B-O2]. If A is an eight-dimensional irreducible $su(3)$-module, A was shown to be a flexible generalized pseudo-octonion algebra.

For a real division algebra A with $\mathrm{Der}(A) \cong su(3)$ which is reducible as $su(3)$-module, a multiplication table was given [B-O2, (4.2)] and it was shown that every real algebra defined by this table admits $su(3)$ as derivation algebra [B-O2, Theorem 4.1]. The multiplication table contains 16 different scalars, and the authors admitted that "the question of whether a real algebra with multiplication given by (4.2) is a division algebra is a formidable one because of the large number of scalars in the multiplication table." They presented one family of division algebras as an example [B-O1, Theorem 20, Corollary 21]. Another family was discussed in [J-P], Section 4.3. In [Do-Z2], Dokovich and Zhao gave three necessary conditions for an algebra with such a multiplication table to be a real division algebra and achieved the partial result that A is division if and only if a certain subalgebra is [Do-Z2, Proposition 4.1]. They also determined the possible automorphism groups of such a division algebra and when two such algebras are isomorphic. In a list of still open questions, they asked for necessary and sufficient conditions for the algebras with multiplication table [B-O2, (4.2)] to be division algebras. Such conditions were obtained in [Do-Z1] in a special case

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where the equalities [Do-Z2, (4.1)] hold. The algebras were shown to be an enlargement of the truncated algebra of strictly pure octonions and $\mathrm{Aut}(A) \cong SU(3)Z_2$, apart from one special case where $\mathrm{Aut}(A) \cong G_2$ [Do-Z1]. In [P-I], Pérez-Izquierdo classified division composition algebras via their derivation algebras and exhibited among others families of composition division algebras with derivation algebras isomorphic to $su(3)$ over a field of characteristic not 2 or 3.

In this paper, we construct unital algebras over a field F by generalizing a known construction method for octonion algebras using a hermitian form over a quadratic étale algebra. This method is presented in Section 2. The description of the algebras is straightforward and base free. They are not third-power associative (therefore not quadratic) and contain a quadratic étale algebra as a subalgebra. Over the reals, we obtain a family of unital eight-dimensional division algebras with automorphism group isomorphic to $SU(3)$ and derivation algebra isomorphic to $su(3)$, whose multiplication fits into table [B-O2, (4.2)], see Section 3. This family is different from the ones given in [B-O2] and [Do-Z1]. All our real division algebras contain $\mathbb{C}$ and a nonassociative quaternion subalgebra, which is unique up to isomorphism. The nonassociative quaternion subalgebra corresponds to the subalgebra mentioned in [Do-Z2, Proposition 4.1]. By strictly truncating our algebras we obtain the real algebra $(\mathbb{C}^3, \times)$, however our algebras are not enlargements of $(P, \times) = (\mathbb{C}^3, \times)$ in the sense of [Do-Z1, 4.1], because (making free use of their terminology here) the restriction homomorphism $Z_G(\pi) \to \mathrm{Aut}(P, \mu_S)$ is only onto for the subgroup $SU(3)$ of $\mathrm{Aut}(P, \mu_{\mathbb{S}}) \cong SU(3)Z_2$.

We then use our family of unital division algebras to construct families of eight-dimensional non-unital division algebras in Section 4, which over $F = \mathbb{R}$ again satisfy the multiplication table [B-O2, (4.2)]. Their automorphism group is again isomorphic to $SU(3)$ and their derivation algebra to $su(3)$. As a byproduct, we obtain conditions for certain scalar constants in the multiplication table [B-O2, (4.2)], where the equalities [Do-Z2, (4.1)] do not hold, to be the scalar constants of a division algebra.

1. Preliminaries

1.1. **Nonassociative algebras.** Let F be a field. By "F-algebra" we mean a finite dimensional unital nonassociative algebra over F.

A nonassociative algebra A is called a *division algebra* if for any $a \in A$, $a \neq 0$, the left multiplication with a, $L_a(x) = ax$, and the right multiplication with a, $R_a(x) = xa$, are bijective. A is a division algebra if and only if A has no zero divisors [Sch, pp. 15, 16].

For an F-algebra A, associativity is measured by the *associator* $[x, y, z] = (xy)z - x(yz)$. A is called *alternative* if its associator $[x, y, z]$ is alternating. An anti-automorphism $\sigma : A \to A$ of period 2 is called an *involution* on A. If F has characteristic not 2, we have $A = \mathrm{Sym}(A, \sigma) \oplus \mathrm{Skew}(A, \sigma)$ with $\mathrm{Skew}(A, \sigma) = \{x \in A \mid \sigma(x) = -x\}$ the set of skew-symmetric elements and $\mathrm{Sym}(A, \sigma) = \{x \in A \mid \sigma(x) = x\}$ the set of symmetric elements in A with respect to σ. An involution is called *scalar* if all *norms* $\sigma(x)x$ are elements of $F1$. For every scalar involution σ, $N_A(x) = \sigma(x)x$ (resp. the *trace* $T_A(x) = \sigma(x) + x$) is a quadratic (resp. a linear) form on A. A is called *quadratic*, if there exists a quadratic form $N \colon A \to F$ such that $N(1_A) = 1$ and $x^2 - N(1_A, x)x + N(x)1_A = 0$ for all $x \in A$, where

$N(x,y)$ denotes the induced symmetric bilinear form $N(x,y)=N(x+y)-N(x)-N(y)$. The form N is uniquely determined and called the *norm* $N=N_A$ of the quadratic algebra A [Pu2]. The existence of a scalar involution on an algebra A implies that A is quadratic [M1].

Let S be a quadratic étale algebra over F (i.e., a separable quadratic F-algebra in the sense of [Knu, p. 4]) with canonical involution $\sigma\colon S\to S$, also written as $\sigma={}^-$, and with nondegenerate norm $N_{S/F}\colon S\to F$, $N_{S/F}(s)=s\overline{s}=\overline{s}s$. S is a two-dimensional unital commutative associative algebra over F. With the diagonal action of F, $F\times F$ is a quadratic étale algebra with canonical involution $(x,y)\mapsto(y,x)$.

An F-algebra C is called a *unital composition algebra* or a *Hurwitz algebra* if it has a unit element and carries a quadratic form $n\colon C\to F$ whose induced symmetric bilinear form $N(x,y)$ is *nondegenerate*, i.e., determines an F-vector space isomorphism $C\to C^\vee=\mathrm{Hom}_F(C,F)$, and which satisfies $N(xy)=N(x)N(y)$ for all $x,y\in C$. Hurwitz algebras are quadratic alternative; any nondegenerate quadratic form N on the Hurwitz algebra which permits composition is uniquely determined up to isometry. It is called the *norm* of C and is denoted by $N_{C/F}$. A quadratic alternative algebra is a Hurwitz algebra if and only if its norm is nondegenerate [M, 4.6]. Hurwitz algebras only exist in ranks 1, 2, 4 or 8. Those of dimension 2 are exactly the quadratic étale F-algebras, those of dimension 4 exactly the well-known quaternion algebras. The ones of dimension 8 are called *octonion algebras.* A Hurwitz algebra C has a *canonical involution* ${}^-$ given by $\overline{x}=T_{C/F}(x)1_C-x$, where $T_C\colon C\to F$, $T_{C/F}(x)=N_{C/F}(1_C,x)$, is the *trace* of C. This involution is scalar.

1.2. **The generalized Cayley-Dickson doubling process.** (cf. [Pu2])

Let D be a unital algebra over F with an involution $\sigma:D\to D$. Let $c\in D$ be an invertible element not contained in F such that $\sigma(c)\neq c$. Then the F-vector space $A=D\oplus D$ can be made into a unital algebra over F via the multiplications

(1) $$(u,v)(u',v')=(uu'+c(\sigma(v')v),v'u+v\sigma(u'))$$

(2) $$(u,v)(u',v')=(uu'+\sigma(v')(cv),v'u+v\sigma(u'))$$

or

(3) $$(u,v)(u',v')=(uu'+(\sigma(v')v)c,v'u+v\sigma(u'))$$

for $u,u',v,v'\in D$. The unit element of the new algebra A is given by $1=(1,0)$ in each case.

A is called the *Cayley-Dickson doubling of D (with scalar c on the left hand side, in the middle, or on the right hand side)* and denoted by $\mathrm{Cay}(D,c)$ for multiplication (1), by $\mathrm{Cay}_m(D,c)$ for multiplication (2) and by $\mathrm{Cay}_r(D,c)$ for multiplication (3). We call every such algebra obtained from a Cayley-Dickson doubling of D, with the scalar c in the middle, resp. on the left or right hand side, a *Dickson algebra* over F.

1.3. **Flexible quadratic algebras.** (cf. [Pu1])

Let M be a finite dimensional F-vector space. An alternating F-bilinear map $\times:M\times M\to M$ is called a *cross product* on M. Let D be an associative F-algebra with a scalar involution $\sigma={}^-$. Let P be a locally free right D-module of constant finite rank s together with a sesquilinear form $h:P\times P\to D$ (i.e., h is a biadditive map such that $h(ua,vb)=$

$\bar{a}h(u,v)b$ for all $a,b \in D$, $u,v \in P$). Let $\times$ be a cross product on P, where now P is viewed as an F-vector space. I.e., together with $\times : P \times P \to P$, P is an alternating F-algebra (and anticommutative if char $F \neq 2$). The F-vector space $A = D \oplus P$ becomes a unital F-algebra denoted by $A = (D, P, h, \times)$ via the multiplication

$$(a,u)(b,v) = (ab - h(v,u), va + u\bar{b} + v \times u),$$

for all $a,b \in D$, $u,v \in P$. D is a subalgebra of $(D, F, h, \times)$.

