Isometries between JB*-triples

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Abstract Let Z and W be JB*-triples and let T be a linear isometry from Z into W. For any $z \in Z$ with $\|z\| < 1$, we show that

$$T\{z,z,z\} = \{T(z),T(z),T(z)\}$$

if the Möbius transform induced by $T(z)$ preserves the unit ball of $T(Z)$. We show further that T is, *locally*, a triple homomorphism via a tripotent: for any $z \in Z$, there is a tripotent u in W^{**} such that

$$\{u, T\{a,b,c\}, u\} = \{u, \{T(a),T(b),T(c)\}, u\}$$

for all a, b, c in the smallest subtriple Z_z of Z containing z, and also, $\{u, T(\cdot), u\} : Z_z \longrightarrow W^{**}$ is an isometry.

1 Introduction

Jordan algebraic structures play an important role in the geometry of infinite dimensional Banach manifolds. Indeed, as shown by Kaup [14], every bounded symmetric domain gives rise to a Jordan triple product $\{\cdot,\cdot,\cdot\}$ on its tangent space and a surjective linear map T between these spaces is an isometry if, and only if, it preserves the Jordan triple product:

$$T\{a,b,c\} = \{T(a),T(b),T(c)\}\,.$$

These tangent spaces form an important class of complex Banach spaces, called JB*-*triples*. One can therefore study the geometry of symmetric domains via the algebraic structures of JB*-triples. We remark that Jordan methods were

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first introduced by Koecher [15] into the theory of finite dimensional bounded symmetric domains and they were also discussed in detail in [16].

Although a Jordan triple monomorphism is necessarily an isometry, a *non-surjective* linear isometry between two JB*-triples need not preserve the Jordan triple product. It is natural to ask to what extent can a non-surjective linear isometry preserve the Jordan triple product. The object of this paper is to address this question. We note that, by polarization, a linear map $T : Z \longrightarrow W$ between JB*-triples preserves the Jordan triple product if, and only if,

$$T\{a, a, a\} = \{T(a), T(a), T(a)\} \qquad (a \in Z)\,.$$

To answer the above question, our first task is to understand what makes a surjective linear isometry preserve the Jordan triple product. Upon a closer study of the geometry behind the proof of this fact in [14, Proposition 5.5], we found that the condition needed is a certain invariant property of the Möbius transformation. In Section 3 we discuss this in detail and show that, given a linear isometry $T : Z \longrightarrow W$ between JB*-triples, not necessarily surjective, one has

$$T\{a, a, a\} = \{T(a), T(a), T(a)\}$$

for $\|a\| < 1$ if the Möbius transformation g_{Ta} induced by $T(a)$ preserves the open unit ball of the image $T(Z)$. In Section 4, we show that, although a non-surjective linear isometry $T : Z \longrightarrow W$ between JB*-triples need not be a triple homomorphism, it is, nevertheless, *locally* a triple homomorphism, that is, for any $a \in Z$, there is a tripotent $u \in W^{**}$ such that $\|\{u, T(z), u\}\| = \|z\|$ and

$$\{u, T\{z, z, z\}, u\} = \{u, \{T(z), T(z), T(z)\}, u\}$$

for every z in the JB*-triple generated by a. The tripotent u above depends on the given element $a \in Z$, but if Z admits a character, then one can find a tripotent $v \in W^{**}$ such that $\{v, T(\cdot), v\} \neq 0$ and

$$\{v, T\{z, z, z\}, v\} = \{v, \{T(z), T(z), T(z)\}, v\}$$

for all $z \in Z$. Without any condition on Z, such a tripotent v may not exist. Finally in Section 5, we prove more specialized results in the setting of JB*-algebras. In particular, we show that, if $T : Z \longrightarrow W$ is a linear isometry from a JB*-triple Z into a JB*-algebra $(W, \circ)$, then there is a *largest* projection $p \in W^{**}$ such that, for all $a \in Z$,

$$T\{a, a, a\} \circ p = \{T(a), T(a), T(a)\} \circ p$$

and p operator commutes with $T(a) \circ T(a)^*$.

The results in this paper generalize those in [7] for C*-algebras. We begin in the next section with some basic definitions and results concerning JB*-triples.

2 JB*-triples

Throughout this paper, an isometry $T : Z \longrightarrow W$ between Banach spaces is *not* assumed to be surjective and we often write Ta for the image $T(a)$ for convenience. We first recall that a *JB*-triple* Z is a complex Banach space equipped with a continuous Jordan triple product $\{\cdot,\cdot,\cdot\} : Z^3 \longrightarrow Z$ which is symmetric and linear in the outer variables, and conjugate linear in the middle variable such that for $a, b, c, x, y \in Z$, we have

(i) $\{a, b, \{c, x, y\}\} = \{\{a, b, c\}, x, y\} - \{c, \{b, a, x\}, y\} + \{c, x, \{a, b, y\}\}$;
(ii) the map $z \in Z \mapsto \{a, a, z\} \in Z$ is hermitian with nonnegative spectrum;
(iii) $\|\{a, a, a\}\| = \|a\|^3$.

For later reference, we define two fundamental linear operators on a JB*-triple Z. For $x, y \in Z$, the *box operator* $x \,\square\, y : Z \longrightarrow Z$ and the *Bergman operator* $B(x, y) : Z \longrightarrow Z$ are defined by

$$(\,x \,\square\, y\,)(z) = \{x, y, z\}$$
$$B(x, y)(z) = z - 2\{x, y, z\} + \{x, \{y, z, y\}, x\}.$$

Every C*-algebra A is a JB*-triple with the following Jordan triple product

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a) \qquad (a, b, c \in A).$$

A closed subspace of a JB*-triple is called a *subtriple* if it is closed with respect to the triple product. A linear map $T : Z \longrightarrow W$ between JB*-triples is called a *triple homomorphism* if it preserves the triple product in which case, the kernel J of T is a *triple ideal* of Z, that is, $\{Z, Z, J\} + \{Z, J, Z\} \subset J$ and the range $T(Z)$ is a subtriple of W. We refer to [3, 6, 18–20] for expositions as well as recent surveys of JB*-triples and symmetric Banach manifolds. In the sequel, we write $a^{(3)} = \{a, a, a\}$ and use frequently the polarization formula

$$\{a, b, c\} = \frac{1}{8} \sum_{\alpha^4 = \beta^2 = 1} \alpha\beta(a + \alpha b + \beta c)^{(3)}.$$

An element u in a JB*-triple is called a *tripotent* if $u^{(3)} = u$. If a JB*-triple Z has a predual (which is necessarily unique), then it is called a *JBW*-triple* in which case, Z has an abundance of tripotents. Each tripotent $u \in Z$ induces a splitting of Z, $Z = Z_0 \oplus Z_1 \oplus Z_2$, known as the *Peirce decomposition*, into a direct sum of the 0, 1 and 2-eigenspaces of the operator $2\, u \,\square\, u$. The Peirce projections $P_i(u) : Z \to Z_i$ onto the eigenspaces Z_i, for $i = 0, 1, 2$, are given in terms of the triple product,

$$P_0(u)(z) = B(u, u)(z)$$
$$P_1(u)(z) = 2(\{u, u, z\} - \{u, \{u, z, u\}, u\})$$
$$P_2(u)(z) = \{u, \{u, z, u\}, u\}.$$

