Minimal Ideals of Jordan Systems [1]

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Abstract: We show that minimal ideals of Jordan systems (algebras, triple systems, and pairs) are either simple or trivial, so answering the question posed by Nam and McCrimmon in 1983.

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0. Introduction and Preliminaries

Minimal ideals, in particular, the heart, appear naturally when studying Jordan systems [10, III.6]. Indeed, having a nonzero heart can be considered as a strong version of primeness for semiprime or nondegenerate systems.

Using Andrunakievich's Lemma, it is very easy to prove that a minimal ideal of an associative algebra is either simple or trivial. However, a Jordan version of Andrunakievich's Lemma is false even for linear Jordan algebras (cf. [12]) and thus the study of minimal ideals in Jordan systems requires different techniques.

For linear Jordan algebras the fact that minimal ideals are either simple or trivial is obtained by Medvedev [12, p. 933] and Skosyrskii [14, Cor. 3.1]. The techniques are mainly combinatorial and strongly dependent on the linearity, i.e., on the existence of 1/2 in the ring of scalars. Prior to these papers, Nam and McCrimmon study in [13] minimal ideals in quadratic Jordan algebras. In this more general setting, they show that these ideals are either $\mathcal{D}$-simple or trivial. In [5], Block shows that $\mathcal{D}$-simple algebras can be described in terms of simple algebras under the additional assumption of having a minimal ideal. However, nothing is known without this additional condition, so that Nam and McCrimmon leave open the general question on the simplicity of nontrivial minimal ideals. With a different approach, mainly based on the structure theory, in [4] it is shown that the heart of a nondegenerate Jordan (quadratic) algebra, triple system, or pair is simple when nonzero.

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In this paper we settle the question posed in [13] for quadratic Jordan algebras, obtaining also analogous results for pairs and triple systems, which were unknown even in the linear setting.

Indeed, we focus first on triple systems, and derive from that the corresponding results for algebras and pairs, which gives the shortest approach. However, this generates some combinatorial difficulties due, among other things, to the fact that, in a Jordan triple system, the cube of an ideal need not be an ideal. This is particularly evident in the first section of the paper, devoted to study the multiplication by the elements of a radical ideal, so that a reader only interested in algebras could skip some technicalities. In the second section we obtain the main results of the paper. We first show that minimal ideals are always hearts of certain systems, so that the problem reduces to the degenerate case by using [4]. The proof that the heart of a degenerate Jordan system is trivial is partly inspired by the arguments given by Medvedev in [12]. A crucial fact is the local nilpotency of the McCrimmon radical of a Jordan system, which is a consequence of a result on Lie sandwiches due to Kostrikin and Zelmanov [7].

0.1 We will deal with associative and Jordan algebras, pairs and triple systems over an arbitrary ring of scalars Φ. The reader is referred to [6, 8, 10, 11] for basic results, notation, and terminology, though we will stress some notions. The identities JPx listed in [8] will be quoted with their original numbering without explicit reference to [8].

—Given a Jordan algebra J, its products will be denoted by x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively.

—For a Jordan pair $V = (V^+, V^-)$, we have products $Q_x y \in V^\sigma$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z} y = \{x, y, z\} = D_{x,y} z$.

—A Jordan triple system J is given by its products $P_x y$, for any $x, y \in J$, with linearizations denoted by $P_{x,z} y = \{x, y, z\} = L_{x,y} z$.

A Jordan algebra or triple system J is said to be *trivial* if its cube ($J^3 = U_J J$ or $P_J J$, respectively) is zero. A Jordan pair V is called *trivial* if its cube $V^3 = (Q_{V^+} V^-, Q_{V^-} V^+)$ is zero.

