HOW TO OBTAIN NON-FLEXIBLE QUADRATIC ALGEBRAS AS MUTATION ALGEBRAS OF HURWITZ ALGEBRAS

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Abstract. We present an elementary way to obtain new families of division algebras of degree n out of division algebras with a multiplicative norm. Families of four- and eight-dimensional non-flexible quadratic division algebras over a field F of characteristic not 2 are constructed elementary as a-mutation algebras of Hurwitz division algebras A over F, choosing $a \in A \setminus F$. In particular, we obtain real eight-dimensional division algebras with derivation algebra $su(3)$ or G_2.

Introduction

Since there does not exist any general theory of nonassociative algebras, it is desirable to get a more unified point of view at least on certain types. Shavarevich [Sh, p. 201] suggested the structure of a real division algebra as a test problem for a possible future understanding of various types of algebras from a unified point of view. It thus seems only natural to try to find division algebra constructions which work over any base field and not just over the reals. It is usually beneficial to formulate such constructions in a base-free way, since this makes the structure of the algebras easier to understand.

For a unital algebra A over a field F and an element $a \in A$ in the center of A, the *a-mutation algebra* $A^{(a)}$ is obtained by assigning the underlying vector space of A the new multiplication $u \circ v = auv + (1-a)vu$.

For an associative algebra A, the mutation algebras $A^{(a)}$, also called *quasi-associative* algebras, appear in the structure theory of noncommutative simple Jordan algebras [M2].

There is no doubt that a-mutation algebras of alternative algebras play a prominent role in the classification of real division algebras: If, in a non-commutative four- or eight-dimensional real Jordan division algebra, any two elements not in the same two-dimensional subalgebra generate a four-dimensional subalgebra, then that algebra is either an a-mutation algebra of Hamilton's quaternion algebra or of Cayley's octonion algebra, with $a \in \mathbb{R}$, $a \neq \frac{1}{2}$. Furthermore, it is well-known that all four-dimensional real flexible quadratic division algebras are a-mutations $\mathbb{H}^{(a)}$ of Hamilton's quaternion algebra, where $a \in \mathbb{R}$, $a \neq \frac{1}{2}$.

The situation becomes more complex in dimension eight: eight-dimensional real flexible quadratic division algebras do not all arise from a-mutations with $a \in \mathbb{R}$ or even from a combination of such mutations and generalized Cayley-Dickson doublings. They can be obtained out of Cayley's octonion algebra $\mathbb{O}$ by vectorial isotopy, however, cf. [C-V-K-R].

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In [L], those eight-dimensional real quadratic division algebras which arise as the Cayely-Dickson doubling of a quadratic division algebra are listed. The classification is broken down into classifying the subcategories of eight-dimensional real quadratic division algebras of degree 1, 3 or 5 in [D3]. The major tools in this classification are dissident maps [Da, D1, 2, 3].

In this paper, we construct families of non-flexible quadratic division algebras over an arbitrary field of characteristic not 2 out of Hurwitz division algebras of dimension 4 or 8. The advantage of our construction is that it is straightforward and base free. The algebras are constructed as a-mutations $A^{(a)}$ of Hurwitz division algebras. More precisely, we generalize the concept of an a-mutation and allow $a \in A$ (and not just in the center of A) when defining the multiplication

$$u \circ v = a(uv) + (1-a)(vu)$$

of the a-mutation algebra $A^{(a)}$ of a unital algebra A.

How to obtain new division algebras via general mutations out of non-commutative division algebras with a multiplicative norm form is described in Section 2.1. We then focus on some special cases, among them the a-mutation algebras $A^{(a)}$. Our main result is contained in Theorem 10: For a Hurwitz division algebra C of dimension 4 or 8 and all $a \in C$, $a \neq \frac{1}{2}$, $C^{(a)}$ is a quadratic division algebra which is flexible if and only if $a \in F$. If C is a quaternion division algebra and $a \in C$, $a \neq \frac{1}{2}$, these four-dimensional unital division algebras $C^{(a)}$ are not isotopic to associative or nonassociative quaternion algebras. Cayley-Dickson doublings of four-dimensional mutation algebras are briefly considered in Section 2.2. In Section 3, mutation algebras of associative algebras with higher degree multiplicative norm forms are considered. Section 4 deals with the automorphisms and derivations of mutation algebras.

1. Preliminaries

Let F be a field.

1.1. **Nonassociative algebras.**

By "F-algebra" we mean a finite dimensional nonassociative algebra over F. A nonassociative algebra $A \neq 0$ is called a *division algebra* if for any $a \in A$, $a \neq 0$, the left multiplication with a, $L_a(x) = ax$, and the right multiplication with a, $R_a(x) = xa$, are bijective. A is a division algebra if and only if A has no zero divisors [Sch, pp. 15, 16].

For an F-algebra A, commutativity is measured by the *commutator* $[x, y] = xy - yx$ and associativity is measured by the *associator* $[x, y, z] = (xy)z - x(yz)$. The *nucleus* of A is given by $\mathrm{Nuc}(A) = \{x \in A \,|\, [x, A, A] = [A, x, A] = [A, A, x] = 0\}$. The nucleus is an associative subalgebra of A containing $F1$ and $x(yz) = (xy)z$ whenever one of the elements x, y, z is in $\mathrm{Nuc}(A)$. The *center* of A is defined as $\mathrm{C}(A) = \{x \in A \,|\, x \in \mathrm{Nuc}(A) \text{ and } xy = yx \text{ for all } y \in A\}$.

Let A be an F-algebra is which is unital and strictly power-associative. Fixing a basis $(u_i)_{1 \leq i \leq r}$ of A and taking indeterminates $x_1, \dots, x_r$, there is a generic element

$$x = \sum x_i u_i \in A \otimes_F F(x_1, \dots, x_r)$$

and a unique monic polynomial

$$P_{A,x}(X) = X^m - S_1(X) + \cdots + (-1)^m S_m(X)1_A$$

of least degree which has x as a root. The coefficients S_i are homogeneous polynomials in the x_i's, $S_1 = T_A$ is the *(generic) trace*, $S_m = N_A$ *the (generic) norm* and m is called *the degree* of A [KMRT, p. 451 ff.].

A is called *alternative* if its associator $[x, y, z]$ is alternating. An anti-automorphism $\sigma : A \to A$ of period 2 is called an *involution* on A. An involution on a unital algebra is called *scalar* if all *norms* $\sigma(x)x$ are elements of $F1$. For every scalar involution σ, $N_A(x) = \sigma(x)x$ (resp. the *trace* $T_A(x) = \sigma(x) + x$) is a quadratic (resp. a linear) form on A. An algebra A together with a nondegenerate quadratic form $N : A \to F$ is called a *mono-composition algebra*, if $N(x^2) = N(x)^2$ for all $x \in A$.

1.2. **Quadratic algebras.** A unital algebra A is called *quadratic*, if there exists a quadratic form $N\colon A \to F$ such that $x^2 - N(1_A, x)x + N(x)1_A = 0$ for all $x \in A$, where $N(x, y) = N(x+y) - N(x) - N(y)$ denotes the symmetric bilinear form induced by N. This automatically implies that $N(1_A) = 1$. The form N is uniquely determined and called the *norm* $N = N_A$ of the quadratic algebra A. The existence of a scalar involution on a unital algebra A implies that A is quadratic [M1, Theorem 1.1]. Moreover, the unital algebras with scalar involution are precisely those quadratic algebras with normal trace [M1, Theorem 1.1].

Every quadratic algebra is a unital mono-composition algebra [M1, p. 86].

