Trivial Minimal Ideals of Jordan Systems [1]

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Dedicated to Professor Michel Racine on the occasion of his retirement

Abstract: We show that in Jordan systems (algebras, triple systems, and pairs) monomials containing two elements of a trivial minimal ideal vanish, so improving the answer given by Anquela and Cortés in *Inventiones Mathematicae*, **168** (2007) 83-90 to the problem posed in 1968 by Zhevlakov in the *Dniester Notebook: Unsolved Problems in the Theory of Rings and Modules*, extended by Nam and McCrimmon in *Proc. Amer. Math. Soc* **88** (4) (1983), 579-583.

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Introduction

Andrunakievich's Lemma readily implies that a minimal ideal of an associative algebra is either simple or has zero multiplication. However, a Jordan version of Andrunakievich's Lemma is false even for linear Jordan algebras (cf. [12]) and thus the problem of knowing whether minimal ideals in Jordan systems were always either simple or trivial remained open for decades.

The question, for linear Jordan algebras, was posed in 1969 by Zhevlakov (see [5]) and extended to quadratic Jordan algebras by Nam and McCrimmon in 1983 [14]. A positive answer in the case of linear Jordan algebras was obtained independently by Medvedev [12, p. 933] and Skosyrskii [15, Cor. 3.1]. The techniques were mainly combinatorial and strongly dependent on the linearity, i.e., on the existence of 1/2

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in the ring of scalars. Prior to these papers, Nam and McCrimmon had studied in [14] minimal ideals in quadratic Jordan algebras, showing that they should be either $\mathcal{D}$-simple or trivial. In [4], Block had shown that $\mathcal{D}$-simple algebras could be described in terms of simple algebras under the additional assumption of having a minimal ideal. However, nothing was known about $\mathcal{D}$-simple algebras, hence about minimal ideals of quadratic Jordan algebras, without this additional condition. With a different approach, mainly based on the structure theory, in [3] it is shown that the heart of a nondegenerate Jordan (quadratic) algebra, triple system, or pair is simple when nonzero. That is the starting point to show in [2] that a minimal ideal of a quadratic Jordan system must be either simple or trivial, so fully answering Nam-McCrimmon's question. However, due to the quadratic nature of this answer, some questions remained open.

This paper deals with trivial minimal inner ideals of Jordan systems, indeed with their "level of triviality". In the general quadratic setting [14] "trivial" was defined as having zero cube, so that the results obtained in [2] do not imply those in the linear setting due to Medvedev and Skosyrskii, where triviality meant having zero square. The problem comes from the fact that the square of an ideal need not be an ideal.

We will show that, if a minimal ideal I of Jordan system J has zero cube, then it is more than trivial in the sense that any monomial in J containing two elements of I vanishes. In particular, this shows that the square of I vanishes when J is a Jordan algebra, so that I is indeed trivial as a Jordan algebra, which implies the result of Medvedev and Skosyrskii. This new notion of triviality which depends obviously on the enveloping system J will be called J-triviality.

The paper is organized as follows: after a preliminary section we start with some combinatorial lemmas dealing with multiplication operators acting on trivial minimal ideals. Then, in the second section, we focus on triple systems because the notation is less cumbersome, and derive from that the corresponding results for algebras and pairs in the third section. Those readers who are only interested in algebras can skip the first three sections and go directly to the fourth section where we sketch an alternative proof for Jordan algebras in which most of the technicalities of triple systems are avoided. The final fifth section is devoted to explaining the connection of the problem we are dealing with with that of the definition of a Baer radical in linear Jordan algebras.

0. Preliminaries

0.1 We will deal with associative and Jordan algebras, pairs and triple systems over an arbitrary ring of scalars Φ. The reader is referred to [6, 7, 10, 11] for basic results, notation, and terminology, though we will stress some of them.

—Given a Jordan algebra J, its products will be denoted by x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y$, $U_{x,z}y = \{x, y, z\} = V_{x,y}z$, respectively.

—For a Jordan pair $V = (V^+, V^-)$, we have products $Q_x y \in V^\varepsilon$, for any $x \in V^\varepsilon$, $y \in V^{-\varepsilon}$, $\varepsilon = \pm$, with linearizations $Q_{x,z}y = \{x, y, z\} = D_{x,y}z$.

—A Jordan triple system J is given by its products $P_x y$, for any $x, y \in J$, with linearizations denoted by $P_{x,z}y = \{x, y, z\} = L_{x,y}z$.

0.2 Philosophically, triviality of a Jordan system J should mean that all products of elements of J vanish. In triples or pairs this means cubic triviality, $P_J J = 0$ or $Q_V V = 0$. In Jordan algebras it means squares-and-cubes triviality, $U_J J = J^2 = 0$. Then all linearized products $\{J, J, J\}$ and $J \circ J$ vanish as well. If I is an ideal in a Jordan triple J, pair V, or algebra J we will call I *trivial* if it is intrinsically trivial as a subsystem,

$$P_I I = 0, \quad \text{resp.} \quad Q_{I^\varepsilon} I^{-\varepsilon} = 0, \quad \text{resp.} \quad U_I I = I^2 = 0.$$

We warn the reader that, in the case of algebras, the above notion of triviality differs from that used in [14] and [2], which was just cube triviality.

