A Note on Moduli of Inner Ideals in Jordan Systems

FERNANDO MONTANER*

Departamento de Matemáticas, Universidad de Zaragoza,
50009 Zaragoza, Spain

Abstract: We prove a conjecture of D'Amour and McCrimmon concerning moduli of inner ideals of Jordan systems. The techniques developped for that also provide a proof of the equivalence of $(+)$ and $(-)$-primitivity of Jordan systems based on elementary computations, and not making use of the structure theory

Introduction

Modular inner ideals of unital linear Jordan algebras were first introduced in [6] (latter generalized to quadratic Jordan algebras in [3]) and used to define primitive algebras in a way that parallels the intrinsic characterization of primitive associative algebras by means of modular one-sided ideals. That notion of primitivity allowed the developpement of a structure theory based on the Jacobson radical, since factoring out that radical produces a subdirect product of primitive algebras [3].

That notion of primitivity was extended to Jordan pairs and triple systems in [7], again having a notion of modular inner ideal as a starting point. However, moduli of inner ideals in Jordan pairs and triple systems are more difficult to deal with than moduli of inner ideals in algebras, since they are pairs of elements rather than single elements. As a consequence, the notion of primitivity for Jordan systems other that algebras is in a sense "local" (see [1]).

It is therefore important in the study of primitive Jordan systems to have results that ease the computations with modular inner ideals, and, in particular, that allow to find new moduli from known ones. That is illustrated in D'Amour and McCrimmon's work [1], where they rise the following question:

If (a, b) and (a, b') are moduli for an inner ideal K of a Jordan system, is $(a, b' + P_b k)$ a modulus for any $k \in K$?

The present note was motivated by that conjecture, that we generalize and prove in Section 3. The ideas that we introduce also provide an elementary proof of a result that is a consequence of a more general result of Anquela and Cortes [2], namely, a Jordan pair is $(+)$-primitive if and only if it is $(-)$-primitive.

* Partially supported by the DGES, PB97-1069-C02-02

The paper is organized as follows. After recording in Section 0 some well known facts from Jordan theory, we devote Section 1 to recall mostly known results on modularity of Jordan systems. In particular we include a proof of the characterizations of the Jacobson radical of a primitive pair by means of maximal-modular inner ideals that follows from the corresponding characterization for Jordan triple systems (and which does not seem to be explicity displayed in the literature). In Section 2 we introduce our main technical construction. To every (a, b)-modular inner ideal K, we attach in a natural way a (b, a)-modular inner ideal that we call the a-dual of K. Finally, we apply this construction in Section 3 to prove the above mentioned conjecture of D'Amour and McCrimmon.

0. Preliminaries

Throughout Φ will be a fixed unital commutative ring.

0.1 We work with Jordan pairs, triple systems and algebras over Φ. We refer to [5, 4] for notation, terminology, and basic results. We record in this section some of those notations and results.

A Jordan Pair $V = (V^+, V^-)$ has products $Q_x y$ for $x \in V^\sigma$ and $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z}y = D_{x,y}z = \{x, y, z\} = Q_{x+z}y - Q_x y - Q_z y$. The Bergmann operators is defined by $B_{x,y}z = z - D_{x,y}z + Q_x Q_y z$ for $x, z \in V^\sigma$, $y \in V^{-\sigma}$, which satisfies $Q_{B_{x,y}z} = B_{x,y}Q_z B_{y,x}$.

A Jordan triple system T has product $P_x y$ whose linearizations are $P_{x,y}z = L_{x,z}y = \{x, z, y\} = P_{x+y}z - P_x z - P_y z$. The Bergmann operator in triple systems is given by $B_{x,y}z = z - L_{x,y}z + P_x P_y z$.

A Jordan algebra J has products x^2 and $U_x y$ with linearizations $V_x y = x \circ y = (x + y)^2 - x^2 - y^2$ and $U_{x,y}z = V_{x,z}y = \{x, z, y\} = U_{x+y}z - U_x z - U_y z$. A Jordan algebra is unital if there exists an element $1 \in J$ with $U_1 = \mathrm{Id}$ and $U_x 1 = x^2$ for any $x \in J$. If J is a Jordan algebra, we can always construct a unital hull: an algebra $\hat{J} = J + \Phi 1$ having J as a subalgebra, which is unital with unit 1.

0.2 Doubling a Jordan triple system T produces a Jordan pair $V(T) = (T, T)$ with $Q_x y = P_x y$. Reciprocally, each Jordan pair $V = (V^+, V^-)$ gives rise to a polarized triple system $T(V) = V^+ \oplus V^-$ with product $P_{x^+ \oplus x^-} y^+ \oplus y^- = Q_{x^+} y^- \oplus Q_{x^-} y^+$.

The double $V(T)$ of a Jordan triple system allows the use of pair identities for triple systems. In particular, we will make use of the identities proved in [4] to which we will refer with the labelling JPn used there without making reference to [4].

0.3 A Φ-submodule $K \subseteq J$ of a Jordan algebra J is an *inner ideal* if $U_K \hat{J} \subseteq K$, where $\hat{J} = J + \Phi 1$ is a unital hull of J.

A Φ-submodule $K \subseteq V^\sigma$, $\sigma = \pm$ of a Jordan pair $V = (V^+, V^-)$ is an *inner ideal* if $Q_K V^{-\sigma} \subseteq V^\sigma$.

A Φ-submodule $K \subseteq J$ of a Jordan triple system J is an *inner ideal* if $P_K J \subseteq K$. (Equivalently, K is an inner ideal of the pair $V(J)$.)

