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The Magic Square and Symmetric Compositions

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Abstract: The construction of the exceptional classical simple Lie algebras by Barton and Sudbery will be interpreted and extended by using a pair of symmetric composition algebras, instead of the classical unital composition algebras.

1. Introduction.

The exceptional simple Lie algebras were constructed in a unified way by Tits [T] (see also [S,F]), who used composition algebras and simple Jordan algebras of degree 3 to build a Magic Square containing all these simple Lie algebras. A more symmetric approach was taken by Vinberg [V, OV]. Recently, these approaches have been interpreted by Barton and Sudbery [BS1,2] as a construction depending on two composition algebras and closely related to the triality principle. A similar construction has been given by Landsberg and Manivel [LM1,2], inspired by previous work of Allison and Faulkner [AF].

In most known constructions of the exceptional simple Lie algebras from simpler constituents, the difficult task is to check that the construction gives indeed a Lie algebra. The aim of this paper is to reinterpret, and extend, the beautiful approach by Barton and Sudbery from a different perspective. A Lie algebra will be defined from scratch out of two symmetric composition algebras. These are nonunital (unless the dimension is 1) composition algebras which provide very simple formulas for triality (see [KMRT, Ch. VIII]). The simplicity of these formulas leads to a very easy proof that the new algebra constructed here is indeed a Lie algebra.

The next section will review the definition and basic properties needed about symmetric composition algebras over fields of characteristic $\neq 2$. Then, in Section 3, given two such algebras S and S', a Lie algebra $\mathfrak{g} = \mathfrak{g}(S, S')$ will be constructed. With a few exceptions in characteristic three, $\mathfrak{g}$ will be either simple or semisimple and the Magic Square of Lie algebras will be obtained by varying S and S'. The symmetry of the construction shows a natural automorphism of order 3 of $\mathfrak{g}$. The different possibilities for symmetric composition algebras of dimension 8 give different

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nonconjugate such automorphisms for the exceptional Lie algebras. All this will be considered in Section 4. In the final section, the approach here will be related to the ones by Barton and Sudbery, which has been the inspiration for this work, by Allison and Faulkner and by Landsberg and Manivel. It will turn out that the constructions given by these authors give, up to isomorphism, the Lie algebras $\mathfrak{g} = \mathfrak{g}(S, S')$, with S and S' para-Hurwitz algebras. Any four dimensional (and essentially any two dimensional) symmetric composition algebra is a para-Hurwitz algebra, but this is no longer true for eight-dimensional symmetric composition algebras, due to the existence of the so called Okubo algebras.

2. Symmetric composition algebras.

A *symmetric composition algebra* is a triple $(S, *, q)$, where $(S, *)$ is a (nonassociative) algebra over a field F with multiplication denoted by $x * y$ for $x, y \in S$, and where $q : S \to F$ is a regular quadratic form satisfying for any $x, y, z \in S$:

$$q(x * y) = q(x)q(y), \tag{2.1}$$
$$q(x * y, z) = q(x, y * z), \tag{2.2}$$

where $q(x, y) = q(x + y) - q(x) - q(y)$ is the polar of q. Throughout the paper, the notations and conventions in [KMRT, Ch. VIII] will be followed. If no confusion arises, we will speak of the symmetric composition algebra S, with $*$ and q tacitly assumed.

In the presence of (2.1), (2.2) is equivalent to

$$(x * y) * x = x * (y * x) = q(x)y \tag{2.3}$$

for any $x, y \in S$ ([KMRT, (34.1)]).

The classification of the symmetric composition algebras was obtained in [EM] (for characteristic $\neq 2, 3$, although the results can be slightly changed to cover characteristic 2, see also [KMRT, Ch. VIII]) and in [E1] (for characteristic 3).

Given any unital composition algebra (or Hurwitz algebra, or Cayley-Dickson algebra) C with norm q and standard involution $x \mapsto \bar{x}$, the new algebra defined on C but with multiplication

$$x * y = \bar{x}\bar{y},$$

is a symmetric composition algebra, called the associated *para-Hurwitz algebra*. In dimensions 1, 2 or 4, any symmetric composition algebra is a para-Hurwitz algebra, with a few exceptions in dimension 2 which are, nevertheless, forms of para-Hurwitz algebras; while in dimension 8, apart from the para-Hurwitz algebras, there is a new family of symmetric composition algebras termed *Okubo algebras*.

If $(S, *, q)$ is any symmetric composition algebra, consider the corresponding orthogonal Lie algebra $\mathfrak{o}(S,q) = \{d \in \operatorname{End}_F(S) : q\big(d(x), y\big) + q\big(x, d(y)\big) = 0\ \forall x, y \in S\}$, and the subalgebra of $\mathfrak{o}(S,q)^3$ defined by

$$\begin{aligned}\mathfrak{tri}(S,*,q) &= \{(d_0,d_1,d_2) \in \mathfrak{o}(S,q)^3 : d_0(x*y) = d_1(x)*y + x*d_2(y)\ \forall x,y \in S\}\\ &= \{(d_0,d_1,d_2) \in \mathfrak{o}(S,q)^3 : \langle d_0(x),y,z\rangle + \langle x,d_1(y),z\rangle + \langle x,y,d_2(z)\rangle\ \forall x,y,z \in S\},\end{aligned}$$

where $\langle x,y,z\rangle = q(x, y*z)$. By (2.2), $\langle x,y,z\rangle = \langle z,x,y\rangle = \langle y,z,x\rangle$. Hence the map

$$\begin{aligned}\theta : \mathfrak{tri}(S,*,q) &\to \mathfrak{tri}(S,*,q)\\ (d_0,d_1,d_2) &\mapsto (d_2,d_0,d_1)\end{aligned} \tag{2.4}$$

is an automorphism of $(S,*,q)$ of order 3. Its fixed subalgebra is (isomorphic to) the derivation algebra of $(S,*)$ which, if the dimension is 8 and the characteristic of the ground field is $\neq 2, 3$, is a simple Lie algebra of type G_2 in the para-Hurwitz case and a simple Lie algebra of type A_2 (a form of $\mathfrak{sl}_3$) in the Okubo case.

Assume from now on that the characteristic of the ground field is $\neq 2$.

A straightforward computation (see [EO] for a more general setting) using (2.3) shows that for any $x, y \in S$, the triple

$$t_{x,y} = \left(\sigma_{x,y}, \frac{1}{2}q(x,y)I - r_x l_y, \frac{1}{2}q(x,y)I - l_x r_y\right) \tag{2.5}$$

is in $\mathfrak{tri}(S,*,q)$, where $\sigma_{x,y}(z) = q(x,z)y - q(y,z)x$, $r_x(z) = z*x$, and $l_x(z) = x*z$ for any $x,y,z \in S$.