These projections are contractive. Each eigenspace Z_i is a subtriple of Z. Indeed we have $\{Z_i, Z_j, Z_k\} \subset Z_{i-j+k}$ for $i, j, k \in \{0, 1, 2\}$ where $Z_r := \{0\}$ for $r \notin \{0, 1, 2\}$. In particular, $Z_2 = P_2(u)(Z)$ is a JB*-algebra with identity u, and with respect to the following non-associative product and involution:

$$x \circ y = \{x, u, y\}, \qquad x^* = \{u, x, u\}.$$

We note that a JB*-*algebra*, that is, a Jordan Banach algebra $(A, \circ)$ equipped with an isometric involution $*$ satisfying $\|x \circ y\| \leq \|x\|\|y\|$ and $\|\{x, x, x\}\| = \|x\|^3$, is also a JB*-triple with the Jordan triple product

$$\{a, b, c\} = (a \circ b^*) \circ c - (a \circ c) \circ b^* + (b^* \circ c) \circ a.$$

For example, a C*-algebra is a JB*-algebra with the Jordan product

$$a \circ b = \frac{1}{2}(ab + ba).$$

A JB*-algebra is called a JC*-*algebra* if it can be embedded as a norm-closed subspace of a C*-algebra, closed with respect to the involution and the above Jordan product. A JB*-algebra having a (necessarily unique) predual is called a JBW*-*algebra*, it is called a JW*-*algebra* if it is also a JC*-algebra. We refer to [10] for a detailed exposition of Jordan Banach algebras including JB*-algebras and JBW*-algebras.

Each tripotent u in a JBW*-triple Z has a support face $F(u)$ in the predual Z_* of Z, given by

$$F(u) = \{\varphi \in Z_* : \|\varphi\| = 1 = \varphi(u)\}$$

which is a norm-exposed face of the closed unit ball Z_{*1} of Z_*. One can introduce a partial ordering $\leq$ to the set $\mathcal{T}(Z)$ of tripotents in a JBW*-triple Z. For any two tripotents u and v in Z, one defines $u \leq v$ if $v - u$ is orthogonal to u which means that

$$\{u, v - u, x\} = 0$$

for all $x \in Z$. With this partial ordering, it has been shown in [8] that given a family of tripotents $\{u_\alpha\}_{\alpha \in Q}$ in Z, either the lattice supremum $\bigvee_{\alpha \in Q} u_\alpha$ exists in $\mathcal{T}(Z)$, or $Z_{*1} = \bigvee_{\alpha \in Q} F(u_\alpha)$, that is, the smallest norm-exposed face of Z_{*1} containing the union $\bigcup_\alpha F(u_\alpha)$ is Z_{*1} itself. By [5], Z embeds as a subtriple of a JBW*-algebra A such that the predual Z_* is a 1-complemented subspace of the predual A_* of A, where we recall that a closed subspace of a Banach space E is called 1-*complemented* if it is the range of a contractive projection on E. In particular, faces of Z_{*1} are faces of the closed unit ball A_{*1} and, every face F of A_{*1} is either disjoint from Z_{*1} or the intersection $F \cap Z_{*1}$ is a face of Z_{*1}. It follows that, if $\{u_\alpha\}_{\alpha\in Q}$ is a family of tripotents in $\mathcal{T}(Z)$ such that $Z_{*1} = \bigvee_\alpha F(u_\alpha)$, then we also have $A_{*1} = \bigvee'_\alpha F(u_\alpha)$, where $\bigvee'$ denotes the supremum in A_{*1}, for otherwise, $F = \bigvee'_\alpha F(u_\alpha)$ is a proper norm-exposed face of A_{*1} and the intersection $F \cap Z_{*1}$ is a norm-exposed face of Z_{*1} containing $\bigcup_\alpha F(u_\alpha)$, giving $F \cap Z_{*1} = Z_{*1}$ which is impossible since $0 \notin F$.

By [8, p.322], every element z in a JBW*-triple Z admits a support tripotent $u_z \in \mathcal{T}(Z)$ satisfying

$$z = \{u_z, z, u_z\} = \{u_z, u_z, z\}\,.$$

3 Isometries and Möbius transformation

In this section, we reveal the role of Möbius transformations in the preservation of Jordan structures by a linear isometry. We first introduce the relevant geometric and holomorphic aspects of JB*-triples. A map $g : D \longrightarrow U$ between open sets in complex Banach spaces Z and W, respectively, is called *holomorphic* if the Fréchet derivative $g'(a) : Z \longrightarrow W$ exists for every $a \in D$, where $g'(a)$ is a linear map satisfying

$$\lim_{t\to 0} \frac{\|g(a+t) - g(a) - g'(a)(t)\|}{\|t\|} = 0.$$

A holomorphic map $g : D \longrightarrow U$ is called *biholomorphic* if it is bijective and the inverse g^{-1} is also holomorphic. The open unit ball of a Banach space Z will be denoted by Z_0. Let $Aut\, Z_0$ be the automorphism group of Z_0, consisting of all biholomorphic maps from Z_0 onto itself. Upmeier [20] has shown that $Aut\, Z_0$ is a real Banach-Lie group and by a deep result of Kaup [14], a complex Banach space Z is a JB*-triple if, and only if, $Aut\, Z_0$ acts transitively on Z_0, in which case, the Jordan triple product is constructed via the Lie algebra of $Aut\, Z_0$. For a JB*-triple Z, the basic elements in $Aut\, Z_0$ are the Möbius transformations. Given $a \in Z_0$, we define the *Möbius transformation of* Z_0, *induced by* a, to be the biholomorphic map $g_a : Z_0 \longrightarrow Z_0$ given by

$$g_a(z) = a + B(a,a)^{1/2}(I + z\,\square\, a\,)^{-1}(z)$$

where I is the identity operator. We have $g_a(0) = a$, $g_a^{-1} = g_{-a}$ and, the Fréchet derivatives $g'_a(0) = B(a,a)^{1/2}$ and $g'_{-a}(a) = B(a,a)^{-1/2}$ (cf. [14]). If Z is a C*-algebra, we have the following formula for the Möbius transformation which was due to Potapov [17] and Harris [11]:

$$g_a(z) = (1 - aa^*)^{-1/2}(a+z)(1 + a^*z)^{-1}(1 - a^*a)^{1/2}.$$

Lemma 1. *Let $T : Z \longrightarrow W$ be a linear isometry between JB*-triples Z and W. Let $a \in Z_0$ and let $\psi \in Aut\, T(Z)_0$ be such that $\psi(T(a)) = 0$. Then*

$$\psi(0) = -\psi'(T(a))(T(B(a,a)^{1/2}(a))).$$

Proof. Let $h = \psi T g_a : Z_0 \longrightarrow T(Z)_0$. Then h is biholomorphic and $h(0) = 0$. Hence h is linear by Cartan's uniqueness theorem and on Z_0, $h = h'(0) = (\psi T g_a)'(0) = (\psi T)'(g_a(0))g_a'(0) = (\psi T)'(a)B(a,a)^{1/2}$. Evaluating h at $-a$, we get the formula.