For all $a \in D$, $u \in P$, define

$$\begin{aligned} &\sigma_A : A \to A, \quad (a,u) \to (\overline{a}, -u), \\ &N_{A/F} : A \to D, \quad N_{A/F}((\alpha,u)) = \sigma_A(a,u)(a,u), \\ &T_{A/F} : A \to R, \quad T_{A/F}((a,u)) = \sigma_A(a,u) + (a,u) = (T_D(a), 0). \end{aligned}$$

Obviously, $\ker(T_{A/F}) = \ker(T_{D/F}) \oplus P$ and $u \times v = uv - \frac{1}{2}N_{A/F}(u,v)$. σ_A is a scalar involution if and only if h is a hermitian form (i.e., $h(u,v) = \overline{h(v,u)}$ for all $u,v \in F$). If h is a (perhaps degenerate) hermitian form, then $T : A \times A \to F, T(x,y) = T_{A/F}(xy)$ is a symmetric F-bilinear form and $A = (D, F, h, \times)$ is a quadratic F-algebra with scalar involution σ_A and norm $N_{A/F}$, where $N_{A/F}((a,u)) = N_{D/F}(a) + h(u,u)$. Moreover, in that case $N_{A/F}$ is isotropic iff A has zero divisors.

If $h : F \times F \to D$ is a hermitian form, then $(D, P, h, \times)$ is flexible if and only if

$$h(u \times v, u) + \overline{h(u \times v, u)} = N_{A/F}(u \times v, u) = 0$$

and

$$(u \times v) \times u = u \times (v \times u)$$

for all $u,v \in P$. Moreover, then $(D, P, h, \times)$ is alternative if and only if $h(u, u \times v) = 0$ and $u \times (u \times v) = -h(u,u)v + h(v,u)u$ for all $u,v \in P$, if and only if $h(u \times v, v) = 0$ and $(u \times v) \times v = h(v,v)u - h(u,v)v$ for all $u,v \in P$.

If $(D, F, h, \times)$ is flexible with a scalar involution, then it is a noncommutative Jordan algebra, i.e. we have $(xy)x^2 = x(yx^2)$ for all x,y [M1, (3.3)]. If $\times$ is the zero-map, then $(D, F, h, 0)$ is trivially flexible.

If D is a composition algebra of dimension ≤ 4 over F with canonical involution $^-$ and $h : D \times D \to D$ a nondegenerate $^-$-hermitian form, then there is $c \in F^\times$ such that $h(u,u) = cN_{D/F}(u)$ for all $u \in D$ and

$$\mathrm{Cay}(D, -c) = (D, D, h, 0).$$

1.4. **A construction method for octonion algebras.** (cf. Petersson-Racine [P-R, 3.8] or Thakur [T])

Let S be a quadratic étale F-algebra with canonical involution $^-$. Let (P,h) be a ternary nondegenerate $^-$-hermitian space (P a projective S-module) such that $\bigwedge^3(P,h) \cong \langle 1 \rangle$. Choose an isomorphism $\alpha : \bigwedge^3(P,h) \to \langle 1 \rangle$ and define a cross product $\times_\alpha : P \times P \to P$ via

$$h(u \times_\alpha v, w) = \alpha(u \wedge v \wedge w)$$

as in [T, p. 5122]. The F-vector space $\mathrm{Cay}(S,P,h,\times_\alpha)=S\oplus P$ becomes an octonion algebra under the multiplication

$$(a,u)(b,v)=(ab-h(v,u),va+u\bar{b}+u\times_\alpha v)$$

for all $u,v\in P$ and $a,b\in S$, with norm

$$N((a,u))=n_S(a)+h(u,u).$$

So $\mathrm{Cay}(S,P,h,\times_\alpha)=(S,P,h,-\times_\alpha)$. If the ternary hermitian space (P,h) is orthogonally decomposable (which is always the case if S is a separable quadratic field extension) this construction is independent of the choice of the isomorphism α and we may simply write $\mathrm{Cay}(S,P,h)$. Any octonion algebra over F can be constructed like this. For $h=\langle e\rangle\perp h_2$ and $D=\mathrm{Cay}(S,-e)$,

$$\mathrm{Cay}(S,P,\langle e\rangle\perp h_2)\cong\mathrm{Cay}(D,-q_{h_2})$$

with $q_{h_2}(x)=h_2(x,x)$ for all $x\in P_2$.

1.5. **Nonassociative quaternion algebras.** A *nonassociative quaternion algebra* is a four-dimensional unital F-algebra A whose nucleus is a quadratic étale algebra over F. Let S be a quadratic étale algebra over F with canonical involution $^-$. For every $b\in S\setminus F$, the vector space

$$\mathrm{Cay}(S,b)=S\oplus S$$

becomes a nonassociative quaternion algebra over F with unit element $(1,0)$ and nucleus S under the multiplication

$$(u,v)(u',v')=(uu'+b\overline{v}'v,v'u+v\overline{u}')$$

for $u,u',v,v'\in S$. This means that $\mathrm{Cay}(S,b)=(S,S,-bh,0)$ with $h(v',v)=\overline{v}'v$. Given any nonassociative quaternion algebra A over F with nucleus S, there exists an element $b\in S\setminus F$ such that $A\cong\mathrm{Cay}(S,b)$ [As-Pu, Lemma 1]. $\mathrm{Cay}(S,b)$ is a division algebra if and only if S is a separable quadratic field extension of F [W, p. 369]. Two nonassociative quaternion algebras $\mathrm{Cay}(K,b)$ and $\mathrm{Cay}(L,c)$ can only be isomorphic if $L\cong K$. Moreover,

$$\mathrm{Cay}(K,b)\cong\mathrm{Cay}(K,c)\text{ iff }g(b)=N_{K/F}(d)c$$

for some automorphism $g\in\mathrm{Aut}(K)$ and some non-zero $d\in K$ [W, Theorem 2] (see also [Al-H-K, Thm. 14] for $F=\mathbb{R}$).

2. The generalized construction

2.1. Let D be an associative F-algebra with a scalar involution $^-$. Let P be a locally free right D-module of constant finite rank s together with a sesquilinear form $h:P\times P\to D$. Let $\times$ be a cross product on P and let $c\in D^\times$ and not in F. The F-vector space $A=D\oplus P$ becomes a unital F-algebra denoted by $(D,P,ch,\times)$ via the multiplication

$$(a,u)(b,v)=(ab-ch(v,u),va+u\bar{b}+v\times u),$$

and if D is not commutative, also a unital F-algebra denoted by $(D,P,hc,\times)$ via

$$(a,u)(b,v)=(ab-h(v,u)c,va+u\bar{b}+v\times u),$$

for all $a,b\in D$, $u,v\in P$.

Lemma 1. *Let $A=(D,P,ch,\times)$ or $A=(D,P,hc,\times)$.*
(i) D is a subalgebra of A.
(ii) Suppose that $\ker(T_{D/F})=F$. If P is torsion-free and h a hermitian form such that $h(u,u)\neq 0$ for some $u\in P$ then A is not third power-associative and not quadratic.
(iii) Let D be a composition algebra of dimension 2 or 4 and $h=\langle e\rangle\perp b$ a hermitian form. Suppose that $\times|_{D\times D}=0$, e.g. if already $\times=0$ or if $D=S$ is quadratic étale, $\mathrm{Cay}(S,P,h,\times_\alpha)$ an octonion algebra (see 1.3.) and $A=\mathrm{Cay}(S,P,ch,\times_\alpha)$. Then

$$\mathrm{Cay}(D,-ce)$$

is a subalgebra of $A=(D,P,c(\langle e\rangle\perp b),\times)$ and

$$\mathrm{Cay}_r(D,-ce)$$

a subalgebra of $A=(D,P,(\langle e\rangle\perp b)c,\times)$.

Proof. (i) and (iii) are trivial.
(ii) Let $u\in P$ such that $h(u,u)\neq 0$. For $l=(0,u)$, we have $l^2=(-ch(u,u),0)$ and so $ll^2=(0,-uh(u,u)\bar{c})$ while $l^2l=(0,-uch(u,u))$. Thus $l^2l=ll^2$ if and only if $uh(u,u)\bar{c}=uch(u,u)$. If P is torsion-free, this is equivalent to $\bar{c}=c$. Hence A is not third power-associative. Every quadratic unital algebra is clearly power-associative, so A is not quadratic. □

In particular, if S is a quadratic étale algebra, the nonassociative quaternion algebras

$$\mathrm{Cay}(S,-ce),\mathrm{Cay}(S,-cf) \text{ and } \mathrm{Cay}(S,-cef)$$

are subalgebras of $A=\mathrm{Cay}(S,S^3,c\langle e,f,ef\rangle)$.

Theorem 2. *(i) Let D be a Hurwitz division algebra of dimension $n=2$ or 4 and h an anisotropic hermitian form. Then D is the only Hurwitz subalgebra of $A=(D,P,ch,\times)$, resp. $A=(D,P,hc,\times)$, of dimension n.*
(ii) Suppose that $C=\mathrm{Cay}(S,P,h)$ is an octonion algebra with S a separable field extension, h anisotropic, and assume $c\in S\setminus F$. If $\mathrm{Cay}(K,d)$ is a nonassociative quaternion subalgebra of $A=\mathrm{Cay}(S,P,ch)$ then $K=S$ and there is $u\in P$ such that $d=-ch(u,u)$.
For $F=\mathbb{R}$, $\mathrm{Cay}(\mathbb{C},-c)$ is up to isomorphism the only nonassociative quaternion subalgebra of $\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle)$.