It is well-known that if $\operatorname{char} F \neq 2$, every quadratic algebra over F can be obtained out of an anti-commutative algebra $(V, \wedge)$ and an F-bilinear form $(,) : V \times V \to V$ by defining a multiplication on the F-vector space $A = F \oplus V$ via

$$(a, u)(b, v) = (ab + (u, v), av + bu + u \wedge v)$$

for all $a, b \in F$ and $u, v \in V$. The resulting algebra A is denoted by $(V, (,), \wedge) = F \oplus V$.

1.3. **Composition algebras.** A quadratic form $N\colon A \to F$ on an algebra A is *multiplicative* if $N(uv) = N(u)N(v)$ for all $u, v \in A$. An algebra A is called a *composition algebra* over F if it admits a *nondegenerate* multiplicative quadratic form $N\colon A \to F$; i.e., its induced symmetric bilinear form $N(u, v) = N(u+v) - N(u) - N(v)$ determines a module isomorphism $C \xrightarrow{\sim} A^\vee = Hom_F(A, F)$. The form N is unique [KMRT, p. 454 ff.]. It is called the *norm* of A and we also write $N = N_A$. A unital composition algebra is called a *Hurwitz algebra.* Hurwitz algebras are quadratic alternative and $N(1_A) = 1$; the norm of a Hurwitz algebra C is the unique nondegenerate quadratic form on A that is multiplicative. A quadratic alternative algebra is a Hurwitz algebra if and only if its norm is nondegenerate [M1, 4.6]. Hurwitz algebras exist only in dimensions 1, 2, 4 or 8. Those of dimension 2 are exactly the quadratic étale F-algebras, those of dimension 4 exactly the well-known quaternion algebras. The ones of dimension 8 are called *octonion algebras.* The conjugation $\overline{x} = T_{C/F}(x)1_A - x$ of a Hurwitz algebra C is a scalar involution, called the *canonical involution* of C, where $T_A\colon A \to F$, $T_{A/F}(x) = N_{A/F}(1_A, x)$, is the *trace* of A.

Let D be a Hurwitz algebra over F with canonical involution $^- : D \to D$. Let $c \in F^\times$. Then the F-vector space $A = D \oplus D$ can be made into a unital algebra over F via the multiplications

$$(u,v)(u',v') = (uu' + c\bar{v}'v, v'u + v\bar{u}')$$

for $u, u', v, v' \in D$. The unit element of A is given by $1 = (1, 0)$. A is called the *Cayley-Dickson* doubling of D.

Every composition algebra A is a principal Albert isotope of a Hurwitz algebra: There are isometries φ_1, φ_2 of the norm N_C for a suitable Hurwitz algebra C over F such that the multiplication $\star$ of A can be written as

$$x \star y = \varphi_1(x)\varphi_2(y).$$

Given a Hurwitz algebra C over F of dimension ≥ 2 with canonical involution $^{-}$, the multiplications

$$x \star y = \bar{x}\bar{y}, \quad x \star y = \bar{x}y, \quad x \star y = x\bar{y}$$

for all $x, y \in C$ define the *para-Hurwitz algebra*, resp. the *left-* and *right composition algebra* associated to C. Together with C these are called the *standard composition algebras*.

Standard composition algebras of dimension eight have a derivation algebra isomorphic to G_2.

2. General mutation algebras

2.1. Let F be a field and A an algebra over F with underlying vector space V and let $f, g, f_i \in \mathrm{Gl}(V)$ for $i = 1, \dots, 6$.

Define the algebra $(A, \circ_{(f_1,\dots,f_6)})$ over F as the algebra with underlying vector space V and multiplication given by

$$u \circ_{(f_1,\dots,f_6)} v = f_3(f_1(u)f_2(v)) + f_6(f_4(v)f_5(u)).$$

Theorem 1. *Let A be an algebra over F with an anisotropic multiplicative norm $N : A \to F$ of degree n and f_i a similarity of N with similarity factor α_i, $i = 1, \dots, 6$. If*

$$\alpha_1\alpha_2\alpha_3 \neq \alpha_4\alpha_5\alpha_6$$

then $(A, \circ_{(f_1,\dots,f_6)})$ is a division algebra.

Proof. Since A has an anisotropic multiplicative norm, it must itself be a division algebra: $uv = 0$ implies $N(u)N(v) = 0$, thus $N(u) = 0$ or $N(v) = 0$, yielding $u = 0$ or $v = 0$. Let $u, v \in A$ be non-zero. Suppose that $u \circ_{(f_1,\dots,f_6)} v = 0$, then $f_3(f_1(u)f_2(v)) = -f_6(f_4(v)f_5(u))$. Applying the norm on both sides yields $\alpha_1\alpha_2\alpha_3 N(u)N(v) = \alpha_4\alpha_5\alpha_6 N(v)N(u)$, thus $\alpha_1\alpha_2\alpha_3 = \alpha_4\alpha_5\alpha_6$, a contradiction. $\square$

Remark 2. Let A be a principal Albert isotope $A_0^{(f,g)}$ of an algebra A_0 over F with multiplicative norm N_{A_0} and f, g isometries of the norm. Then A has the multiplicative norm $N_A = N_{A_0}$ and $(A, \circ_{(f_1,\dots,f_6)}) = (C, \circ_{(f\circ f_1, g\circ f_2, f_3, f\circ f_4, g\circ f_5, f_6)})$.
If A is a division algebra over F with multiplicative norm $N : A \to F$ of degree n and $f_i \in \mathrm{Gl}(V)$ similarities of N with similarity factors α_i, $i = 1, \dots, 6$, then we can also define a new multiplication on V via

$$u \diamond_{(f_1,\dots,f_6)} v = f_3(f_1(u)f_2(v)) + f_6(f_5(u)f_4(v)).$$

If

$$\alpha_1\alpha_2\alpha_3 \neq \alpha_4\alpha_5\alpha_6$$

then $(A, \diamond_{(f_1,\dots,f_6)})$ is a division algebra by an analogous argument as in the proof of Theorem 1. We will not pursue this idea here. There is some overlap with this type of multiplication and a possible way to generalize the construction of twisted fields by Albert (cf. [Pu]).

We will focus on the special case

$$u \circ_{(f,g)} v = f(uv) + g(vu),$$

in particular on

$$u \circ_f v = f(uv) + (id - f)(vu),$$

assuming here that $f \in \mathrm{Gl}(V)$ is chosen such that $id - f \in \mathrm{Gl}(V)$ as well. Moreover, define $A^{(a,b)}$ to be the algebra with V as underlying vector space and new multiplication given by

$$u \circ v = a(uv) + b(vu),$$

$A_r^{(a,b)}$ be the algebra with V as underlying vector space and new multiplication given by

$$u \circ v = (uv)a + (vu)b$$

$A_{l,r}^{(a,b)}$ be the algebra with V as underlying vector space and new multiplication given by

$$u \circ v = a(uv) + (vu)b,$$

$A_{r,l}^{(a,b)}$ be the algebra with V as underlying vector space and new multiplication given by

$$u \circ v = (uv)a + b(vu)$$

and $A_m^{(a,b)}$ be the algebra with V as underlying vector space and new multiplication given by

$$u \circ v = (ua)v + (vb)u.$$

Note that $(A^{opp})_m^{(b,a)}$ has the multiplication

$$u \circ v = u(av) + v(bu),$$

so that we do not need to study this multiplication separately. Obviously, also

$$(A_{l,r}^{(a,b)})^{op} = A_{r,l}^{(b,a)}.$$

For a unital algebra A, we define $A^{(a)} = A^{(a,1_A-a)}$, $A_r^{(a)} = A_r^{(a,1_A-a)}$, $A_m^{(a)} = A_m^{(a,1_A-a)}$ and $A_{l,r}^{(a)} = A_{l,r}^{(a,1_A-a)}$. Obviously,

$$(A^{op})^{(f,g)} = A^{(g,f)} \quad (A^{op})^{(a,b)} = A_r^{(b,a)}, \quad (A^{op})_r^{(a,b)} = A^{(b,a)},$$

so that in the following it suffices to consider $A^{(a)}$, $A_m^{(a)}$ and $A_{l,r}^{(a)}$, for instance.