On the other hand, triviality of I *as ideal* of a Jordan system should mean more than just triviality as a subsystem, it should mean that *all* monomial products in the system vanish as soon as there are at least two factors from I. Because this depends on the enveloping system J, we will say that such an ideal is *J-trivial.* In a Jordan triple J, a simple argument by induction on the degree shows that I being J-trivial means

$$P_I J = \{I, I, J\} = 0 \qquad (\text{hence} \quad \{I, J, I\} = 0),$$

in a Jordan pair V, a V-trivial ideal I satisfies

$$Q_{I^\varepsilon} V^{-\varepsilon} = \{I^\varepsilon, I^{-\varepsilon}, V^\varepsilon\} = 0, \qquad (\text{hence} \quad \{I^\varepsilon, V^{-\varepsilon}, I^\varepsilon\} = 0),$$

while in a Jordan algebra J, J-triviality of I means

$$U_I J = \{I, I, J\} = I^2 = 0 \qquad (\text{hence} \quad \{I, J, I\} = 0).$$

In linear Jordan algebras, where $\frac{1}{2} \in \Phi$, all products can be built from the bullet or circle $x \cdot y = \frac{1}{2}(x \circ y)$, so intrinsic triviality of I reduces to $I \circ I = 0$ since then

$2I^2 = I \circ I = 0$ and $2U_I I \subseteq I \circ (I \circ I) + I^2 \circ I = 0$. But even more, in this case I remains J-trivial for *any* enveloping algebra J since $2U_I J \subseteq I \circ (I \circ J) + I^2 \circ J = 0$ and $2\{I, I, J\} \subseteq I \circ (I \circ J) + J \circ (I \circ I) = 0$. In quadratic Jordan algebras, or in triple systems or pairs, there is no way to reduce the quadratic products $U_x y, P_x y, Q_{x^\varepsilon}(y^{-\varepsilon})$ to products of lower degree, so is unlikely that intrinsic triviality implies enveloping triviality. It is not known if a trivial ideal I in J always contains a J-trivial ideal; we will prove this when I is minimal.

0.3 A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P = U$. By doubling any Jordan triple system T one obtains the *double Jordan pair* $V(T) = (T, T)$ with products $Q_x y = P_x y$, for any $x, y \in T$. From a Jordan pair $V = (V^+, V^-)$ one can get a (*polarized*) Jordan triple system $T(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+} y^- \oplus Q_{x^-} y^+$ [7, 1.13, 1.14]

0.4 An *ideal* I of a Jordan triple system J is a Φ-submodule of J such that it is both an *inner ideal* ($P_I J \subseteq I$) and an *outer ideal* ($P_J I + \{J, J, I\} \subseteq I$). Similar notions are defined for Jordan algebras and pairs.

0.5 An element x of a Jordan system J (algebra, pair, or triple system) is called *invisible* if every Jordan monomial of degree > 1 in J containing x vanishes.

0.6 We recall the following identities valid for arbitrary Jordan triple systems which will be needed in the sequel:

(i) $P_x\{y, z, t\} = \{\{x, y, z\}, t, x\} - \{z, y, P_x t\}$,

(ii) $\{x, P_y z, t\} = \{x, y, \{z, y, t\}\} - \{x, P_y t, z\}$,

(iii) $\{x, \{y, z, t\}, u\} = \{\{x, y, z\}, t, u\} + \{\{u, y, z\}, t, x\} - \{z, y, \{x, t, u\}\}$,

(iv) $\{P_x y, z, t\} + \{P_x z, y, t\} = \{x, \{y, x, z\}, t\}$,

(v) $\{\{x, y, u\}, z, t\} + \{\{x, z, u\}, y, t\} = \{x, \{y, u, z\}, t\} + \{u, \{y, x, z\}, t\}$

(vi) $P_{P_x y} = P_x P_y P_x$.

Indeed, (i)—(vi) are respectively JP12, JP9, JP15, JP8, JP16, JP3 of [7].

0.7 For a Jordan triple system J, $\mathcal{M}(J)$ will denote its *multiplication algebra*, i.e., the unital subalgebra of $\mathrm{End}_\Phi(J)$ generated by all multiplication operators P_x (hence containing $P_{x,y}$), $L_{x,y}$ for $x, y \in J$. Equivalently, $\mathcal{M}(J)$ is generated by all $T = P_x, B_{x,y} := Id - L_{x,y} + P_x P_y$ ([7, 2.11]), which have the advantage of being *structural transformations* ($P_{T(x)} = T P_x T^*$). If A, B are Φ submodules of J, $L_{A,B}$, $P_{A,B} = P_{B,A}$ will denote respectively the spans of all $L_{a,b}$, $P_{a,b} = P_{b,a}$ for $a \in A$, $b \in B$, while $M_{A,B} \subseteq \mathrm{End}_\Phi J$ will denote the sum

$$M_{A,B} = M_{B,A} := L_{A,B} + L_{B,A} + P_{A,B}.$$

For a given element $x \in J$, let $L_{A,x}$, $L_{x,A}$, $P_{x,A} = P_{A,x}$, denote respectively the spans (in this case just the set) of all $L_{a,x}$, $L_{x,a}$, $P_{x,a} = P_{a,x}$ for $a \in A$ and

$$M_{A,x} = M_{x,A} := L_{A,x} + L_{x,A} + P_{x,A}.$$

Also $\mathcal{M}_A$ will denote the unital subalgebra of $\mathcal{M}(J)$ generated by all P_a, $L_{a,a'}$ for $a, a' \in A$ (hence containing all $P_{a,a'}$).