Proof. (i) Let $n=2$ and $D=S$. Let F have characteristic not 2 and let $K=F(\sqrt{e})$ be a quadratic field extension contained in A. Then there is an element $X\in A$, $X=(a,u)$ with $a\in S$, $u\in P$ such that $X^2=e\in F^\times$, i.e.,

$$a^2-ch(u,u)=e \text{ and } u(a+\bar{a})=0.$$

If $u\neq 0$ then $\bar{a}=-a$ hence $a^2=-N_{S/F}(a)\in F$, thus $c=-h(u,u)^{-1}(N_{S/F}(a)+e)\in F$ which is a contradiction since the left hand side lies in S and not in F, while the right hand side lies in F. Therefore $u=0$, $a^2=e$ and $X=(a,0)\in(S,0)$ implies $K=S$.
Let F have characteristic 2 and suppose K is a separable quadratic field extension of F

contained in A. Hence there is an element $X = (a, u) \in A$, $a \in S$, $u \in P$ such that $X^2 + X = e \in F^\times$. This implies

$$a^2 - ch(u,u) + a = e \text{ and } u(a + \bar{a} + 1) = 0.$$

If $u \neq 0$ then $\bar{a} + a + 1 = 0$. This implies $a = -(\bar{a} + 1)$, thus $\overline{a^2} + \bar{a} - ch(u,u) = e$ i.e. we get $\overline{a^2 + a} + ch(u,u) = a^2 + a + ch(u,u)$ which yields $\overline{a^2 + a} = a^2 + a$. Hence $a^2 + a \in F$. This implies that $ch(u,u) = e - (a^2 + a) \in F$, a contradiction. Hence $u = 0$ which implies $a^2 + a = e$ and $X = (a, 0) \in D$. Thus the basis $1, X = a$ for K lies in S and we obtain $S = K$.

Suppose next that $n = 4$. Let F have characteristic not 2. Suppose there is a quaternion subalgebra $B = (e,f)_F$ in A. Then there is an element $X \in A$, $X = (a, u)$ with $a \in D$, $u \in P$, such that $X^2 = e \in F^\times$ and an element $Y \in A$, $Y = (b, v)$ with $b \in D$, $v \in P$, such that $Y^2 = f \in F^\times$ and $XY + YX = 0$. The first equation implies $X = (a, 0)$ with $a^2 = e$, the second that $Y = (b, 0)$ with $b^2 = f$ as in (i). Hence $(0, 0) = XY + YX = (ab + bu, 0)$ means $ab + bu = 0$ and so the standard basis $1, X = a, Y = b, XY = ab$ for the quaternion algebra $(e,f)_F$ lies in D and we obtain $D = (e,f)_F$.

Let F have characteristic 2. Suppose there is a quaternion subalgebra $B = [e,f)$ in A. Then there is an element $X \in A$, $X = (a, u)$ with $a \in D$, $u \in P$, such that $X^2 + X = e \in F$ and an element $Y \in A$, $Y = (b, v)$ with $b \in D$, $v \in P$, such that $Y^2 = f \in F^\times$ and $XY = YX + Y$. Analogously as before, $Y^2 = f \in F^\times$ implies $b^2 = f$ and $Y = (b, 0)$, $b \in D$. The first equation implies $X = (a, 0) \in D$, $a^2 + a = e$, as in (i) Now $XY = YX + Y$ means $ab = ba + b$ and the standard basis $1, X = a, Y = b$ for the quaternion algebra $[e,f)$ lies in D. We obtain $D = [e,f)$.

(ii) Analogously as in the proof of (i) we obtain $K = S$ with $S = (S, 0) \subset A = S \oplus P$, hence $d \in S \setminus F$. The fact that $\mathrm{Cay}(S,d) \subset A$ implies that there is an element $Z = (a, u) \in A$ such that $Z^2 = (d, 0)$ and $(b, 0)(a, u) = (a, u)(\bar{b}, 0)$ for all $b \in S$. This is equivalent to

$$a^2 - ch(u,u) = d, \quad u(a + \bar{a}) = 0 \text{ and } ab = a\bar{b}$$

for all $b \in S$. If $u = 0$ then $a^2 = e$ means $a \neq 0$ and $ab = a\bar{b}$ for all $b \in S$ implies a contradiction. Thus $u \neq 0$. Since $ab = a\bar{b}$ for all $b \in S$ implies $a = 0$ we obtain $d = -ch(u,u)$.

Suppose that $F = \mathbb{R}$ and $A = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle)$. Build a basis of $\mathbb{C}^3$ starting with u, which is orthogonal with respect to the anisotropic form $h = \langle 1,1,1\rangle$. Then $h(u,u) = e \in \mathbb{R}_{>0}$. Thus $d = -ce = -cN_{\mathbb{C}/\mathbb{R}}(x)$ for some $x \in \mathbb{C}$ and $\mathrm{Cay}(\mathbb{C}, d) = \mathrm{Cay}(\mathbb{C}, -cN_{\mathbb{C}/\mathbb{R}}(x)) \cong \mathrm{Cay}(\mathbb{C}, -c)$ (Section 1.5) is up to isomorphism the only nonassociative quaternion subalgebra of A. □

Corollary 3. *Let D and B be two Hurwitz algebras over F of dimension n, and let $n = 2$ or 4, $(D, P, h, \times)$ and $(B, P', h', \times')$ two division algebras over F and h, h' hermitian forms. Let $c \in D^\times \setminus F$ and $d \in B^\times \setminus F$, and suppose*

$$(D, P, ch, \times) \cong (B, P', c'h', \times') \text{ or } (D, P, hc, \times) \cong (B, P', h'c', \times').$$

Then $D \cong B$.

Proof. By Theorem 2, any isomorphism maps the unique subalgebra D of $(D, P, ch, \times)$ to the unique subalgebra B of $(B, P', c'h', \times')$, respectively of $(B, P', h'c', \times')$, hence $D \cong B$. □

Let D be a Hurwitz division algebra of dimension 2 or 4 and h an anisotropic hermitian form. Then there exists an orthogonal basis for the hermitian space h, so that $h \cong \langle e_1, \dots, e_n \rangle$ [Knu, p. 30] and for $n \geq 2$, $(D, P, h, 0)$ is not a division algebra over F: take z_1, z_2 orthogonal with respect to h, then

$$(0, z_1)(0, z_2) = (-h(z_1, z_2), 0) = (0, 0),$$

therefore the algebra has zero divisors.

Theorem 4. *Let D be a Hurwitz algebra of dimension 2 or 4, h a hermitian form and $C = (D, P, h, \times)$ a division algebra over F. Suppose that either $\times$ is the zero map or that $h(v \times u, u) = 0$ for all $u, v \in P$. Then $A = (D, P, ch, \times)$, resp. $A = (D, P, hc, \times)$, is a division algebra over F, for any choice of $c \in D^\times$ not in F.*

Proof. (i) We show that $A = (D, P, ch, \times)$ has no zero divisors: suppose

$$(0, 0) = (a, u)(b, v) = (ab - ch(v, u), va + u\bar{b} + v \times u)$$

for $a, b \in D$, $u, v \in P$. This is equivalent to

$$ab - ch(v, u) = 0 \text{ and } va + u\bar{b} + v \times u = 0.$$

If $a = 0$ then $h(v, u) = 0$, thus

$$(a, u)(b, v) = (ab - ch(v, u), va + u\bar{b} + v \times u) = (a, u) \cdot_C (b, v)$$

in this case. Since C is a division algebra, this implies that $(a, u) = (0, 0)$ or $(b, v) = (0, 0)$. If $a \neq 0$ then $ab = ch(v, u)$ means $b = a^{-1}ch(v, u)$, $\bar{b} = \overline{h(v, u)}\bar{c}\bar{a}^{-1}$, therefore $va + u\overline{h(v, u)}\bar{c}\bar{a}^{-1} + v \times u = 0$.

For $\times = 0$ we get $v = -u\bar{b}a^{-1}$ and substituting this into the first equation gives $ab + ch(u\bar{b}a^{-1}, u) = ab + c\bar{a}^{-1}bh(u, u) = 0$, i.e $(a + c\bar{a}^{-1}bh(u, u))b = 0$. Thus $b = 0$ or $a + c\bar{a}^{-1}bh(u, u) = 0$. If $b = 0$ then $h(v, u) = 0$ and $(b, v) = (0, 0)$, since by assumption C is a division algebra and in this case the multiplications $(a, u)(b, v)$ and $(a, u) \cdot_C (b, v)$ are identical again. If $a + c\bar{a}^{-1}bh(u, u) = 0$ then $N_{D/F}(a) + h(u, u)c = 0$ which means $h(u, u)c = -N_{D/F}(a) \in F^\times$ and hence $c \in F^\times$, a contradiction.

If we have $h(v \times u, u) = 0$ for all $u, v \in P$, then $va = -u\bar{b} - v \times u$, so $v = -u\bar{b}a^{-1} - (v \times u)a^{-1}$. Substituting this into the first equation gives $ab + ch(u\bar{b}a^{-1}, u) + ch((v \times u)a^{-1}, u) = ab + c\bar{a}^{-1}bh(u, u) + c\bar{a}^{-1}h((v \times u), u) = ab + c\bar{a}^{-1}bh(u, u) = 0$ as above, and again we obtain $(b, v) = (0, 0)$.

(ii) $A = (D, P, hc, \times)$ has no zero divisors: suppose

$$(0, 0) = (a, u)(b, v) = (ab - h(v, u)c, va + u\bar{b} + v \times u)$$

for $a, b \in D$, $u, v \in P$. This is equivalent to

$$ab - h(v, u)c = 0 \text{ and } va + u\bar{b} + v \times u = 0.$$

If $a = 0$ then $h(v,u)c = 0$ and the same proof as in (i) shows that $(a,u) = (0,0)$ or $(b,v) = (0,0)$.
If $a \neq 0$ and $\times = 0$ we get $v = -u\bar{b}a^{-1}$ and substituting this into the first equation gives $ab + h(u\bar{b}a^{-1}, u)c = ab + \bar{a}^{-1}bh(u,u)c = 0$, multiply by $\bar{a}$ from the left to obtain $N_{D/F}(a)b + bh(u,u)c = b(N_{D/F}(a)b + h(u,u)c)$. Thus $b = 0$ or $N_{D/F}(a)b + h(u,u)c = 0$, both cases leading to the same conclusions as in (i). □

Among others, this result contains [W, p. 369] and [Pu2, Theorem 11] as special cases:

Corollary 5. *Let $C = \mathrm{Cay}(S,P,h)$ be an octonion division algebra, or $C = \mathrm{Cay}(D,e)$ a Hurwitz division algebra with D a Hurwitz algebra of dimension 2 or 4. Then $\mathrm{Cay}(S,P,ch)$, $\mathrm{Cay}(D,ce)$ and $\mathrm{Cay}(D,ec)$ are division algebras over F, for any choice of $c \in D^\times$ not in F.*

Proof. Since C is alternative, $h(v \times u, u) = 0$ for all $u, v \in P$ by 1.3. □

Example 6. Let $F = \mathbb{Q}$ and $C = \mathrm{Cay}(\mathbb{Q}, a, b, e) = \mathrm{Cay}(\mathbb{Q}(\sqrt{a}), \langle -b, -e, be\rangle)$ an octonion algebra. Suppose $a, b, e < 0$, then C is a division algebra and so is the unital algebra

$$\mathrm{Cay}(\mathbb{Q}(\sqrt{a}), c\langle -b, -e, be\rangle)$$

for all $c \in \mathbb{Q}(\sqrt{a}) \setminus \mathbb{Q}$.