We will mostly focus on unital algebras A.

Remark 3. (i) $A_m^{(a,b)}$ generalizes a quantum mechanical version of a generalized Hamilton mechanics algebra and has been extensively studied for alternative algebras A by Elduque, Montaner and Myung [E-M], [E-My1, 2].
(ii) For $a,b\in F$, obviously $A^{(a,b)}=A_r^{(a,b)}=A_{l,r}^{(a,b)}=A_m^{(a,b)}$ and if A is unital, $\mathrm{Der}(A)\subset\mathrm{Der}(A^{(a,b)})$.
(iii) If $G:A\to B$ is an algebra isomorphism then

$$G(u\circ v)=G(a(uv))+G(b(vu))=G(a)(G(u)G(v))+G(b)(G(v)G(u))=G(u)\circ G(v),$$

therefore G is an isomorphism between $A^{(a,b)}$ and $B^{(G(a),G(b))}$. An analogous argument implies that G is an isomorphism between $A_r^{(a,b)}$ and $B_r^{(G(a),G(b))}$, $A_m^{(G(a),G(b))}$ and $B_m^{(G(a),G(b))}$ and also between $A_{l,r}^{(a,b)}$ and $B_{l,r}^{(G(a),G(b))}$. Similarly, an anti-isomorphism $G:A\to B$ induces isomorphisms between $A^{(a,b)}$ and $B_r^{(G(b),G(a))}$, $A_r^{(a,b)}$ and $B^{(G(b),G(a))}$, $A_m^{(a,b)}$ and $B_m^{(G(b),G(a))}$, and $A_{l,r}^{(a,b)}$ and $B_{l,r}^{(G(b),G(a))}$.

The following observations are obvious:

(1) $u\circ_f u=u^2$. If A carries a multiplicative norm N of degree n then $N(u\circ_f u)=N(u)^2$. If $n=2$ then $(A,\circ_f)$ is a mono-composition algebra.
(2) For all $a,b\in A$, $u\circ u=(a+b)u^2$ in $A^{(a,b)}$.If A carries a multiplicative norm N of degree n then
$$N(u\circ u)=N(a+b)N(u)^2$$
and so
$$N(u\circ u)=N(u)^2 \text{ iff } N(a+b)=1.$$
Thus if $n=2$ and $N(a+b)=1$ then $A^{(a,b)}$ is a mono-composition algebras.
(3) If A carries a multiplicative norm N of degree 2 then $A=C^{(h_1,h_2)}$ for a suitable Hurwitz algebra C over F, with h_1,h_2 isometries of the norm $N_C=N$, and $(A,\circ_{(f,g)})$ with $f(x)=ax$, $g(x)=(1_C-a)x$, is a mono-composition algebra.
(4) $(A,\circ_{(f,f)})$ is isotopic to A^+ with multiplication $u\cdot v=\frac{1}{2}(uv+vu)$.

Corollary 4. *Let A be an algebra over F together with an anisotropic multiplicative form N of degree n.*
(i) If f, g are similarities of N with similarity factors $\alpha\neq\beta$ then $(A,\circ_{(f,g)})$ is a division algebra.
(ii) If $a,b\in A$ such that $N(a)\neq N(-b)$, then $A^{(a,b)}$, $A_{l,r}^{(a,b)}$ and $A_m^{(a,b)}$ are division algebras. If additionally $N(a+b)=1$ then $N(u\circ u)=N(u)^2$ in $A^{(a,b)}$.

This follows from Theorem 1 (the proof of $A_m^{(a,b)}$ being a division algebra was already given in [E-My1, 2]).

From now on, let

$$F \text{ be a field of characteristic not 2.}$$

For a unital algebra A and $a,b\in A$,

(4) $(A,\circ_f)$ has unit element 1_A.
(5) $A^{(a,b)}$ is a unital algebra with unit element 1_A iff $b=1_A-a$.
(6) $A^{(\frac{1}{2})}=A^+$ with $u\circ v=\frac{1}{2}(uv+vu)$.

(7) In $A^{(a)}$, $u \circ v = vu + a[u,v]$.
(8) In $A^{(a)}$ and $A_m^{(a)}$, $u \circ u = u^2$. If A carries a multiplicative norm N of degree n then $N(u \circ u) = N(u)^2$ in $A^{(a)}$ and $A_m^{(a)}$. In particular, for $n = 2$, $A^{(a)}$ and $A_m^{(a)}$ are mono-composition algebras.
(9) If A is power-associative (resp., strictly power-associative), then so is $A^{(a)}$ and $u^{\circ k} = u^k$ in $A^{(a)}$ for all $k \geq 2$. If A also carries a multiplicative norm, then $N(u^{\circ k}) = N(u)^k$ in $A^{(a)}$ for all $k \geq 2$.
(10) For all $a \in A^\times$, $a \neq \frac{1}{2}$, if A is not commutative then $A^{(a)}$ is not commutative.
(11) If A has a scalar involution $^-$, i.e. $\bar{1}_A = 1_A$ and $u\bar{u} \in F1_A$ then in $A^{(a)}$, $\bar{u} \circ u = u \circ \bar{u} = N(u)1_A$.
(12) For all $a \in F$, $a \neq \frac{1}{2}$, we have $\mathrm{Der}(A^{(a)}) = \mathrm{Der}(A)$.

Corollary 5. *(i) Let A be a Hurwitz division algebra. For all $a \in A$, $a \neq \frac{1}{2}$, $A^{(a)}$ is a unital mono-composition division algebra over F.*
(ii) Let A be a composition division algebra, i.e. $A = C^{(h_1,h_2)}$ for a suitable Hurwitz algebra C over F and isometries h_1, h_2 of its norm N_C. Define $f(x) = ax$, $g(x) = (1_C - a)x$ with $a \in A$, $a \neq 1_C$. Then for all $a \in A$, $a \neq \frac{1}{2}$, $(A, \circ_{(f,g)})$ is a mono-composition division algebra over F.