For any Φ submodule S of $\mathrm{End}_\Phi J$, $\widehat{S}$ will denote $S + \Phi Id_J$.

1. Combinatorial Lemmas

1.1 Throughout this section J will denote a Jordan triple system, and I will be an ideal of J. We will write X for the Φ-submodule of J spanned by a given finite set $\{x_1, \ldots, x_n\} \subseteq J$.

Let $\mathcal{Z}$ denote the set of elements M in $\mathcal{M}(J)$ that annihilate I, $M(I) = 0$, which is obviously an ideal of $\mathcal{M}(J)$, and let $\equiv$ denote congruence modulo $\mathcal{Z}$.

1.2 $M_{I,X}$-MIGRATION LEMMA. $M_{I,X}\mathcal{M}_X \subseteq \mathcal{M}_X M_{I,X}$.

PROOF: It suffices if $UM \in \mathcal{M}_X M_{I,X}$ for the spanning elements $U = L_{x,a}$, $L_{a,x}$, $P_{x,a}$ for $x \in X$, $a \in I$ and the generating elements $M = L_{y,z}, P_y$ for $y, z \in X$. The resulting 6 cases are handled as follows:

(1) $L_{x,a}P_y = P_{\{x,a,y\},y} - P_y L_{a,x}$ (by (0.6)(i) acting on t, replacing $x, y, z, t \mapsto y, a, x, \bullet$) $\in P_{I,X} + \mathcal{M}_X L_{I,X} \subseteq \mathcal{M}_X M_{I,X}$;

(2) $L_{a,x}P_y = P_{\{a,x,y\},y} - P_y L_{x,a}$ (by (0.6)(i) acting on t, replacing $x, y, z, t \mapsto y, x, a, \bullet$, i.e. switching x, a in (1)) $\in P_{I,X} + \mathcal{M}_X L_{X,I} \subseteq \mathcal{M}_X M_{I,X}$;

(3) $P_{x,a}P_y = L_{x,y}L_{a,y} - L_{x,P_y a}$ (by (0.6)(ii) acting on z, replacing $x, y, z, t \mapsto x, y, \bullet, a$) $\in \mathcal{M}_X L_{I,X} + L_{X,I} \subseteq \mathcal{M}_X M_{I,X}$;

(4) $L_{x,a}L_{y,z} = L_{y,z}L_{x,a} + L_{\{x,a,y\},z} - L_{y,\{a,x,z\}}$ (by (0.6)(iii) acting on x, replacing $x, y, u, z, t \mapsto \bullet, a, y, x, z$) $\in \mathcal{M}_X L_{X,I} + L_{I,X} + L_{X,I} \subseteq \mathcal{M}_X M_{I,X}$;

(5) $L_{a,x}L_{y,z} = L_{y,z}L_{a,x} + L_{\{a,x,y\},z} - L_{y,\{x,a,z\}}$ (by (0.6)(iii) acting on x, replacing $x, y, u, z, t \mapsto \bullet, x, y, a, z$, i.e. switching x, a in (4)) $\in \mathcal{M}_X L_{I,X} + L_{I,X} + L_{X,I} \subseteq \mathcal{M}_X M_{I,X}$;

(6) $P_{a,x}L_{y,z} = L_{x,y}P_{a,z} + P_{\{a,y,z\},x} - P_{z,x}L_{y,a}$ (by (0.6)(v) acting on y, replacing $x, y, u, z, t \mapsto a, \bullet, z, y, x$) $\in \mathcal{M}_X P_{I,X} + P_{I,X} + \mathcal{M}_X L_{X,I} \subseteq \mathcal{M}_X M_{I,X}$. ∎

1.3 x, y-ALTERNATION LEMMA. *If I is trivial, then for any $x, y \in J$ we have*

$$M_{I,x}M_{I,y} \subseteq \widehat{M}_{I,y}M_{I,x} + \mathcal{Z}.$$

More specifically, for any $x, y \in X$, we have

(C1) $L_{x,I}L_{y,I} \subseteq L_{y,I}L_{x,I} + \mathcal{Z}$, (C1)$_0$ $L_{x,I}L_{x,I} \subseteq \mathcal{Z}$,

(C2) $L_{x,I}L_{I,y} \subseteq \widehat{L}_{I,y}L_{x,I} + \mathcal{Z}$,

(C3) $L_{x,I}P_{y,I} \subseteq L_{y,I}P_{x,I} + \mathcal{Z}$, (C3)$_0$ $L_{x,I}P_{x,I} \subseteq \mathcal{Z}$,

(C4) $L_{I,x}L_{y,I} \subseteq P_{y,I}P_{x,I} + \mathcal{Z}$,

(C5) $L_{I,x}L_{I,y} \subseteq L_{I,y}L_{I,x} + \mathcal{Z}$, (C5)$_0$ $L_{I,x}L_{I,x} \subseteq \mathcal{Z}$,