0.4 Every element $b \in V^\sigma$ of a Jordan pair $V = (V^+, V^-)$ gives rise to a Jordan algebra $V^{-\sigma(b)}$, called the *b-homotope* of V, by defining on the Φ-module $V^{-\sigma}$ the operations:

$$x^{(2,b)} = Q_x b \qquad \text{and} \qquad U_x^{(b)} y = Q_x Q_b y,$$

for any $x, y \in V^{-\sigma}$.

If J is a Jordan triple system and $b \in J$, the *b-homotope* $J^{(b)}$ is defined as the b-homotope of the Jordan pair $V(J)$ at $b \in J = V(J)^+$.

0.5 An element x of a Jordan algebra J is called *quasi-invertible* if $1 - x$ is invertible in a unital hull $\hat{J} = J + \Phi 1$ of J. This is equivalent to the surjectivity operator $U_{1-x} = \mathrm{Id} - V_x + U_x$ on J, and to the bijectivity of that operator.

If $V = (V^+, V^-)$ is a Jordan pair, a pair of elements $(x, y) \in V^\sigma \times V^{-\sigma}$ is called *quasi-invertible* if x is quasi-invertible in the homotope $V^{\sigma(y)}$. This is equivalent to $B_{x,y}$ being surjective, and to $B_{x,y}$ being bijective. An element $x \in V^\sigma$ is called *properly quasi-invertible* if (x, y) is quasi-invertible for any $y \in V^{-\sigma}$, i. e., if x is quasi-invertible in every homotope $V^{\sigma(y)}$.

If J is a Jordan pair or a Jordan algebra, an element $x \in J$ is *properly quasi-invertible* if $x \in J = V(J)^+$ is properly quasi-invertible in the Jordan pair $V(J)$.

The set of properly quasi-invertible elements of a Jordan system J (algebra, pair or triple) forms an ideal $\mathrm{Jac}(J)$ called the Jacobson radical of J.

1. Modular inner ideals

1.1 The Jordan algebra version of the notion of primitivity mimics the intrinsic characterization of primitivity in associative algebras by means of modular ideals, so it requires a Jordan version of modularity. Recall [3] that an inner ideal K of a Jordan algebra J is called *e-modular*, for an element $e \in J$ which is called a *modulus* for K, if it satisfies:

(i) $U_{1-e} J \subseteq K$,

(ii) $\{K, J, 1 - e\} \subseteq K$,

(iii) $(1-e)\circ K\subseteq K$,

(iv) $e-e^2\in K$.

This is equivalent to the fact that $\hat{K}=K+\Phi(1-e)$ be an inner ideal of the unital hull $\hat{J}$ of J. Moreover, an inner ideal of a unital algebra J is e-modular if and only if it contains $1-e$ [3, 2.9].

1.2 Modularity of inner ideals in Jordan pairs and triple systems is defined as modularity in some homotope (see [3]). Let $V=(V^+,V^-)$ be a Jordan pair, and $(a^+,a^-)\in V^+\times V^-$. An inner ideal $K\subseteq V^\sigma$, $\sigma=\pm$, is called *a^σ-modular at $a^{-\sigma}$*, or *(a^+,a^-)-modular*, and (a^+,a^-) is called a *modulus* for K, if K is an a^σ-modular inner ideal of the homotope $V^{\sigma(a^{-\sigma})}$, i.e. if it satisfies:

(i) $B_{a^\sigma,a^{-\sigma}}V^\sigma\subseteq K$,

(ii) $\{k,a^{-\sigma},x\}-\{k,Q_{a^{-\sigma}}x,a^\sigma\}\in K$, for all $x\in V^\sigma$, $k\in K$,

(iii) $\{K,a^{-\sigma},a^\sigma\}\subseteq K$,

(iv) $a^\sigma-Q_{a^\sigma}a^{-\sigma}\in K$.

Modularity of Jordan triple systems are defined in the same way, so that an inner ideal K of J is *(a,b)-modular*, where $a,b\in J$, if it is a-modular in the homotope $J^{(b)}$. In this case (a,b) is called a *modulus* for K.

1.3 Proper modular inner ideals can not contain their moduli: an (a,b)-modular inner ideal K will be proper if and only if $a\notin K$. (If $a\in K$, for any $x\in J$, $\{a,b,x\}\equiv\{a,P_bx,a\}\equiv 0$ (mod. K) by 1.2(ii) and $a\in K$, hence $x=B_{a,b}x+\{a,b,x\}-P_aP_bx\in K$, by 1.2(i), $a\in K$ and $\{a,b,x\}\in K$.)

1.4 To apply 1.3 it is important that we can get new moduli from kown ones [1, 5.13]:

(1) *If K is an inner ideal of the Jordan system J, and (a,b) is a modulus for K, then $(a^{(n,b)},b^{(m,a)})$, $(a+k,b)$ and $(a,b+P_bk)$ are moduli of K.*

We recall here a question raised by D'Amour and McCrimmon [1, 5.13]:

Question. If (a,b) and (a,b') are moduli for an inner ideal K, is $(a,b'+P_bk)$ also a modulus for any $k\in K$?

As remarked in [1] this would provide a more direct proof that $(a,b^{(2,a)}+P_bk)$ is a modulus than the one in [1, 5.13]. This, together with the congruences $a^{(n,b)}\equiv a$ mod K, and $b^{(m,a)}\equiv b^{(2,a)}$ mod K, $m\geq 2$, is used in [1] to prove that all $(a^{(n,b)},b^{(m,a)})$ are moduli. We also record here the comment of D'Amour and McCrimmon that the assertion $b^{(m,a)}\equiv b$ mod P_bK, symmetric to $a^{(n,b)}\equiv a$ mod K, is too restrictive, since it would require b to be regular.