We note that $Tg_{-a}T^{-1}$ is an automorphism of $T(Z)_0$ and maps $T(a)$ to 0. For a C*-algebra, we have $B(a,a)^{1/2}(a) = (1-aa^*)^{1/2}a(1-a^*a)^{1/2} = a - aa^*a$ since $(1-aa^*)^{1/2}a = a(1-a^*a)^{1/2}$. Therefore we have $B(a,a)^{1/2}(a) = a - \{a,a,a\}$ in a JB*-triple by considering the subtriple generated by a which is linearly isometric to an abelian C*-algebra. Since $B(a,a) = B(-a,-a)$, we have $g_{-a}(-z) = -g_a(z)$. It follows that, if $D = -D$ is a subset of the open unit ball of a JB*-triple, invariant under g_a, then it is also invariant under g_{-a} and $g_a(D) = D$.

By refining Kaup's result in [14, Proposition 5.5] (see also [11]), we now show how the Möbius transformation and surjectivity effect the preservation of the triple product by a linear isometry.

Proposition 1. *Let $T : Z \longrightarrow W$ be a linear isometry between JB*-triples Z and W. Let $a \in Z_0$ and let $g_{Ta} \in Aut\, W_0$ be the Möbius transformation induced by $T(a)$. If $g_{Ta}(T(Z)_0) \subset T(Z)_0$, then we have*

$$T\{a,a,a\} = \{T(a), T(a), T(a)\}.$$

In particular, if T is surjective, then T is a triple isomorphism.

Proof. Let ψ be the restriction to $T(Z)_0$ of the Möbius transformation $g_{-Ta} \in Aut\, Z_0$. Then $\psi \in Aut\, T(Z)_0$, $\psi(T(a)) = 0$ and the derivative $\psi'(T(a)) : T(Z) \longrightarrow T(Z)$ is the restriction of the derivative $g'_{T(a)}(T(a)) : W \longrightarrow W$ which is equal to $B(T(a),T(a))^{-1/2}$. By Lemma 1, we have

$$-T(a) = \psi(0) = -\psi'(T(a))(T(B(a,a)^{1/2}(a)) = -B(T(a),T(a))^{-1/2}T(a - a^{(3)}).$$

It follows that $T(a) - T(a)^{(3)} = B(T(a),T(a))^{1/2}(T(a)) = T(a - a^{(3)})$ which gives $T(a)^{(3)} = T(a^{(3)})$.

Finally, if T is surjective then $T(Z)_0 = W_0$ is invariant under g_{Ta} for all $a \in A_0$. Hence T preserves the triple product.

Remark 1. The above result subsumes Kadison's seminal result for surjective isometries between C*-algebras. It has also been discussed in [4] in the setting of JB*-algebras. We note from [7] that, for a fixed a, the condition $T(a^{(3)}) = (Ta)^{(3)}$ alone does not imply $T(a^{(n)}) = (Ta)^{(n)}$ for any odd integer $n > 3$.

The following corollary is immediate.

Corollary 1. *Let $T : Z \longrightarrow W$ be a linear isometry between JB*-triples. Then $T(Z)$ is a subtriple of W if, and only if, $T(Z)_0$ is invariant under the Möbius transformation $g_{T(a)}$ for all $\|a\| < 1$.*

Example 1. Let M_n be the JB*-triple of $n \times n$ complex matrices. Let $T : \mathbb{C} \longrightarrow M_2$ be defined by

$$T(a) = \begin{pmatrix} 0 & \frac{a}{2} \\ a & 0 \end{pmatrix}.$$

Then T is a linear isometry and $T(\mathbb{C})$ is not a subtriple of M_2. Also $T(1)$ is not unitary and $T(\mathbb{C})$ contains no nontrivial positive element. For $a \neq 0$, we have $T(a^{(3)}) \neq (Ta)^{(3)}$ and in fact, for $0 < |a| < 1$,

$$g_{Ta}(Tx) = \begin{pmatrix} 0 & \frac{2(a+x)}{4+\bar{a}x} \\ \frac{a+x}{1+\bar{a}x} & 0 \end{pmatrix}$$

which is outside $T(\mathbb{C})$.

Example 2. A Hilbert space H is a JB*-triple with Jordan triple product

$$\{x, y, z\} = \frac{1}{2}\left(\langle x, y\rangle z + \langle z, y\rangle x\right)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in H. Hence, given a linear isometry $T : H \longrightarrow K$ between Hilbert spaces, the range $T(H)$ is a subtriple of K and T is a triple isomorphism onto $T(H)$.

Given $a \in H_0$, the Möbius transformation $g_a : H_0 \longrightarrow H_0$ is given by

$$g_a(x) = \frac{a + E_a(x) + \sqrt{1 - \|a\|^2}(I - E_a)(x)}{1 + \langle x, a\rangle}$$

where E_a is the projection from H onto the subspace $\mathbb{C}a$. Given a linear isometry T on H, we have $\langle Tx, Ta\rangle = \langle x, a\rangle$ and $E_{Ta}(Tx) = E_a(x)Ta$. It follows that

$$g_{Ta}(Tx) = T(g_a(x))$$

and indeed, $T(H)_0$ is invariant under g_{Ta} for all $\|a\| < 1$.

Example 3. Let $C(\Omega)$ and $C(\Omega \cup \{\beta\})$ be the C*-algebras of continuous functions on the closed unit disc $\Omega \subset \mathbb{C}$ and $\Omega \cup \{\beta\}$ respectively, where $\beta \in \mathbb{C} \backslash \Omega$. Define $T : C(\Omega) \longrightarrow C(\Omega \cup \{\beta\})$ by

$$(Ta)(x) = \begin{cases} a(x) & \text{if } x \in \Omega \\ \frac{1}{2}(a(1) + a(0)) & \text{if } x = \beta. \end{cases}$$

Then T is a linear isometry and $T(C(\Omega)) = \{h \in C(\Omega \cup \{\beta\}) : 2h(\beta) = h(1) + h(0)\}$ which is not a subtriple of $C(\Omega \cup \{\beta\})$. It is easy to see that T is not a triple isomorphism onto its range, but for $a \in C(\Omega)_0$ with $a(1) = a(0) = 0$, we have $g_{Ta}(T(C(\Omega))_0) \subset T(C(\Omega))_0$. Indeed, if $h \in T(C(\Omega))_0$, then

$$\begin{aligned} g_{Ta}(h)(\beta) &= \left(\frac{Ta + h}{1 + \overline{Ta}h}\right)(\beta) = \frac{Ta(\beta) + h(\beta)}{1 + \overline{Ta(\beta)}h(\beta)} \\ &= \frac{2(a(1) + a(0) + h(1) + h(0))}{4 + \overline{a(1) + a(0)}(h(1) + h(0))} = \frac{1}{2}(h(1) + h(0)) \\ &= \frac{1}{2}(g_{Ta}(h)(1) + g_{Ta}(h)(0)) \end{aligned}$$

which gives $g_{Ta}(h) \in T(C(\Omega))_0$. It is clear that $T(a^{(3)}) = T(a)^{(3)}$.

4 Isometries and Jordan triple product

Our goal in this section is to show that a non-surjective linear isometry $T : Z \longrightarrow W$ between JB*-triples preserves, at least *locally*, the Jordan triple product, via a tripotent. Since the JB*-subtriple generated by an element $z \in Z$ is Jordan isomorphic to the JB*-triple $C_0(X)$ of complex continuous functions on a locally compact Hausdorff space X, vanishing at infinity, it suffices to study the case in which $Z = C_0(X)$.