(C6) $L_{I,x}P_{I,y} \subseteq P_{y,I}L_{x,I} + \mathcal{Z}$,

(C7) $P_{y,I}L_{x,I} \subseteq L_{I,x}P_{I,y} + \mathcal{Z}$,

(C8) $P_{x,I}L_{I,y} \subseteq P_{y,I}L_{I,x} + \mathcal{Z}$, (C8)$_0$ $P_{x,I}L_{I,x} \subseteq \mathcal{Z}$,

(C9) $P_{x,I}P_{y,I} \subseteq L_{I,y}L_{x,I} + \mathcal{Z}$,

PROOF: These follow from the following formulas taking into account that $P_{I,I}$ and $L_{I,I}$ are contained in the ideal $\mathcal{Z}$ since I is trivial. Let $a, b \in I$,

(C1)$_0$: $L_{x,a}L_{x,b} = P_xP_{b,a} + L_{P_xa,b}$ (by (0.6)(i) acting on z, replacing $x, y, z, t \mapsto x, b, \bullet, a) \in P_xP_{I,I} + L_{I,I} \subseteq \mathcal{Z}$,

(C3)$_0$: $L_{x,a}P_{x,b} = P_{P_xa,b} + P_xL_{a,b}$ (by (0.6)(i) acting on y, replacing $x, y, z, t \mapsto x, \bullet, b, a) \in P_{I,I} + P_xL_{I,I} \subseteq \mathcal{Z}$,

(C5)$_0$: $L_{a,x}L_{b,x} = P_{a,b}P_x + L_{a,P_xb}$ (by (0.6)(ii) acting on z, replacing $x, y, z, t \mapsto a, x, \bullet, b) \in P_{I,I}P_x + L_{I,I} \subseteq \mathcal{Z}$,

(C8)$_0$: $P_{x,a}L_{b,x} = P_{P_xb,a} + L_{a,b}P_x$ (by (0.6)(iv) acting on z, replacing $x, y, z, t \mapsto x, b, \bullet a) \in P_{I,I} + L_{I,I}P_x \subseteq \mathcal{Z}$.

Notice that (C1), (C3), (C5), (C8) follow from (C1)$_0$, (C3)$_0$, (C5)$_0$, (C8)$_0$ by linearization.

(C4) , (C9) : $L_{a,x}L_{y,b} - P_{y,a}P_{x,b} = L_{a,\{x,y,b\}} - L_{a,b}L_{y,x}$ (by (0.6)(v) acting on u, replacing $x, y, u, z, t \mapsto y, x, b, \bullet, a) \in L_{I,I} + L_{I,I}L_{y,x} \subseteq \mathcal{Z}$,

(C2) : $L_{x,a}L_{b,y} = P_{x,b}P_{y,a} + L_{x,\{a,b,y\}} - L_{x,y}L_{b,a}$ (by (0.6)(v) acting on u, replacing $x, y, u, z, t \mapsto b, y, \bullet, a, x) \equiv P_{x,b}P_{y,a} + L_{x,\{a,b,y\}} \in P_{x,I}P_{y,I} + L_{x,I} \subseteq L_{I,y}L_{x,I} + L_{x,I} + \mathcal{Z}$ by (C9),

(C6) , (C7) : $L_{a,x}P_{b,y} - P_{y,a}L_{x,b} = P_{a,b}L_{x,y} - P_{a,\{y,x,b\}}$ (by (0.6)(v) acting on z, replacing $x, y, u, z, t \mapsto y, x, b, \bullet, a) \in P_{I,I}L_{x,y} + P_{I,I} \subseteq \mathcal{Z}$. ■

2. Trivial Minimal Ideals of Jordan Triple Systems

2.1 LEMMA. *If I is a minimal ideal of a Jordan triple system J, then $\{I, I, J\} \subseteq$*

$\{I, J, I\}$.

PROOF: There are two possibilities: either $P_J I = 0$ or $P_J I \neq 0$. In the first case $\{I, I, J\} \subseteq P_J I = 0$. In the second case, $P_J I$ is a nonzero ideal of J (it is a semiideal [9, 6.2(a)] and $P_J P_J I \subseteq P_J I$ by idealness of I) which is contained in I, thus $I = P_J I$ by minimality of I; hence

$$\{I, I, J\} = \{I, P_J I, J\} \subseteq_{(0.6)(\text{ii})} \{I, J, \{I, J, J\}\} + \{I, P_J J, I\} \subseteq \{I, J, I\}.$$

In either case, $\{I, I, J\} \subseteq \{I, J, I\}$. ∎

2.2 PROPOSITION. *If a minimal ideal I of a Jordan triple system J is trivial as a subsystem, $P_I I = 0$, and $P_I J \neq 0$, then $I = P_I J = \{I, J, I\}$.*