In Section 3, we will answer the previous question in the affirmative, and moreover will find the right symmetric condition on the powers of b: $b^{(m,a)} \equiv b \bmod L$ for a (b,a)-modular inner ideal L which is naturally attached to K.

1.5 Primitive Jordan systems (algebras, triple systems or pairs) [3, 7, 1] are defined as systems J having a *primititzer*, a proper modular inner ideal K which complements nonzero ideals: $K + I = J$ for all nonzero $I \lhd J$ if J is an algebra or a triple system, and $K + I^\sigma = V^\sigma$ for all nonzero $I = (I^+, I^-) \lhd J$, and $K \subseteq V^\sigma$, if $J = (V^+, V^-)$ is a Jordan pair. The definition of primitivity for pairs splits into two notions, $(+)$-primitivity and $(-)$-primitivity, according to whether the primitizer K is contained in V^+ or V^-, respectively. However, it has been proved in [2] that these are equivalent notions (and we will reprove this later).

The fact that primitizers K of a Jordan system J complement nonzero ideals means that they are core-less: $\mathrm{Core}(K) = 0$, where $\mathrm{Core}(K)$ is defined as the sum of all ideals contained in K if J is an algebra or a triple system, and $\mathrm{Core}(K)$ is the sum of all nonzero ideals $I = (I^+, I^-)$ with $I^\sigma \subseteq K$ if $J = (V^+, V^-)$ is a Jordan pair, and $K \subseteq V^\sigma$ for $\sigma = +$ or $-$.

Since, for a nonzero ideal I, the inner ideal $K + I$ (resp. $K + I^\sigma$ in the pair case) is again modular with the same modulus, the properness criterion 1.3 implies that a core-less K will be a primitizer if it is maximal among proper modular inner ideals with the given modulus.

A modular inner ideal is called *maximal-modular* if, for some of its modulus (a,b) (e, in the algebra case), it is maximal among all proper (a,b)-modular (e-modular, respectively) inner ideals. By an easy argument using Zorn's Lemma, it can be shown that any proper modular inner ideal K is contained in a proper maximal-modular inner ideal M, and if K complements nonzero ideals, M is core-less. Therefore, primitivity is equivalent to the existence of a core-less maximal-modular inner ideal.

1.6 Modularity in Jordan pairs and triple systems can be related by means of the functors T and V.

A Jordan triple system J gives rise to the Jordan pair $V(J)$. If $\sigma = \pm$, $S \subseteq J$ and $x \in J$, we write S^σ for S as a subset of $V(J)^\sigma$, and x^σ for x viewed as an element of $V(J)^\sigma$. We have the obvious properties:

Let J be a Jordan triple system, K be an inner ideal of J, $a, b \in J$, and $\sigma = \pm$. Then:

(a) *K is (a,b)-modular if and only if K^σ is $(a^\sigma, b^{-\sigma})$-modular.*

(b) *K is maximal-modular for the modulus (a,b) if and only if K^σ is maximal-modular for the modulus $(a^\sigma, b^{-\sigma})$.*

We note that K being core-less or complementing nonzero ideals does not imply the same properties for K^σ, and thus K^σ need not be a primitizer if K is (see [1, 5.6]).

Consider now a Jordan pair $V = (V^+, V^-)$. This gives rise to a Jordan triple system $T(V) = V^+ \oplus V^-$. We denote by π^σ the projection of $T(V)$ onto V^σ, $\sigma = \pm$, and identify $V^\sigma \subseteq T(V)$. Also, if $S = (S^+, S^-)$ is a pair submodule of V, we write $T(S) = S^+ \oplus S^- \subseteq T(V)$.

Let V be a Jordan pair, $\sigma = \pm$, $K \subseteq V^\sigma$ be an inner ideal, $a \in V^\sigma$ and $b \in V^{-\sigma}$. Write $\tilde{K} = K \oplus V^{-\sigma} \subseteq T(V)$. Then:

(c) *For any $a', b' \in T(V)$ with $\pi^\sigma(a') = a$ and $\pi^{-\sigma}(b') = b$, K is (a, b)-modular if and only if $\tilde{K}$ is (a', b')-modular.*

(d) $T(\mathrm{Core}(K)) = \mathrm{Core}(\tilde{K})$, *hence K is core-less if and only if $\tilde{K}$ is core-less.*

(e) *The inner ideal $\tilde{K}$ is maximal-modular for the modulus (a, b) if and only if K is maximal-modular for the modulus (a, b).*

(f) *K is a primitizer if and only if $\tilde{K}$ is a primitizer.*

Proof: Assertions (c) and (f) are in [1, 5.5.1]. (d) is straightforward. For (e), the “if” is obvious, and for the “only if” it suffices to note that any inner ideal N of $T(V)$ containing $\tilde{K}$ contains $V^{-\sigma}$, hence has $N = N^\sigma \oplus V^{-\sigma}$ for $N^\sigma = N \cap V^\sigma$, and N^σ is an inner ideal containing K which is (a, b)-modular if N is, by (c). ■

These results allow the lifting of properties from V to $T(V)$. In the reverse way we have:

Let V be a Jordan pair, K be an inner ideal of $T(V)$, and $a = a^+ \oplus a^-, b = b^+ \oplus b^- \in T(V)$. Denote $K^\sigma = \pi^\sigma(K)$, $\sigma = \pm$, and $\tilde{K} = K^+ \oplus K^-$, the polarization of K. Then:

