Local Nilpotency of the McCrimmon Radical of a Jordan System

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Abstract: Using the fact that absolute zero divisors in Jordan pairs become Lie sandwiches of the corresponding TKK Lie algebras, we prove local nilpotency of the McCrimmon radical of a Jordan system (algebra, triple system or pair) over an arbitrary ring of scalars. As an application, we get that simple Jordan systems are always nondegenerate.

Keywords: Jordan system, absolute zero divisor, McCrimmon radical, Lie algebra, sandwich, local nilpotency

MSC2010: 17C10, 17C17, 17C20.

The paper is dedicated to Vladimir Petrovich Platonov on the occasion of his 75th birthday.

0. Introduction and Preliminaries

Much of the structure theory of infinite dimensional linear Jordan systems is based on the local nilpotency of the McCrimmon radical (see [17, 18, 19]). In this paper we prove this fact for Jordan systems over an arbitrary ring of scalars. As an application, we conclude that simple Jordan systems are always nondegenerate. Two more applications to Kurosh-type problems and to Moufang loops will appear in subsequent papers.

The key observation that absolute zero divisors of a Jordan pair become Lie sandwiches of the corresponding TKK Lie algebra allows us to translate our problem into the Lie setting. Thus, our proof is based on Kostrikin-Zelmanov's theorem on the local nilpotency of Lie algebras generated by sandwiches [6]. Different proofs of this result can be found in the literature [2, 20].

[1] Partially supported by the Spanish Ministerio de Economía y Competitividad and Fondos FEDER, MTM2014-52470-P.

[2] Partially supported by the National Science Foundation of the USA.

0.1 We will deal with Jordan systems (algebras, triple systems and pairs), and associative and Lie algebras over an arbitrary ring of scalars Φ. We warn the reader that even when dealing with Lie algebras, we will not make any additional assumption on the ring of scalars. In particular, we will NOT assume $1/2 \in \Phi$.

The reader is referred to [3, 4, 8, 13, 14] for basic results, notation, and terminology, though we will stress some notions.

— When dealing with an associative algebra, the (associative) products will be denoted by juxtaposition.

— The product in a Lie algebra L will be denoted by square brackets $[x, y] = Ad_x(y) = Ad(x)(y)$, for any $x, y \in L$.

— Given a Jordan algebra J, its products will be denoted by x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y = V_x y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively.

— For a Jordan pair $V = (V^+, V^-)$, we have products $Q_x y = Q_x^\sigma y \in V^\sigma$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z} y = Q_{x,z}^\sigma y =: \{x, y, z\} =: D_{x,y}^\sigma z = D_{x,y} z$.

— A Jordan triple system T is given by its products $P_x y$, for any $x, y \in T$, with linearizations denoted by $P_{x,z} y =: \{x, y, z\} =: L_{x,y} z$.

0.2 (i) A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P = U$. By doubling any Jordan triple system T one obtains the *double Jordan pair* $\mathrm{V}(T) = (T, T)$ with products $Q_x^\sigma y = P_x y$, $\sigma = \pm$, for any $x, y \in T$. From a Jordan pair $V = (V^+, V^-)$ one can get a (*polarized*) Jordan triple system $\mathrm{T}(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+}^+ y^- \oplus Q_{x^-}^- y^+$ [8, 1.13, 1.14].

(ii) An associative system R gives rise to a Jordan system $R^{(+)}$ by *symmetrization*: over the same Φ-module (the same pair of Φ-modules in the pair case), we define, for any $x, y \in R$, $x^2 = xx$, $U_x y = xyx$ in the case of algebras, $P_x y = xyx$ in the case of triple systems, and, for any $x \in R^\sigma, y \in R^{-\sigma}$, $\sigma = \pm$, $Q_x^\sigma y = xyx$ in the pair case, where juxtaposition denotes the associative product in R.

0.3 A Jordan system J is called *special* if it is a subsystem of $R^{(+)}$, for some associative system R. Otherwise J is said to be *exceptional*.

0.4 For a Jordan triple system T, a *derivation* is a map $\Delta \in \mathrm{End}_\Phi T$ satisfying $\Delta(P_x y) = P_{\Delta(x),x} y + P_x \Delta(y)$, for any $x, y \in T$. For a Jordan algebra, besides $\Delta(U_x y) = U_{\Delta(x),x} y + U_x \Delta(y)$ it must satisfy $\Delta(x^2) = x \circ \Delta(x)$.

Given a Jordan pair V, a *derivation* of V is any pair of Φ-linear maps $\Delta = (\Delta^+, \Delta^-) \in \mathrm{End}_\Phi V^+ \times \mathrm{End}_\Phi V^-$ such that

$$\Delta^\sigma(Q_x^\sigma y) = \{\Delta^\sigma(x), y, x\} + Q_x^\sigma \Delta^{-\sigma}(y),$$

for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$. The set $\mathrm{Der}(V)$ of all derivations of V is a (Lie) subalgebra of $(\mathrm{End}_\Phi V^+ \times \mathrm{End}_\Phi V^-)^{(-)}$ (see [8, 1.4]). For any $x \in V^+$, $y \in V^-$, we define $\delta(x, y) := (D_{x,y}, -D_{y,x})$ which turns out to be a derivation of V by [8, JP12], and is called an *inner derivation* of V. The Φ-submodule of $\mathrm{Der}(V)$ spanned by all inner derivations of V will be denoted $\mathrm{InDer}(V)$ and is an ideal of the Lie algebra $\mathrm{Der}(V)$ since

$$[\Delta, \delta(x, y)] = \delta(\Delta^+(x), y) + \delta(x, \Delta^-(y)),$$

for any $\Delta \in \mathrm{Der}(V)$, and any $x \in V^+$, $y \in V^-$.