PROOF: We recall that $P_I J$ is a nonzero semiideal of J [9, 6.2(a)], hence $P_I J + P_J P_I J$ is a nonzero ideal of J [9, 6.2(b)] contained in I. By minimality of I,

$$I = P_I J + P_J P_I J, \tag{1}$$

hence,

$$P_I J \subseteq \{I, J, I\}. \tag{2}$$

Indeed, $P_I J =_{(1)} P_{P_I J + P_J P_I J} J$ is spanned (for $a_i, b_i \in I$ and $x_j, y_j.z_j \in J$) by

$$\begin{aligned} P_{\sum P_{a_i} x_i + \sum P_{y_j} P_{b_j} z_j} J &\subseteq_{(0.6)(\text{vi})} \sum \big(P_{a_i} P_{x_i} P_{a_i}\big) J + \sum \big(P_{y_j} P_{b_j} P_{z_j} P_{b_j} P_{y_j}\big) J + P_{I,I} J \\ &\subseteq P_I P_J P_I J + P_J P_I P_J P_I P_J J + \{I, J, I\} \\ &\subseteq P_I I + P_J P_I I + \{I, J, I\} = \{I, J, I\}, \end{aligned}$$

using triviality of I. Now,

$$\begin{aligned} P_J P_I J \subseteq_{(2)} P_J \{I, J, I\} &\subseteq_{(0.6)(\text{i})} \{\{J, I, J\}, I, J\} + \{J, I, P_J I\} \\ &\subseteq \{I, I, J\} + \{J, I, I\} = \{I, I, J\} \subseteq_{(2.1)} \{I, J, I\}. \end{aligned} \tag{3}$$

Now (1), (2), and (3) yield $I \subseteq \{I, J, I\}$. But $\{I, J, I\} \subseteq P_I J \subseteq I$. ∎

2.3 THEOREM. *If a minimal ideal I of a Jordan triple system J is trivial as a subsystem, $P_I I = 0$, then it is J-trivial, $P_I J + \{I, I, J\} = 0$.*

PROOF: Let us assume that $P_I J \neq 0$. Taking any $a \in I$, $y \in J$ such that $z := P_a y \neq 0$, we have a nonzero absolute zero divisor $z \in I$ ($P_z J = P_{P_a y} J = P_a P_y P_a J$ (by (0.6)(vi)) $\subseteq P_I P_J P_I J \subseteq P_I I = 0$ since I is a trivial ideal). Hence Φz is an inner ideal of J and the ideal of J generated by z is just the outer hull $\mathcal{M}(J)(z)$ ([14, 1.9] can be easily extended to triple systems by replacing the operators U_x by structural

P_x and $B_{x,y}$ so that each monomial $M_1 \cdots M_r(z)$ remains an absolute zero divisor), and $I = \mathcal{M}(J)(z)$ by minimality. We have

$$z \in I = \{I, J, I\} \text{ (by (2.2)) } = L_{I,J} I = L_{I,J} \mathcal{M}(J) z, \tag{1}$$

and there are finitely many elements $\{x_1, \ldots, x_n\} \subseteq J$ involved in (1) to express $z = T(z)$ for $T \in L_{I,J}\mathcal{M}(J)$. If X is the Φ-module spanned by those elements, we indeed have $T \in L_{I,X}\mathcal{M}_X \subseteq M_{I,X}\mathcal{M}_X$.

For any positive integer m,

$$T^m \in \overbrace{(M_{I,X}\mathcal{M}_X) \cdots (M_{I,X}\mathcal{M}_X)}^{m} \subseteq \mathcal{M}_X \overbrace{M_{I,X} \cdots M_{I,X}}^{m}, \tag{2}$$

by $M_{I,X}$-Migration Lemma (1.2). But $\overbrace{M_{I,X} \cdots M_{I,X}}^{m}$ consists of the sum of the submodules $M_{I,x_{i_1}} \cdots M_{I,x_{i_m}}$ for all choices of $i_1, \ldots i_m \in \{1, \ldots, n\}$.

Let $\mathcal{Z}$ as in (1.1) denote the set of elements in $\mathcal{M}(J)$ annihilating I. We claim that

(3) when $m \geq 4n+1$, $M_{I,x_{i_1}} \cdots M_{I,x_{i_m}} \subseteq \mathcal{Z}$, hence $\overbrace{M_{I,X} \cdots M_{I,X}}^{m} \subseteq \mathcal{Z}$.

Indeed, in this case one of the different x_i must occur at least 5 times. By x,y-Alternation Lemma (1.3), we can move this $x := x_i$ to the right modulo $\mathcal{Z}$, so that, $M_{I,x_{i_1}} \cdots M_{I,x_{i_m}} \subseteq M_{I,x_{j_1}} \cdots M_{I,x_{j_r}} \overbrace{M_{I,x} \cdots M_{I,x}}^{5} + \mathcal{Z}$ (where it might be $r \leq 4n-4$, recalling possible attrition in (1.3)(C2)). Now, we just need to show that $\overbrace{M_{I,x} \cdots M_{I,x}}^{5} \subseteq \mathcal{Z}$. Each string of length 5 consisting of elements of $L_{x,I}$, $L_{I,x}$, $P_{x,I}$ can be normalized modulo $\mathcal{Z}$ as follows:

(I) $L_{I,x}L_{x,I}$ can be replaced by $P_{x,I}P_{x,I}$ (1.3)(C4),

(II) $L_{I,x}P_{x,I}$ can be replaced by $P_{x,I}L_{x,I}$ (1.3)(C6),

(III) $L_{x,I}L_{x,I}$, $L_{x,I}P_{x,I}$, $L_{I,x}L_{I,x}$ can be replaced by zero by (1.3)(C1)$_0$, (C3)$_0$, (C5)$_0$, respectively.