Proof. (i) Suppose A is Hurwitz. Let $u, v \in A$ such that $u \neq 0$, $v \neq 0$ and $u \circ_f v = f(uv) + (id - f)(vu) = a(uv) + (1_A - a)(vu)$. Then $N_A(a)N_A(u)N_A(v) = N_A(1-a)N_A(v)N_A(u)$ and thus $N_A(a) = N_A(a-1)$. Suppose A is a quaternion algebra. Then for $a = x_1 + x_2 i + x_3 j + x_4 k$, $N(a) = x_1^2 - cx_2^2 - dx_3^2 + cdx_4^2$ for suitable $c, d \in F^\times$ and $N_A(a-1) = (x_1 - 1)^2 - cx_2^2 - dx_3^2 + cdx_4^2$, so that $N_A(a) = N_A(a-1)$ implies $a = \frac{1}{2}$. An analogous calculation implies the assertion if A is an octonion algebra. If A is a separable field extension, it is trivial.
(ii) Let A be a composition algebra. Then there exists a suitable Hurwitz algebra C over F and isometries h_1, h_2 of its norm N_C, such that $A = C^{(h_1,h_2)}$ [KMRT] and with respect to the multiplication in C, we have

$$f(x) = h_1(a)h_2(x), \quad g(x) = h_1(1_C - a)h_2(x)$$

and

$$\begin{aligned} u \circ v &= f(h_1(u)h_2(v)) + g(h_1(v)h_2(u)) \\ &= h_1(a)h_2(h_1(u)h_2(v))) + h_1(1_C - a)h_2(h_1(v)h_2(u)) \end{aligned}$$

Thus, if there are $u, v \in (A, \circ_f)$ such that $u \neq 0$, $v \neq 0$ and $u \circ v = 0$ then $h_1(a)h_2(h_1(u)h_2(v)) = -h_1(1_C - a)h_2(h_1(v)h_2(u))$ in C, therefore $N_C(a) = N_C(a - 1_C)$ and hence for all $a \in A$ with $N_C(a) \neq N_C(a - 1_C)$, $(A, \circ_f)$ is a division algebra over F as in (i). $\square$

Remark 6. If A is a quadratic algebra and $a \in A$ we know that every proper ideal of $A^{(a)}$ contains an idempotent. Moreover, every $f \in \mathrm{Aut}(A^{(a)})$ is an isometry of N_A. So

$$\mathrm{Aut}(A^{(a)}) \subset O(N_A)$$

(cf. Gainov's results on mono-composition algebras [G1], [G2]).

If A is unital, we call $A^{(a)}$ an *a-mutation* of A, generalizing the common terminology used in the literature.

Proposition 7. *(i) Let A be a quadratic algebra with norm N. Then $(A, \circ_f)$ is a quadratic algebra over F with norm N and unit element 1_A.*
(ii) Let A be a unital algebra of degree n with norm N. Then $A^{(a)}$ is an algebra of degree n over F for all $a \in A$. If N is also multiplicative and A a division algebra then $A^{(a)}$ is a division algebra for all $a \in A$ with $N(a) \neq N(a-1)$.

Proof. (i) A is unital and therefore $(A, \circ_f)$ is unital with unit element 1_A. We have

$$u \circ_f u - N(1_A, u) \circ_f u + N(u) \circ_f 1_A = u^2 - uN(1_A, u)u + N(u)1_A = 0.$$

(ii) Since A is unital and strictly power-associative, so is $A^{(a)}$. A straightforward calculation yields:

$$u^{\circ\, n} - S_1(u) \circ u^{\circ\, n-1} + \cdots + (-1)^n N(u) \circ 1_A = u^n - S_1(u)u^{n-1} + \cdots + (-1)^n N(u)1_A = 0.$$

$\square$

Lemma 8. *(i) If A is flexible and a, b both lie in the center of A then $A^{(a,b)}$ is flexible.*
(ii) Let $A = \mathrm{Cay}(D, c)$ be a Hurwitz algebra of dimension 4 or 8 and $a = (x, y), b = (x', y') \in D \oplus D$. Then $D^{(x,x')}$ is a subalgebra of $A^{(a,b)}$, $D_m^{(x,x')}$ a subalgebra of $A_m^{(a,b)}$ and $D_{l,r}^{(x,x')}$ a subalgebra of $A_{l,r}^{(a,b)}$).
(iii) Let $A = (L/F, \sigma, c)$ be a cyclic algebra over F of degree n and $a = (x, y, z), b = (x', y', z') \in L \oplus L \oplus L = (L/F, \sigma, c)$. Then $L^{(x,x')}$, with multiplication $u \circ v = (x + x')uv$, is a subalgebra of $A^{(a,b)}$, $A_m^{(a,b)}$ and $A_{l,r}^{(a,b)}$.

Proof. All three proofs are straightforward calculations. We demonstrate (ii) for the sake of the reader and compute the multiplication in $A^{(a)}$ as example:

$$(u, v) \circ (u', v') = (x, y)(uu' + c\bar{v}'v, v'u + v\bar{u}') + (1 - x, -y)(u'u + c\bar{v}v', vu' + v'\bar{u})$$

$$= (xuu' + xc\bar{v}'v - c\bar{u}\bar{v}'y - cu'\bar{v}y, v'ux + v\bar{u}'x - y\bar{u}'\bar{u} - y\bar{v}v'c)$$

$$+((1 - x)u'u + (1 - x)c\bar{v}v' - c\bar{u}'\bar{v}y - cu\bar{v}'y, vu'(1 - x) + v'\bar{u}(1 - x) - y\bar{u}\bar{u}' - y\bar{v}'vc)$$

$$= (u \circ_x u' + xc\bar{v}'v - c\bar{u}\bar{v}'y - cu'\bar{v}y + (1 - x)c\bar{v}v' - c\bar{u}'\bar{v}y - cu\bar{v}'y,$$

$$v'ux + v\bar{u}'x - y\bar{u}'\bar{u} - y\bar{v}v'c + vu'(1 - x) + v'\bar{u}(1 - x) - y\bar{u}\bar{u}' - y\bar{v}'vc)$$

$$= (u \circ_x u' + xc(\bar{v}'v - \bar{v}v') - c\bar{u}\bar{v}'y - cu'\bar{v}y + c\bar{v}v' - c\bar{u}'\bar{v}y - cu\bar{v}'y,$$

$$v'ux + v\bar{u}'x - y\bar{u}'\bar{u} - y\bar{v}v'c + vu' - xvu' + v'\bar{u} - v'\bar{u}x - y\bar{u}\bar{u}' - y\bar{v}'vc).$$

Thus

$$(u, v) \circ (u', v') = (u \circ_x u' + c[x(\bar{v}'v - \bar{v}v') + \bar{v}v'] - c[\bar{u} + u]\bar{v}'y - c[u' + \bar{u}']\bar{v}y,$$

$$[v'ux + v'\bar{u} - v'\bar{u}x] + [v\bar{u}'x + vu' - xvu'] - y[\bar{u}'\bar{u} + \bar{u}\bar{u}'] - y[\bar{v}'v + \bar{v}v']c).$$

$\square$

Let $A = \mathrm{Cay}(D, c)$ be a Hurwitz algebra of dimension 4 or 8 and $a = (x, y) \in D \oplus D$. Then, by Lemma 8, $D^{(x)}$ is a subalgebra of $A^{(a)}$. For $A = (L/F, \sigma, c)$, L is a subalgebra of $A^{(a)}$, of $A_m^{(a)}$ and of $A_{l,r}^{(a)}$.

Proposition 9. *Let $A = (V, (,), \wedge) = F \oplus V$ be a quadratic algebra. Let $a = (\alpha, x) \in A$, $\alpha \in F, x \in V$.*
(i) $A^{(a)} = (V, (/)_a, \wedge_a)$ is a quadratic algebra with norm N_A, F-bilinear form

$$(u/v)_a := \alpha((u,v) - (v,u)) + (v,u) + 2(x, u \wedge v)$$

and $\wedge_a$ given by

$$u \wedge_a v := ((u,v) - (v,u))x + 2x \wedge (u \wedge v) + (2\alpha - 1)(u \wedge v).$$

(ii) If A has a scalar involution, then $(u/v)_a = (v,u) + 2(x, u \wedge v)$.
(iii) If A is a Hurwitz algebra and $a \notin F$ then $A^{(a)}$ is not flexible.
(iv) If A is a quaternion division algebra and $a \neq \frac{1}{2}$ then $A^{(a)}$ is not isotopic to an associative or nonassociative quaternion algebra.