Lemma 2.1. *Let $(S,*,q)$ be an eight-dimensional symmetric composition algebra. Then:*

(i) *(Principle of Local Triality) The projection $\pi_0 : \mathfrak{tri}(S,*,q) \to \mathfrak{o}(S,q)$, $(d_0,d_1,d_2) \mapsto d_0$ is an isomorphism.*

(ii) $\mathfrak{tri}(S,*,q) = t_{S,S}$ $(:= \operatorname{span}\langle t_{x,y} : x,y \in S\rangle)$.

(iii) *For any $a,b,x,y \in S$, $[t_{a,b}, t_{x,y}] = t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}$.*

Proof: (i) is proved in [KMRT, (45.5)] (see also [E3]); (ii) follows immediately from it; and (iii) follows from (i) too, since the zero component of $[t_{a,b}, t_{x,y}]$ and of $t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}$ coincide. □

Corollary 2.2. *Let $(S,*,q)$ be a symmetric composition algebra. Then for any $a,b,x,y \in S$:*

$$[t_{a,b}, t_{x,y}] = t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}. \tag{2.6}$$

Proof: $(S,*,q)$ can be embedded in an eight-dimensional symmetric composition algebra and then Lemma 2.1 applies. Alternatively, one can use (2.1-3) to check (2.6) directly. □

Lemma 2.3. *Let $(S,*,q)$ be a symmetric composition algebra, then for any $a,b,x,y\in S$:*

$$\begin{aligned}(a*x)*(y*b)&+(a*y)*(x*b)\\&=q(b*a,x)y+q(b*a,y)x-q(x,y)b*a.\end{aligned}\tag{2.7}$$

Proof: Use (2.3) twice to get

$$\begin{aligned}(a*x)*(x*b)&=q(a*x,b)x-b*(x*(a*x))\\&=q(b*a,x)x-q(x)b*a,\end{aligned}$$

and linearizing this, one gets (2.7). □

3. Construction of Lie algebras from pairs of symmetric composition algebras.

Let $(S,*,q)$ and $(S',*,q')$ be two symmetric composition algebras and define $\mathfrak{g}=\mathfrak{g}(S,S')$ to be the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded anticommutative algebra such that

$$\begin{aligned}\mathfrak{g}_{(\bar 0,\bar 0)}&=\mathfrak{tri}(S,*,q)\oplus\mathfrak{tri}(S',*,q'),\\ \mathfrak{g}_{(\bar 1,\bar 0)}&=\mathfrak{g}_{(\bar 0,\bar 1)}=\mathfrak{g}_{(\bar 1,\bar 1)}=S\otimes S'.\end{aligned}$$

(Unadorned tensor products are considered over the ground field F.) For any $a\in S$ and $x\in S'$, denote by $\iota_i(a\otimes x)$ the element $a\otimes x$ in $\mathfrak{g}_{(\bar 1,\bar 0)}$ (respectively $\mathfrak{g}_{(\bar 0,\bar 1)}$, $\mathfrak{g}_{(\bar 1,\bar 1)}$) if $i=0$ (respectively, $i=1,2$). The anticommutative multiplication of $\mathfrak{g}$ is defined by means of:

- $\mathfrak{g}_{(\bar 0,\bar 0)}$ is a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0,d_1,d_2),\iota_i(a\otimes x)]=\iota_i\big(d_i(a)\otimes x\big)$, $[(d_0',d_1',d_2'),\iota_i(a\otimes x)]=\iota_i\big(a\otimes d_i'(x)\big)$, for any $(d_0,d_1,d_2)\in\mathfrak{tri}(S,*,q)$, $(d_0',d_1',d_2')\in\mathfrak{tri}(S',*,q')$, $a\in S$ and $x\in S'$.
- $[\iota_i(a\otimes x),\iota_{i+1}(b\otimes y)]=\iota_{i+2}\big((a*b)\otimes(x*y)\big)$ (indices modulo 3), for any $a,b\in S$, $x,y\in S'$.
- $[\iota_i(a\otimes x),\iota_i(b\otimes y)]=q'(x,y)\theta^i(t_{a,b})+q(a,b)\theta'^i(t'_{x,y})$, for any $i=0,1,2$, $a,b\in S$ and $x,y\in S'$, where $t_{a,b}\in\mathfrak{tri}(S,*,q)$ (respectively $t'_{x,y}\in\mathfrak{tri}(S',*,q')$) is the element in (2.5) for $a,b\in S$ (resp. $x,y\in S'$) and θ (resp. θ') is the automorphism of $\mathfrak{tri}(S,*,q)$ (resp. $\mathfrak{tri}(S',*,q')$) given in (2.4).

Theorem 3.1. *With this multiplication, $\mathfrak{g}=\mathfrak{g}(S,S')$ is a Lie algebra.*

Proof: The linear maps $\mathfrak{o}(S,q)\otimes S\to S$: $d\otimes a\mapsto d(a)$; $S\otimes S\to\mathfrak{o}(S,q)$: $a\otimes b\mapsto\sigma_{a,b}$; and $S\otimes S\to F$: $a\otimes b\mapsto q(a,b)$ are $\mathfrak{o}(S,q)$-invariant, and similarly for S'. This, together with the cyclic symmetry of the definition of $\mathfrak{tri}(S,*,q)$, $\mathfrak{tri}(S',*,q')$ and of

$\mathfrak{g}$, and with (2.5), shows that the only instances of the Jacobi identity that have to be checked are:

i) $J(\iota_0(a_0\otimes x_0),\iota_1(a_1\otimes x_1),\iota_2(a_2\otimes x_2))=0$,

ii) $J(\iota_0(a_0\otimes x_0),\iota_0(a_1\otimes x_1),\iota_0(a_2\otimes x_2))=0$, and

iii) $J(\iota_0(a_1\otimes x_1),\iota_0(a_2\otimes x_2),\iota_1(b\otimes y))=0$,

for any $a_0,a_1,a_2,b\in S$ and $x_0,x_1,x_2,y\in S'$.