Applying (I-III) to $L_{I,x}L_{x,I}$, $L_{I,x}L_{I,x}$, $L_{I,x}P_{x,I}$, we can assume that any $L_{I,x}$ appears at the very end (and there is at most one of them), so that the string has an initial substring of at least 4 terms consisting only of $L_{x,I}$'s and $P_{x,I}$'s. But then $L_{x,I}L_{x,I}, L_{x,I}P_{x,I} \subseteq \mathcal{Z}$ by (III) implies that we can assume that there is at most one $L_{x,I}$ at the very end, so that there must be a string of at least three $P_{x,I}$, and

$$P_{x,I}P_{x,I}P_{x,I} \subseteq_{(1.3)(C9)} P_{x,I}L_{I,x}L_{x,I} + \mathcal{Z} \subseteq_{(1.3)(C8)_0} \mathcal{Z}.$$

Hence, (2) and (3) imply $T^{4n+1} \in \mathcal{Z}$, and $z = T(z) = T^{4n+1}(z) = 0$, which is a contradiction coming from our assumption that $P_I J \neq 0$. Then it must be $P_I J = 0$, hence $\{I, J, I\} \subseteq P_I J = 0$ and $\{I, I, J\} = 0$ using (2.1). ■

3. Trivial Minimal Ideals of Jordan Algebras and Pairs

We will use the functors linking Jordan algebras and pairs with triple systems to obtain analogues of (2.3) for algebras and pairs.

3.1 THEOREM. *If a minimal ideal I of a Jordan algebra J is trivial as a triple subsystem, $I^3 = U_I I = 0$, then it is trivial as a subalgebra, and even J-trivial, $I^2 + U_I J + \{I, I, J\} = 0$.*

PROOF: Notice that I remains a trivial minimal ideal of the unitization $\hat{J}$ of J, hence we may assume that J is unital. But the ideals of a unital Jordan algebra and those of its underlying triple system coincide since $x^2 = P_x 1$ and $x \circ y = \{x, 1, y\}$ are now triple products, so that I is a trivial minimal ideal of J as a triple system and we can apply (2.3) noticing $I^2 = P_I 1$. ■

3.2 REMARK: The functor $T(\ )$ (0.3) linking pairs and triple systems does not interact with ideals nicely enough to provide a straightforward pair version of (2.3). In [2, 2.5], a similar problem was solved by applying [1, Sect. 5] and [3, 3.7(ii)] to a suitable quotient of the given Jordan pair where the minimal ideal under study turned to be the heart. In our case, we are looking for a property involving not only the minimal ideal I of the Jordan pair V, but also V. So, rather than modifying V, we can use the argument given in [1, Sect. 5] without assuming semiprimeness of V, which proves:

> For a given Jordan pair V and a nonzero triple ideal L of $T(V)$, either there exists a nonzero pair ideal K of V such that $T(K) \subseteq L$, or the $(+/-)$ projections $\pi^+(L)$ and $\pi^-(L)$ of L consist of invisible elements.

3.3 PROPOSITION. *Let $I = (I^+, I^-)$ be a pair of Φ-submodules of a Jordan pair V. Then I is a minimal ideal of V if and only if $T(I)$ is a minimal ideal of $T(V)$.*

PROOF: From the definition of $T(\ )$ (0.3), it is clear (and indeed well known) that I is an ideal of V if and only if $T(I)$ is an ideal of $T(V)$.

Assume first that I is minimal as an ideal of V, and let L be any nonzero triple ideal of $T(V)$ such that $L \subseteq T(I)$. We claim that $L = T(I)$, proving that $T(I)$ is minimal in $T(V)$.

If there exists a nonzero pair ideal K of V with $T(K) \subseteq L \subseteq T(I)$ then by minimality $K = I$ and $T(I) \subseteq L \subseteq T(I)$ yields the desired equality $L = T(I)$.

Otherwise, by (3.2) both $\pi^+(L)$ and $\pi^-(L)$ consist of invisible elements. In this case $\pi^\varepsilon(L) \neq 0$ for $\varepsilon = +$ or $-$, and $K = (\pi^\varepsilon(L), 0) \subseteq (I^\varepsilon, 0) \subseteq I$ would be an ideal of V, hence by minimality $I = K = (\pi^\varepsilon(L), 0)$, and $L \subseteq T(I) = \pi^\varepsilon(L) \oplus 0$ forces $L = \pi^\varepsilon(L) \oplus 0 = T(I)$ as desired.

Conversely, if $T(I)$ is a minimal ideal of $T(V)$ then I is a minimal ideal of V since any ideal $0 \neq M \subseteq I$ has $0 \neq T(M) \subseteq T(I)$, hence $T(M) = T(I)$ by minimality in $T(V)$, hence $M = I$ in V. ∎

3.4 THEOREM. *If a minimal ideal I of a Jordan pair V is trivial as a subpair, $Q_{I^\varepsilon} I^{-\varepsilon} = 0$, $\varepsilon = \pm$, then it is V-trivial, $Q_{I^\varepsilon} V^{-\varepsilon} + \{I^\varepsilon, I^{-\varepsilon}, V^\varepsilon\} = 0$, $\varepsilon = \pm$.*

PROOF: Since $T(I)$ is a minimal ideal of $T(V)$ (3.3), and it is trivial since I is so, $P_{T(I)} T(V) + \{T(I), T(I), T(V)\} = 0$ by (2.3), which readily implies $Q_{I^\varepsilon} V^{-\varepsilon} + \{I^\varepsilon, I^{-\varepsilon}, V^\varepsilon\} = 0$, $\varepsilon = \pm$. ∎

4. An Alternative Simpler Approach to the Algebra Case

This section is devoted to sketch an alternative direct proof of (3.1) without making use of triple systems.