Proof. (i) The proof of the identities is a simple computation.
(ii) If A has a scalar involution, then $(\, , \,)$ is symmetric, hence $(u/v)_a = (v,u) + 2(x, u \wedge v)$.
(iii) If A is a Hurwitz algebra then $(\, , \,)$ is symmetric and $(u/v)_a = (v,u) + 2(x, u \wedge v)$. Hence $(\,/\,)_a$ is symmetric iff $(v/u)_a = (u/v)_a$ iff $(x, v \wedge u) = (x, u \wedge v)$ iff $(x, u \wedge v) = 0$ for all $u, v \in A$. Since $(\, , \,)$ is nondegenerate, this implies $x = 0$. Hence for $x \neq 0$, $(\,/\,)_a$ is not symmetric and thus $A^{(a)}$ not flexible by [R, Lemme 2.2].
(iv) Suppose $A^{(a)}$ is isotopic to a quaternion division algebra D over F, then this implies it must have $\mathrm{Nuc}(A^{(a)}) = D$, so that $A^{(a)}$ is associative, a contradiction. Suppose $A^{(a)}$ is isotopic to a nonassociative quaternion division algebra $\mathrm{Cay}(K, c)$ over F [W], [A-P], K a separable quadratic field extension of F, then this implies it must have $\mathrm{Nuc}(A^{(a)}) = K$, so that $A^{(a)}$ is itself isomorphic to a nonassociative quaternion algebra by Waterhouse's classification [W], again a contradiction since $A^{(a)}$ is quadratic. □

Suppose A is a Hurwitz division algebra and $a \in A$, $a \neq \frac{1}{2}$. Then $\wedge_a$ is a dissident map [O]. In particular, if $a \in F$, we obtain the classical situation that

$$(u,v)_a = a((u,v) - (v,u)) + (v,u), \quad u \wedge_a v = (2a - 1)(u \wedge v).$$

Using Lemma 8 (i), we obtain from this our main result:

Theorem 10. *Let C be a Hurwitz division algebra of dimension 4 or 8. For all $a \in C$, $a \neq \frac{1}{2}$, $C^{(a)}$ is a quadratic division algebra which is flexible if and only if $a \in F$.*

2.2. Cayley-Dickson doublings of quadratic algebras. Let D be a quadratic algebra over F with trace involution $^-$ (which need not be an algebra involution and may only be a linear map $A \to A$). Let $c \in F^\times$. Then the F-vector space $A = D \oplus D$ can be made into a quadratic algebra over F via the multiplications

$$(u,v)(u',v') = (uu' + c\bar{v}'v, v'u + v\bar{u}')$$

for $u, u', v, v' \in D$. The unit element of A is given by $1 = (1,0)$. A is called the *Cayley-Dickson* doubling of D and denoted by $\mathrm{Cay}(D,c)$. A is a quadratic algebra with norm $N_A = N_D \perp (-c)N_D$.

Note that the doubling process referred to in [L] only employs the scalar $c = -1$.

Remark 11. Suppose D is a quaternion algebra.
(i) $A = \mathrm{Cay}(D^{(a)}, c)$ with $a \in A \setminus \{\frac{1}{2}\}$, $c \in F^\times$ is a quadratic algebra with subalgebra $D^{(a)}$ and flexible iff $a \in F$. Moreover, $A = \mathrm{Cay}(D^{(a)}, c) = (V, (/), \mu)$ is a division algebra iff $N_D \perp (-c)N_D$ is anisotropic and $\mu : V \wedge V \to V$ is a dissident map, cf. [O], i.e. iff $c \notin N_D(D^\times)$ and u, v and $\mu(u \wedge v)$ are linearly independent whenever u and v are linearly independent.
(ii) For $A^{(a)}$, $a = (x, y) \in A = \mathrm{Cay}(D, c)$, the multiplication is given by

$$(u, v) \circ (u', v') = (u \circ_x u' + c[x(\bar{v}'v - \bar{v}v') + \bar{v}v'] - c[\bar{u} + u]\bar{v}'y - c[u' + \bar{u}']\bar{v}y,$$
$$[v'ux + v'\bar{u} - v'\bar{u}x] + [v\bar{u}'x + vu' - xvu'] - y[\bar{u}'\bar{u} + \bar{u}\bar{u}'] - y[\bar{v}'v + \bar{v}v']c).$$

$A^{(a)}$ is a quadratic algebra with subalgebra $D^{(x)}$ (Lemma 8). In particular, for $a = (x, 0)$, the multiplication is given by

$$(u, v) \circ (u', v') = (u \circ_x u' + c[x(\bar{v}'v - \bar{v}v') + \bar{v}v'], [v'ux + v'\bar{u} - v'\bar{u}x] + [v\bar{u}'x + vu' - xvu']).$$

For $B = \mathrm{Cay}(D^{(x)}, c)$, however, the multiplication is given by

$$(u, v) \star (u', v') = (u \circ_x u' + c\bar{v}' \circ_x v, v' \circ_x u + v \circ_x \bar{u}')$$
$$= (u \circ_x u' + c(x(\bar{v}'v) + (1 - x)(v\bar{v}')), x(v'u) + (1 - x)(uv') + x(v\bar{u}') + (1 - x)(\bar{u}'v)).$$

Again, B is a quadratic algebra with subalgebra $D^{(x)}$. We leave it open for now if an algebra obtained through a Cayley-Dickson doubling of a mutation algebra again is a mutation algebra.

Example 12. For Hamilton's quaternion algebra $\mathbb{H}$ over $\mathbb{R}$, the Cayley-Dickson doubling $A = \mathrm{Cay}(\mathbb{H}^{(a)}, -1)$ is a quadratic division algebra for all $a \in \mathbb{H}$ with $a \neq \frac{1}{2}$ [L, 1.7]. $\mathbb{H}^{(a)}$ is flexible iff $a \in \mathbb{R}$. Hence

$$A = \mathrm{Cay}(\mathbb{H}^{(a)}, -1)$$

is a non-flexible quadratic division algebra over $\mathbb{R}$ for all $a \in \mathbb{H} \setminus \mathbb{R}$.

3. Division algebras of degree n

Theorem 1 has the following consequences:

Proposition 13. *Let F have characteristic not 3. Let $A = (L/F, \sigma, c)$ be a cyclic division algebra of degree 3 with norm N and $a = (u, v, w) \in L \oplus L \oplus L = (L/F, \sigma, c)$. If $N_L(u) \neq N_L(u - 1) + 3cT_L(vw)$ then $A^{(a)}$ is a unital division algebra of degree 3 such that $N(u^{\circ k}) = N(u)^k$ for all $k \geq 2$.*

Proof. If $N(a) \neq N(a - 1)$ then $A^{(a)}$ is a division algebra. Now

$$N(a) = N((u, v, w)) = N_L(u) + cN_L(v) + c^2N_L(w) - 3cT_L(uvw)$$

and

$$N(a - 1) = N((u - 1, v, w)) = N_L(u - 1) + cN_L(v) + c^2N_L(w) - 3cT((u - 1)vw),$$

so that $N(a) = N(a - 1)$ if and only if

$$N_L(u) + cN_L(v) + c^2N_L(w) - 3cT(uvw) = N_L(u - 1) + cN_L(v) + c^2N_L(w) - 3cT_L((u - 1)vw)$$

iff

$$N_L(u) - 3cT(uvw) = N_L(u-1) - 3cT(uvw) + 3cT_L(vw)$$

iff

$$N_L(u) = N_L(u-1) + 3cT_L(vw).$$

□

Proposition 14. *Let L be a separable field extension over F of degree n with norm N_L. Let h_1, h_2 be isometries of N_L, such that the principal Albert isotope $L^{(h_1,h_2)} = (L, \star)$, $u \star v = h_1(u)h_2(v)$, is a non-unital non-commutative algebra. $A = L^{(h_1,h_2)}$ has the multiplicative norm N_L. For all $a, b \in L$ with $N_L(-b) \neq N_L(a)$, $A^{(a,b)}$, $A_{l,r}^{(a,b)}$ and $A_m^{(a,b)}$ are division algebras. In particular, this holds for all $a \in L$ with $N_L(a - 1_L) \neq N_L(a)$ and $b = 1_L - a$.*