We recall that for any functional φ in the predual of a JBW*-triple W, there is a unique tripotent $u_\varphi \in W$, called the *support tripotent* of φ, such that $\varphi = \varphi \circ P_2(u_\varphi)$ and $\varphi|_{P_2(u_\varphi)(W)}$ is a faithful normal positive functional on the JBW*-algebra $P_2(u_\varphi)(W)$ [9, Proposition 2]. The JBW*-algebra $P_2(u_\varphi)(W)$ becomes an inner product space with respect to the inner product

$$\langle a, b\rangle = \varphi\{a, b, u_\varphi\}.$$

Moreover φ is an extreme point of the closed unit ball of the predual if, and only if, u_φ is a minimal tripotent, that is, $\{u_\varphi, W, u_\varphi\} = \mathbb{C}u_\varphi$. We denote by ∂E the set of extreme points of the closed unit ball of a Banach space E.

As usual, we embed and regard a JB*-triple Z as a subtriple of its second dual Z^{**} which is a JBW*-triple. The following theorem generalizes the results in [7,12].

Theorem 1. *Let W be a JB*-triple and let $T : C_0(X) \longrightarrow W$ be a linear isometry. Then either T is a triple homomorphism or there is a tripotent $u \in W^{**}$ such that*

$$\{u, T(f^3), u\} = \{u, T(f)^3, u\}$$

for all $f \in C_0(X)$ and

$$\{u, T(\cdot), u\} : C_0(X) \longrightarrow W^{**}$$

is an isometry.

Proof. Let $E = T(C_0(X))$. Then the dual map $T^* : E^* \to C_0(X)^*$ of $T : C_0(X) \to E$ is a surjective linear isometry. We also denote by T^* the dual map of $T : C_0(X) \longrightarrow W$ since no confusion is likely. Let

$$Q = \{\varphi \in \partial W^* : \varphi|_E \in \partial E^*\}.$$

Then Q is non-empty since each extreme point $\psi \in \partial E^*$ extends to an extreme point $\varphi \in \partial W^*$.

Let $\varphi \in Q$ with $\psi = \varphi|_E \in \partial E^*$. Then $T^*\varphi = T^*\psi$ is an extreme point of the closed unit ball of $C_0(X)^*$ and hence there exists $x_\varphi \in X$ such that $T^*\psi = \alpha\delta_{x_\varphi}$ with $|\alpha| = 1$. Let $u_\varphi \in W^{**}$ be the support tripotent of φ.

Since u_φ is a minimal tripotent and $\varphi\{u_\varphi, \cdot, u_\varphi\} = \overline{\varphi \circ P_2(u_\varphi)}(\cdot) = \bar{\varphi}(\cdot)$, where the bar '$-$' denotes complex conjugation, we have

$$\{u_\varphi, b, u_\varphi\} = \overline{\varphi(b)} u_\varphi \qquad (b \in W^{**}).$$

From $\varphi \circ T(f) = (T^*\varphi)(f) = (T^*\psi)(f) = \alpha f(x_\varphi)$, we obtain, in W^{**},

$$\{u_\varphi, T(f), u_\varphi\} = \overline{\alpha f(x_\varphi)} u_\varphi \qquad (f \in C_0(X))$$

and $\{u_\varphi, T(\cdot), u_\varphi\}$ is a triple homomorphism. In particular,

$$\begin{aligned}\overline{\alpha f^{(3)}(x_\varphi)} u_\varphi &= \{u_\varphi, Tf, u_\varphi\}^{(3)} = \{u_\varphi, \{Tf, P_2(u_\varphi)(Tf), Tf\}, u_\varphi\} \\ &= \alpha f(x_\varphi)\{u_\varphi, \{Tf, u_\varphi, Tf\}, u_\varphi\}\end{aligned}$$

and hence $\varphi\{u_\varphi, \{Tf, u_\varphi, Tf\}, u_\varphi\} = (\overline{\alpha f(x_\varphi)})^2$ or

$$\varphi\{Tf, u_\varphi, Tf\} = (\alpha f(x_\varphi))^2. \tag{1}$$

We prove that

$$\{u_\varphi, T(f^{(3)}), u_\varphi\} = \{u_\varphi, (Tf)^{(3)}, u_\varphi\} \qquad (f \in C_0(X)).$$

It suffices to show that

$$\varphi\{u_\varphi, (Tf)^{(3)}, u_\varphi\} = \overline{\alpha f^{(3)}(x_\varphi)}.$$

We first show that

$$\{u_\varphi, u_\varphi, Th\} = u_\varphi$$

for $h \in C_0(X)$ satisfying $\|h\| = 1$ and $h(x_\varphi) = \bar{\alpha}$. We have, by the Schwarz inequality [2, Proposition 1.2],

$$\begin{aligned}1 &= |\varphi(Th)|^2 = |\varphi\{u_\varphi, Th, u_\varphi\}|^2 \\ &\leq \varphi\{u_\varphi, u_\varphi, u_\varphi\}\varphi\{Th, Th, u_\varphi\} \leq \|Th\|^2 = \|h\|^2 = 1\end{aligned}$$

giving $\varphi\{Th, Th, u_\varphi\} = 1$. Let

$$N_\varphi = \{b \in W^{**} : \varphi\{b, b, u_\varphi\} = 0\}.$$

Then we have

$$N_\varphi = P_0(u_\varphi)(W^{**}) \tag{2}$$

by [2, p.516]. We show $Th - u_\varphi \in N_\varphi$. Indeed, we have

$$\begin{aligned}&\varphi\{Th-u_\varphi, Th-u_\varphi, u_\varphi\}\\ &= \varphi\{Th,Th,u_\varphi\} - \varphi\{u_\varphi,Th,u_\varphi\} + \varphi\{u_\varphi,u_\varphi,u_\varphi\} - \varphi\{Th,u_\varphi,u_\varphi\} = 0\end{aligned}$$

where $\varphi\{Th,u_\varphi,u_\varphi\} = \overline{\varphi\{u_\varphi,Th,u_\varphi\}} = 1$. Hence, by (2), we have $\{u_\varphi,u_\varphi,Th-u_\varphi\} = 0$ and $\{u_\varphi,u_\varphi,Th\} = u_\varphi$.

We next show that $\varphi\{Tg,Tg,u_\varphi\} = 0$ whenever $g \in C_0(X)$ satisfies $g(x_\varphi) = 0$. We may assume, by Urysohn's lemma, that g vanishes on a neighbourhood of x_φ, in which case, we can choose $k \in C_0(X)$ such that $\|k\| = 1$, $k(x_\varphi) = \alpha$ and $kg = 0$. Then $\|k+g\| = 1$ and $(k+g)(x_\varphi) = \alpha$. Therefore, by the above, we have $T(k+g) + N_\varphi = u_\varphi + N_\varphi = Tk + N_\varphi$ which yields $Tg \in N_\varphi$, that is, $\varphi\{Tg,Tg,u_\varphi\} = 0$.