0.2 A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P = U$. By doubling any Jordan triple system T one obtains the *double Jordan pair* $V(T) = (T, T)$ with products $Q_x y = P_x y$, for any $x, y \in T$. From a Jordan pair $V = (V^+, V^-)$ one can get a (*polarized*) Jordan triple system $T(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+} y^- \oplus Q_{x^-} y^+$ [8, 1.13, 1.14]

0.3 The *heart* $\mathrm{Heart}(S)$ of a Jordan system S is the intersection of all nonzero ideals of S.

1. Multiplication by elements of some degenerate ideals

The algebra analogues of the next two results are immediate consequences of the fact that the cube of an ideal is an ideal, which fails to hold for Jordan triple systems.

1.1 Lemma. *$P_J\{I,I,I^3\} \subseteq I^3$ for any ideal I of a Jordan triple system J.*

Proof: Let $x \in J$, $a,b,c,d \in I$. Then $u := \{b,a,x\}, v := P_xP_cd \in I$, and

$$\begin{aligned} P_x\{a,b,P_cd\} &= P_xL_{a,b}P_cd =_{\text{by JP12}} \left(P_{x,\{b,a,x\}} - L_{b,a}P_x\right)P_cd = P_{x,u}P_cd + L_{b,a}v \\ &= P_{u,x}P_cd + \{b,a,v\} =_{\text{by JP9}} (L_{u,c}L_{x,c} - L_{u,P_cx})\,d + \{b,a,v\} \\ &= \{u,c,\{x,c,d\}\} - \{u,P_cx,d\} + \{b,a,v\} \in \{I,I,I\} \subseteq I^3, \end{aligned}$$

which proves our assertion since I^3 is spanned by all P_cd with $c,d \in I$. ∎

1.2 Lemma. *Any minimal ideal I of a Jordan triple system J is either trivial or idempotent (satisfies $I^3 = I$).*

Proof: Let us suppose that, on the contrary, $I^3 \neq 0, I$. Recall that if H,K are semiideals of J, then so is P_HK [9, 6.2(a)], with ideal hull $P_HK + P_JP_HK$ [9, 6.2(b)]. Thus I^3 is a nonzero semiideal with nonzero hull $I^3 + P_JI^3 \subseteq I$, so that minimality of I yields

$$I = I^3 + K, \text{ where } K := P_JI^3 \text{ is a semiideal such that } K \subseteq I, K \not\subseteq I^3. \tag{1}$$

If the semiideal P_II^3 is nonzero, then $I = P_II^3 + P_JP_II^3$ as above, so that semiidealness of I^3 would lead to $I \subseteq P_II + P_JP_JI^3 \subseteq I^3 \subseteq I$, contrary to our assumption. Therefore $P_II^3 = 0$, hence $I^3 = P_II = P_I(I^3+K) = 0 + P_IK$ using (1). On the other hand, $P_IP_JK = P_IP_JP_JI^3 \subseteq P_II^3$ again by semiidealness of I^3. Thus

$$P_II^3 = P_IP_JK = 0, \quad I^3 = P_IK, \tag{2}$$

so that $K = P_JI^3 =_{\text{by (2)}} P_JP_IK =_{\text{by (1)}} P_JP_{I^3+K}K$. But

— $P_{I^3,K}K = \{I^3,K,K\} \subseteq_{\text{by (1)}} \{I^3,I,I\}$,

— $P_{I^3}K \subseteq_{\text{by JP3}} P_IP_IP_IK + P_{I^3,I^3}K \subseteq_{\text{by (2)}} 0 + \{I^3,K,I^3\} \subseteq_{\text{by (1)}} \{I^3,I,I\}$,

— $P_KK = P_{P_JI^3}K \subseteq_{\text{by JP3}} P_JP_{I^3}P_JK + P_{K,K}K \subseteq P_JP_IP_JK + \{K,K,K\} \subseteq_{\text{by (2)}} 0 + \{K,K,K\} \subseteq \{J,J,K\} = L_{J,J}K \subseteq K$ by semiidealness of K, and thus $K = P_J P_{I^3+K} K \subseteq P_J(\{I^3, I, I\} + K) = P_J\{I^3, I, I\} + P_J P_J I^3 \subseteq I^3$ by (1.1) and semiidealness of I^3, which contradicts (1). ■