(g) *K is (a, b)-modular if and only if $\tilde{K}$ is (a, b)-modular, and, in this case, K is proper if and only if $\tilde{K}$ is proper.*

(h) *K is (a, b)-modular if and only if K^σ is $(a^\sigma, b^{-\sigma})$-modular for $\sigma = +$ and $-$.*

(i) *K is maximal-modular for the modulus (a, b) if and only if $K = \tilde{K}$ is polarized, and, for some $\sigma = \pm$, K^σ is maximal-modular for the modulus $(a^\sigma, b^{-\sigma})$, and $K^{-\sigma} = V^{-\sigma}$.*

Proof: Assertions (g) and (h) are in [1, 5.5.2]. To prove (i) note that $K = \tilde{K}$ by maximality of K and (a, b)-modularity of $\tilde{K}$ (by (g)). Now, one of K^+, K^- is proper by (g), say $K^+ \neq V^+$. Then K^+ is (a^+, b^-)-modular by (h), hence $K^+ \oplus V^-$ is (a, b)-modular by (c). Thus, by maximality, $K = K^+ \oplus V^-$. The reciprocal is (e). ■

Maximal modular inner ideals are important for the characterization of the Jacobson radical through modular inner ideals:

1.7 THEOREM. *Let J be a Jordan system, then:*

(a) *If J is an algebra or a triple system,* $\mathrm{Jac}(J)$ *is the intersection of all proper maximal-modular inner ideals, and the intersection of the cores of all proper maximal-modular inner ideals.*

(b) *If $J = (V^+, V^-)$ is a Jordan pair, and $\sigma = \pm$, the σ-part of the Jacobson radical* $\mathrm{Jac}(J)^\sigma$ *is the intersecction of all proper maximal-modular inner ideals contained in V^σ, and the intersection of the σ-part of the cores of all proper maximal-inner ideals.*

(c) $J/\mathrm{Jac}(J)$ *is a subdirect product of primitive Jordan systems.*

PROOF: If K is a maximal modular inner ideal of J, then $J/\mathrm{Core}(K)$ is primitive with primitizer $K/\mathrm{Core}(K)$, hence (c) follows from (a) and (b). Assertion (a) is proved in [3, 5.6] for algebras, and in [7, Lemma 6] for triple systems.

A generalization of (b) is left to the reader in [8, Theorem 8]. Here, we derive it from its triple version (a). Indeed, the polarized triple $T(V)$ has $\mathrm{Jac}(V(T)) = T(\mathrm{Jac}(V)) = \mathrm{Jac}(V^+) \oplus \mathrm{Jac}(V^-)$ [1, 5.3.4]. By (a), $T(\mathrm{Jac}(V))$ is the intersection of all proper maximal-modular inner ideals of $T(V)$, but by 1.6(i), those are precisely the inner ideals $K^+ \oplus V^-$ or $V^+ \oplus K^-$ where $K^\sigma \subseteq V^\sigma$ is a proper maximal-modular inner ideal of V. Therefore $\mathrm{Jac}(V^\sigma)$ is the intersection of all proper maximal-modular inner ideals contained in V^σ. On the other hand, if $K = K^+ \oplus V^-$ or $V^+ \oplus K^-$, $\mathrm{Core}(K) = T(\mathrm{Core}(K^+))$ or $T(\mathrm{Core}(K^-))$, respectively. Therefore, the intersection of all the ideals $T(\mathrm{Core}(L))$ where L is a proper maximal-inner ideal of V coincides with the intersection of the cores of all proper maximal-modular inner ideals of $T(V)$, and this is the Jacobson radical of $T(V)$ by (a). Thus, the σ-part of the intersection of the cores of all proper maximal-inner ideal of V is the σ-part of the polarized ideal $\mathrm{Jac}(T(V))$, that is $\mathrm{Jac}(V^\sigma)$. ■

REMARK. A question that naturally arises in view of this theorem is whether, for a Jordan pair $V = (V^+, V^-)$, $\mathrm{Jac}(V^\sigma)$ is the σ-part of the intersection of all cores of proper maximal modular inner ideals *contained in* V^σ (instead of *all* proper maximal-modular ideals, as in 1.7). We will prove that later.

2. Dual modular inner ideals

Let J be a Jordan pair, $(a, b) \in V^\sigma \times V^{-\sigma}$, and let $K \subseteq V^\sigma$ be an (a, b)-modular inner ideal of J. We define the *a-dual* of K, denoted $K^{\langle a \rangle}$, as the set of elements $x \in V^{-\sigma}$ such that

(1) $Q_a x \in K$, $Q_a Q_x V^\sigma \subseteq K$,

(2) $\{K, x, a\} + \{K, Q_x V^\sigma, a\} \subseteq K$.

(In other words, $Q_a(x) + \{K, (x), a\} \subseteq K$, where $(x) = \Phi x + Q_x V^\sigma$ is the inner ideal generated by x.)

For a Jordan triple system J and an (a, b)-modular inner ideal K of J, we define the *a-dual* $K^{\langle a\rangle}$ of K in the same way, by deleting the superscripts.

2.1 Lemma. *Let J be a Jordan system, $a, b \in J$, and K be an (a, b)-modular inner ideal. If $P_a y \in K$ and $\{K, y, a\} \subseteq K$ (in particular, if there is $x \in K^{\langle a\rangle}$, such that $y \in (x)$), $z \in J$, and $k \in K$, then:*

(3) $\{a, y, z\} - P_a\{y, z, b\} \in K$,

(4) $\{a, \{y, z, b\}, k\} - \{z, y, k\} \in K$.