For i), since $q(x_0*x_1,x_2)$ is invariant under cyclic permutations, the component in $\mathfrak{tri}(S,*,q)$ of $J(\iota_0(a_0\otimes x_0),\iota_1(a_1\otimes x_1),\iota_2(a_2\otimes x_2))$ is $q(x_0*x_1,x_2)$ times

$$\theta^2\left(t_{a_0*a_1,a_2}\right)+t_{a_1*a_2,a_0}+\theta\left(t_{a_2*a_0,a_1}\right).$$

The three components of this element of $\mathfrak{tri}(S,*,q)$ present the form

$$\sigma_{b_0*b_1,b_2}+\left(\frac{1}{2}q(b_1*b_2,b_0)I-l_{b_1*b_2}r_{b_0}\right)+\left(\frac{1}{2}q(b_2*b_0,b_1)I-r_{b_2*b_0}l_{b_1}\right),\qquad(3.1)$$

where (b_0,b_1,b_2) is a cyclic permutation of (a_0,a_1,a_2). But (2.7) gives

$$\begin{aligned}\Bigl(l_{b_1*b_2}r_{b_0}&+r_{b_2*b_0}l_{b_1}\Bigr)(c)\\&=q(b_0*b_1,b_2)c+q(b_0*b_1,c)b_2-q(c,b_2)b_0*b_1\\&=\sigma_{b_0*b_1,b_2}(c)-q(b_0*b_1,b_2)c,\end{aligned}$$

for any $b_0,b_1,b_2,c\in S$ and this shows that (3.1) is identically zero, thus proving i).

For ii), just note that

$$\begin{aligned}&[[\iota_0(a_0\otimes x_0),\iota_0(a_1\otimes x_1)],\iota_0(a_2\otimes x_2)]\\&\qquad=\sigma_{a_0,a_1}(a_2)\otimes q'(x_0,x_1)x_2+q(a_0,a_1)a_2\otimes\sigma'_{x_0,x_1}(x_2),\end{aligned}$$

and the cyclic sum of this latter expression is trivial.

Finally, using (2.3)

$$\begin{aligned}&[[\iota_0(a_1\otimes x_1),\iota_0(a_2\otimes x_2)],\iota_1(b\otimes y)]\\&\quad=\iota_1\biggl(\Bigl(\frac{1}{2}q(a_1,a_2)b-(a_2*b)*a_1\Bigr)\otimes q'(x_1,x_2)y\\&\qquad\quad+q(a_1,a_2)b\otimes\Bigl(\frac{1}{2}q'(x_1,x_2)y-(x_2*y)*x_1\Bigr)\biggr)\\&\quad=\iota_1\Bigl(q(a_1,a_2)b\otimes q'(x_1,x_2)y-\bigl((a_2*b)*a_1\bigr)\otimes q'(x_1,x_2)y\\&\qquad\quad-q(a_1,a_2)b\otimes\bigl((x_2*y)*x_1\bigr)\Bigr)\\&\quad=\iota_1\Bigl(\bigl((a_1*b)*a_2\bigr)\otimes q'(x_1,x_2)y-q(a_1,a_2)b\otimes\bigl((x_2*y)*x_1\bigr)\Bigr),\end{aligned}$$

while

$$\begin{aligned}
&[[\iota_0(a_1\otimes x_1),\iota_1(b\otimes y)],\iota_0(a_2\otimes x_2)]+[\iota_0(a_1\otimes x_1),[\iota_0(a_2\otimes x_2),\iota_1(b\otimes y)]]\\
&\quad=[\iota_2\bigl((a_1*b)\otimes(x_1*y)\bigr),\iota_0(a_2\otimes x_2)]+[\iota_0(a_1\otimes x_1),\iota_2\bigl((a_2*b)\otimes(x_2*y)\bigr)]\\
&\quad=\iota_1\Bigl(((a_1*b)*a_2)\otimes((x_1*y)*x_2)-((a_2*b)*a_1)\otimes((x_2*y)*x_1)\Bigr)\\
&\quad=\iota_1\Bigl(((a_1*b)*a_2)\otimes\bigl(q'(x_1,x_2)y-(x_2*y)*x_1\bigr)\\
&\qquad\qquad\qquad-((a_2*b)*a_1)\otimes((x_2*y)*x_1)\Bigr)\\
&\quad=\iota_1\Bigl(((a_1*b)*a_2)\otimes q'(x_1,x_2)y-q(a_1,a_2)b\otimes((x_2*y)*x_1)\Bigr).\quad\square
\end{aligned}$$

Remark 3.2. The definition of $\mathfrak{g}=\mathfrak{g}(S,S')$ can be generalized by taking three nonzero scalars $0\neq\alpha_i\in F$ $(i=0,1,2)$ and modifying the multiplication of elements in $\mathfrak{g}_{(\bar 1,\bar 0)}\oplus\mathfrak{g}_{(\bar 0,\bar 1)}\oplus\mathfrak{g}_{(\bar 1,\bar 1)}$ as follows:

$$\begin{aligned}
[\iota_i(a\otimes x),\iota_{i+1}(b\otimes y)]&=\alpha_{i+2}\iota_{i+2}\bigl((a*b)\otimes(x*y)\bigr),\\
[\iota_i(a\otimes x),\iota_i(b\otimes y)]&=\alpha_{i+1}\alpha_{i+2}\bigl(q'(x,y)\theta^i(t_{a,b})+q(a,b)\theta'^i(t'_{x,y})\bigr),
\end{aligned}$$

(indices modulo 3) for any $a,b\in S$, $x,y\in S'$. Denote the resulting Lie algebra by $\mathfrak{g}_{\underline{\alpha}}(S,S')$, with $\underline{\alpha}=(\alpha_0,\alpha_1,\alpha_2)$. If F is quadratically closed, then the new algebra thus obtained is isomorphic to the original one $\mathfrak{g}(S,S')$. More specifically, by scaling the elements $\iota_i(a\otimes x)$ by nonzero scalars $\mu_i\in F$ $(i=0,1,2)$ it is checked that $\mathfrak{g}_{\underline{\alpha}}(S,S')$ is isomorphic to $\mathfrak{g}_{\underline{\nu\alpha}}(S,S')$ where $\underline{\nu\alpha}=(\nu_0\alpha_0,\nu_1\alpha_1,\nu_2\alpha_2)$ and $\nu_i=\mu_i^{-1}\mu_{i+1}\mu_{i+2}$, indices modulo 3 (which implies that $\mu_i^2=\nu_{i+1}\nu_{i+2}$ $\forall i$ and $\mu_0\mu_1\mu_2=\nu_0\nu_1\nu_2$).

In order to study some properties of the Lie algebras $\mathfrak{g}(S,S')$, the next result is useful.