We study now how the elements of certain ideals interact by outer multiplication with the whole multiplication algebra of the system. Given an ideal I of a Jordan triple system J, $\mathcal{M}(I)$ denotes the unital subalgebra of $\mathrm{End}(J)$ generated by $L_{a,b}, P_c$ with $a, b, c \in I$, whereas $L_{I,I}$, $L_{I,J}$, $L_{J,I}$, $L_{J,J}$, P_J, P_I, $P_{I,I}$, $P_{J,I}$ and their products are the submodules of $\mathrm{End}(J)$ spanned by the obvious elements. We will also use $T_I := P_I + L_{I,I}$ and $T_J := P_J + L_{J,J}$.

1.3 LEMMA. *For any ideal I of a Jordan triple system J,*

(i) $L_{I,I}L_{J,J} \subseteq L_{J,J}L_{I,I} + L_{I,I} \subseteq L_{J,J}\mathcal{M}(I) + \mathcal{M}(I)$,

(ii) $P_I L_{J,J} \subseteq L_{J,J}P_I + P_I \subseteq L_{J,J}\mathcal{M}(I) + \mathcal{M}(I)$,

(iii) $L_{I,I}P_J \subseteq P_J L_{I,I} + P_J \subseteq P_J\mathcal{M}(I)$,

(iv) $L_{I^3,J} + L_{J,I^3} \subseteq L_{I,I} \subseteq \mathcal{M}(I)$,

(v) $P_{I^3}P_J \subseteq P_I P_I + L_{I,I} + L_{I,I}L_{I,I} \subseteq \mathcal{M}(I)$,

(vi) $P_{I^3}L_{J,J} \subseteq P_J\mathcal{M}(I)$,

(vii) $P_{I^3}P_{I^3}L_{J,J} \subseteq \mathcal{M}(I)$,

(viii) $\mathcal{M}(I^3)T_J \subseteq T_J\mathcal{M}(I) + \mathcal{M}(I)$,

(ix) *If $I^3 = I$, then $\mathcal{M}(I)T_J \subseteq T_J\mathcal{M}(I) + \mathcal{M}(I)$, and $P_I P_I T_J \subseteq \mathcal{M}(I)$.*

PROOF: (i) $[L_{J,J}, L_{I,I}] \subseteq_{\text{by JP15}} L_{\{J,J,I\},I} + L_{I,\{J,J,I\}} \subseteq L_{I,I}$.

(ii) $L_{J,J}P_I + P_I L_{J,J} \subseteq_{\text{by JP12}} P_{I,\{J,J,I\}} \subseteq P_{I,I} \subseteq P_I$.

(iii) $L_{I,I}P_J + P_J L_{I,I} \subseteq_{\text{by JP12}} P_{J,\{I,I,J\}} \subseteq P_J$.

(iv) This is just a triple system version of [13, 1.11]: for $a, b \in I$,

$$L_{P_a b,J} \subseteq_{\text{by JP8}} L_{a,\{b,a,J\}} + L_{P_a J,b} \in L_{I,I}, \quad L_{J,P_a b} \subseteq_{\text{by JP7}} L_{\{J,a,b\},a} + L_{b,P_a J} \in L_{I,I}.$$