First of all, one can obtain a unital algebra version of the $M_{I,X}$-Migration Lemma (1.2) using the following additional algebra identities.

4.1 Let J be a unital Jordan algebra, for any $x, y, z \in J$

(i) $(x \circ y) \circ z = \{x, y, z\} + \{y, x, z\}$,

(ii) $V_x = V_{x,1} = V_{1,x} = U_{x,1}$.

Indeed, (i) is the linearization of [6, QJ12, p. 2.16], and (ii) can be found in [6, p. 1.11].

4.2 $V_{I,X}$-MIGRATION LEMMA. *Let J be a unital Jordan algebra, I be an ideal of J, and X be a Φ-submodule of J spanned by a finite set of elements $\{x_0, x_1, \ldots, x_n\}$, where $x_0 = 1$. Then*

$$V_{I,X} \mathcal{M}_X \subseteq \mathcal{M}_X V_{I,X}.$$

PROOF: Noticing that $\mathcal{M}_X$ is generated by operators of the form U_x, for $x \in X$, the result is obtained by repeatedly using the fact that for $w \in I, x, y \in X$ we have

$$V_{w,y} U_x \subseteq \mathcal{M}_X V_{I,X}. \tag{1}$$

Indeed,

$$\begin{aligned} V_{w,y}U_x &= U_{\{w,y,x\},x} - U_xV_{y,w} \text{ (by (0.6)(i) in algebra form)} \\ &\subseteq U_{I,X} + U_xV_{X,I} \end{aligned} \tag{2}$$

But

$$\begin{aligned} V_{X,I} &\subseteq V_{X\circ I} - V_{I,X} \text{ (by (4.1)(i))} \\ &\subseteq V_I + V_{I,X} = V_{I,x_0} + V_{I,X} \text{ (by (4.1)(ii))} \\ &\subseteq V_{I,X} \end{aligned} \tag{3}$$

and

$$\begin{aligned} U_{I,X} &\subseteq V_XV_I + V_{X,I} \text{ (by (4.1)(i))} \\ &= U_{X,x_0}V_{I,x_0} + V_{I,X} \text{ (by (4.1)(ii) and (3))} \ , \\ &\subseteq U_XV_{I,X} + V_{I,X} \end{aligned} \tag{4}$$

and (1) follows from (2), (3), and (4). ■

4.3 LEMMA. *Let J be a Jordan algebra, and I be an ideal of J such that $\{I,I,I\} = 0$. Then, for any $x \in J$, $V_{I,x}V_{I,x}I = 0$.*

PROOF: For any $a_1, a_2, b \in I$, $V_{a_1,x}V_{a_2,x}b = \{a_1, U_xa_2, b\} + \{a_1, U_xb, a_2\}$ (by (0.6)(ii) in algebra form) $\in \{I,I,I\} = 0$. ■

Now we can obtain (3.1).

ALTERNATIVE PROOF OF (3.1): We can replace J by its unitization and assume that J is unital. We will prove that $U_IJ = 0$ by showing that $U_IJ \neq 0$ leads to a contradiction. Otherwise U_IJ is a nonzero ideal of J (see [8, p. 221]) which coincides with I by minimality, hence $I = U_IJ = U_{U_IJ}J$ is spanned for $w_i \in I$, $a_i \in J$ by elements

$$\begin{aligned} U_{\sum_i U_{w_i}a_i}J &= \sum_i U_{U_{w_i}a_i}J + \sum_{i<j} U_{U_{w_i}a_i, U_{w_j}a_j}J \\ &\subseteq 0 + \sum_{i<j} U_{I,I}J = \{I,J,I\} \end{aligned}$$

since all $z_i := U_{w_i}a_i$ are absolute zero divisors [$U_{z_i}J = U_{w_i}U_{a_i}U_{w_i}J$ (by (0.6)(vi) in algebra form) $\subseteq U_I(U_JU_IJ) \subseteq U_II = 0$ by cubelessness] leading to

$$I = \{I,J,I\} = V_{I,J}I. \tag{1}$$

Choose $0 \neq z \in I$ of the form $z = U_wa$ for $w \in I$, $a \in J$ (we are assuming $U_IJ \neq 0$). As above, z is an absolute zero divisor of J, then Φz is an inner ideal of J and the ideal of J generated by z is just the outer hull $\mathcal{M}(J)z$ [14, 1.9]. By minimality of I, $I = \mathcal{M}(J)z$, hence $I = V_{I,J}I = V_{I,J}\mathcal{M}(J)z$. In particular, $z = T(z)$ for a multiplication operator T of the form

$$T = \sum_i V_{w_i,y_i} U_{x_{i1}} \cdots U_{x_{in_i}} \tag{2}$$

for some $w_i \in I$, $y_i, x_{ij} \in J$. Let X be the finitely generated Φ-submodule of J spanned by all y_i, x_{ij} appearing in (2), together with $x_0 = 1$. We have $T \in V_{I,X}\mathcal{M}_X$. Thus

$$T^m \in \overbrace{(V_{I,X}\mathcal{M}_X)\cdots(V_{I,X}\mathcal{M}_X)}^{m} \subseteq \mathcal{M}_X \overbrace{V_{I,X}\cdots V_{I,X}}^{m} \tag{3}$$

using (4.2). By (4.3), for fixed $a_1, \ldots, a_m \in I$ each $V_{a_1,z_1} \cdots V_{a_m,z_m}$ is an alternating multilinear function of $z_1, \ldots, z_m$ modulo the ideal $\mathcal{Z}$ of $\mathcal{M}(J)$ of multiplication operators annihilating I. This alternating function must vanish modulo $\mathcal{Z}$ on the finitely-spanned submodule X as soon as m exceeds the cardinality of the spanning system of X, so for suitably large m we have $z = T(z) = T^2(z) = \cdots T^m(z) \subseteq T^m(I) = 0$, the desired contradiction. ■