Lemma 3.3. *Let $(S,*,q)$ be a symmetric composition algebra. Then $\mathfrak{tri}(S,*,q)=\ker\pi_0\oplus t_{S,S}$ (direct sum of ideals) and $t_{S,S}$ is isomorphic to $\mathfrak{o}(S,q)$ (by means of π_0).*

Proof: It is clear that $\pi_0:\mathfrak{tri}(S,*,q)\to\mathfrak{o}(S,q)$ takes the subalgebra $t_{S,S}$ of $\mathfrak{tri}(S,*,q)$ isomorphically onto $\mathfrak{o}(S,q)=\sigma_{S,S}$. Moreover, inside $\mathfrak{g}=\mathfrak{g}(S,F)$,

$$[\ker\pi_0,t_{S,S}]=[\ker\pi_0,[\mathfrak{g}_{(\bar 1,\bar 0)},\mathfrak{g}_{(\bar 1,\bar 0)}]]\subseteq[[\ker\pi_0,\mathfrak{g}_{(\bar 1,\bar 0)}],\mathfrak{g}_{(\bar 1,\bar 0)}]=0.\quad\square$$

Recall ([EP, Lemma 3.3]) that any four-dimensional symmetric composition algebra $(S,*,q)$ is a para-Hurwitz algebra. If e is the unit of the corresponding Hurwitz algebra (it is called the para-unit of $(S,*,q)$), then $S=Fe\oplus(Fe)^\perp=Fe\oplus[S,S]$, where $[S,S]=\operatorname{span}\langle a*b-b*a:a,b\in S\rangle$ is a three dimensional simple Lie algebra under the commutator in S (or in the associative Hurwitz algebra –an algebra of quaternions–). Actually, any three dimensional simple Lie algebra appears in this way.

The Principle of Local Triality (Lemma 2.1) asserts that $\ker\pi_0=0$ for eight dimensional symmetric composition algebras. However:

Corollary 3.4. *Let $(S,*,q)$ be a symmetric composition algebra.*

(i) *If* $\dim S=1$, *then* $\mathfrak{tri}(S,*,q)=0$.

(ii) *If* $\dim S=2$, *then* $\mathfrak{o}(S,q)$ *is one dimensional. If d spans* $\mathfrak{o}(S,q)$, *then* $\mathfrak{tri}(S,*,q)=\{(\alpha d,\beta d,\gamma d):\alpha,\beta,\gamma\in F,\ \alpha+\beta+\gamma=0\}$.

(iii) *If* $\dim S=4$, *then* $\mathfrak{tri}(S,*,q)=\ker\pi_0\oplus\ker\pi_1\oplus\ker\pi_2$ *and* $\ker\pi_0=\{(0,l_a\tau,-r_a\tau):a\in[S,S]\}\cong[S,S]$, *where τ is the reflection through the para unit e of S ($\tau(e)=e$, $\tau(a)=-a$ for any $a\in(Fe)^\perp$). Moreover,* $t_{S,S}=\ker\pi_1\oplus\ker\pi_2$. *(Note that* $\ker\pi_i=\theta^i(\ker\pi_0)$, $i=0,1,2$.*)*

Proof: Item (i) is clear since $\mathfrak{o}(S,q)=0$ if $\dim S=1$.

Let us consider next the four dimensional case. We have $a*b=\bar a\bar b$ for any $a,b\in S$ for a suitable product on S that makes it a quaternion (four dimensional Hurwitz) algebra. Moreover, $\tau(a)=\bar a$ for any $a\in S$. Then

$$\ker\pi_0=\{(0,d_1,d_2):d_1(x)*y+x*d_2(y)=0\,\forall x,y\in S\}.$$

With $x=e$ and then $y=e$ (the para-unit) one obtains that $d_1=l_a\tau$ and $d_2=-r_a\tau$ for some $a\in S$. Since $d_1\in\mathfrak{o}(S,q)$, it follows that $a\in[S,S]=(Fe)^\perp$. Lemma 3.3 shows here that $\dim\mathfrak{tri}(S,*,q)$ is 9 and the simple ideals $\ker\pi_i$ $(i=0,1,2)$ are different and of dimension 3. This proves (iii).

Finally, if $\dim S=2$, we may extend scalars and then assume that S is para-Hurwitz. The argument above shows then that $\ker\pi_0=\{(0,l_a\tau,-r_a\tau):a\in(Fe)^\perp\}=\{(0,l_a\tau,-l_a\tau):a\in(Fe)^\perp\}$, by commutativity. But $(Fe)^\perp=Fa$ for some a and $\mathfrak{o}(S,q)=F(l_a\tau)$. Lemma 3.3 then shows that $\mathfrak{tri}(S,*,q)$ is two dimensional and hence $\mathfrak{tri}(S,*,q)=\ker\pi_0+\ker\pi_1=\{(\alpha d,\beta d,\gamma d):\alpha,\beta,\gamma\in F,\ \alpha+\beta+\gamma=0\}$ for any d spanning $\mathfrak{o}(S,q)$. □

The $\mathbb{Z}_2\times\mathbb{Z}_2$-grading of the Lie algebra $\mathfrak{g}=\mathfrak{g}(S,S')$ constructed above induces a $\mathbb{Z}_2$ gradation given by $\mathfrak{g}_{\bar 0}=\mathfrak{g}_{(\bar 0,\bar 0)}\oplus\mathfrak{g}_{(\bar 1,\bar 0)}$ and $\mathfrak{g}_{\bar 1}=\mathfrak{g}_{(\bar 0,\bar 1)}\oplus\mathfrak{g}_{(\bar 1,\bar 1)}$. The structure of $\mathfrak{g}_{\bar 0}$ is given by:

Corollary 3.5 *Let $(S,*,q)$ and $(S',*,q')$ be two symmetric composition algebras and let $\mathfrak{g}=\mathfrak{g}(S,S')$ and $\mathfrak{g}_{\bar 0}$ as above. Then $\mathfrak{g}_{\bar 0}$ is the direct sum of the ideals* $\ker\pi_0$, $\ker\pi_0'$ *and an ideal isomorphic to the orthogonal Lie algebra* $\mathfrak{o}(S\oplus S',q\perp q')$.