Corollary 15. *Let L be a separable field extension of F and $A = L^{(h_1,h_2)}$, where h_1, h_2 are isometries of the norm N_L.*
(i) Let F be a field which contains a primitive third root of unity and $[L : F] = 3$. Let $a \in L$, $a \neq 1$, $\frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ and $b = 1_L - a$. Then $A^{(a,b)}$, $A_{l,r}^{(a,b)}$ and $A_m^{(a,b)}$ are division algebras over F with $N(u \circ u) = N(u)^2$.
(ii) Let F be a field which contains a primitive 5th root of unity and $[L : F] = 5$ where $L = F(\alpha)$ with $\alpha^5 = c$. Supposes $a = u_1 + u_2\alpha + \cdots + u_5\alpha^4 \in L$ such that

$$\begin{aligned}
&u_1^3[u_1^2 - 5cu_2u_5 - 5cu_3u_4] \neq \\
&u_1[(u_1-1)^4 - 5c(u_1-1)^2u_2u_5 - 5c(u_1-1)^2u_3u_4 - u_2^2u_4 - u_2u_3^2 - u_5^2u_3 - u_5u_4^2] \\
&-(u_1-1)^4 + 5c(u_1-1)^2u_2u_5 + 5c(u_1-1)^2u_3u_4 - 5c(u_1-1)u_2^2u_4 - 5c(u_1-1)u_2u_3^2 - 5c^2(u_1-1)u_5^2u_3 \\
&-5c^2(u_1-1)u_5u_4^2 + 5cu_2^3u_3 - 5c^2u_2^2u_5^2 + 5c^2u_2u_4^3 + 5c^3u_5^3u_4 + 5c^2u_5u_3^3 - 5c^2u_3^2u_4^2 + 5c^2u_2u_3u_4u_5
\end{aligned}$$

and $b = 1_L - a$. Then $A^{(a,b)}$, $A_{l,r}^{(a,b)}$ and $A_m^{(a,b)}$ are division algebras over F with $N(u \circ u) = N(u)^2$.
In particular, if $a = 1 + u_2\alpha + \cdots + u_5\alpha^4 \in L$, $a \neq 1$, such that

$$\begin{aligned}
&1 = 5c[u_2u_5 + u_3u_4 + u_2^3u_3 - cu_2^2u_5^2 + cu_2u_4^3 + c^2u_5^3u_4 + cu_5u_3^3 - cu_3^2u_4^2 + cu_2u_3u_4u_5] \\
&-u_2^2u_4 - u_2u_3^2 - u_5^2u_3 - u_5u_4^2
\end{aligned}$$

then $A^{(a,b)}$, $A_{l,r}^{(a,b)}$ and $A_m^{(a,b)}$ are division algebras over F.

Proof. (i) Let $L = F(\alpha)$ with $\alpha^3 = c$ be a cubic field extension of F with norm N. Then

$$N(u_1 + u_2\alpha + u_3\alpha^2) = u_1^3 + cu_2^3 + c^2u_3^3 - 3cu_1u_2u_3$$

for $u_i \in F$, so that $N(a) = N(a-1)$ for $a = u_1 + u_2\alpha + u_3\alpha^2$ implies

$$u_1^3 + cu_2^3 + c^2u_3^3 - 3cu_1u_2u_3 = (u_1-1)^3 + cu_2^3 + c^2u_3^3 - 3cu_1u_2u_3,$$

which is equivalent to $u_1^3 = u_1^3 - 3u_1^2 + 3u_1 - 1$, i.e. to $0 = 3u_1^2 - 3u_1 + 1$, which means $u_1 = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$.

(ii) Let $L=F(\alpha)$ with $\alpha^5=c$ be a quintic field extension of F with norm N. Then

$$
\begin{aligned}
&N_L(u_1+u_2\alpha+\cdots+u_5\alpha^4)=u_1^5+cu_2^5+c^2u_3^5+c^3u_4^5+c^4u_5^5\\
&-5cu_1^3u_2u_5-5cu_1^3u_3u_4+5cu_1^2u_2^2u_4+5cu_1^2u_2u_3^2+5c^2u_1^2u_5^2u_3\\
&+5c^2u_1^2u_5u_4^2-5cu_1u_2^3u_3+5c^2u_1u_2^2u_5^2-5c^2u_1u_2u_4^3-5c^3u_1u_5^3u_4\\
&-5c^2u_1u_5u_3^3+5c^2u_1u_3^2u_4^2-5c^2u_2^3u_5u_4+5c^2u_2^2u_5u_3^2+5c^2u_2^2u_3u_4^2-5c^2u_2u_3^3u_4\\
&-5c^3u_2u_5^3u_3+5c^3u_2u_5^2u_4^2+5c^3u_5^2u_3^2u_4-5c^3u_5u_3u_4^3-5c^2u_1u_2u_3u_4u_5
\end{aligned}
$$

for $u_i\in F$, and

$$
\begin{aligned}
&N_L((u_1-1)+u_2\alpha+\cdots+u_5\alpha^4)=(u_1-1)^5+cu_2^5+c^2u_3^5+c^3u_4^5+c^4u_5^5\\
&-5c(u_1-1)^3u_2u_5-5c(u_1-1)^3u_3u_4+5c(u_1-1)^2u_2^2u_4+5c(u_1-1)^2u_2u_3^2+5c^2(u_1-1)^2u_5^2u_3\\
&+5c^2(u_1-1)^2u_5u_4^2-5c(u_1-1)u_2^3u_3+5c^2(u_1-1)u_2^2u_5^2-5c^2(u_1-1)u_2u_4^3-5c^3(u_1-1)u_5^3u_4\\
&-5c^2(u_1-1)u_5u_3^3+5c^2(u_1-1)u_3^2u_4^2-5c^2u_2^3u_5u_4+5c^2u_2^2u_5u_3^2+5c^2u_2^2u_3u_4^2-5c^2u_2u_3^3u_4\\
&-5c^3u_2u_5^3u_3+5c^3u_2u_5^2u_4^2+5c^3u_5^2u_3^2u_4-5c^3u_5u_3u_4^3-5c^2(u_1-1)u_2u_3u_4u_5
\end{aligned}
$$

so that $N(a)=N(a-1)$ for $a=u_1+u_2\alpha+\cdots+u_5\alpha^4$ is equivalent to

$$
\begin{aligned}
&u_1[u_1^4-5cu_1^2u_2u_5-5cu_1^2u_3u_4+5cu_1u_2^2u_4+5cu_1u_2u_3^2+5c^2u_1u_5^2u_3\\
&+5c^2u_1u_5u_4^2-5cu_2^3u_3+5c^2u_2^2u_5^2-5c^2u_2u_4^3-5c^3u_5^3u_4-5c^2u_5u_3^3+5c^2u_3^2u_4^2-5c^2u_2u_3u_4u_5]=\\
&(u_1-1)[(u_1-1)^4-5c(u_1-1)^2u_2u_5-5c(u_1-1)^2u_3u_4+5c(u_1-1)u_2^2u_4+5c(u_1-1)u_2u_3^2\\
&+5c^2(u_1-1)u_5^2u_3+5c^2(u_1-1)u_5u_4^2-5cu_2^3u_3+5c^2u_2^2u_5^2-5c^2u_2u_4^3-5c^3u_5^3u_4-5c^2u_5u_3^3\\
&+5c^2u_3^2u_4^2-5c^2u_2u_3u_4u_5]
\end{aligned}
$$