Proof: We write $a \equiv b$ for $a - b \in K$.

Since $B_{a,b}z \in K$ by modularity, we have $0 \equiv \{a, y, B_{a,b}z\}$ (since $\{K, y, a\} \subseteq K$) $= \{a, y, z\} - \{a, y, \{a, b, z\}\} + \{a, y, P_a P_b z\} = \{a, y, z\} - \{P_a y, b, z\} - P_a\{y, b, z\} + \{P_a y, P_b z, a\}$ (by JP13 and JP4) $\equiv \{a, y, z\} - P_a\{y, b, z\}$ (since $P_a y \in K$, and $\{P_a y, b, z\} - \{P_a y, P_b z, a\} \in K$ by modularity of K). This proves (3).

Now, $\{B_{a,b}z, y, k\} \in K$, hence $\{z, y, k\} \equiv \{\{a, b, z\}, y, k\} - \{P_a P_b z, y, k\} = \{a, \{b, z, y\}, k\} - \{a, y, \{z, b, k\}\} + \{\{a, y, k\}, b, z\} - \{P_a P_b z, y, k\}$ (by JP14) $\equiv \{a, \{b, z, y\}, k\} - \{a, y, \{z, b, k\}\} + \{\{a, y, k\}, P_b z, a\} - \{P_a P_b z, y, k\}$ (since, by hypothesis, $\{a, y, k\} \in K$, and $\{\{a, y, k\}, b, z\} - \{\{a, y, k\}, P_b z, a\} \in K$ by modularity of K) $\equiv \{a, \{b, z, y\}, k\} - \{a, y, \{a, P_b z, k\}\} + \{\{a, y, k\}, P_b z, a\} - \{P_a P_b z, y, k\}$ (since we have $\{a, y, \{a, z, k\} - \{a, P_b z, k\}\} \in \{a, y, K\} \subseteq K$, by modularity of K, and by $\{K, y, a\} \subseteq K$) $= \{a, \{b, z, y\}, k\} - \{P_a y, P_b z, k\}$ (since, setting $u = P_b z$, $\{a, y, \{a, u, k\}\} + \{P_a u, y, k\} = \{a, u, \{a, y, k\}\} + \{P_a y, u, k\}$ by JP6) $\equiv \{a, \{b, z, y\}, k\}$ (since $P_a y \in K$). This proves (4). ■

2.2 Proposition. *Let J be a Jordan system, $a, b \in J$, and K be an (a, b)-modular inner ideal. Then the a-dual of K is a (b, a)-modular inner ideal, and is contained in the b-dual of the a-dual of K.*

Proof: Clearly, if $x \in K^{\langle a\rangle}$, $P_x J \subseteq K^{\langle a\rangle}$. Therefore, $K^{\langle a\rangle}$ will be an inner ideal if for all $x, y \in K^{\langle a\rangle}$, $x + y \in K^{\langle a\rangle}$. We obviously have $P_a(x + y) \in K$ and $\{K, x + y, a\} \subseteq K$, hence we must show that for any $z \in J$, $P_a P_{x+y} z \in K$, and $\{K, P_{x+y} z, a\} \subseteq K$. Moreover, since $P_a P_x z, P_a P_y z \in K$, and $\{K, P_x z, a\} + \{K, P_y z, a\} \subseteq K$, this reduces to $P_a\{x, z, y\} \in K$, and $\{K, \{x, z, y\}, a\} \subseteq K$. As before we write $u \equiv v$ for $u - v \in K$.

Now, $P_a\{x, z, y\} = \{a, x, \{z, y, a\}\} - \{P_a x, y, z\}$ (by JP12) $= \{a, x, P_a\{y, z, b\}\} -$

$\{P_a x, y, z\}$ (by (4) and (2)) $= \{P_a x, \{y, z, b\}, a\} - \{P_a x, y, z\}$ (by JP4) $\equiv 0$ (by (4), since $P_a x \in K$ by (1)).

On the other hand, if $k \in K$, $\{k, \{x, z, y\}, a\} = \{k, y, \{z, x, a\}\} + \{\{z, x, k\}, y, a\} - \{z, x, \{k, y, a\}\}$ (by JP14) $\equiv \{k, y, P_a\{x, z, b\}\} + \{\{z, x, k\}, y, a\} - \{z, x, \{k, y, a\}\}$ (by (3) and (2)) $\equiv \{k, y, P_a\{x, z, b\}\} + \{\{a, \{x, z, b\}, k\}, y, a\} - \{z, x, \{k, y, a\}\}$ (by (4) and (2)) $\equiv \{k, y, P_a\{x, z, b\}\} + \{\{a, \{x, z, b\}, k\}, y, a\} - \{a, \{x, z, b\}, \{k, y, a\}\}$ (by (4), since $\{k, y, a\} \in K$ by (2)) $= \{k, y, P_a t\} + \{\{a, t, k\}, y, a\} - \{a, t, \{k, y, a\}\}$ (setting $t = \{k, y, a\}$) $= \{P_a y, t, k\}$ (by JP5) $\in K$ (since $P_a y \in K$ by (1)).

Therefore $K^{\langle a \rangle}$ is an inner ideal.