Now let $f \in C_0(X)$ with $\|f\| = 1$. Pick $h \in C_0(X)$ with $\|h\| = 1$ and $h(x_\varphi) = \bar{\alpha}$. Then $(f - \alpha f(x_\varphi)h)(x_\varphi) = 0$ and therefore we have $Tf - \alpha f(x_\varphi)Th \in N_\varphi$ and by (2) again,

$$\{u_\varphi,u_\varphi,Tf - \alpha f(x_\varphi)Th\} = 0$$

giving

$$\{u_\varphi,u_\varphi,Tf\} = \alpha f(x_\varphi)\{u_\varphi,u_\varphi,Th\} = \alpha f(x_\varphi)u_\varphi.$$

Moreover, we have

$$\begin{aligned}&\alpha f(x_\varphi)\{u_\varphi,Tf,u_\varphi\} = \{u_\varphi,Tf,\{u_\varphi,u_\varphi,Tf\}\}\\ &= \{\{u_\varphi,Tf,u_\varphi\},u_\varphi,Tf\} - \{u_\varphi,\{Tf,u_\varphi,u_\varphi\},Tf\} + \{u_\varphi,u_\varphi,\{u_\varphi,Tf,Tf\}\}\\ &= \overline{\alpha f(x_\varphi)}\{u_\varphi,u_\varphi,Tf\} - \overline{\alpha f(x_\varphi)}\{u_\varphi,u_\varphi,Tf\} + \{u_\varphi,u_\varphi,\{u_\varphi,Tf,Tf\}\}\\ &= \{u_\varphi,u_\varphi,\{u_\varphi,Tf,Tf\}\}\end{aligned}$$

and hence $\varphi\{u_\varphi,Tf,Tf\} = \varphi(\{u_\varphi,u_\varphi,\{u_\varphi,Tf,Tf\}\}) = \alpha f(x_\varphi)\varphi\{u_\varphi,Tf,u_\varphi\}$. Therefore we have

$$\begin{aligned}&\overline{\varphi\{u_\varphi,(Tf)^{(3)},u_\varphi\}} = \varphi\{u_\varphi,u_\varphi,\{Tf,Tf,Tf\}\}\\ &= \varphi(\{\{u_\varphi,u_\varphi,Tf\},Tf,Tf\} - \{Tf,\{u_\varphi,u_\varphi,Tf\},Tf\} + \{Tf,Tf,\{u_\varphi,u_\varphi,Tf\}\})\\ &= 2\alpha f(x_\varphi)\varphi\{u_\varphi,Tf,Tf\} - \overline{\alpha f(x_\varphi)}\varphi\{Tf,u_\varphi,Tf\}\\ &= \alpha f^{(3)}(x_\varphi)\end{aligned}$$

using (1). It follows that

$$\varphi\{u_\varphi,(Tf)^{(3)},u_\varphi\} = \overline{\alpha f^{(3)}(x_\varphi)} = \varphi\{u_\varphi,T(f^{(3)}),u_\varphi\}.$$

By the remarks in Section 2, we have two cases :

(i) the lattice supremum $u = \bigvee_{\varphi \in Q} u_\varphi$ is a tripotent in W^{**};

(ii) $W_1^* = \bigvee_{\varphi \in Q} F(u_\varphi) = \bigvee_{\varphi \in Q} \{\varphi\}$.

Case (i). The tripotent $u = \bigvee_{\varphi \in Q} u_\varphi$ has support face

$$F(u) = \{\psi \in W^* : \|\psi\| = \psi(u) = 1\}$$

which is the normal state space of the atomic JBW*-algebra $P_2(u)(W^{**})$. Let ρ be an extreme point of $F(u)$ with support tripotent u_ρ which is a minimal projection in $P_2(u)(W^{**})$. If we select from $\{u_\varphi\}_{\varphi \in Q}$ a maximal subfamily $\{u_\varphi\}_{\varphi \in Q'}$ with mutually orthogonal central supports $\{c(u_\varphi)\}_{\varphi \in Q'}$, then $u = \sum_{\varphi \in Q'} c(u_\varphi)$ where each $P_2(c(u_\varphi))(W^{**})$ is a type I JBW*-factor. It follows from [10, Lemma 5.3.2] that u_ρ is exchanged by a symmetry $s \in P_2(u)(W^{**})$ to some u_φ with $\varphi \in Q'$, that is, $u_\rho = \{s, u_\varphi, s\}$, with $T^*\varphi = \alpha\delta_{x_\varphi}$ as before. Then we have $\rho\{s, \cdot, s\} = \overline{\varphi}(\cdot)$.

Let $S : C_0(X) \to W^{**}$ be the isometry defined by

$$S(f) = \{s, Tf, s\}^* \qquad (f \in C_0(X))$$

where $*$ is the involution in $P_2(u)(W^{**})$. By the above argument, we have $\varphi(S(f^{(3)})) = \varphi((Sf)^{(3)})$. As φ is a state of $P_2(u)(W^{**})$, it follows that

$$\begin{aligned}
\rho(T(f^{(3)})) &= \overline{\varphi}\{s, T(f^{(3)}), s\} = \varphi(\{s, T(f^{(3)}), s\}^*) \\
&= \varphi((\{s, Tf, s\}^*)^{(3)}) \\
&= \varphi(\{s, \{Tf, \{s, \{s, Tf, s\}, s\}, Tf\}, s\}^*) \\
&= \varphi(\{s, (Tf)^{(3)}, s\}^*) \\
&= \rho((Tf)^{(3)}).
\end{aligned}$$

Since $\rho \in F(u)$ was arbitrary, we obtain

$$\{u, T(f^{(3)}), u\} = \{u, (Tf)^{(3)}, u\}.$$

Finally, for any $f \in C_0(X)$, pick $x \in X$ with $\|f\| = |f(x)|$. Let $\psi \in \partial E^*$ with $T^*\psi = \delta_x$, and let $\varphi \in \partial W^*$ be an extension of ψ. Then $\varphi \in Q$ and $T^*\varphi = \delta_x$. Hence

$$\begin{aligned}
\|Tf\| \geq \|\{u, Tf, u\}\| &\geq \|\{u_\varphi, \{u_\varphi, \{u, Tf, u\}, u_\varphi\}, u_\varphi\}\| \\
&= \|\{u_\varphi, Tf, u_\varphi\}\| \\
&= \|\overline{f(x)} u_\varphi\| = |f(x)| = \|f\|
\end{aligned}$$

which gives $\|\{u, Tf, u\}\| = \|f\|$.

Case (ii). Let W^{**} be embedded as a subtriple of a JBW*-algebra B such that W^* is 1-complemented in the predual B_*. As remarked in Section 2, we have $B_{*1} = \bigvee_{\varphi \in Q} \{\varphi\}$. It follows that there is a subfamily $\{\varphi\}_{\varphi \in Q''}$ such that the atomic part B_a of B is a direct sum

$$B_a = \bigoplus_{\varphi \in Q''} B(u_\varphi)$$

where $B(u_\varphi)$ is the weak*-closed ideal in B generated by u_φ and is a type I JBW*-factor. Given an extreme point $\rho \in \partial W^*$, it is also an extreme point of B_{*1} and its support tripotent u_ρ is in some $B(u_\varphi)$. As before, u_ρ is equivalent to u_φ via a symmetry in B and it follows that

$$\rho(T(f^{(3)})) = \rho((Tf)^{(3)}).$$

As $\rho \in \partial W^*$ was arbitrary, we have

$$T(f^{(3)}) = (Tf)^{(3)}$$

for all $f \in C_0(X)$, that is, T is a triple homomorphism. This completes the proof.

Remark 2. We note that the map $\{u, T(\cdot), u\}$ in Theorem 1 is complex conjugate linear and it is equivalent to state that the complex linear map $P_2(u) \circ T$ is an isometry.