5. On the Baer Radical of Jordan Algebras

5.1 In 1968 Zhevlakov posed the following problem [5, problem 1.44]: *Do there exist solvable prime Jordan rings?* A negative answer was obtained by Medvedev and Zelmanov in [13, Th. 3] for Jordan algebras over a ring of scalars having 1/2. Indeed, they show that any nonzero solvable algebra J, i.e., satisfying that the series of subalgebras

$$J^{(0)} := J \supseteq J^{(1)} := J^2 \supseteq \cdots \supseteq J^{(n)} := (J^{(n-1)})^2 \supseteq \cdots$$

terminates ($J^{(n)} = 0$ for some n), necessarily contains a nonzero ideal with zero multiplication. The crucial point of this problem is the same as that of the one we have dealt with in the previous sections: the fact that the square of an ideal is not necessarily an ideal. This also causes problems with the definition of the radical related to solvability in linear Jordan algebras, i.e., with the analogue of the Baer radical of associative algebras.

We will say that a quadratic Jordan algebra J is *semiprime* if it has no nonzero ideals which are trivial as subalgebras I, i.e., square-cube trivial, $U_I I = I^2 = 0$ (although in the literature, the notion of semiprimeness is defined by imposing the stronger condition of absence of nonzero ideals with zero cube). The Baer radical $B(J)$ of J is built by transfinite induction eliminating trivial ideals of J; it is the smallest of the ideals L such that J/L is semiprime. Then J is semiprime iff $B(J) = 0$.

Transfinite induction can be avoided by simply defining $B(J)$ to be the intersection of all such ideals L: this intersection B is again such an ideal, since if I/B is trivial in J/B so is its homomorphic image $I/L \cong (I/B)/(L/B)$ modulo any larger L, but J/L is semiprime, so $I/L = 0$ and $I \subseteq L$ for all L, i.e. $I \subseteq B$ and $I/B = 0$. The following open problem has become a part of the Jordan folklore.

5.2 Can a semiprime J have nonzero nilpotent ideals?

For a (not necessarily linear) Jordan system S, being *nilpotent of index* n means that any Jordan monomial of degree n or bigger vanishes when evaluated in S. Obviously $J/B(J)$ does not have nonzero ideals which are nilpotent of index 2, but what about other indexes of nilpotency?

We will show how the results obtained in the previous sections solve the above problems under the assumption that we deal with algebras with the minimal condition on ideals.

We say that J is *free* of some sort of ideal if it has no *nonzero* such ideals.

5.3 COROLLARY. (i) *Let J be a Jordan algebra such that any nonzero ideal contains a minimal ideal (for example, when J satisfies the minimal condition on ideals). If J is semiprime, then J is free of ideals of zero cube, then also free of nilpotent ideals.*

(ii) *Let J be a Jordan algebra satisfying the minimal condition on ideals. Then $B(J)$ is the smallest of the ideals L of J such that J/L is free of nilpotent ideals.*

PROOF: (i) If L is a nonzero ideal of zero cube of J, by hypothesis L contains a minimal ideal L_0 which has zero cube too. Thus L_0 is trivial as a triple subsystem, and by (3.1), L_0 is J-trivial, hence trivial as subalgebra, which contradicts semiprimeness of J.

If I is a nonzero nilpotent ideal of J of index n for some $n \geq 3$, then we can use the fact that the cubes of ideals are again ideals to find another nonzero ideal L of J with cube zero, which is impossible as shown above.

(ii) Notice that $J/B(J)$, as any quotient of J, also satisfy the minimal condition on ideals, so that we can apply (i), and the assertion readily follows. ■

5.4 REMARKS: (i) We remark that the minimal condition on ideals is weaker than other more usual finiteness conditions, so that (5.3) can be applied to broad families of examples. In particular, the descending chain condition on inner ideals clearly implies the minimal condition on ideals.

(ii) The problems concerning solvability and nilpotency have a long tradition in the study of linear Jordan algebras (see, Chapter 4 of [16]). This is because, in this setting, where any product can be expressed in terms of the bilinear product, the analogues of associative algebra results and notions are easy to express and the problems come out naturally.

(iii) Notice that, when dealing with linear algebras where intrinsic triviality reduces to zero square (see (0.2)), (5.3) gives an answer to the classical formulation of the problem stated in (5.2): Let J be a Jordan algebra such that any nonzero ideal contains a minimal ideal (for example, when J satisfies the minimal condition on ideals). If J is free of ideals of zero square, then it is also free of ideals of zero cube and free of nilpotent ideals.

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