(v) First note that $P_{I^3}P_J$ is spanned by elements of the form $P_{P_a b}P_x, P_{c,d}P_x$ for $a, b \in I, x \in J, c, d \in I^3$. On the one hand,

$$\begin{aligned}
P_{P_a b}P_x &=_{\text{by JP3}} P_a P_b P_a P_x =_{\text{by JP21}} P_a\left[P_{\{x,a,b\}} + P_{P_x P_a b,b} - P_x P_a P_b - L_{x,a}P_b L_{a,x}\right] \\
&=_{\text{by JP12}} P_a\left[P_{\{x,a,b\}} + P_{P_x P_a b,b} - P_x P_a P_b - L_{x,a}\left(P_{b,\{x,a,b\}} - L_{x,a}P_b\right)\right] \\
&=_{\text{by JP13}} P_a\left[P_{\{x,a,b\}} + P_{P_x P_a b,b} - P_x P_a P_b - L_{x,a}P_{b,\{x,a,b\}}\right. \\
&\quad \left. + \left(L_{P_x a,a} + P_x P_{a,a}\right)P_b\right] \\
&= P_a\left[P_{\{x,a,b\}} + P_{P_x P_a b,b} + P_x P_a P_b - L_{x,a}P_{b,\{x,a,b\}} + L_{P_x a,a}P_b\right] \\
&=_{\text{by JP3 and JP4}} P_a P_{\{x,a,b\}} + P_a P_{P_x P_a b,b} + P_{P_a x}P_b - P_{a,P_a x}P_{b,\{x,a,b\}} \\
&\quad + P_{a,P_a P_x a}P_b \in P_I P_I,
\end{aligned}$$

and, on the other hand,

$$P_{c,d}P_x =_{\text{by JP9}} L_{c,x}L_{d,x} - L_{c,P_x d} \in L_{I^3,J}L_{I^3,J} + L_{I,I} \subseteq_{\text{by (iv)}} L_{I,I}L_{I,I} + L_{I,I}.$$

(vi) Linearizing JP12 one gets

$$L_{a,x}P_{b,y} = P_{b,\{a,x,y\}} + P_{y,\{a,x,b\}} - P_{b,y}L_{x,a}. \tag{1}$$

If $c \in I^3, x, y \in J$, then

$$\begin{aligned} P_c L_{x,y} &=_{\text{by JP11}} L_{c,x}P_{c,y} - P_{P_c x,y} \\ &=_{\text{by (1)}} P_{c,\{c,x,y\}} + P_{y,\{c,x,c\}} - P_{c,y}L_{x,c} - P_{P_c x,y} \\ &\subseteq_{\text{by (iv)}} P_J + P_J L_{I,I}. \end{aligned}$$

(vii) $P_{I^3}P_{I^3}L_{J,J} \subseteq_{\text{by (vi)}} P_{I^3}P_J\mathcal{M}(I) \subseteq_{\text{by (v)}} \mathcal{M}(I)$.

(viii) It is enough to notice that $L_{J,J}\mathcal{M}(I) + \mathcal{M}(I)$ is invariant under left multiplication by the generators of $\mathcal{M}(I)$ by (i) and (ii), whereas $P_J\mathcal{M}(I) + \mathcal{M}(I)$ is invariant under left multiplication by the generators of $\mathcal{M}(I^3)$ by (iii) and (v).

Finally, (ix) follows from (viii), (v), and (vii). ■

2. Minimal ideals of Jordan Systems

2.1 Lemma. *Any minimal ideal of a Jordan system J (algebra, triple system, or pair) is isomorphic to the heart of a homomorphic image of J.*

Proof: Given a minimal ideal I of the Jordan system J, let us consider

$$\mathcal{L} := \{ \text{ ideals } L \text{ of } J \text{ such that } I \cap L = 0 \ \},$$

which is easily seen to be a nonempty inductive set, and let L be a maximal element of $\mathcal{L}$. Then, $I \cong I/I \cap L \cong (I + L)/L$, which is a nonzero ideal $\tilde{I}$ of $\tilde{J} := J/L$, and we only have to prove that $\tilde{I}$ is the heart of $\tilde{J}$, i.e., that it is contained in all nonzero ideals of $\tilde{J}$.