Theorem 2. *Let $T : Z \longrightarrow W$ be a linear isometry between JB*-triples Z and W. Then for any $z \in Z$, there is a tripotent $u_z \in W^{**}$ such that*

$$\{u_z, T(a^{(3)}), u_z\} = \{u_z, (Ta)^{(3)}, u_z\}$$

for all a in the subtriple Z_z generated by z, and that

$$\{u_z, T(\cdot), u_z\} : Z_z \longrightarrow W^{**}$$

is an isometry.

Proof. Let $z \in Z$. If the restriction $T : Z_z \longrightarrow W$ is a triple homomorphism, one can take $u_z \in W^{**}$ to be the support tripotent of $T(z)$; otherwise, Theorem 1 furnishes the required tripotent u_z.

Example 4. Let $T : \mathbb{C} \longrightarrow M_2$ be the isometry defined in Example 1:

$$T(a) = \begin{pmatrix} 0 & \frac{a}{2} \\ a & 0 \end{pmatrix}.$$

Then the tripotent

$$u = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfies the conditions in Theorem 1.

In Theorem 2, the tripotent u_z depends on the given element $z \in Z$. Extending the arguments in the proof of Theorem 1, we show below that, if Z admits a character, then one can find a tripotent $v \in W^{**}$ such that $\{v, T(\cdot), v\} \neq 0$ and

$$\{v, T(a^{(3)}), v\} = \{v, (Ta)^{(3)}, v\}$$

for all $a \in Z$. Without any condition on Z, such a tripotent v may not exist.

A *character* φ of a JB*-triple Z is a non-zero triple homomorphism $\varphi : Z \longrightarrow \mathbb{C}$.

Lemma 2. *Let φ be a character of a JB*-triple Z. Then φ is an extreme point of the closed unit ball of Z^*.*

Proof. Since $\varphi : Z \longrightarrow \mathbb{C}$ is a triple homomorphism, the induced quotient map $\widetilde{\varphi} : Z/\ker\varphi \longrightarrow \mathbb{C}$ is a triple isomorphism and hence an isometry. In particular, $\|\varphi\| = 1$. Let $e \in Z^{**}$ be the support tripotent of φ. For $f \in Z^*$, we denote by $P_2(e)f$ the composite function $f \circ P_2(e) \in Z^*$ and $P_2(e)Z^*$ is defined accordingly.

If $\varphi = \frac{1}{2}f + \frac{1}{2}g$ with $\|f\|, \|g\| \leq 1$, then we have $\|f\| = \|g\| = 1$ and

$$\begin{aligned} 1 = \|\varphi\| &= \|P_2(e)\varphi\| \\ &\leq \frac{1}{2}\|P_2(e)f\| + \frac{1}{2}\|P_2(e)g\| \leq 1 \end{aligned}$$

which yields $\|P_2(e)f\| = 1 = \|P_2(e)g\|$ and so $P_2(e)f = f$ and $P_2(e)g = g$ by [9, Proposition 1]. It follows that φ is an extreme point of the unit ball of Z^* if, and only if, it is an extreme point of the unit ball of of $P_2(e)Z^*$.

Now consider the character $\varphi : P_2(e)Z^{**} \longrightarrow \mathbb{C}$ as a weak* continuous functional. The kernel $\ker\varphi$ is a weak*-closed Jordan ideal in the JBW*-algebra $(P_2(e)Z^{**}, \circ)$. Hence there is a central projection q in $P_2(e)Z^{**}$ such that $\ker\varphi = P_2(e)Z^{**} \circ q$ [10, 4.3.6]. The projection $e - q$ has a weak*-closed support face in $P_2(e)Z^*$, namely,

$$F_{e-q} = \{\psi \in P_2(e)Z^* : \|\psi\| = \psi(e) = 1 = \psi(e-q)\}.$$

Pick an extreme point ρ from F_{e-q}. Then $\rho(\ker\varphi) = \{0\}$ implies that $\varphi = \rho$ which is an extreme point of the unit ball of $P_2(e)Z^*$.

Proposition 2. *Let $T : Z \to W$ be a linear isometry between JB*-triples. If Z admits a character, then there is a tripotent u in W^{**} such that $\{u, T(\cdot), u\} : Z \longrightarrow W^{**}$ is a nonzero triple homomorphism and*

$$\{u, T(a^{(3)}), u\} = \{u, (Ta)^{(3)}, u\} \qquad (a \in Z).$$

Proof. Let η be a character of Z and consider the isometry $T^* : T(Z)^* \longrightarrow Z^*$. Since η is the pre-image of an extreme point of the unit ball of $T(Z)^*$, and since the extreme points in the unit ball of $T(Z)^*$ can be extended to the extreme points in the unit ball of W^*, we see that there is an extreme point φ of the unit ball of W^* such that $\varphi \circ T = \eta$. Let $u \in W^{**}$ be the minimal tripotent supporting φ. Then

$$\{u, T(\cdot), u\} = \varphi \circ T(\cdot)u = \eta(\cdot)u$$

implies that $\{u, T(\cdot), u\}$ is a nonzero triple homomorphism, and as in the proof of Theorem 1, we have

$$\{u, T(a^{(3)}), u\} = \{u, (Ta)^{(3)}, u\} \qquad (a \in Z).$$

The converse of Proposition 2 holds if W is abelian.

Proposition 3. *Let $T : Z \longrightarrow W$ be a linear isometry between JB*-triples where W is an abelian C*-algebra. The following conditions are equivalent:*

(i) there is a tripotent $u \in W^{**}$ such that $\{u, T(\cdot), u\} \neq 0$ and $\{u, T(a^{(3)}), u\} = \{u, (Ta)^{(3)}, u\}$ for $a \in Z$;
(ii) Z admits a character.

Proof. Let u be the tripotent in (i) such that $\{u, T(\cdot), u\} \neq 0$. Then there exists a character ρ of W which does not vanish on $\{u, T(Z), u\}$, and hence the composite $\rho \circ \{u, T(\cdot), u\} : Z \longrightarrow \mathbb{C}$ is a non-zero triple homomorphism.

Example 5. Let $T : M_2 \longrightarrow C(Y)$ be the natural linear isometry into the continuous functions on the closed unit ball Y of M_2^*. Since M_2 has no character, there is no tripotent in $C(Y)^{**}$ satisfying Proposition 2.

5 Isometries in JB*-algebras

In this section, we consider a linear isometry from a JB*-triple into a JB*-algebra. This is motivated by the fact that, given a linear isometry $T : Z \longrightarrow W$ between JB*-triples, by considering the second dual map, we may assume that W is a JBW*-triple which is, via an isometric embedding [5], a subtriple of a JBW*-algebra. This leads to the case in which the range W can be taken as a JB*-algebra. We will prove a more general result for linear contractions from JB*-triples into JB*-algebras. In this case, they may still preserve a fair amount of Jordan structure, after scaling down by a projection.