Proposition 7. *Let D be a Hurwitz algebra of dimension 2 or 4. Let (P,h) and (P',h') be two hermitian spaces over D and $c, c' \in D^\times$. Let $\times$ be a cross product on P and $\times'$ be a cross product on P'.*
(i) Suppose that there is a D-module isomorphism $f: P \to P'$ such that $c'h'(f(v), f(u)) = ch(v,u)$ (resp., $h'(f(v), f(u))c' = ch(v,u)c$ or $c'h'(f(v), f(u)) = h(v,u)c$) for all $u, v \in P$. If $f(v \times u) = f(v) \times' f(u)$ for all $u, v \in P$, then

$$(D, P, ch, \times) \cong (D, P', c'h', \times')$$

and resp.,

$$(D, P, hc, \times) \cong (D, P', h'c', \times') \text{ or } (D, P, hc, \times) \cong (D, P', c'h', \times').$$

(ii) Suppose $(P,h) \cong (P',h')$ with isometry f. If $f(v \times u) = f(v) \times' f(u)$ for all $u, v \in P$, e.g. if $\times = \times' = 0$, then

$$(D, P, ch, \times) \cong (D, P', ch', \times').$$

Proof. (i) We show the first case in (i): Let $f: P \to P'$ be a D-module isomorphism.
Take the F-linear map $G(a,u) = (a, f(u))$. Then

$$G((a,u)(b,v)) = G(ab - ch(v,u), va + u\bar{b} + v \times u) = (ab - ch(v,u), f(v)a + f(u)\bar{b} + f(v \times u))$$

and

$$G(a,u)G(b,v) = (a, f(u))(b, f(v)) = (ab - ch'(f(v), f(u)), f(v)a + f(u)\bar{b} + f(v) \times f(u)),$$

hence G is multiplicative iff for all $u, v \in P$:

$$ch(v,u) = c'h'(f(v), f(u)) \text{ and } f(v \times u) = f(v) \times f(u).$$

The other two cases in (i) are shown analogously.
(ii) follows from (i). □

The case that D is a Hurwitz division algebra of dimension 2 or 4, $h : D \times D \to D$, $h(u,v) = \bar{v}au$ the hermitian form and $\times = 0$ has been dealt with already in [Pu2] and [W].

It remains to closer investigate the division algebras of the type $\mathrm{Cay}(S, S^3, ch)$ with h an anisotropic hermitian form.

2.2. Eight-dimensional division algebras and their automorphism group. From now on let S be a quadratic étale algebra over F with canonical involution $^-$. For $c \in S \setminus F$, we consider the eight-dimensional unital algebra

$$\mathrm{Cay}(S, P, ch, \times_\alpha) = S \oplus P,$$

$$(a,u)(b,v) = (ab - ch(v,u), va + u\bar{b} + u \times_\alpha v)$$

for $a, b \in S$, $u, v \in P$. Note that $(P, \times_\alpha)$ is the strictly truncated (anticommutative) algebra (P, μ_S) [Do-Z1, 2] obtained from $\mathrm{Cay}(S, P, ch, \times_\alpha)$. Every isometry $f : (P, h) \to (P, h)$ yields an automorphism of $(P, \times_\alpha)$, thus $SU(3) \subset \mathrm{Aut}(P, \times_\alpha)$.

For $A = \mathrm{Cay}(S, S^3, c\langle e_1, e_2, e_1e_2\rangle)$ define $^- : S^3 \to S^3$ via

$$u = (u_1, u_2, u_3) \to \overline{u} = \overline{(u_1, u_2, u_3)} = (\overline{u_1}, \overline{u_2}, \overline{u_3}).$$

Clearly, $^- \in \mathrm{Aut}(S^3, \times_\alpha)$.

Let $\mathrm{Cay}(S, P, h)$ be an octonion division algebra and $G : \mathrm{Cay}(S, P, ch) \longrightarrow \mathrm{Cay}(S, P', c'h')$ be an algebra isomorphism. We have $G((S,0)) \subset (S,0)$ (this can be checked directly by a similar calculation as in the proof of Theorem 2), thus $G((S,0)) = (S,0)$.

Proposition 8. *Let $\mathrm{Cay}(S, P, h)$ and $\mathrm{Cay}(S, P', h')$ be two octonion division algebras and $c, c' \in S \setminus F$.*
(i) Let $G : \mathrm{Cay}(S, P, ch) \longrightarrow \mathrm{Cay}(S, P', c'h')$ be an algebra isomorphism with $G((0,P)) = (0, P')$. If $G((a,u)) = (a, g(u))$ with $g = G|_P$, then $g(v \times u) = g(u) \times' g(v)$ for all $u, v \in P$, $\bar{c}/c = \bar{c}'/c'$ and $(P, ch) \cong (P', c'h')$ as ε-hermitian forms with isometry g, where $\varepsilon = \bar{c}/c$.
If $G((a,u)) = (\bar{a}, g(u))$ with $g = G|_P$, then $g(v \times u) = g(u) \times' g(v)$ for all $u, v \in P$, $g(va) = g(v)\bar{a}$ for all $a \in S$, $v \in P$ and $c' = \alpha\bar{c}$ for some $\alpha \in F^\times$.
(ii) Suppose $(P, h) \cong (P', h')$. Then $\mathrm{Cay}(S, P, ch) \cong \mathrm{Cay}(S, P, ch')$. In particular,

$$\mathrm{Cay}(S, S^3, c\langle e_1, e_2, e_1e_2\rangle) \cong \mathrm{Cay}(S, S^3, cd^2\langle e_1, e_2, e_1e_2\rangle)$$

for all $e_i, d \in F^\times$.
(iii) If $(P, ch) \cong (P', c'h')$ as ε-hermitian spaces with isometry f, $\varepsilon = \bar{c}/c$, and if $f(v \times u) = f(v) \times' f(u)$ for all $u, v \in P$, then $\mathrm{Cay}(S, P, ch, \times_\alpha) \cong \mathrm{Cay}(S, P, c'h', \times_\alpha)$.
(iv) $\mathrm{Cay}(S, S^3, c\langle e_1, e_2, e_1e_2\rangle) \cong \mathrm{Cay}(S, S^3, \bar{c}\langle e_1, e_2, e_1e_2\rangle)$ for all $e_i \in F^\times$.

Proof. (i) Suppose first that $G|_{(S,0)} = id$, then $G((a,u)) = (a, g(u))$ with $g = G|_P$ and $G((a,u)(b,v)) = G(a,u)G(b,v)$ is equivalent to

$$G(ab - ch(v,u), va + u\bar{b} + u \times v) = (ab - ch(v,u), g(va) + g(u\bar{b}) + g(u \times v))$$

$$= (ab - c'h'(g(v), g(u)), g(v)a + g(u)\bar{b} + g(u) \times' g(v))$$

for all $a,b\in S$, $u,v\in P$ which implies that $ch(v,u)=c'h(g(v),g(u))$ and $g(v\times u)=g(u)\times' g(v)$ for all $u,v\in P$ (put $a=b=0$). Moreover, $g(va)=g(v)a$ for all $a\in S$, $v\in P$ (just put $u=0$). This means that ch and $c'h'$ are isometric ε-hermitian forms with $\varepsilon=\bar c/c=\bar c'/c'$.

Suppose now that $G|_{(S,0)}={}^-$ and $G((0,P))=(0,P')$, so that $G((a,u))=(\bar a,g(u))$ with $g=G|_P$. Then $G((a,u)(b,v))=G(a,u)G(b,v)$ is equivalent to

$$G(ab-ch(v,u),va+u\bar b+v\times u)=(\bar a\bar b-\bar c\overline{h(v,u)},g(va)+g(u\bar b)+g(u\times v))$$
$$=(\bar a\bar b-c'h'(g(v),g(u)),g(v)\bar a+g(u)b+g(u)\times' g(v))$$

for all $a,b\in S$, $u,v\in P$ which implies that $\bar c\overline{h(v,u)}=c'h'(g(v),g(u))$ and $g(v\times u)=g(u)\times' g(v)$ for all $u,v\in P$. Moreover, $g(va)=g(v)\bar a$ for all $a\in S$, $v\in P$. Now $\bar ch(u,u)=c'h'(g(u),g(u))$ implies that $c'^{-1}\bar c\in F^\times$, i.e. $c'=\alpha\bar c$ for some $\alpha\in F^\times$ and so $\overline{h(v,u)}=\alpha h'(g(v),g(u))$ for all $u,v\in P$.
(ii) This follows directly from the proof of (i) employing [T, Section 2] which implies that $f(v\times_\alpha u)=f(v)\times_{\alpha'} f(u)$ if $(P,h)\cong(P',h')$ with isometry f. Use that for all $d\in F^\times$, $\langle e_1,e_2,e_1e_2\rangle\cong d^2\langle e_1,e_2,e_1e_2\rangle$ for the second part of the assertion.
(iii) is trivial.
(iv) A straightforward calculation using that $h(\bar v,\bar u)=\overline{h(v,u)}$ shows that $F((a,u))=(\bar a,\bar u)$ yields an isomorphism. □

Proposition 9. *Let* $\mathrm{Cay}(S,P,h,\times_\alpha)$ *be an octonion algebra and* $A=\mathrm{Cay}(S,P,ch,\times_\alpha)$. *Then* $SU(3)\subset\mathrm{Aut}(A)$.