Proof: From Lemma 3.3, $\mathfrak{tri}(S,*,q)=t_{S,S}\oplus\ker\pi_0\cong\mathfrak{o}(S,q)\oplus\ker\pi_0$, and $\mathfrak{tri}(S',*,q')=t_{S',S'}\oplus\ker\pi_0'\cong\mathfrak{o}(S',q')\oplus\ker\pi_0'$. Both $\ker\pi_0$ and $\ker\pi_0'$ are trivially ideals of $\mathfrak{g}_{\bar 0}$ and

$$\mathfrak{g}_{\bar 0}=\ker\pi_0\oplus\ker\pi_0'\oplus\bigl(t_{S,S}\oplus t_{S',S'}\oplus S\otimes S'\bigr)$$

(direct sum of three ideals). Now the linear map:

$$\begin{aligned} t_{S,S}\oplus t_{S',S'}\oplus S\otimes S' &\to \mathfrak{o}(S\oplus S',q\perp q')\\ t_{a,b} &\mapsto \gamma_{a,b}\\ t'_{x,y} &\mapsto \gamma_{x,y}\\ a\otimes x &\mapsto \gamma_{a,x} \end{aligned}$$

where $\gamma_{u,v} = Q(u,-)v - Q(v,-)u$ for any $u, v \in S \oplus S'$ and $Q = q \perp q'$, is an isomorphism of Lie algebras. $\square$

Note that the same result with the same proof applies to $\mathfrak{g}_{(\bar{0},\bar{0})} \oplus \mathfrak{g}_{(\bar{0},\bar{1})}$ (respectively, $\mathfrak{g}_{(\bar{0},\bar{0})} \oplus \mathfrak{g}_{(\bar{1},\bar{1})}$), with $\ker \pi_1$ and $\ker \pi_1'$ (resp. $\ker \pi_2$ and $\ker \pi_2'$) replacing $\ker \pi_0$ and $\ker \pi_0'$, because of the cyclic symmetry of $\mathfrak{g}$.

If the characteristic of the ground field is $\neq 2, 3$, then $\mathfrak{g} = \mathfrak{g}(S, S') = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and Corollary 3.5 gives the structure of $\mathfrak{g}_{\bar{0}}$. From here it is easy to check that $\mathfrak{g}$ is simple as a $\mathbb{Z}_2$-graded algebra, since any nonzero ideal I of $\mathfrak{g}_{\bar{0}}$ satisfies $[I, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$, unless $\dim S = \dim S' = 2$. This leads quickly to the conclusion that $\mathfrak{g}$ is simple with the possible exception of $\dim S = \dim S' = 2$. The type of the Lie algebra obtained is summarized in the next table (the Magic Square):

		$\dim S$			
		1	2	4	8
$\dim S'$	1	A_1	A_2	C_3	F_4
	2	A_2	$A_2 \oplus A_2$	A_5	E_6
	4	C_3	A_5	D_6	E_7
	8	F_4	E_6	E_7	E_8

Let us check a couple of instances of this table:

• Assume $\dim S' = 1$, $\dim S = 4$. Then after a scalar extension if necessary, $S' = F$ and S is the para-Hurwitz algebra associated to the split quaternion algebra $\mathrm{Mat}_2(F)$. By Corollary 3.4 $\mathfrak{tri}(S', *, q') = 0$ and $\mathfrak{tri}(S, *, q) \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, where the three copies of $\mathfrak{sl}_2$ correspond to $\ker \pi_i$ ($i = 0, 1, 2$). Take $\mathfrak{h}_i$ a Cartan subalgebra of $\ker \pi_i$ and $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Let $\varepsilon_i \in \mathfrak{h}_i^*$ (identified as a subspace of $\mathfrak{h}^*$) such that the roots of $\mathfrak{h}$ on $\mathfrak{g}_{(\bar{0},\bar{0})}$ are $\pm 2\varepsilon_i$ ($i = 0, 1, 2$). The weights of $\mathfrak{h}$ on $\iota_i(S \otimes S')$ are $\pm\varepsilon_{i+1} \pm \varepsilon_{i+2}$ (indices modulo 3). Hence $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and the set of roots is

$$\{\pm\varepsilon_i \pm \varepsilon_j : 0 \leq i < j \leq 2\} \cup \{\pm 2\varepsilon_i : i = 0, 1, 2\}.$$

This is the root system of type C_3. A set of simple roots is given by $\Pi = \{\varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, 2\varepsilon_2\}$.

• Assume now that $\dim S' = 2$, $\dim S = 8$. Then after a scalar extension if necessary, q and q' have maximal Witt index, so

$$\mathfrak{tri}(S, *, q) \cong \mathfrak{o}(4,4) = \left\{ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} : a, b, c \in \mathrm{Mat}_4(F),\ b = -b^t, c = -c^t \right\}.$$

Also $\mathfrak{tri}(S', *, q') = Fk_1 \oplus Fk_2$, where $k_1 = (d, -d, 0)$ and $k_2 = (d, 0, -d)$, $d = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{o}(1,1) \cong \mathfrak{o}(S', q')$. The diagonal matrices in $\mathfrak{o}(4,4)$ form a Cartan subalgebra of $\mathfrak{o}(4,4)$ with roots $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq 4$ (where $\varepsilon_i\big(\mathrm{diag}(\alpha_1, \ldots, \alpha_4, -\alpha_1, \ldots, -\alpha_4)\big) =$

α_i) and system of simple roots $\{\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_3,\varepsilon_3-\varepsilon_4,\varepsilon_3+\varepsilon_4\}$. Also consider $\{\delta_1,\delta_2\}$ the dual basis in $(Fk_1\oplus Fk_2)^*$ of $\{k_1,k_2\}$. The direct sum $\mathfrak{h}$ of the given Cartan subalgebra in $\mathfrak{tri}(S,*,q)$ and of $\mathfrak{tri}(S',*,q')$ is a Cartan subalgebra in $\mathfrak{g}=\mathfrak{g}(S,S')$. The roots in $\mathfrak{g}_{(\bar 0,\bar 0)}$ are $\pm\varepsilon_i\pm\varepsilon_j$, $1\leq i<j\leq 4$. Since $\mathfrak{g}_{(\bar 1,\bar 0)}=S\otimes S'$ with the natural action of $\mathfrak{o}(4,4)\cong\mathfrak{tri}(S,*,q)$ on S; the roots in $\mathfrak{g}_{(\bar 1,\bar 0)}$ are $\pm\varepsilon_i\pm(\delta_1+\delta_2)$, $1\leq i\leq 4$, and the roots in $\mathfrak{g}_{(\bar 0,\bar 1)}$ and in $\mathfrak{g}_{(\bar 1,\bar 1)}$ are obtained by applying to the roots in $\mathfrak{g}_{(\bar 1,\bar 0)}$ the triality automorphisms on the Dynkin diagram of D_4 that fixes $\varepsilon_2-\varepsilon_3$ and permutes cyclically $\varepsilon_1-\varepsilon_2,\varepsilon_3-\varepsilon_4$ and $\varepsilon_3+\varepsilon_4$, while substituting $\delta_1+\delta_2$ by δ_1 and δ_2. As a consequence, the roots in $\mathfrak{g}_{(\bar 0,\bar 1)}$ (respectively $\mathfrak{g}_{(\bar 1,\bar 1)}$) are $\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)\pm\delta_1$ (resp. $\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)\pm\delta_2$) with an odd (resp. even) number of minus signs in the ε's. A system of simple roots is given by

$$\{\delta_2-\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4),\varepsilon_1-\varepsilon_2,\varepsilon_3+\varepsilon_4,\varepsilon_2-\varepsilon_3,\varepsilon_3-\varepsilon_4,\delta_1-\frac{1}{2}(\epsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)\}$$

which is a system of type E_6.