Let us see now that $K^{\langle a \rangle}$ is (b, a)-modular. First $P_a B_{b,a} J = B_{a,b} P_a J \subseteq K$, and for all $k \in K$, $\{k, B_{b,a} z, a\} = \{k, z, a\} - \{k, \{b, a, z\}, a\} + \{k, P_b P_a z, a\} = \{k, z, a\} - \{k, b, P_a z\} - \{k, z, P_a b\} + \{k, P_b P_a z, a\}$ (by JP10) $= \{k, z, a - P_a b\} - (\{k, b, P_a z\} - \{k, P_b P_a z, a\}) \in K$ (by modularity of K). Therefore $B_{b,a} J \subseteq K^{\langle a \rangle}$.

Now, if $x \in K^{\langle a \rangle}$ and $z \in J$, denote $W(x, z) = \{x, a, z\} - \{x, P_a z, b\}$. We claim that, for any $t \in J$,

$$P_{W(x,z)} t = P_x P_a P_z B_{a,b} t + B_{b,a} P_z P_a P_x t -$$

$$\{P_x P_a z, t, B_{b,a} z\} + W(x, P_z(\{x, t, a\} - P_a\{b, t, x\})).$$

Indeed, by JP21,

$(*)$ $P_{\{x,a,z\}} t = P_x P_a P_z t + P_z P_a P_x t + \{x, a, P_z\{t, x, a\}\} - \{P_x P_a z, t, z\}$.

Next, from JP20,

$(**)$ $P_{\{x,P_a z,b\}} t = P_x P_a P_z P_a P_b t + P_b P_a P_z P_a P_x t + \{x, P_a P_z P_a\{b, t, x\}, b\} -$
$\{P_x P_a z, t, P_b P_a z\}$.

Now, the linearization of JP21 in x gives the identity

$$Q(Q_{x,v} Q_y z, \{x, y, z\}) + Q(Q_x Q_y z, \{v, y, z\}) =$$

$$Q_x Q_y Q_z D_{y,v} + Q_{x,v} Q_y Q_z D_{y,x} +$$

$$D_{x,y} Q_z Q_y Q_{x,v} + D_{v,y} Q_z Q_y Q_x.$$

Hence, setting $x = x$, $y = a$, $z = z$, and $v = b$, and applying the operators to t, we obtain,

$(***)$ $\{\{x, a, z\}, t, \{x, P_a z, b\}\} = -\{P_x P_a z, t, \{b, a, z\}\} + P_x P_a P_z\{a, b, t\} +$
$\{x, P_a P_z\{a, x, t\}, b\} + \{x, a, P_z P_a\{x, t, b\}\} + \{b, a, P_z P_a P_x t\}$.

The claim then follows by substituting $(*), (**)$ and $(***)$ in

$$P_{w(x,z)}t = P_{\{x,a,z\}}t + P_{\{x,P_az,b\}} - \{\{x,a,z\},t,\{x,P_az,b\}\}.$$

The expression for $P_{W(x,z)}t$ gives $P_{W(K^{\langle a\rangle},J)}J \subseteq K^{\langle a\rangle} + W(K^{\langle a\rangle},J)$, hence to have that $W(K^{\langle a\rangle},J) \subseteq K^{\langle a\rangle}$ it suffices to have

$$P_aW(K^{\langle a\rangle},J) \subseteq K \qquad \text{and} \qquad \{a,W(K^{\langle a\rangle},J),K\} \subseteq K.$$

Now, $P_aW(x,a) = P_a\{x,a,z\} - P_a\{x,P_az,b\} = \{P_ax,z,a\} - \{P_ax,z,P_ab\}$ (by JP1 and JP3) $= \{P_ax,z,a-P_ab\} \in K$ (by modularity of K since $P_ax \in K$ by (1)), and $\{k,\{x,P_az,b\},a\} \equiv \{k,x,P_az\}$ (by (4)) $\equiv \{k,x,P_az\} + \{k,z,P_ax\}$ (since $P_ax \in K$ by (1)) $= \{k,\{x,a,z\},a\}$ (by JP12), hence $\{k,W(x,z),a\} \in K$.

Next, $P_{b-P_ba}J = B_{b,a}P_bJ \subseteq K^{\langle a\rangle}$, $P_a(b-P_ba) = a - P_ab - B_{a,b}a \in K$, and, if $k \in K$, $\{a,b-P_ba,k\} = \{a,b,k\} - \{a,P_ba,k\} \in K$ (by modularity). Therefore $b - P_ba \in K^{\langle a\rangle}$.

Now, P_bK is an inner ideal of J, $P_aP_bK \subseteq K$ by modularity, and, if $k \in K$, $\{a,P_bK,k\} \equiv \{K,b,k\}$ (by modularity) $\subseteq K$. Thus $P_bK \subseteq K^{\langle a\rangle}$. Then $P_bP_aK^{\langle a\rangle} \subseteq P_bK \subseteq K^{\langle a\rangle}$, and for any $x \in K^{\langle a\rangle}$, $\{x,b,a\} = x + P_bP_ax - B_{b,a}x \in K^{\langle a\rangle}$, hence $\{K^{\langle a\rangle},a,b\} \subseteq K^{\langle a\rangle}$, and this finishes the proof of the (b,a)-modularity of $K^{\langle a\rangle}$.