Proof. Every isometry $f:(P,h)\to(P,h)$ yields an F-linear bijection $F((a,u))=(a,f(u))$ on A. F is multiplicative if and only if $F((a,u)(b,v))=F(a,u)F(b,v)$ which is equivalent to

$$(ab-ch(v,u),f(v)a+f(u)\bar b+f(u\times_\alpha v))$$
$$=(ab-ch(f(v),f(u)),f(v)a+f(u)\bar b+f(u)\times_\alpha f(v)),$$

i.e. equivalent to $h(v,u)=h(f(v),f(u))$ and $f(u\times_\alpha v)=f(u)\times_\alpha f(v)$ for all $u,v\in P$. Hence F is multiplicative if and only if $f(u\times_\alpha v)=f(u)\times_\alpha f(v)$ which is satisfied for every isometry f, cf. [T, Section 2]. □

Remark 10. Let $c\in S^\times$. We observe that our previous results easily carry over to the opposite algebra of $A=\mathrm{Cay}(S,P,ch,\times_\alpha)$. If S is a separable quadratic field extension, then S is the only field extension contained in A^{op}. In particular, A^{op} has the same derivation algebra as A.

Let $A=\mathrm{Cay}(S,S^3,c\langle e_1,e_2,e_1e_2\rangle)$. Suppose that S is a separable quadratic field extension and $h=\langle e_1,e_2,e_1e_2\rangle$ anisotropic, $e_1,e_2\in F^\times$. Let u,v be an F-basis of S and $z_1,\dots,z_6$ an orthogonal basis of $P=S^3$ with respect to the nondegenerate quadratic form q_h associated to h. Let $C=\mathrm{Cay}(S,S^3,h)$ be the octonion algebra associated to $A=\mathrm{Cay}(S,S^3,ch)$. Then

$$(0,z_i)(0,z_j)=(0,z_i\times z_j)\text{ hence }(0,z_i)(0,z_j)=(0,z_i)\cdot_C(0,z_j),$$
$$(a,z_i)(0,z_j)=(0,z_ja+z_i\times z_j)\text{ hence }(a,z_i)(0,z_j)=(a,z_i)\cdot_C(0,z_j),$$

$$(0, z_i)(b, z_j) = (0, z_i\bar{b} + z_i \times z_j) \text{ hence } (0, z_i)(b, z_j) = (0, z_i) \cdot_C (b, z_j),$$

for all $i \neq j$ and

$$(0, z_i)(0, z_i) = (-ch(z_i, z_i), 0) = (c, 0)(-h(z_i, z_i), 0) \text{ hence } (0, z_i)(0, z_i) = (c, 0)\cdot_C[(0, z_i)\cdot_C(0, z_j)]$$

for all $z_i, z_j \in P$, $i \neq j$. By strictly truncating our algebras we hence obtain the algebra $(P, \times) = (S^3, \times)$. Moreover, $SP \subset P$ and $PS \subset P$. However, our algebras are not enlargements of $(P, \mu_S) = (S^3, \times)$ in the sense of [Do-Z1, 4.1], because (making free use of their terminology here) as we will see in the next section for $F = \mathbb{R}$, the restriction homomorphism $Z_G(\pi) \to \mathrm{Aut}(P, \mu_S)$ is only onto for the subgroup $SU(3)$ of $\mathrm{Aut}(P, \mu_S) \cong SU(3)Z_2$.

3. The case $F = \mathbb{R}$

For $F = \mathbb{R}$, $\mathbb{O} = \mathrm{Cay}(\mathbb{H}, -1) = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, \langle 1, 1, 1\rangle)$ is up to isomorphism the only octonion division algebra over $\mathbb{R}$. With the choice of basis as given in [B-O2], $\mathbb{O}$ fits into multiplication table (4.2) in [B-O2] which is our Table 2. We will choose the basis $u = 1$, $v = i$ and

$$z_1 = (0, (1, 0, 0)), z_2 = (0, (0, 1, 0)), z_3 = (0, (i, 0, 0)), z_4 = (0, (0, 0, 1)), z_5 = (0, (0, 0, i)), z_6 = (0, (0, i, 0))$$

for $\mathbb{O}$. Using this basis, the argument in [B-O2, p. 278] yields Table 1 instead of (4.2) in [B-O2]. I.e., by choosing this basis, we have to slightly adjust multiplication table [B-O2, (4.2)]: instead of the 6×6-matrix in the lower right hand corner of the multiplication table given by

$\cdot$	z_1	z_2	z_3	z_4	z_5	z_6
z_1	$-u$	z_4	v	$-z_2$	z_6	$-z_5$
z_2	$-z_4$	$-u$	z_5	z_1	$-z_3$	v
z_3	$-v$	$-z_5$	$-u$	z_6	z_2	$-z_4$
z_4	z_2	$-z_1$	$-z_6$	$-u$	v	z_3
z_5	$-z_6$	z_3	$-z_2$	$-v$	$-u$	z_1
z_6	z_5	$-v$	z_4	$-z_3$	$-z_1$	$-u$

we have instead the multiplication table

$\cdot$	z_1	z_2	z_3	z_4	z_5	z_6
z_1	$-u$	z_4	v	$-z_2$	$-z_6$	$-z_5$
z_2	$-z_4$	$-u$	z_5	z_1	z_3	v
z_3	$-v$	$-z_5$	$-u$	$-z_6$	z_2	$-z_4$
z_4	z_2	$-z_1$	z_6	$-u$	v	$-z_3$
z_5	z_6	$-z_3$	$-z_2$	$-v$	$-u$	z_1
z_6	z_5	$-v$	z_4	z_3	$-z_1$	$-u$

The rest of the table stays the same. An algebra fits into multiplication table (4.2) in [B-O2], i.e. Table 2, if and only if it fits into Table 1. We point out that our basis and the equivalent Table 1 was already used in [Do-Z1].

This table contains 16 parameters. In this case, the parameters satisfy

$$(*) \quad \eta_2 = \eta_3 = \theta_1 = \theta_4 = \sigma_2 = \sigma_3 = \tau_2 = \tau_3 = 0$$

and

$$(**) \quad \theta_2 = \sigma_1, \quad \theta_3 = \tau_1, \quad \sigma_4 = 1, \quad \eta_4 = \tau_4 = -1.$$

Note that $(*)$ is [Do-Z2, (4.1)], $(**)$ is [Do-Z2, (4.2)].

For every non-real $c \in \mathbb{C}$,

$$A = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle)$$

is a division algebra over $\mathbb{R}$ and $\mathrm{Cay}(\mathbb{C}, -c)$ is up to isomorphism the only nonassociative quaternion subalgebra of A (Theorem 2). A fits into multiplication table 1 as follows:
We need a real basis $u, v, z_1, \dots z_6$ of A such that the vectors u, v span the subalgebra $S = \mathbb{C}$ and $z_1, \dots, z_6$ are an orthonormal basis of the quadratic form induced by $h = \langle 1,1,1\rangle$. More precisely, choose $z_1, \dots, z_6$ as above. Let $c = x + iy$ with $x, y \in \mathbb{R}$, $y \neq 0$. The multiplication table now forces the choice of $u = c$ and $v = ic$. Since

$$(0, z_i)(0, z_j) = (0, z_i \times z_j) = (0, z_i) \cdot_{\mathbb{O}} (0, z_j)$$

unless $z_i = sz_j$ for some $s \in \mathbb{C}$, the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $\mathbb{O}$, with the exception of its diagonal entries being $-u$ because of

$$(0, z_i)(0, z_i) = (-ch(z_i, z_i), 0) = (-c, 0)$$

and the entries of the form

$$(0, z_1)(0, z_3) = (-ch(z_1, z_3), 0) = (ic, 0) = -(0, z_3)(0, z_1),$$
$$(0, z_2)(0, z_6) = (-ch(z_1, z_3), 0) = (ic, 0) = -(0, z_6)(0, z_2),$$
$$(0, z_4)(0, z_5) = (-ch(z_1, z_3), 0) = (ic, 0) = -(0, z_5)(0, z_4).$$

The choice of $u = c$, $v = ic$ gives the following structure constants in Table 1:

$$\eta_1 = x, \quad \eta_2 = -y = \eta_3, \quad \eta_4 = -x,$$
$$\theta_1 = y, \quad \theta_2 = x = \theta_3, \quad \theta_4 = -y,$$
$$\sigma_1 = x, \quad \sigma_2 = y, \quad \sigma_3 = -y, \quad \sigma_4 = x,$$
$$\tau_1 = x, \quad \tau_2 = -y, \quad \tau_3 = -y, \quad \tau_4 = -x.$$

The algebra generated by u, v, z_1, z_3 is the nonassociative quaternion subalgebra $\mathrm{Cay}(\mathbb{C}, -c)$.

Remark 11. For all non-real c, both $\langle u, v, z_1, z_3\rangle \cong \mathrm{Cay}(\mathbb{C}, -c)$ and $A = \mathrm{Cay}(\mathbb{C}, c\langle 1,1,1\rangle)$ are division algebras. Hence [Do-Z2, Proposition 4.1] is a trivial observation for our family of algebras.

Since our division algebras fit into Table 1 we obtain:

Theorem 12. *For all nonreal c and $A = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle)$,*

$$\mathrm{Der}(A) \cong su(3)$$

and A is the direct sum of two irreducible 1-dimensional modules and the irreducible 6-dimensional module $P = \mathbb{C}^3$. $SU(3)$ is the identity component of $\mathrm{Aut}(A)$.