Any nonzero ideal of $\tilde{J}$ has the form $\widetilde{M} = M/L$, where M is an ideal of J strictly containing L. By maximality of L we get $M \cap I \neq 0$, and hence, $M \cap I = I$ by minimality of I. Thus, M contains I and L, so $\tilde{I} = (I + L)/L \subseteq M/L = \widetilde{M}$. ■

2.2 Theorem. *The heart of a Jordan triple system J is either simple or trivial.*

Proof: We may assume that $I := \mathrm{Heart}(J)$ is nonzero. If J is nondegenerate, we have the simplicity of I by [4, 3.6], so we only have to deal with a degenerate J. In this case, the McCrimmon radical $\mathcal{M}c(J)$ ([8, §4]) is a nonzero ideal of J, so that $I \subseteq \mathcal{M}c(J)$ and then $I = \mathcal{M}c(I)$ by [8, 4.13], which is locally nilpotent by [3, 2.4(i); 7]. We will show that $I^3 = 0$.

Otherwise, since the nonzero heart of a Jordan system is a minimal ideal, $I^3 = I$ by (1.2).

We claim that I is spanned by absolute zero divisors of J. Indeed, $I = I^3$ gives that I is spanned by elements of the form $P_a b$ with $a, b \in I$. But [8, 4.6, 4.7] implies that the span of all absolute zero divisors of J is a nonzero ideal of J then containing I. If we write any $b \in I$ as a finite sum of absolute zero divisors of J, $b = z_1 + \cdots + z_n$, we get $P_a b = P_a z_1 + \cdots + P_a z_n = x_1 + \cdots + x_n$, where $x_i := P_a z_i \in I$ and it is an absolute zero divisor of J by JP3, for all $i = 1, \ldots, n$.

Let $0 \neq x \in I$ be an absolute zero divisor of J (for example, one the x_i's above). Obviously, Φx is an inner ideal of J, so that the ideal of J generated by x is its outer hull $0 \neq \mathrm{Id}_J(x) = \mathrm{Out}_J(x) = \sum_{n=0}^{\infty} \overbrace{T_J \cdots T_J}^{n} x$, where $\overbrace{T_J \cdots T_J}^{0} x$ means Φx ([13, 1.9] can be easily extended to triple systems by replacing the operators U_x by operators P_x and transformations $B_{x,y}$ [8, 2.11]). Here, $x \in I$ implies $\mathrm{Id}_J(x) \subseteq I$, and since a nonzero ideal of J contains its heart I, we get $I = \mathrm{Id}_J(x) = \sum_{n=0}^{\infty} \overbrace{T_J \cdots T_J}^{n} x$.

On the other hand, since $x \in I = I^3 = P_I I = P_I P_I I$, it must be $x = \sum_{\text{finite}} P_{a_i} P_{b_i} c_i$ for some $a_i, b_i, c_i \in I$. Let F be the finite set of elements of J appearing in the expression of all the a_i's, b_i's, and c_i's as elements in $\sum_{n=0}^{\infty} \overbrace{T_J \cdots T_J}^{n} x$.

Let us consider the subsystem J_0 of J generated by $F \cup \{x\}$, and its ideal $I_0 := \mathrm{Id}_{J_0}(x) = \mathrm{Out}_{J_0}(x)$, so that $a_i, b_i, c_i \in I_0$ for all i, and then $x \in P_{I_0} P_{I_0} I_0$.

Let us prove $I_0 = I_0^3$. Indeed, we have

$$I_0 = \mathrm{Id}_{J_0}(x) \subseteq \mathrm{Id}_{J_0}(P_{I_0} P_{I_0} I_0) = P_{I_0} P_{I_0} I_0 + P_{J_0} P_{I_0} P_{I_0} I_0 \subseteq P_{I_0} I_0 = I_0^3 \subseteq I_0,$$

using semi-idealness of $P_{I_0} P_{I_0} I_0$ and $P_{I_0} I_0$ [9, 6.2] , and idealness of I_0.