We have already noted that $P_bK \subseteq K^{\langle a\rangle}$. Now, if $k \in K$, $x \in K^{\langle a\rangle}$, $\{x,k,b\} \in K^{\langle a\rangle}$. Indeed, $P_a\{x,k,b\} \equiv \{a,x,k\}$ (by (3)) $\in K$ (by (2)), and $\{a,\{x,k,b\},K\} \equiv \{k,x,K\}$ (by (4)) $\subseteq K$. Therefore $K \subseteq K^{\langle a\rangle\langle b\rangle}$. ■

2.3 Remark. The above result applies to Jordan pairs with formally the same computations. Alternatively, one can deduce the result from pairs from the result from triple systems: If $V = (V^+,V^-)$ is a Jordan pair, $(a,b) \in V^\sigma \times V^{-\sigma}$, and $K \subseteq V^\sigma$ is an (a,b)-modular inner ideal, take $\tilde{K} = K \oplus V^{-\sigma} \subseteq T(V)$. Then $\tilde{K}$ is (a,b)-modular by 1.6(c) (we identify $a = a\oplus 0$, $b = 0\oplus b \in V^\sigma \oplus V^{-\sigma} = T(V)$), and it is easy to see that $\tilde{K}^{\langle a\rangle} = V^\sigma \oplus K^{\langle a\rangle}$. Then, by the result for triples, $V^\sigma \oplus K^{\langle a\rangle}$ is (b,a)-modular, hence $K^{\langle a\rangle}$ is (b,a)-modular by 1.6(c). Also, $\tilde{K} \subseteq \tilde{K}^{\langle a\rangle\langle b\rangle} = K^{\langle a\rangle\langle b\rangle} \oplus V^{-\sigma}$, hence $K \subseteq K^{\langle a\rangle\langle b\rangle}$.

3. Consequences

In this section we draw some consequences from the construction and properties of the a-dual of an (a,b)-modular inner ideal. The first one is a consequence of a much stronger result of Anquela and Cortés [2, Theorem 2.2] asserting that primitive Jordan systems are primitive at each element. The interest of obtaining that result from what we have proved before is that the proof given here is elementary (although not trivial), while the original proof depended on the structure theory of Jordan pairs.

3.1 THEOREM. *Let V be a Jordan pair, and $\sigma = +$ or $-$. If $K \subseteq V^\sigma$ is a primitizer with modulus $(a,b) \in V^\sigma \times V^{-\sigma}$, then $K^{\langle a\rangle}$ is also a primitizer. Therefore, a Jordan pair is $(+)$-primitive if and only if it is $(-)$-primitive.*

PROOF: Let $I = (I^+, I^-)$ be a nonzero ideal of V. Since K is a primitizer, $I^\sigma + K = V^\sigma$. Thus there exist $y \in I^\sigma$ and $k \in K$ such that $y + k = a$. Now $P_b y \in P_{V^{-\sigma}} I^\sigma \subseteq I^{-\sigma}$, and $P_b k \in P_b K \subseteq P_b K^{\langle a\rangle\langle b\rangle}$(by 2.2) $\subseteq K^{\langle a\rangle}$ (by the definition of the b-dual of $K^{\langle a\rangle}$). Therefore, $b = P_b a + (b - P_b a) = P_b y + P_b k + (b - P_b a) \in I^{-\sigma} + K^{\langle a\rangle}$ (since $b - P_b a \in K^{\langle a\rangle}$ by modularity). Now, $I^{-\sigma} + K^{\langle a\rangle}$ is again (b,a)-modular, hence $V^{-\sigma} = I^{-\sigma} + K^{\langle a\rangle}$, and $K^{\langle a\rangle}$ complements nonzero ideals.

On the other hand, if $K^{\langle a\rangle} = V^{-\sigma}$ is not proper, then $P_a b \in P_a V^{-\sigma} = P_a K^{\langle a\rangle} \subseteq K$, hence K contains one of its b-moduli, contradicting properness of K. Therefore $K^{\langle a\rangle}$ is proper, and so it is a primitizer. ∎

As a consequence, we obtain the following characterization of the Jacobson radical of a Jordan pair in terms of cores of maximal modular inner ideals which improves 1.7(b) (and answers the question raised in the remark after that theorem).

3.2 COROLLARY. *Let V be a Jordan pair, and $\sigma = \pm$, then $\mathrm{Jac}(V^\sigma)$ is the intersection of the σ-parts of the cores of all maximal modular inner ideals contained in V^σ.*

PROOF: In view of 1.7(b) it suffices to show that the core of any maximal-modular inner ideal contained in $V^{-\sigma}$ is contained in the core of some maximal-modular inner ideal contained in V^σ. Now, if $K \subseteq V^{-\sigma}$ is proper and maximal (a,b)-modular for some modulus $(a,b) \in V^{-\sigma} \times V^\sigma$, $K^{\langle a\rangle} \subseteq V^\sigma$ is (b,a)-modular, and proper by the argument of the proof of 3.1. Using Zorn's Lemma we can find a proper maximal (b,a)-modular inner ideal $L \subseteq V^\sigma$ containing $K^{\langle a\rangle}$, so it suffices to prove that $\mathrm{Core}(K) \subseteq \mathrm{Core}(L)$.

Set $(I^+, I^-) = \mathrm{Core}(K)$, Then $P_a I^\sigma \subseteq I^{-\sigma} \subseteq K$, and $\{K, I^\sigma, a\} \subseteq I^{-\sigma} \subseteq K$, since (I^+, I^-) is an ideal. Thus $I^\sigma \subseteq K^{\langle a\rangle} \subseteq L$, hence $\mathrm{Core}(K) \subseteq \mathrm{Core}(L)$. ∎

We next answer the question raised by D'Amour and McCrimmon that was mentioned in 1.4. This is contained in the more general result:

3.3 PROPOSITION. *Let J be a Jordan system, $a, b \in J$, and let K be an (a,b)-modular inner ideal of J. Then K is $(a, b+x)$-modular if and only if $x \in K^{\langle a\rangle}$. In particular, if (a,b') is another modulus of K, then $(a, b' + P_b k)$ is a modulus for K*

for any $k \in K$.