$\cdot$	u	v	z_1	z_2	z_3	z_4	z_5	z_6
u	$\eta_1 u+\theta_1 v$	$\eta_2 u+\theta_2 v$	$\sigma_1 z_1+\sigma_2 z_3$	$\sigma_1 z_2+\sigma_2 z_6$	$-\sigma_2 z_1+\sigma_1 z_3$	$\sigma_1 z_4+\sigma_2 z_5$	$-\sigma_2 z_4+\sigma_1 z_5$	$-\sigma_1 z_2+\sigma_2 z_6$
v	$\eta_3 u+\theta_3 v$	$\eta_4 u+\theta_4 v$	$\sigma_3 z_1+\sigma_4 z_3$	$\sigma_3 z_2+\sigma_4 z_6$	$-\sigma_4 z_1+\sigma_3 z_3$	$\sigma_3 z_4+\sigma_4 z_5$	$-\sigma_4 z_4+\sigma_3 z_5$	$-\sigma_4 z_2+\sigma_3 z_6$
z_1	$\tau_1 z_1+\tau_2 z_3$	$\tau_3 z_1+\tau_4 z_3$	$-u$	z_4	v	$-z_2$	$-z_6$	$-z_5$
z_2	$\tau_1 z_2+\tau_2 z_6$	$\tau_3 z_2+\tau_4 z_6$	$-z_4$	$-u$	z_5	z_1	z_3	v
z_3	$-\tau_2 z_1+\tau_1 z_3$	$-\tau_4 z_1+\tau_3 z_3$	$-v$	$-z_5$	$-u$	$-z_6$	z_2	$-z_4$
z_4	$\tau_1 z_4+\tau_2 z_5$	$\tau_3 z_4+\tau_4 z_5$	z_2	$-z_1$	z_6	$-u$	v	$-z_3$
z_5	$-\tau_2 z_4+\tau_1 z_5$	$-\tau_4 z_4+\tau_3 z_5$	z_6	$-z_3$	$-z_2$	$-v$	$-u$	z_1
z_6	$-\tau_2 z_2+\tau_1 z_6$	$-\tau_4 z_2+\tau_3 z_6$	z_5	$-v$	z_4	z_3	$-z_1$	$-u$

Table 1

$\cdot$	u	v	z_1	z_2	z_3	z_4	z_5	z_6
u	$\eta_1 u+\theta_1 v$	$\eta_2 u+\theta_2 v$	$\sigma_1 z_1+\sigma_2 z_3$	$\sigma_1 z_2+\sigma_2 z_6$	$-\sigma_2 z_1+\sigma_1 z_3$	$\sigma_1 z_4+\sigma_2 z_5$	$-\sigma_2 z_4+\sigma_1 z_5$	$-\sigma_1 z_2+\sigma_2 z_6$
v	$\eta_3 u+\theta_3 v$	$\eta_4 u+\theta_4 v$	$\sigma_3 z_1+\sigma_4 z_3$	$\sigma_3 z_2+\sigma_4 z_6$	$-\sigma_4 z_1+\sigma_3 z_3$	$\sigma_3 z_4+\sigma_4 z_5$	$-\sigma_4 z_4+\sigma_3 z_5$	$-\sigma_4 z_2+\sigma_3 z_6$
z_1	$\tau_1 z_1+\tau_2 z_3$	$\tau_3 z_1+\tau_4 z_3$	$-u$	z_4	v	$-z_2$	z_6	$-z_5$
z_2	$\tau_1 z_2+\tau_2 z_6$	$\tau_3 z_2+\tau_4 z_6$	$-z_4$	$-u$	z_5	z_1	$-z_3$	v
z_3	$-\tau_2 z_1+\tau_1 z_3$	$-\tau_4 z_1+\tau_3 z_3$	$-v$	$-z_5$	$-u$	z_6	z_2	$-z_4$
z_4	$\tau_1 z_4+\tau_2 z_5$	$\tau_3 z_4+\tau_4 z_5$	z_2	$-z_1$	$-z_6$	$-u$	v	z_3
z_5	$-\tau_2 z_4+\tau_1 z_5$	$-\tau_4 z_4+\tau_3 z_5$	$-z_6$	z_3	$-z_2$	$-v$	$-u$	z_1
z_6	$-\tau_2 z_2+\tau_1 z_6$	$-\tau_4 z_2+\tau_3 z_6$	z_5	$-v$	z_4	$-z_3$	$-z_1$	$-u$

Table 2

Proof. The first statement is [B-O2], Proposition 4.1. A is not irreducible as $su(3)$-module, or else our algebras would be generalized pseudo-octonion algebras, which they are not. $SU(3)$ is the identity component of $\mathrm{Aut}(A)$ [Do-Z2, Proof of Proposition 4.3., p. 768]. □

Furthermore, for every $G \in \mathrm{Aut}(A)$, $G((\mathbb{C},0)) = (\mathbb{C},0)$ and $G((0,\mathbb{C}^3)) = (0,\mathbb{C}^3)$, cf. [Do-Z2, Proposition 4.3., p. 768]. This however implies that $G \in \mathrm{Aut}(A)$ is of the type $G((a,u)) = (a, f(u))$ with f an isometry of the hermitian form $\langle 1,1,1\rangle$ by an argument as in the proof of Proposition 8 (i).

Note that the structure constants of our algebras do not satisfy equation $(*)$ which is [Do-Z2, (4.1)] and hence [Do-Z2, Proposition 4.3] yields that $\mathrm{Aut}(A) \cong SU(3)$. We give a more direct proof as well:

Corollary 13. *For every non-real c, the automorphisms of $\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle)$ are given by $G((a,u)) = (a, f(u))$ where f is an isometry of $h = \langle 1,1,1\rangle$ and $\mathrm{Aut}(A) = SU(3)$.*

Proof. Let $G : \mathrm{Cay}(\mathbb{C},\mathbb{C}^3,ch) \longrightarrow \mathrm{Cay}(\mathbb{C},\mathbb{C}^3,ch)$ be an algebra automorphism. Then $G((0,\mathbb{C}^3)) = (0,\mathbb{C}^3)$, cf. the proof of Proposition 4.3. in [Do-Z2], and $G|_P$ is an isometry of $\langle 1,1,1\rangle$. Conversely, every isometry $f : (P,h) \to (P,h)$ yields an automorphism $G((a,u)) = (a, f(u))$. □

Proposition 14. $\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle) \cong \mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c'\langle 1,1,1\rangle)$ *if and only if $c' = c$ or $c' = \bar{c}$.*

This follows from [Do-Z2, Proposition 4.4.], it is also straightforward to prove directly applying Proposition 8.

4. Some families of non-unital division algebras

Following the notation introduced in [P, Section 1], denote the set of possibly non-unital algebra structures on an F-vector space by $\mathrm{Alg}(V)$. Given $A \in \mathrm{Alg}(V)$, we write xAy for the product of $x, y \in V$ in the algebra, if it is not clear from the context which multiplication is used. Let $G = \mathrm{Gl}(V) \times \mathrm{Gl}(V)$ be the direct product of two copies of the full linear group of V. It acts on $\mathrm{Alg}(V)$ by means of *principal Albert isotopes*: For $f, g \in \mathrm{Gl}(V)$ define the algebra $A^{(f,g)}$ as V together with the new multiplication

$$xA^{(f,g)}y = f(x)g(y) \qquad x, y \in V.$$

This defines a right action of G on $\mathrm{Alg}(V)$ which is compatible with passing to the opposite algebra, i.e., $(A^{(f,g)})^{op} = (A^{op})^{(f,g)}$. If A is a division algebra, so is $A^{(f,g)}$. Regular, thus in particular division algebras, are principal Albert isotopes of unital algebras [P, 1.5].

Every composition algebra is a principal Albert isotope of a Hurwitz algebra: There are isometries φ_1, φ_2 of the norm N_C for a suitable Hurwitz algebra C over F such that its multiplication can be written as

$$x \star y = \varphi_1(x) C \varphi_2(y).$$

Given a Hurwitz algebra C over F of dimension ≥ 2 with canonical involution $^-$, the multiplications

$$x \star y = \bar{x}\bar{y}, \quad x \star y = \bar{x}y, \quad x \star y = x\bar{y}$$

for all $x, y \in C$ define the *para-Hurwitz algebra*, resp. the *left-* and *right composition algebra* associated to C. Together with C these are called the *standard composition algebras*.

Standard composition algebras of dimension eight satisfy Table 1 and have derivation algebra isomorphic to G_2. The automorphism group of the para-octonion algebra is isomorphic to G_2 [P-I].

In light of the above, we look at some principal Albert isotopes of our algebras $A = \mathrm{Cay}(S, P, ch, \times_\alpha)$ with $c \in S^\times$. Denote the multiplication in A by $\cdot$ or just juxtaposition as before.

If $V = U \oplus W$ with U the underlying two-dimensional vector space of S, W the underlying six-dimensional vector space of P, for $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathrm{Gl}(V)$ with $f_1, g_1 \in \mathrm{Gl}(U)$, $f_2, g_2 \in \mathrm{Gl}(W)$, the algebra $A^{(f,g)}$ contains the two-dimensional subalgebra $S^{(f,g)} = S^{(f_1,g_1)}$. If $f_1, g_1 \in \mathrm{Gl}(U)$ are isometries of the norm $N_{S/F}$ then $A^{(f,g)}$ contains the two-dimensional composition subalgebra $S^{(f_1,g_1)}$. For $c \in F^\times$, the multiplication

$$(u, v)A^{(f,g)}(u', v') = (f_1(u), f_2(v))(g_1(u'), g_2(v'))$$

yields a composition algebra. Our algebras $A^{(f,g)}$ could thus be considered as division algebras which are generalizations of these 'associated' composition algebras.