We first need to develop some basic results for JB*-algebras in which one can make good use of projections apart from tripotents. The Jordan product in a JB*-algebra will be denoted by $\circ$. We note that every JBW*-algebra A has an identity **1** [10, 4.1.7] and a continuous linear functional φ on A is positive if, and only if, $\|\varphi\| = \varphi(\mathbf{1})$. If φ is a positive functional and if $\varphi(p) = \varphi(\mathbf{1})$ for some projection p in A, then we have

$$\varphi(a \circ p) = \varphi(a) \qquad (a \in A).$$

Indeed, if $a = a^*$, then the Schwarz inequality [10, 3.6.2] gives

$$0 \leq \varphi(a \circ (\mathbf{1} - p))^2 \leq \varphi(a^2)\varphi((\mathbf{1} - p)^2) = 0$$

and therefore $\varphi(a \circ (\mathbf{1} - p)) = 0$. We also have

$$\varphi\{p, a, p\} = \varphi(2p \circ (p \circ a) - p \circ a) = \varphi(a).$$

Let φ be a normal state of A. Since the projections in A form a complete lattice [10, 4.2.8], there is a smallest projection $p_\varphi \in A$ such that $\varphi(p_\varphi) = 1$. We call p_φ the *support projection* of φ. For any positive normal functional φ, its *support projection* is the smallest projection p_φ in A satisfying $\varphi(p_\varphi) = \varphi(\mathbf{1})$. More generally, a norm-closed face of the normal state space of A also admits a support projection shown in the following lemma.

Lemma 3. *Let F be a norm-closed face of the normal state space S of a JBW*-algebra A. Then there is a projection $p \in A$ such that*

$$F = \{\varphi \in S : \varphi(p) = 1\}.$$

Proof. Since F is a norm-closed face of the closed unit ball of the predual A_* of A, it follows from [8, Corollary 4.5] that F is a norm-exposed face of S. By [1], every norm-exposed face of S is of the above form.

Given a JB*-algebra A, we let

$$Q(A) = \{\varphi \in A^* : \varphi \geq 0 \text{ and } \|\varphi\| \leq 1\}$$

be the quasi-state space of A. Given a projection p in A^{**}, the set

$$F^+(p) = \{\varphi \in Q(A) : \varphi(\mathbf{1} - p) = 0\}$$

is a face of $Q(A)$ containing 0. We show below that all weak* closed faces of $Q(A)$ containing 0 are of this form.

Lemma 4. *Let A be a JB*-algebra and let $F \subset Q(A)$ be a weak* closed face of $Q(A)$ containing 0. Then there is a projection p in A^{**} such that*

$$F = F^+(p) = \{\varphi \in Q(A) : \varphi(\mathbf{1} - p) = 0\}.$$

Proof. Let $S = \{\varphi \in A^* : \varphi(\mathbf{1}) = 1 = \|\varphi\|\}$ be the normal state space of A^{**}. We have $F = co(F' \cup \{0\})$ where $F' = F \cap S$ is a weak* closed face of S and by Lemma 3, there is a projection $p \in A^{**}$ such that

$$F' = \{\varphi \in S : \varphi(p) = 1\}$$

and it follows that $F = F^+(p)$.

Lemma 5. *Let A be a JC*-algebra and let $p \in A^{**}$ be a projection. Then for all $x \in A$, we have $x \circ p = 0$ if, and only if, $\varphi(x^* \circ x) = 0$ for all $\varphi \in F^+(p)$.*

Proof. The second dual A^{**} is a JW*-algebra and we may assume that it is a unital Jordan subalgebra of a von Neumann algebra $\mathcal{A}$, with the same identity. Let $\varphi \in F^+(p)$. Then $\varphi(p) = \varphi(\mathbf{1})$ and by previous remarks, we have $\varphi(x) = \varphi(p \circ x) = \varphi(\{p, x, p\})$ for all $x \in A$. The condition $0 = x \circ p = xp + px$ implies that $pxp = -px = -xp$ and so $px = xp = 0$. Hence $\varphi(x^* \circ x) = \varphi(\{p, x^* \circ x, p\}) = \frac{1}{2}\varphi(p(x^*x + xx^*)p) = \frac{1}{2}\varphi(0) = 0$.

For the converse, choose $\psi \in Q(A)$ and let $\widetilde{\psi}$ be a norm-preserving extension of ψ to $\mathcal{A}$. Then $\widetilde{\psi}$ is positive on $\mathcal{A}$. Define $\varphi(\cdot) = \psi\{p, (\cdot)^*, p\}$. Then $\varphi \in F^+(p)$ and so $\varphi(x^* \circ x) = 0$. The Schwarz inequality gives

$$\begin{aligned}|\widetilde{\psi}(px)|^2 + |\widetilde{\psi}(xp)|^2 &\leq \widetilde{\psi}(pxx^*p) + \widetilde{\psi}(px^*xp) = 2\widetilde{\psi}(p(x^* \circ x)p) \\ &= 2\psi\{p, (x^* \circ x), p\} = 2\varphi(x^* \circ x) = 0.\end{aligned}$$

Hence $\widetilde{\psi}(px) = \widetilde{\psi}(xp) = 0$ and $\psi(x \circ p) = \widetilde{\psi}(x \circ p) = 0$. As ψ was arbitrary in $Q(A)$, it follows that $x \circ p = 0$.

Proposition 4. *Let B be a JB*-algebra and let $p \in B^{**}$ be a projection. Then for $x \in B$, the following conditions are equivalent:*

(i) $x \circ p = 0$ *;*
(ii) $\varphi(x^* \circ x) = 0$ *for all* $\varphi \in F^+(p)$.

Proof. Let B_{sa} be the self-adjoint part of B. First, let $x \in B_{sa}$ and let A be the JBW*-subalgebra of B^{**} generated by x, p and $\mathbf{1}$. Then A is a JW*-algebra and by Lemma 5, we have $x \circ p = 0$ if, and only if, $\psi(x^2) = 0$ for all $\psi \in F_A^+(p)$, where

$$F_A^+(p) = \{\psi \in A_* : \psi(\mathbf{1}) = \|\psi\| \text{ and } \psi(\mathbf{1} - \mathrm{p}) = 0\}.$$

Since every $\varphi \in F^+(p)$ restricts to a quasi-state $\varphi|_A \in F_A^+(p)$ and since every $\psi \in F_A^+(p)$ extends to a quasi-state $\widetilde{\psi} \in F^+(p)$, we have $x \circ p = 0$ if, and only if, $\varphi(x^2) = 0$ for all $\varphi \in F^+(p)$.

Now for any $x \in B$, write $x = x_1 + ix_2$ with $x_1, x_2 \in B_{sa}$. Then $x \circ p = 0$ if, and only if, $x_1 \circ p = 0$ and $x_2 \circ p = 0$. This is equivalent to $\varphi(x_1^2) = 0 = \varphi(x_2^2)$ for all $\varphi \in F^+(p)$ which is the same as $\varphi(x_1^2 + x_2^2) = 0 = \varphi(x^* \circ x)$ for every $\varphi \in F^+(p)$.

Two self-adjoint elements a and p in a JB*-algebra B are said to *operator commute* if they generate an associative subalgebra of B. If p is a projection, this is equivalent to $\{p,a,p\} = a \circ p$, and to $\{a,p,a\} = a^2 \circ p$ (cf. [10, Lemma 2.5.5]). Condition (ii) below is an operator commuting condition.

Theorem 3. *Let Z be a JB*-triple, B be a JB*-algebra and let $T : Z \longrightarrow B$ be a linear contraction. Then there is a largest projection p in B^{**} such that for all $a,b,c \in Z$, we have*

(i) $T\{a,b,c\} \circ p = \{Ta,Tb,Tc\} \circ p$;

(ii) $\{p, T(a)^* \circ T(b), p\} = (T(a) \circ T(b)^*) \circ p$.