In characteristic 3, the algebra obtained by means of tensoring the $\mathbb{Z}$-span of a Chevalley basis of the complex simple Lie algebras of type A_2 or E_6 with the ground field are no longer simple, but have a one-dimensional center such that the quotients modulo this center are simple. This is reflected on our Lie algebras $\mathfrak{g}=\mathfrak{g}(S,S')$ as follows. Let $(S,*,q)$ be a symmetric composition algebra of dimension 2. After a scalar extension we may assume that S has a basis $\{e_1,e_2\}$ with $q(e_1)=q(e_2)=0$, $q(e_1,e_2)=1$ and $e_1*e_1=e_2$, $e_2*e_2=e_1$ and $e_1*e_2=e_2*e_1=0$ (the para-Hurwitz algebra associated to the Hurwitz algebra $F\times F$). Then $t_{e_j,e_j}=0$, $j=1,2$, while

$$\begin{aligned}
t_{e_1,e_2}&=\left(\sigma_{e_1,e_2},\frac{1}{2}I-r_{e_1}l_{e_2},\frac{1}{2}I-l_{e_1}r_{e_2}\right)\\
&\cong\left(\begin{pmatrix}-1&0\\0&1\end{pmatrix},\begin{pmatrix}1/2&0\\0&-1/2\end{pmatrix},\begin{pmatrix}1/2&0\\0&-1/2\end{pmatrix}\right)\\
&=\left(\begin{pmatrix}-1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\0&1\end{pmatrix}\right)
\end{aligned}$$

so that $\theta(t_{e_1,e_2})=t_{e_1,e_2}$ and hence, for any other symmetric composition algebra $(S',*,q')$,

$$[\iota_i(S\otimes S'),\iota_i(S\otimes S')]\in t_{S,S}\oplus\mathfrak{tri}(S',*,q')$$

for any $i=0,1,2$. Thus $[\mathfrak{g},\mathfrak{g}]$ has codimension 1 if $\dim S=2\neq\dim S'$ and codimension 2 if $\dim S=\dim S'=2$. All the other entries of the Magic Square remain unchanged.

4. Related finite order automorphisms.

Given two symmetric composition algebras $(S,*,q)$ and $(S',*,q')$, the triality automorphisms θ of $\mathfrak{tri}(S,*,q)$ and θ' of $\mathfrak{tri}(S',*,q')$, given in (2.4), immediately define an automorphism Θ of order 3 of the Lie algebra

$$\mathfrak{g} = \mathfrak{g}(S,S') = \Bigl(\mathfrak{tri}(S,*,q) \oplus \mathfrak{tri}(S',*,q')\Bigr) \oplus \Bigl(\oplus_{i=0}^{2}\iota_i(S\otimes S')\Bigr),$$

defined by means of

$$\begin{cases} \text{a) } \Theta\vert_{\mathfrak{tri}(S,*,q)} = \theta,\ \Theta\vert_{\mathfrak{tri}(S',*,q')} = \theta', \\ \text{b) } \Theta\bigl(\iota_i(a\otimes x)\bigr) = \iota_{i+1}(a\otimes x)\ \forall a\in S, x\in S' \quad \text{(indices modulo 3)} \end{cases}$$

The fixed subalgebra of Θ is

$$\tilde{\mathfrak{g}} = \Bigl(\operatorname{Der}(S,*) \oplus \operatorname{Der}(S',*)\Bigr) \oplus \iota(S\otimes S'),$$

where $\iota = \iota_0 + \iota_1 + \iota_2$. If the characteristic of the ground field is $\neq 2,3$, $\operatorname{Der}(S,*)$ is trivial if $\dim S = 1$ or 2, a simple three dimensional Lie algebra if $\dim S = 4$, a central simple Lie algebra of type A_2 if $(S,*,q)$ is an Okubo algebra, and a central simple Lie algebra of type G_2 if $(S,*,q)$ is an eight-dimensional para-Hurwitz algebra. (In characteristic 3 this is no longer true [E2,AEMP].)

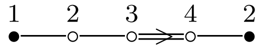
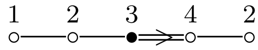
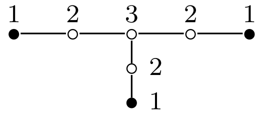
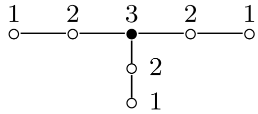
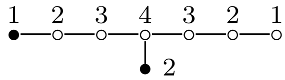
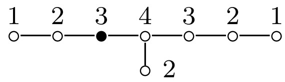
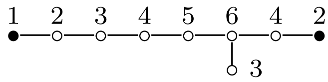
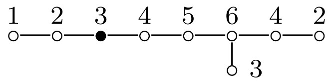
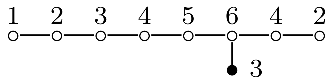
For the exceptional Lie algebras that appear in the Magic Square, the possibilities are summarized in the next table, where pH_n stands for a para-Hurwitz algebra of dimension n and Ok for an Okubo algebra (eight dimensional). The right most column represents the corresponding extended Dynkin diagram $X_N^{(1)}$ with marked nodes, where X_N is the entry in the third column. Recall that over an algebraically closed field of characteristic 0, any order three automorphism of a simple Lie algebra of type X_N is determined, up to conjugation, by some marked nodes in $X_N^{(1)}$ such that the sum of the labels of the nodes is 3 [K, Theorem 8.6]. A Z in the third column means a one-dimensional central ideal.