PROOF: Suppose first that $(a, b+x)$ is a modulus for K. Then, $a - P_a(b+x) = (a - P_a b) + P_a x \in K$, hence $P_a x \in K$. Also, for any $k \in K$, $\{k, b+x, a\} = \{k, b, a\} + \{k, x, a\} \in K$, hence $\{k, x, a\} \in K$. From 2.1 then follows that, for any $z \in J$ and any $k \in K$, $\{a, x, z\} - P_a\{x, z, b\} \in K$ and $\{a, \{y, z, b\}, k\} - \{z, y, k\} \in K$. Now, if $z \in J$, $B_{a,b+x}z = B_{a,b}z - (\{a, x, z\} - P_a\{x, z, b\}) + P_a P_x z \in K$, hence $P_a P_x z \in K$, and we have $P_a(x) \subseteq K$. Next, $(\{k, b, z\} - \{k, P_b z, a\}) + (\{k, x, z\} - \{k, \{b, z, x\}, a\}) - \{k, P_x z, a\} = \{k, b+x, z\} - \{k, P_{b+x}z, a\} \in K$, hence $\{k, P_x z, a\} \in K$, and $\{K, (x), a\} \subseteq K$. Therefore $x \in K^{\langle a \rangle}$.

Reciprocally, if $x \in K^{\langle a \rangle}$, then $B_{a,b+x}z = B_{a,b}z - (\{a, x, z\} - P_a\{x, z, b\}) + P_a P_x z \in K$ by (a, b)-modularity, (1) and (3); $\{k, b+x, z\} - \{k, P_{b+x}z, a\} = (\{k, b, z\} - \{k, P_b z, a\}) + (\{k, x, z\} - \{k, \{b, z, x\}, a\}) - \{k, P_x z, a\} \in K$ by (a, b)-modularity, (2) and (4); $\{k, b+x, a\} = \{k, b, a\} + \{k, x, a\} \in K$ by (a, b)-modularity and (2); $a - P_a(b+x) = a - P_a + P_a x \in K$ by (a, b)-modularity of K and (1). Thus K is $(a, b+x)$-modular.

Finally, $P_b K \subseteq P_b K^{\langle a \rangle \langle b \rangle}$ (by 2.2) $\subseteq K^{\langle a \rangle}$, hence if (a, b') is a modulus and $k \in K$, $P_b k \in K^{\langle a \rangle}$, hence $(a, b' + P_b k)$ is again a modulus by what we have just proved (note that $K^{\langle a \rangle}$ does not depend on the particular element b such that (a, b) is a modulus). ■

As a final remark, we show how the ideal $K^{\langle a \rangle}$ restores the "broken symmetry" in the proof that the powers $(a^{(n,b)}, b^{(m,a)})$ of a modulus (a, b) are again moduli (cf. 1.4):

3.4 PROPOSITION. *Let J be a Jordan system, $a, b \in J$, and K be an (a, b)-modular inner ideal of J. Then:*

(a) *If $a \equiv a' \bmod K$ and $b \equiv b' \bmod K^{\langle a \rangle}$, then (a', b') is a modulus for K.*

(b) *For any $n, m \geq 1$, $a^{(n,b)} \equiv a \bmod K$, and $b^{(m,a)} \equiv b \bmod K^{\langle a \rangle}$. Therefore $(a^{(n,b)}, b^{(m,a)})$ is a modulus for K.*

PROOF: (a). Since $b \equiv b' \bmod K^{\langle a \rangle}$, (a, b') is a modulus for K by 3.3. Now, that $a \equiv a' \bmod K$ implies that (a', b') is a modulus is straightforward.

(b). The congruence $a^{(n,b)} \equiv a \bmod K$ follows by an easy induction (see [1, 5.13.1]), and $b^{(m,a)} \equiv b \bmod K^{\langle a \rangle}$ is the corresponding assertion applied to the (b, a)-modular inner ideal $K^{\langle a \rangle}$. Finally, that $(a^{(n,b)}, b^{(m,a)})$ is a modulus for K follows from this and (a). ■

REFERENCES

[1] A. D'AMOUR, K. MCCRIMMON, The local algebra of a Jordan system, *J. Algebra* **177** (1995), 199-239.

[2] J. A. ANQUELA, T. CORTÉS, Primitivity in Jordan Systems is Ubiquitous, *J. Algebra* **202** (1998), 295-314.

[3] L. HOGBEN, K. MCCRIMMON, Maximal Modular Inner Ideals and the Jacobson Radical of a Jordan Algebra, *J. Algebra* **68** (1) (1981), 155-169.

[5] N. JACOBSON, "Structure Theory of Jordan Algebras", The University of Arkansas Lecture Notes in Mathematics, vol. 5, Fayetteville, 1981.

[4] O. LOOS, "Jordan Pairs", Lecture Notes in Mathematics, Vol 460, Springer-Verlag, New York, 1975.

[6] E. I. ZELMANOV, On prime Jordan algebras, *Algebra and Logic* **18** (1979), 162-175.

[7] E. I. ZELMANOV, Prime Jordan triple systems, *Siber. Math. J.* **24** (1983), 509-520.

[8] E. I. ZELMANOV, Primary Jordan triple systems III, *Siber. Math. J.* **26** (1985), 55-64.