iff

$$
\begin{aligned}
&u_1[u_1^4-5cu_1^2u_2u_5-5cu_1^2u_3u_4+5cu_1u_2^2u_4+5cu_1u_2u_3^2+5c^2u_1u_5^2u_3+5c^2u_1u_5u_4^2]=\\
&u_1[(u_1-1)^4-5c(u_1-1)^2u_2u_5-5c(u_1-1)^2u_3u_4+5c(u_1-1)u_2^2u_4+5c(u_1-1)u_2u_3^2\\
&+5c^2(u_1-1)u_5^2u_3+5c^2(u_1-1)u_5u_4^2]\\
&-[(u_1-1)^4-5c(u_1-1)^2u_2u_5-5c(u_1-1)^2u_3u_4+5c(u_1-1)u_2^2u_4+5c(u_1-1)u_2u_3^2\\
&+5c^2(u_1-1)u_5^2u_3+5c^2(u_1-1)u_5u_4^2-5cu_2^3u_3+5c^2u_2^2u_5^2-5c^2u_2u_4^3-5c^3u_5^3u_4-5c^2u_5u_3^3\\
&+5c^2u_3^2u_4^2-5c^2u_2u_3u_4u_5]
\end{aligned}
$$

iff

$$
\begin{aligned}
&u_1^3[u_1^2-5cu_2u_5-5cu_3u_4]=\\
&u_1[(u_1-1)^4-5c(u_1-1)^2u_2u_5-5c(u_1-1)^2u_3u_4-u_2^2u_4-u_2u_3^2-u_5^2u_3-u_5u_4^2]\\
&-(u_1-1)^4+5c(u_1-1)^2u_2u_5+5c(u_1-1)^2u_3u_4-5c(u_1-1)u_2^2u_4-5c(u_1-1)u_2u_3^2\\
&-5c^2(u_1-1)u_5^2u_3-5c^2(u_1-1)u_5u_4^2+5cu_2^3u_3-5c^2u_2^2u_5^2+5c^2u_2u_4^3+5c^3u_5^3u_4\\
&+5c^2u_5u_3^3-5c^2u_3^2u_4^2+5c^2u_2u_3u_4u_5
\end{aligned}
$$

Put $u_1=1$ to obtain

$$
\begin{aligned}
&1=5c[u_2u_5+u_3u_4+u_2^3u_3-cu_2^2u_5^2+cu_2u_4^3+c^2u_5^3u_4+cu_5u_3^3-cu_3^2u_4^2+cu_2u_3u_4u_5]\\
&-u_2^2u_4-u_2u_3^2-u_5^2u_3-u_5u_4^2.
\end{aligned}
$$

□

4. Automorphisms and derivations

Since A is a finite-dimensional algebra over F, the Lie algebra $\mathrm{Aut}(A)$ of automorphisms of A, viewed as algebraic group, is a subalgebra of the derivation algebra $\mathrm{Der}(A)$ and for $F=\mathbb{R}$, we have $\mathrm{dim}\mathrm{Aut}(A)=\mathrm{dim}\mathrm{Der}(A)$. For $a\in A$, define

$$\mathrm{Aut}_a(A)=\{F\in \mathrm{Aut}(A)\,|\,F(a)=a\} \text{ and } \mathrm{Der}_a(A)=\{d\in \mathrm{Der}(A)\,|\,d(a)=0\}.$$

If $a\in F$ then $\mathrm{Aut}_a(A)=\mathrm{Aut}(A)$ and $\mathrm{Der}_a(A)=\mathrm{Der}(A)$.

Lemma 16. *For a unital algebra A with underlying vector space V and $f,g\in \mathrm{Gl}(V)$,*

$$\{F\in \mathrm{Aut}(A)\,|\,F\circ f=f\circ F, F\circ g=g\circ F\}\subset \mathrm{Aut}(A,\circ_{(f,g)}),$$

$$\{D\in \mathrm{Der}(A)\,|\,D\circ f=f\circ D, D\circ g=g\circ D\}\subset \mathrm{Der}(A,\circ_{(f,g)}).$$

In particular, for all $a,b\in F$,

$$\mathrm{Aut}(A)\subset \mathrm{Aut}(A^{(a,b)}),\quad \mathrm{Der}(A)\subset \mathrm{Der}(A^{(a,b)}).$$

Proposition 17. *Let A be an algebra over F and $a,b\in A$.*
(i) $\{F\in \mathrm{Aut}(A)\,|\,F(a)=a,F(b)=b\}\subset \mathrm{Aut}(A^{(a,b)})$, $\mathrm{Aut}(A_m^{(a,b)})$ and $\mathrm{Aut}(A_{l,r}^{(a,b)})$.
In particular, $\mathrm{Aut}_a(A)$ is a subgroup of $\mathrm{Aut}(A^{(a)})$, $\mathrm{Aut}(A_m^{(a,b)})$ and $\mathrm{Aut}(A_{l,r}^{(a)})$.
(ii) $\{d\in \mathrm{Der}(A)\,|\,d(a)=0,d(b)=0\}\subset \mathrm{Der}(A^{(a,b)})$, $\mathrm{Der}(A_m^{(a,b)})$ and $\mathrm{Der}(A_{l,r}^{(a,b)})$.
In particular, $\mathrm{Der}_a(A)$ is a subalgebra of $\mathrm{Der}(A^{(a)})$, $\mathrm{Der}(A_m^{(a,b)})$ and $\mathrm{Der}(A_{l,r}^{(a)})$.

Proof. (i) Let $F\in \mathrm{Aut}(A)$. Then

$$F(u\circ v)=F(a(uv)+b(vu))=F(a)(F(u)F(v))+F(b)(F(v)F(u))$$

and

$$F(u)\circ F(v)=a(F(u)F(v))+b(F(v)F(u)).$$

Thus if $F(a)=a$ and $F(b)=b$ then $F\in \mathrm{Aut}(A^{(a,b)})$.
(ii) If D is a derivation of A then $D(uv)=D(u)v+uD(v)$ for all $u,v\in A$. Thus in $A^{(a,b)}$,

$$D(u\circ v)=D(a)(D(u)v)+D(a)(uD(v))+a(D(u)v)+a(uD(v))$$

$$+D(b)(D(v)u)+D(b)(vD(u))+b(D(v)u)+b(vD(u)),$$

while

$$D(u)\circ v+u\circ D(v)=a(uD(v))+b(vD(u))+a(uD(v))+b(D(v)u).$$

Hence if $D(a)=0$ and $D(b)=0$ then $D\in \mathrm{Der}(A^{(a,b)})$.
The proofs for $A_m^{(a,b)}$ and $A_{l,r}^{(a,b)}$ are analogously. $\square$

Lemma 16 and the classification of real division algebras in [B-O1, 2] yield:

Corollary 18. *Suppose that $a,b\in F$ such that $a\neq \pm b$.*
(i) If A is a quaternion division algebra then $A^{(a,b)}$ is a division algebra and

$$SU(2)\subset \mathrm{Aut}(A^{(a,b)}),\quad su(2)\subset \mathrm{Der}(A^{(a,b)}).$$

In particular, if $F=\mathbb{R}$ then

$$SU(2)=\mathrm{Aut}(\mathbb{H}^{(a,b)}),\quad su(2)=\mathrm{Der}(\mathbb{H}^{(a,b)}).$$