In case the two symmetric composition algebras coincide, there is a natural order 2 automorphism Ψ of $\mathfrak{g} = \mathfrak{g}(S,S)$, defined by means of:

$$\begin{cases} \text{a) } \Psi \text{ interchanges the two copies of } \mathfrak{tri}(S,*,q) \text{ in } \mathfrak{g}_{(\bar 0,\bar 0)}, \\ \text{b) } \Psi\bigl(\iota_i(a\otimes x)\bigr) = \iota_i(x\otimes a)\ \forall i=0,1,2,\ \forall a,x\in S. \end{cases}$$

For the most interesting case of an eight-dimensional S, the fixed subalgebra is a direct sum of a three-dimensional simple Lie algebra and a simple Lie algebra of type E_7.

Order 3 automorphisms

S	S'	$\mathfrak{g}$	$\tilde{\mathfrak{g}}$	Dynkin diagram
pH_1	pH_8	F_4	$B_3 \oplus Z$	
pH_1	Ok	F_4	$A_2 \oplus A_2$	
pH_2	pH_8	E_6	$D_4 \oplus Z \oplus Z$	
pH_2	Ok	E_6	$A_2 \oplus A_2 \oplus A_2$	
pH_4	pH_8	E_7	$A_6 \oplus Z$	
pH_4	Ok	E_7	$A_5 \oplus A_2$	
pH_8	pH_8	E_8	$D_7 \oplus Z$	
pH_8	Ok	E_8	$E_6 \oplus A_2$	
Ok	Ok	E_8	A_8	

5. Relationship with previous constructions.

Let C be a Hurwitz algebra with norm q and define a new multiplication on C by means of

$$x * y = \bar{y}\bar{x} = \overline{xy} \tag{5.1}$$

for any $x, y \in C$. Then $\hat{C} = (C, *, q)$ is the para-Hurwitz algebra associated to the opposite algebra of C.

To relate our construction to Barton and Sudbery's construction, we will deal with this para-Hurwitz algebras $\hat{C} = (C, *, q)$.

Now, for $d_0, d_1, d_2 \in \mathfrak{o}(C, q)$, $d_0(x * y) = d_1(x) * y + x * d_2(y)$ for any $x, y \in C$ if and only if $d_0(\bar{y}\bar{x}) = \bar{y}d_1(x) + d_2(y)\bar{x}$ for any $x, y \in C$, which is equivalent to any of the next two conditions

$$\begin{aligned} d_0(xy) &= \bar{d}_2(x)y + x\bar{d}_1(y) \\ \bar{d}_0(xy) &= d_1(x)y + xd_2(y) \end{aligned} \tag{5.2}$$

for any $x, y \in C$, where $\bar{d}(x) = \overline{d(\bar{x})}$.

Barton and Sudbery [BS2, (4.1)] consider the Lie algebra

$$\operatorname{Tri} C = \{(f, g, h) \in \mathfrak{o}(C, q)^3 : f(xy) = xg(y) + h(x)y \ \forall x, y \in C\},$$

while Allison and Faulkner [AF, Section 3] consider

$$\mathfrak{t}(C) = \{(f,g,h) \in \mathfrak{o}(C,q)^3 : \bar{f}(xy) = g(x)y + xh(y)\ \forall x,y \in C\}.$$

Hence (5.2) shows that $\mathfrak{t}(C) = \mathfrak{tri}(C,*,q)$, while the the linear map

$$\begin{aligned} \Phi : \mathfrak{tri}(C,*,q) &\to \operatorname{Tri} C \\ (d_0,d_1,d_2) &\mapsto (d_0,\bar{d}_1,\bar{d}_2) \end{aligned} \tag{5.3}$$

is an isomorphism of Lie algebras. Besides, for any $(d_0,d_1,d_2) \in \mathfrak{tri}(C,*,q)$,

$$\Phi\theta(d_0,d_1,d_2) = \Phi(d_2,d_0,d_1) = (d_2,\bar{d}_0,\bar{d}_1) = \theta_{BS}^2(d_0,\bar{d}_1,\bar{d}_2) = \theta_{BS}^2\Phi(d_0,d_1,d_2),$$

where θ_{BS} is the order 3 automorphism $\theta_{BS} : \operatorname{Tri} C \to \operatorname{Tri} C$, $(f,g,h) \mapsto (\bar{g},h,\bar{f})$ given in [BS2, Lemma 4.3].

Now for any $x,y \in C$, Barton and Sudbery consider the element

$$T_{x,y} = \Bigl(4S_{x,y}, R_yR_{\bar{x}} - R_xR_{\bar{y}}, L_yL_{\bar{x}} - L_xL_{\bar{y}}\Bigr) \in \operatorname{Tri} C,$$

where $S_{x,y}(z) = \langle x,z\rangle y - \langle y,z\rangle x$, with $\langle x,y\rangle = \frac{1}{2}q(x,y)$, and L_x (resp. R_x) denote the left (resp. right) multiplication by x in C. Thus, $4S_{x,y} = 2\sigma_{x,y}$. Also, from (2.3),

$$\begin{aligned} \overline{R_yR_{\bar{x}} - R_xR_{\bar{y}}}(z) &= \overline{(\bar{z}\bar{x})y - (\bar{z}\bar{y})x} \\ &= \overline{(x*z)y - (y*z)x} \\ &= (x*z)*y - (y*z)*x \\ &= q(x,y)z - 2r_xl_y(z), \end{aligned}$$

and similarly $\overline{L_yL_{\bar{x}} - L_xL_{\bar{y}}} = q(x,y)I - 2l_xr_y$ for any $x,y \in C$. Therefore, for any $x,y \in C$,

$$T_{x,y} = 2\Phi(t_{x,y}). \tag{5.4}$$

Now if C and C' are two Hurwitz algebras and $L_3(C,C')$ is the Lie algebra defined in [BS2, Theorem 4.4], the linear map

$$\Lambda : \mathfrak{g}(\hat{C},\hat{C}') \to L_3(C,C')$$

such that

i) $\Lambda(d_0,d_1,d_2) = \Phi(d_0,d_1,d_2)$ and $\Lambda(d_0',d_1',d_2') = \Phi'(d_0',d_1',d_2')$ for any (d_0,d_1,d_2) in $\mathfrak{tri}(\hat{C},*,q)$ (resp. $(d_0',d_1',d_2') \in \mathfrak{tri}(\hat{C}',*,q')$), where Φ (resp. Φ') is defined by (5.3),

ii) $\Lambda\bigl(\iota_i(a\otimes x)\bigr) = F_{i+1}(a\otimes x)$, for any $i = 0,1,2$, $a \in C$ and $x \in C'$,

takes the multiplication in $\mathfrak{g}(\hat{C},\hat{C}')$ into the multiplication in $L_3(C,C')$ given in [BS2, (4.23)-(4.26)], so that it is an isomorphism.