On the other hand, it is clear that $I_0 = \mathrm{Out}_{J_0}(x) = \mathcal{M}(J_0)x$, where the algebra $\mathcal{M}(J_0)$ has finite length k by [2, 4.2]. So,

$$I_0 = I_0^3 = P_{I_0} I_0 = \cdots = \overbrace{P_{I_0} \cdots P_{I_0}}^{2k+1} I_0 = \overbrace{P_{I_0} \cdots P_{I_0}}^{2k+1} \sum_{i=0}^{k} \overbrace{T_{J_0} \cdots T_{J_0}}^{i} x. \tag{1}$$

If $M_i := \overbrace{T_{J_0} \cdots T_{J_0}}^{i}$, we will prove

$$\overbrace{P_{I_0} \cdots P_{I_0}}^{2i} M_i \subseteq \mathcal{M}(I_0) \tag{2}$$

by induction on i. The containment is obvious for $i = 0$, and if we assume it for indexes less than i,

$$\begin{aligned}
\overbrace{P_{I_0} \cdots P_{I_0}}^{2i} M_i &\subseteq \overbrace{P_{I_0} \cdots P_{I_0}}^{2i} M_{i-1} T_{J_0} \subseteq P_{I_0} P_{I_0} \overbrace{P_{I_0} \cdots P_{I_0}}^{2(i-1)} M_{i-1} T_{J_0} \\
&\subseteq P_{I_0} P_{I_0} \mathcal{M}(I_0) T_{J_0} \\
&\subseteq_{\text{by (1.3)(ix)}} P_{I_0} P_{I_0} (T_{J_0} \mathcal{M}(I_0) + \mathcal{M}(I_0)) \subseteq_{\text{by (1.3)(ix)}} \mathcal{M}(I_0).
\end{aligned}$$

Putting together (1) and (2), and noticing $I_0 \subseteq I$, one gets $x \in I_0 \subseteq P_{I_0} \mathcal{M}(I_0) x \subseteq P_I \mathcal{M}(I) x$, which contradicts the local nilpotency of I. ■

The following result is an immediate consequence of (2.1) and (2.2).

2.3 Corollary. *A minimal ideal of a Jordan triple system is either simple or trivial.* ■

Our next goal is extending the above result to Jordan algebras and Jordan pairs. The algebra case closes the problem posed in [13, p. 582] and it is almost a direct consequence of the relation between algebras and triple systems, whereas the pair situation requires a little more effort, so we study the two cases separately.

2.4 Corollary. *A minimal ideal of a Jordan algebra is either simple or trivial.*

Proof: Given a Jordan algebra J and one of its minimal ideals I, we consider a unital hull $\hat{J}$, where I is a minimal ideal too. This has the advantage that $\hat{J}$ has the same set of ideals when considered as a Jordan triple system, so that I is a minimal ideal of the Jordan triple system $\hat{J}$, and then, either $I^3 = 0$ or I is simple as Jordan triple system by (2.3). In the latter case I is obviously simple as Jordan algebra. ■

2.5 Corollary. *A minimal ideal of a Jordan pair is either simple or trivial.*

Proof: To apply the known relations between the ideals of a Jordan pair V and the triple system $T(V)$, one needs to assume that V is semiprime (cf. [1, Sect. 5]). Hence, to get the result as a corollary of the proved property for Jordan triple systems, we have to get rid of the non semiprime case. By (2.1) it is enough to prove that the heart I of a Jordan pair V is either simple or trivial.

If V is not semiprime, it contains some nonzero trivial ideal L, which forces I to be trivial since $I \subseteq L$.

But if V is semiprime, then $T(I) = T(\mathrm{Heart}(V)) = \mathrm{Heart}(T(V))$ by [4, 3.7(ii)], so that $T(I)$ is either simple or trivial by (2.2) (indeed, $T(I)$ is necessarily simple by semiprimeness of $T(V)$ [1, Sect. 5]), which implies the same condition on I. ■

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