(ii) If A is an octonion division algebra then $A^{(a,b)}$ is division and

$$G_2 \subset \mathrm{Aut}(A^{(a,b)}), \quad G_2 \subset \mathrm{Der}(A^{(a,b)}).$$

In particular, if $F=\mathbb{R}$ then

$$G_2 = \mathrm{Aut}(\mathbb{O}^{(a,b)}), \quad G_2 = \mathrm{Der}(\mathbb{O}^{(a,b)}).$$

Remark 19. If A is an associative central simple algebra, all elements of $\mathrm{Der}(A)$ are inner derivations. For a non-zero $a\in A$, the inner derivation $ad_a(x)=ax-xa$ satisfies $ad_a(a)=0$. For all $b\in F(a)\subset A$ we get $ad_a(b)=0$. Thus for all non-zero $a\in A$ and $b\in F(a)$ we have $ad_a, ad_b\in \mathrm{Der}(A^{(a,b)})$, i.e. $\mathrm{Der}(A^{(a,b)})\neq 0$. Analogously, also $\mathrm{Der}(A_m^{(a,b)})\neq 0$ and $\mathrm{Der}(A_{l,r}^{(a,b)})\neq 0$.
For an alternative algebra A over F, the automorphism group of $A_m^{(a,b)}$ either is much larger than the one of A or there is no relation at all between the two groups [E-My1].
By Remark 6, for a quadratic algebra A and all $a\in A$,

$$\mathrm{Aut}(A^{(a)})\subset O(N_A).$$

In the following, we include results on $A_m^{(a,b)}$ for the sake of completeness, although they have been proved already in [E-My1].

Let A be an octonion division algebra over F and $a\in A^\times$. Then $a\in F1_A$ or a yields a quadratic field extension $S=F1+Fa$ of F. In the latter case, let $S^\perp=\{x\in A\,|\,N_A(x,S)=0\}$ and let $\alpha\in S\setminus F$ such that $\mu=\alpha^2\in F$. Then the form $\langle\, ,\, \rangle$ defined via

$$\langle u,v\rangle = \mu N_A(u,v)-\alpha N_A(\alpha u,v)$$

for all $u,v\in S^\perp$ is an S-hermitian form on $S^\perp$. We know that $\mathrm{Aut}_a(A)\cong SU(S^\perp,\langle\, ,\, \rangle)$ and $\mathrm{Der}_a(A)\cong su(S^\perp,\langle\, ,\, \rangle)$ [E-My1, p. 109] which yields:

Proposition 20. *Let A be an octonion division algebra over F and $a\in A\setminus F$. Then*

$$SU(S^\perp,\langle\, ,\, \rangle)\subset \mathrm{Aut}(A^{(a)}),\ SU(S^\perp,\langle\, ,\, \rangle)\subset \mathrm{Aut}(A_m^{(a)}) \text{ and } SU(S^\perp,\langle\, ,\, \rangle)\subset \mathrm{Aut}(A_{l,r}^{(a)});$$

$$su(S^\perp,\langle\, ,\, \rangle)\subset \mathrm{Der}(A^{(a)}),\ su(S^\perp,\langle\, ,\, \rangle)\subset \mathrm{Der}(A_m^{(a)}) \text{ and } su(S^\perp,\langle\, ,\, \rangle)\subset \mathrm{Der}(A_{l,r}^{(a)}).$$

4.1. The real case. Let us briefly recap what this means in the real case. For all $a,b\in\mathbb{H}$, $N(a)\neq N(b)$, the derivation algebra of the division algebras $\mathbb{H}^{(a,b)}$, $\mathbb{H}_m^{(a,b)}$ and $\mathbb{H}_{l,r}^{(a,b)}$ is either $\mathbb{R}$ or $su(2)$ by the classification in [B-O1, 2]. For all $u\in\mathbb{H}$, $\mathbb{H}^{(uau^{-1},ubu^{-1})}\cong\mathbb{H}^{(a,b)}$, $\mathbb{H}_m^{(uau^{-1},ubu^{-1})}\cong\mathbb{H}_m^{(a,b)}$ and $\mathbb{H}_{l,r}^{(uau^{-1},ubu^{-1})}\cong\mathbb{H}_{l,r}^{(a,b)}$ by Remark 3 (iii).

In particular, for all $a\in\mathbb{H}$, $a\neq\frac{1}{2}$, $\mathbb{H}^{(a)}$ is a real quadratic division algebra with subalgebra $\mathbb{C}$. $\mathbb{H}^{(a)}$ is flexible if and only if $a\in\mathbb{R}$.

For all $a,b\in\mathbb{O}$, $N(a)\neq N(b)$, $\mathbb{O}^{(a,b)}$, $\mathbb{O}_m^{(a,b)}$ and $\mathbb{O}_{l,r}^{(a,b)}$ are division algebras. For $a=(a_0,a_1), b=(b_0,b_1)\in\mathbb{O}=\mathrm{Cay}(\mathbb{H},-1)$, the algebras $\mathbb{O}^{(a,b)}$, $\mathbb{O}_m^{(a,b)}$ and $\mathbb{O}_{l,r}^{(a,b)}$ have $\mathbb{H}^{(a_0,b_0)}$, $\mathbb{H}_m^{(a_0,b_0)}$ and $\mathbb{H}_{l,r}^{(a_0,b_0)}$, respectively, as subalgebras.

In particular, $\mathbb{O}_m^{(a)}$ contains $\mathbb{H}_m^{(a_0)}$ as subalgebra, $\mathbb{O}_{l,r}^{(a)}$ contains $\mathbb{H}_{l,r}^{(a_0)}$ as subalgebra and the quadratic division algebra $\mathbb{O}^{(a)}$ contains $\mathbb{H}^{(a_0)}$ as subalgebra.

If $a,b\in\mathbb{R}$ with $a\neq\pm b$, $\mathbb{O}^{(a,b)}$ contains $\mathbb{H}^{(a,b)}$ as subalgebra and has derivation algebra G_2 by Corollary 18.

For all $a \in \mathbb{O}$, $a \neq \frac{1}{2}$, the algebras $\mathbb{O}^{(a)}$, $\mathbb{O}_m^{(a)}$ and $\mathbb{O}_{l,r}^{(a)}$ are division and by Proposition 20, they have a derivation algebra containing $su(3)$, i.e. their derivation algebra is isomorphic to $su(3)$ or G_2. Real division algebras with derivation algebra $su(3)$ were determined in [B-O2]: If the derivation algebra of an eight-dimensional real division algebra A is isomorphic to $su(3)$, A is either a flexible generalized pseudo-octonion algebra or the direct sum of two one-dimensional modules and a six-dimensional irreducible $su(3)$-module [B-O2].

For all $a \in \mathbb{O}$, $a \neq \frac{1}{2}$, $\mathbb{O}^{(a)}$ is a real quadratic division algebra which is flexible if and only if $a \in \mathbb{R}$.

Hence for all $a \in \mathbb{O}$, $a \notin \mathbb{R}$, the non-flexible real quadratic division algebra $\mathbb{O}^{(a)}$ either has derivation algebra G_2 or it has derivation algebra $su(3)$ and is the direct sum of two one-dimensional modules and a six-dimensional irreducible $su(3)$-module. A multiplication table for algebras of the latter case is given [B-O2, (4.2)] and it is shown that every real algebra defined by this table admits $su(3)$ as derivation algebra [B-O2, Theorem 4.1]. Which case appears as derivation algebra is not clear and will probably depend on the choice of a.

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