Proof. Let

$$F_1 = \bigcap_{a \in Z_1} \left\{ \varphi \in Q(B) : \varphi\left((Ta^{(3)} - (Ta)^{(3)})^* \circ (Ta^{(3)} - (Ta)^{(3)}) \right) = 0 \right\}.$$

Then F_1 is a weak* closed face of $Q(B)$ containing zero. For a in Z_1, we define a weak* continuous affine map $\Phi_a : Q(B) \longrightarrow Q(B)$ by

$$\Phi_a(\varphi)(\cdot) = \overline{\varphi}(\{(Ta)^* \circ Ta,\ \cdot\ , (Ta)^* \circ Ta\})$$

where the bar '−' denotes complex conjugation. For $n = 1, 2, \ldots$, the sets

$$F_{n+1} = \{\varphi \in F_n : \Phi_a(\varphi) \in F_n \text{ for all } a \in Z_1\} = \bigcap_{a \in Z_1} F_n \cap \Phi_a^{-1}(F_n)$$

form a decreasing sequence of weak* closed faces of $Q(B)$. The intersection $F = \bigcap_{n=1}^{\infty} F_n$ is a weak* closed face of $Q(B)$ containing zero. By Lemma 4, there is a projection in $p \in B^{**}$ supporting F:

$$F = F^+(p) = \{\varphi \in Q(B) : \varphi(\mathbf{1} - p) = 0\}.$$

For each a in A_1 and φ in F, we have

$$\Phi_a(\varphi)(\cdot) = \overline{\varphi}(\{(Ta)^* \circ (Ta),\ \cdot\ , (Ta)^* \circ (Ta)\}) \in F,$$

and consequently,

$$\varphi\{(Ta)^* \circ Ta,\ p\ , (Ta)^* \circ Ta\} = \overline{\Phi_a(\varphi)(p)} = \overline{\Phi_a(\varphi)(1)} = \varphi(((Ta)^* \circ Ta)^2).$$

Let $z = (Ta)^* \circ Ta$. Then z is self-adjoint, as is $x = p \circ z - z$. For all $\varphi \in F^+(p)$,

$$\begin{aligned}
\varphi(x^* \circ x) &= \varphi((p \circ z - z)^2) \\
&= \varphi((p \circ z)^2 - 2z \circ (p \circ z) + z^2) \\
&= \varphi((p \circ z)^2 - \{z,p,z\} - z^2 \circ p + z^2) \\
&= \varphi((p \circ z)^2 - \{z,p,z\})
\end{aligned}$$

using the fact that $\varphi(z^2) = \varphi(z^2 \circ p)$. By calculating in the special subalgebra generated by p and z, one obtains

$$(p \circ z)^2 = \frac{1}{2} p \circ \{z, p, z\} + \frac{1}{4}\{p, z^2, p\} + \frac{1}{4}\{z, p, z\}.$$

Hence we have

$$\begin{aligned} 4\varphi(x^* \circ x) &= \varphi(2p \circ \{z, p, z\} + \{p, z^2, p\} + \{z, p, z\} - 4\{z, p, z\}) \\ &= \varphi(2\{z, p, z\} + z^2 - 3\{z, p, z\}) \\ &= \varphi(z^2) - \varphi(\{z, p, z\}) \\ &= \Phi_a(1) - \Phi_a(p) = 0. \end{aligned}$$

By Proposition 4, we have $p \circ x = 0$. As p is a projection, it follows that $\{p, z, p\} - p \circ z = 2(p \circ z) \circ p - 2p \circ z = 2p \circ x = 0$, that is,

$$\{p, (Ta)^* \circ (Ta), p\} = ((Ta)^* \circ (Ta)) \circ p$$

for all $a \in Z_1$. By polarization, we have

$$\{p, (Ta)^* \circ (Tb), p\} = ((Ta) \circ (Tb)^*) \circ p$$

for all $a, b \in Z$. For $a \in Z_1$, we have

$$\varphi\left((Ta^{(3)} - (Ta)^{(3)})^* \circ (Ta^{(3)} - (Ta)^{(3)})\right) = 0$$

for all $\varphi \in F$, hence Proposition 4 yields

$$(Ta^{(3)}) \circ p = (Ta)^{(3)} \circ p\,.$$

Polarization then gives

$$T\{a, b, c\} \circ p = \{Ta, Tb, Tc\} \circ p \qquad (a, b, c \in Z).$$

Finally, if q is a projection in B^{**} satisfying conditions (i) and (ii), then $F^+(q) \subset F_1$. Indeed, for $\varphi \in F^+(q)$, we have

$$\varphi\left((Ta^{(3)} - (Ta)^{(3)})^* \circ (Ta^{(3)} - (Ta)^{(3)})\right) = 0$$

since $(Ta^{(3)} - (Ta)^{(3)}) \circ q = 0$ by (i) and Proposition 4 applies. Further, for all $a \in Z_1$, we have $\Phi_a(F^+(q)) \subset F^+(q)$ since

$$\begin{aligned} \Phi_a(\varphi)(q) &= \overline{\varphi}\left(\{(Ta)^* \circ Ta, q, (Ta)^* \circ Ta\}\right) \\ &= \Phi_a(\varphi)((((Ta)^* \circ Ta)^2 \circ q) \\ &= \Phi_a(\varphi)((((Ta)^* \circ Ta)^2) = \Phi_a(\varphi)(1) \end{aligned}$$

where the second identity follows from (ii). Therefore $F^+(q) \subset \bigcap_{n=1}^{\infty} F_n = F^+(p)$ and $q \leq p$.

Remark 3.(1) We note that condition (i) in Theorem 3 also gives

$$\begin{aligned}\{p, T\{a,a,a\}, p\} &= 2(p \circ T\{a,a,a\}^*) \circ p - p \circ T\{a,a,a\}^* \\ &= \{p, \{Ta, Ta, Ta\}, p\}.\end{aligned}$$

(2) If B is a JC*-algebra in Theorem 3, then condition (i) gives

$$T\{a,a,a\}p + pT\{a,a,a\} = \{Ta, Ta, Ta\}p + p\{Ta, Ta, Ta\}$$

and by (1) above, we have both $T\{a,a,a\}p = \{Ta, Ta, Ta\}p$ and $pT\{a,a,a\} = p\{Ta, Ta, Ta\}$.

(3) If B is a JBW*-algebra in Theorem 3, then p can be chosen in B itself. Indeed, we have $B = z \circ B^{**}$ for some central projection $z \in B^{**}$ and $z \circ p$ is the largest projection satisfying conditions (i) and (ii).

Example 6. The projection p in Theorem 3 could be zero, even if T is an isometry. Indeed, for the isometry $T : \mathbb{C} \longrightarrow M_2$ in Example 1, we have $p = 0$. On the other hand, for the isometry $S : \mathbb{C} \to M_3$ given by

$$S(a) = \begin{pmatrix} 0 & 0 & \frac{a}{2} \\ 0 & a & 0 \\ a & 0 & 0 \end{pmatrix}$$

we have

$$p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover $S(\cdot) \circ p$ is an isometry.

Example 7. Let $T : C(\Omega) \longrightarrow C(\Omega \cup \{\beta\})$ be the non-surjective isometry given in Example 3. Then the characteristic function $p = \chi_\Omega \in C(\Omega \cup \{\beta\})$ is the largest projection satisfying the conclusion of Theorem 3.

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