Let $\varepsilon \in \{1, -1\}$ and $h \in \mathrm{Gl}(U)$. Define $h_\varepsilon : A \to A$ by

$$h_\varepsilon((a, u)) = (h(a), \varepsilon u)$$

and a new multiplication $\star$ via

$$xA^{(h_\varepsilon,h_\varepsilon)}y = h_\varepsilon(x)h_\varepsilon(y), \quad xA^{(h_\varepsilon,id)}y = h_\varepsilon(x)y, \text{ resp. } xA^{(id,h_\varepsilon)}y = xh_\varepsilon(y),$$

for all $x, y \in A$. In particular, we look at the special case $\sigma_\varepsilon : A \to A$ defined by

$$\sigma_\varepsilon((a, u)) = (\bar{a}, \varepsilon u).$$

Moreover, let $\bar{h}_\varepsilon(a, u) = (h(a), \varepsilon \bar{u})$. Then we will also investigate the algebras $A^{(\bar{h}_\varepsilon,\bar{h}_\varepsilon)}$, $A^{(\bar{h}_\varepsilon,id)}$ and $A^{(id,\bar{h}_\varepsilon)}$.

Obviously, each of the above $(A, \star)$ is a (non-unital) division algebra if and only if $(A, \cdot)$ is a division algebra.

For $c \in F^\times$ and σ_{-1} this yields the para-octonion algebra (resp., the left or right octonion algebra) associated to the octonion algebra A. For $c \in S \setminus F$, the algebras $A^{(h_\varepsilon,h_\varepsilon)}$, $A^{(h_\varepsilon,id)}$ and $A^{(id,h_\varepsilon)}$ can hence be considered as generalized para-octonion algebras, resp., generalizations of left or right octonion algebras.

Lemma 15. *For the algebras $A^{(h_\varepsilon,h_\varepsilon)}$, $A^{(h_\varepsilon,id)}$ and $A^{(id,h_\varepsilon)}$, $SU(3)$ is contained in their automorphism group.*

Proof. Let $G \in \mathrm{Aut}(A, \cdot)$ such that $G(a, u)) = (a, f(u))$, f an isometry of h (cf. Proposition 10). Then $G((h(a), \varepsilon u)) = (h(a), \varepsilon f(u)) = h_\varepsilon(G((a, u)))$. Therefore

$$G((a, u)A^{(h_\varepsilon,h_\varepsilon)}(b, v)) = G((h(a), \varepsilon u)(h(b), \varepsilon v)) = G((h(a), \varepsilon u))G((h(b), \varepsilon v))$$
$$= G((a, u))A^{(h_\varepsilon,h_\varepsilon)}G((b, v))$$

and so $G \in \mathrm{Aut}(A^{(h_\varepsilon,h_\varepsilon)})$. The argument is analogous for the other cases. $\square$

From now on let

$$F = \mathbb{R} \text{ and } (A, \cdot) = (\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1, 1, 1\rangle), \cdot)$$

with $c \in \mathbb{C}^\times$. In the following, we will show that the non-unital algebras A, $A^{(h_\varepsilon, h_\varepsilon)}$, $A^{(\bar{h}_\varepsilon, \bar{h}_\varepsilon)}$, $A^{(h_1, id)}$ and $A^{(id, h_1)}$ fit into multiplication Table 1 and that equation $(*)$ is not satisfied for any of them. Therefore we know by [B-O2] and [Do-Z2, Proposition 4.3]:

Theorem 16. *The algebras* $A = (\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1, 1, 1\rangle))$, $A^{(h_\varepsilon, h_\varepsilon)}$, $A^{(\bar{h}_\varepsilon, \bar{h}_\varepsilon)}$, $A^{(h_1, id)}$ *and* $A^{(id, h_1)}$ *all have derivation algebra isomorphic to* $su(3)$ *and automorphism group isomorphic to* $SU(3)$. *They are the direct sum of two irreducible 1-dimensional modules and an irreducible 6-dimensional module.*

Unless stated otherwise, define the vectors $u = c$, $v = ic$, $z_1, \dots, z_6$ as above. We fix the following notation: Let $c = x + iy$ with $x, y \in \mathbb{R}$, $y \neq 0$. Let $h \in \mathrm{Gl}(U)$ such that $h(u) = \alpha + i\beta$ and $h(v) = \delta + i\gamma$, $\alpha, \beta, \delta, \gamma \in \mathbb{R}$. Let $c' = x' + iy'$ with $x', y' \in \mathbb{R}$, $y' \neq 0$. Let $h' \in \mathrm{Gl}(U)$ such that $h'(c') = \alpha' + i\beta'$ and $h'(v) = \delta' + i\gamma'$, $\alpha', \beta', \delta', \gamma' \in \mathbb{R}$.

4.1. The algebra $A^{(h_\varepsilon, h_\varepsilon)}$ fits into Table 1

$$(0, z_i)A^{(h_\varepsilon, h_\varepsilon)}(0, z_j) = (0, (\varepsilon z_i) \times (\varepsilon z_i)) = (0, z_i) \cdot (0, z_j)$$

the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $(A, \cdot)$. We obtain the following structure constants:

$$\eta_1 = \frac{2\alpha\beta y + (\alpha^2 - \beta^2)x}{x^2 + y^2}, \quad \eta_2 = \eta_3 = \frac{(\alpha\gamma + \beta\delta)y + (\alpha\delta - \beta\gamma)x}{x^2 + y^2}, \quad \eta_4 = \frac{2\delta\gamma y + (\delta^2 - \gamma^2)x}{x^2 + y^2}$$

$$\theta_1 = -\frac{(\alpha^2 - \beta^2)y - 2\alpha\beta x}{x^2 + y^2}, \quad \theta_2 = \theta_3 = -\frac{(\beta\gamma - \alpha\delta)y + (\alpha\gamma + \beta\delta)x}{x^2 + y^2}, \quad \theta_4 = \frac{(\gamma^2 - \delta^2)y + 2\delta\gamma x}{x^2 + y^2},$$

$$\sigma_1 = \varepsilon\alpha, \quad \sigma_2 = \varepsilon\beta, \quad \sigma_3 = \varepsilon\delta, \quad \sigma_4 = \varepsilon\gamma,$$

$$\tau_1 = \varepsilon\alpha, \quad \tau_2 = -\varepsilon\beta, \quad \tau_3 = \varepsilon\delta, \quad \tau_4 = -\varepsilon\gamma.$$

The vectors u, v span the subalgebra $\mathbb{C}^{(h,h)}$. The algebra generated by u, v, z_1, z_3 is the subalgebra $\mathrm{Cay}(\mathbb{C}, -c)^{(h_\varepsilon, h_\varepsilon)}$. Using [Do-Z2, Proposition 4.4] we conclude:

$$\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1, 1, 1\rangle)^{(h_\varepsilon, h_\varepsilon)} \cong \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c'\langle 1, 1, 1\rangle)^{(h'_\varepsilon, h'_\varepsilon)}$$

implies that $(\alpha, \beta, \delta, \gamma) = (\alpha', \beta', \delta', \gamma')$ or $(\alpha, \gamma) = (\alpha', \gamma')$, and $(\beta, \delta) = -(\beta', \delta')$. More precisely, we have

$$\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1, 1, 1\rangle)^{(h_\varepsilon, h_\varepsilon)} \cong \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c'\langle 1, 1, 1\rangle)^{(h'_\varepsilon, h'_\varepsilon)}$$

if and only if all the corresponding structure constants are equal, or $(\alpha, \gamma) = (\alpha', \gamma')$, $(\beta, \delta) = -(\beta', \delta')$ and

$$\frac{2\alpha\beta y + (\alpha^2 - \beta^2)x}{x^2 + y^2} = \frac{-2\alpha\beta y' + (\alpha^2 - \beta^2)x'}{x'^2 + y'^2},$$

$$\frac{(\alpha\gamma + \beta\delta)y + (\alpha\delta - \beta\gamma)x}{x^2 + y^2} = -\frac{(\alpha\gamma + \beta\delta)y' + (-\alpha\delta + \beta\gamma)x'}{x'^2 + y'^2},$$

$$\frac{2\delta\gamma y + (\delta^2 - \gamma^2)x}{x^2 + y^2} = \frac{-2\delta\gamma y' + (\delta^2 - \gamma^2)x'}{x'^2 + y'^2},$$

$$-\frac{(\alpha^2 - \beta^2)y - 2\alpha\beta x}{x^2 + y^2} = \frac{(\alpha^2 - \beta^2)y' + 2\alpha\beta x'}{x'^2 + y'^2},$$

$$-\frac{(\beta\gamma-\alpha\delta)y+(\alpha\gamma+\beta\delta)x}{x^2+y^2}=-\frac{(-\beta\gamma+\alpha\delta)y'+(\alpha\gamma+\beta\delta)x'}{x'^2+y'^2},$$
$$-\frac{(\delta^2-\gamma^2)y-2\delta\gamma x}{x^2+y^2}=\frac{(\delta^2-\gamma^2)y'+2\delta\gamma x'}{x'^2+y'^2}.$$

We call this last set of equalities Property (A) for further reference.

Example 17. The vectors u, v span the para-quadratic subalgebra $\mathbb{C}^{(-,-)}$ of the division algebra $(\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle)^{(\sigma_\varepsilon,\sigma_\varepsilon)}$. The algebra generated by u, v, z_1, z_3 is the subalgebra $\mathrm{Cay}(\mathbb{C},-c)^{(\sigma_\varepsilon,\sigma_\varepsilon)}$. For $c=x+iy$, $c'=x'+iy'$,

$$\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle)^{(\sigma_\varepsilon,\sigma_\varepsilon)}\cong \mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c'\langle 1,1,1\rangle)^{(\sigma_\varepsilon,\sigma_\varepsilon)}$$

if and only if $c'=c$ or $c'=\bar{c}$.