Therefore, since any para-Hurwitz algebra can be obtained as the $\hat{C} = (C, *, q)$ of some Hurwitz algebra, the Lie algebras $L_3(C, C')$ are exactly, up to isomorphism, the Lie algebra $\mathfrak{g}(S, S')$ for S and S' para-Hurwitz algebras.

On the other hand, Allison and Faulkner give, in [AF, Section 4], a general construction of Lie algebras starting with structurable algebras. Important examples of these algebras are the tensor products $C \otimes C'$ of two Hurwitz algebras. In this case, one checks easily that the operators in [AF, eq. (I)] take the following form:

$$\begin{aligned}
T_1 &= L_{\bar{b}\otimes\bar{y}} L_{a\otimes x} - L_{\bar{a}\otimes\bar{x}} L_{b\otimes y} \\
&= \frac{1}{2}\left(L_{\bar{b}} L_a - L_{\bar{a}} L_b\right) \otimes q'(x,y)I + q(a,b)I \otimes \frac{1}{2}\left(L_{\bar{y}} L_x - L_{\bar{x}} L_y\right), \\
T_2 &= R_{\bar{b}\otimes\bar{y}} R_{a\otimes x} - R_{\bar{a}\otimes\bar{x}} R_{b\otimes y} \\
&= \frac{1}{2}\left(R_{\bar{b}} R_a - R_{\bar{a}} R_b\right) \otimes q'(x,y)I + q(a,b)I \otimes \frac{1}{2}\left(R_{\bar{y}} R_x - R_{\bar{x}} R_y\right), \\
T_0 &= R_{\big((\bar{a}\otimes\bar{x})(b\otimes y) - (\bar{b}\otimes\bar{y})(a\otimes x)\big)} + L_{b\otimes y} L_{\bar{a}\otimes\bar{x}} - L_{a\otimes x} L_{\bar{b}\otimes\bar{y}} \\
&= \sigma_{a,b} \otimes q'(x,y) + q(a,b)I \otimes \sigma'_{x,y}.
\end{aligned}$$

Therefore, the "inner triple" (T_0, T_1, T_2) is exactly $q'(x,y)\hat{T}_{a,b} + q(a,b)\hat{T}'_{x,y} \in \mathfrak{t}(C) \oplus \mathfrak{t}(C')$, where $\hat{T}_{a,b} = \big(\sigma_{a,b}, \frac{1}{2}\left(L_{\bar{b}} L_a - L_{\bar{a}} L_b\right), \frac{1}{2}\left(R_{\bar{b}} R_a - R_{\bar{a}} R_b\right)\big)$. But $L_{\bar{b}} L_a - L_{\bar{a}} L_b = \overline{R_b R_{\bar{a}} - R_a R_{\bar{b}}}$ and $R_{\bar{b}} R_a - R_{\bar{a}} R_b = \overline{L_b L_{\bar{a}} - L_a L_{\bar{b}}}$. Thus, the argument leading to (5.4) gives that the inner triple (T_0, T_1, T_2) is exactly

$$q'(x,y) t_{a,b} + q(a,b) t'_{x,y}. \tag{5.5}$$

Now, with $\mathcal{V} = \mathfrak{t}(C) \oplus \mathfrak{t}(C')$ and $0 \neq \gamma_0, \gamma_1, \gamma_2 \in F$, Allison and Faulkner construct the Lie algebra

$$\mathcal{K}(C \otimes C'; \gamma_0, \gamma_1, \gamma_2) = \mathcal{V} \oplus (C \otimes C')[01] \oplus (C \otimes C')[12] \oplus (C \otimes C')[20],$$

(the direct sum of $\mathcal{V}$ and three copies of $C \otimes C'$), where $(a \otimes x)[ij] = -\gamma_i \gamma_j^{-1} (\bar{a} \otimes \bar{x})[ji]$, with multiplication given by extending the Lie product in $\mathcal{V}$ by setting

$$\begin{aligned}
\big[(a\otimes x)[ij], (b\otimes y)[jk]\big] &= (ab \otimes xy)[ik] = -\gamma_i \gamma_j^{-1} (\overline{ab} \otimes \overline{xy})[ki], \\
\big[(d_0, d_1, d_2), (a\otimes x)[ij]\big] &= (d_k(a) \otimes x)[ij], \\
\big[(d'_0, d'_1, d'_2), (a\otimes x)[ij]\big] &= (a \otimes d'_k(x))[ij], \\
\big[(a\otimes x)[ij], (b\otimes y)[ij]\big] &= \gamma_i \gamma_j^{-1} \left(q'(x,y)\theta^k(t_{a,b}) + q(a,b)\theta'^k(t'_{x,y})\right),
\end{aligned}$$

where (i, j, k) is a cyclic permutation of $(0, 1, 2)$, for any $a, b \in C$, $x, y \in C'$, $(d_0, d_1, d_2) \in \mathfrak{t}(C)$ and $(d'_0, d'_1, d'_2) \in \mathfrak{t}(C')$. Note that the numbering of the indices in [AF] and here are slightly different.

With $\alpha_i = \gamma_{i+1}^{-1}\gamma_{i+2}$ (indices modulo 3), and using the construction in Remark 3.2, it is straightforward to check that the linear map

$$\Gamma : \mathfrak{g}_{\underline{\alpha}}(\hat{C}, \hat{C}') \to \mathcal{K}(C \otimes C'; \gamma_0, \gamma_1, \gamma_2)$$

such that

i) Γ is the identity on $\mathfrak{tri}(C, *, q) \oplus \mathfrak{tri}(C', *, q) = \mathfrak{t}(C) \oplus \mathfrak{t}(C')$,

ii) $\Gamma\big(\iota_i(a \otimes x)\big) = -(a \otimes x)[i+1, i+2]$ for any $a \in C$ and $x \in C'$ (indices modulo 3),

is an isomorphism of Lie algebras.

Finally, the construction given by Landsberg and Manivel is (isomorphic to) the construction $\mathcal{K}(C \otimes C'; 1, 1, 1)$. This is seen by identifying, in [LM1, Theorem 2.1], $u_1 \otimes v_1$ with $(u \otimes v)[01]$, $u_2 \otimes v_2$ with $(u \otimes v)[12]$ and $u_3 \otimes v_3$ with $(u \otimes v)[02] = -(\bar{u} \otimes \bar{v})[20]$. Therefore, this construction gives again exactly the Lie algebras $\mathfrak{g}(S, S')$ for para-Hurwitz algebras S and S'.

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