Moreover,

$$(\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle)^{(\sigma_{-1},\sigma_{-1})})\not\cong(\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle)^{(\sigma_1,\sigma_1)})$$

for all $x\neq 0$ and

$$(\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle)^{(\sigma_{-1},\sigma_{-1})})\not\cong(\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c'\langle 1,1,1\rangle)^{(\sigma_1,\sigma_1)})$$

for all c, c' with $(x,x')\neq(0,0)$ [Do-Z2, Proposition 4.4]. For $c=iy$, however, all structure constants are equal, thus

$$(\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,iy\langle 1,1,1\rangle)^{(\sigma_{-1},\sigma_{-1})})\cong(\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,iy\langle 1,1,1\rangle)^{(\sigma_1,\sigma_1)}).$$

By applying [Do-Z2, Proposition 4.4], it is straightforward to also investigate possible isomorphisms between the algebras constructed in the following.

4.2. The algebra $A^{(\bar{h}_\varepsilon,\bar{h}_\varepsilon)}$ fits into Table 1: We have

$$\overline{z_1}=z_1,\quad \overline{z_2}=z_2,\quad \overline{z_3}=-z_3,\quad \overline{z_4}=z_4,\quad \overline{z_5}=-z_5,\quad \overline{z_6}=-z_6.$$

To assure that the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $(A,\cdot)$, we change the basis as follows: $u=c$, $v=-ic$, z_1, z_2 and z_4 as before and

$$z_3\to -z_3,\quad z_5\to -z_5,\quad z_6\to -z_6.$$

We then obtain the following structure constants:

$$\eta_1=\frac{2\alpha\beta y+(\alpha^2-\beta^2)x}{x^2+y^2},\quad \eta_2=\eta_3=-\frac{(\alpha\gamma+\beta\delta)y+(\alpha\delta-\beta\gamma)x}{x^2+y^2},\quad \eta_4=\frac{2\delta\gamma y+(\delta^2-\gamma^2)x}{x^2+y^2},$$
$$\theta_1=-\frac{(\alpha^2-\beta^2)y-2\alpha\beta x}{x^2+y^2},\quad \theta_2=\theta_3=\frac{(\beta\gamma-\alpha\delta)y+(\alpha\gamma+\beta\delta)x}{x^2+y^2},\quad \theta_4=\frac{(\gamma^2-\delta^2)y+2\delta\gamma x}{x^2+y^2},$$
$$\sigma_1=\tau_1=\varepsilon\alpha,\quad \sigma_2=-\varepsilon\beta,\quad \tau_2=\varepsilon\beta,\quad \sigma_3=\tau_3=\varepsilon\delta\quad \tau_4=-\sigma_4=\varepsilon\gamma.$$

Using [Do-Z2, Proposition 4.4] we obtain:

$$\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle)^{(\bar{h}_\varepsilon,\bar{h}_\varepsilon)}\cong \mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c'\langle 1,1,1\rangle)^{(\bar{h}'_\varepsilon,\bar{h}'_\varepsilon)}$$

implies that $\alpha=\alpha'$ and $\gamma=\gamma'$ and $(\beta,\delta)=(\beta',\delta')$ or $(\beta,\delta)=(-\beta',-\delta')$. More precisely, we have

$$\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle)^{(\bar{h}_\varepsilon,\bar{h}_\varepsilon)}\cong \mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c'\langle 1,1,1\rangle)^{(\bar{h}'_\varepsilon,\bar{h}'_\varepsilon)}$$

if and only if all the corresponding structure constants are equal, or Property (A) holds.

Example 18. Let $\mu : A \to A$ be defined by

$$\mu_\delta((a,u)) = (\bar{a},\bar{u}).$$

This yields the following structure constants:

$$\eta_1 = \frac{x(x^2-3y^2)}{x^2+y^2} = -\eta_4, \quad \eta_2 = \eta_3 = -\frac{y(y^2-3x^2)}{x^2+y^2},$$

$$\theta_1 = \frac{y(y^2-3x^2)}{x^2+y^2} = -\theta_4, \quad \theta_2 = \theta_3 = \frac{x(x^2-3y^2)}{x^2+y^2},$$

$$\sigma_1 = -\sigma_4 = \tau_1 = \tau_4 = x, \quad \sigma_2 = -\tau_2 = \sigma_3 = \tau_3 = y.$$

For $c = x+iy$, $c' = x'+iy'$,

$$\mathrm{Cay}(\mathbb{C},\mathbb{C}^3, c\langle 1,1,1\rangle)^{(\mu,\mu)} \cong \mathrm{Cay}(\mathbb{C},\mathbb{C}^3, c'\langle 1,1,1\rangle)^{(\mu,\mu)}$$

if and only if $c' = c$ or $c' = \bar{c}$. The vectors u, v span the para-quadratic subalgebra $\mathbb{C}^{(-,-)}$.

Remark 19. It is likely that indeed all Albert isotopes $A^{(f,f)}$, where $f = (f_1, f_2) \in \mathrm{Gl}(V)$ with $f_1 \in \mathrm{Gl}(U)$, $f_2 \in \mathrm{Gl}(W)$, have a multiplicative structure which fits into Table 1. The subalgebra $\mathbb{C}^{(f_1,f_1)}$ always fits into the upper left 2×2 matrix in the table. Since

$$(0,z_i)A^{(f,f)}(0,z_j) = (0, f(z_i)\times f(z_j)) = (0, f(z_i)) \cdot_A (0, f(z_j))$$

the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $(A,\cdot)$, provided we change part of the basis of A from $f(z_1),\dots,f(z_6)$ back to $z_1,\dots,z_6$. However, we do not see at this point how to prove this.

4.3. The algebra $A^{(h_1,id)}$ fits into multiplication table (4.2) in [B-O2]: Since

$$(0,z_i)A^{(h_1,id)}(0,z_j) = (0,(z_i)\times(z_j)) = (0,z_i)\cdot_A (0,z_j)$$

the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $(A,\cdot)$. We have the following structure constants:

$$\eta_1 = \alpha, \quad \eta_2 = -\beta, \quad \eta_3 = \delta, \eta_4 = -\gamma,$$

$$\theta_1 = \beta, \quad \theta_2 = \alpha, \quad \theta_3 = \gamma, \quad \theta_4 = \delta,$$

$$\sigma_1 = \alpha, \quad \sigma_2 = \beta, \quad \sigma_3 = \delta, \quad \sigma_4 = \gamma, \quad \tau_1 = x = -\tau_4, \quad \tau_2 = \tau_3 = -y.$$

The vectors u, v span the subalgebra $\mathbb{C}^{(h,id)}$. Using [Do-Z2, Proposition 4.4] we obtain:

$$\mathrm{Cay}(\mathbb{C},\mathbb{C}^3, c\langle 1,1,1\rangle)^{(h_1,id)} \cong \mathrm{Cay}(\mathbb{C},\mathbb{C}^3, c'\langle 1,1,1\rangle)^{(h_1',id)}$$

if and only if all the corresponding structure constants are equal, or if $(x,\alpha,\gamma) = (x',\alpha',\gamma')$ and $(y,\beta,\delta) = -(y',\beta',\delta')$.

Example 20. The algebra $A^{(\sigma_1,id)}$ has the structure constants

$$\eta_1 = x = \eta_4, \quad \eta_2 = y = -\eta_3, \quad \theta_1 = -y = \theta_4, \quad \theta_2 = x = -\theta_3,$$

$$\sigma_1 = x = -\sigma_4, \quad \sigma_2 = -y = \sigma_3, \quad \tau_1 = x = -\tau_4, \quad \tau_2 = \tau_3 = -y.$$

The vectors u, v span the left quadratic subalgebra $\mathbb{C}^{(-,id)}$. For $c = x+iy$, $c' = x'+iy'$,

$$\mathrm{Cay}(\mathbb{C},\mathbb{C}^3, c\langle 1,1,1\rangle)^{(\sigma_1,id)} \cong \mathrm{Cay}(\mathbb{C},\mathbb{C}^3, c'\langle 1,1,1\rangle)^{(\sigma_1,id)}$$

if and only if $c' = c$ or $c' = \bar{c}$.

4.4. The algebra $A^{(id,h_1)}$ fits into Table 1:

$$\eta_1 = \alpha, \quad \eta_2 = \delta, \quad \eta_3 = -\beta, \quad \eta_4 = -\gamma, \quad \theta_1 = \beta, \quad \theta_2 = \gamma, \quad \theta_3 = \alpha, \quad \theta_4 = \delta,$$

$$\sigma_1 = x, \quad \sigma_2 = y, \quad \sigma_3 = -y, \quad \sigma_4 = x, \quad \tau_1 = \alpha, \qquad \tau_2 = -\beta, \quad \tau_3 = \delta, \quad \tau_4 = -\gamma.$$

The vectors u, v span the subalgebra $\mathbb{C}^{(id,h)}$. Using [Do-Z2, Proposition 4.4] we obtain:

$$\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle)^{(id,h_1)} \cong \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c'\langle 1,1,1\rangle)^{(id,h_1')}$$

if and only if all the corresponding structure constants are equal, or if $(x, \alpha, \gamma) = (x', \alpha', \gamma')$ and $(y, \beta, \delta) = -(y', \beta', \delta')$. We note that $A^{(h_1,id)} = A^{(id,(h^{-1})_1)}$.

Remark 21. (i) The algebra $A^{(h_{-1},id)}$ does not seem to fit into Table 1 since

$$(0, z_i) \star (0, z_j) = (0, (-z_i) \times (z_j)) = -(0, z_i) \cdot_A (0, z_j).$$

Therefore the 6×6-matrix in the lower right hand corner of the multiplication table does not remain the same as for $(A, \cdot)$. It is not clear if a change of basis might change this. The same observation applies to the algebra $A^{(id,h_{-1})} = A^{((h^{-1})_{-1},id)}$ and to the algebras $A^{(\bar{h}_{-1},id)}$ and $A^{(id,\bar{h}_{-1})}$.
(ii) The algebras $A^{(\bar{h}_1,id)}$ and $A^{(id,\bar{h}_1)}$ fit into Table 1 by changing the basis as in 4.2. We leave it to the reader to compute their structure constants if desired.

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