0.5 Given a Jordan pair V and any subalgebra $\mathcal{D}$ of $\mathrm{Der}(V)$ containing $\mathrm{InDer}(V)$, the Φ-module

$$\mathrm{TKK}(V, \mathcal{D}) := V^+ \oplus \mathcal{D} \oplus V^-$$

can be equipped with a product $[\ ,\ ]$ given by

$$\begin{aligned} &[x^+ \oplus c \oplus x^-, y^+ \oplus d \oplus y^-] := \\ &\big(c^+(y^+) - d^+(x^+)\big) \oplus \big([c, d] + \delta(x^+, y^-) - \delta(y^+, x^-)\big) \oplus \big(c^-(y^-) - d^-(x^-)\big) \end{aligned} \tag{1}$$

yielding a Lie algebra over Φ called the *Tits-Kantor-Koecher algebra* of V and $\mathcal{D}$ [15, XI]. When $\mathcal{D} = \mathrm{InDer}(V)$, we obtain the so called *Tits-Kantor-Koecher algebra* of V, denoted $\mathrm{TKK}(V)$. In some references (see [9, 10]), the Tits-Kantor-Koecher construction is understood as the universal central cover of the Lie algebra $\mathrm{TKK}(V)$ above.

Notice that $L := \mathrm{TKK}(V)$ is a $\mathbf{Z}$-graded Lie algebra, where $L_{-1} = V^-$, $L_1 = V^+$, $L_0 = \mathrm{InDer}(V)$, $L_i = 0$, for any $i \in \mathbf{Z} \setminus \{-1, 0, 1\}$.

0.6 An *absolute zero divisor* of a Jordan algebra J (resp. triple system T) is an element x such that $U_x J = 0$ (resp. $P_x T = 0$). An absolute zero divisor in a Jordan pair (V^+, V^-) is any element $x \in V^\sigma$ such that $Q_x V^{-\sigma} = 0$. In special Jordan systems this means all elements y sandwiched between two x's are annihilated, $xyx = 0$. The key to understanding such "Jordan sandwiches" will be understanding Lie sandwiches.

A Jordan system is said to be *nondegenerate* if it does not have nonzero absolute zero divisors.

As a consequence of [4, QJ16] and MacDonald's Theorem [7], we have the following well-known lemma.

0.7 Lemma. *If a, b, c are absolute zero divisors of a Jordan algebra, then a^2, $a \circ b$, and $\{a, b, c\}$ are also absolute zero divisors. If a, b, c are absolute zero divisors of a Jordan triple system, then so is $\{a, b, c\}$.*

Proof: In the algebra case,

$$U_{a^2} = U_a U_a = 0,$$

$$U_{a\circ b} = U_a U_b + U_b U_a + V_b U_a V_b - U_{a,U_b a} = 0,$$

$$U_{\{a,b,c\}} = U_{U_{a+c}b - U_a b - U_c b} = U_{U_{a+c}b} = U_{a+c} U_b U_{a+c} = 0,$$

if all of a, b, c are absolute zero divisors.

The triple system assertion holds putting P instead of U in the last equality. ■

0.8 We recall that the *McCrimmon* or *nondegenerate radical* (also called *small radical* in [8, 4.5]) $\mathrm{Mc}(J)$ of a Jordan system J is the smallest ideal of J which produces a nondegenerate quotient. It can be obtained by a transfinite induction process as follows [8, 4.7]: Let $M_1(J)$ be the span of absolute zero divisors of J, which is an ideal of J by [8, 4.6]. Once we have the ideals $M_\alpha(J)$ for all ordinals $\alpha < \beta$, we define $M_\beta(J)$ by

(i) $M_\beta(J)/M_{\beta-1}(J) = M_1(J/M_{\beta-1}(J))$ when β is not a limit ordinal,

(ii) $M_\beta(J) = \cup_{\alpha<\beta} M_\alpha(J)$ when β is a limit ordinal.

Then $\mathrm{Mc}(J) = \lim_\alpha M_\alpha(J)$, so that for any Jordan system J, $\mathrm{Mc}(J) = M_\alpha(J)$ for some ordinal α (such that $M_1(J/M_\alpha(J)) = 0$, i.e., $J/M_\alpha(J)$ is nondegenerate).

0.9 An element a in a Lie algebra L is called a *sandwich* (see [5]) if

$$\text{(i) } \big[a, [L, a]\big] = 0 \qquad \text{and} \qquad \text{(ii) } \Big[a, \big[L, [L, a]\big]\Big] = 0$$

or

$$\text{(i) } Ad_a^2 = 0 \qquad \text{and} \qquad \text{(ii) } Ad_a\, Ad_L\, Ad_a = 0$$

If L does not have 2-torsion, then (ii) follows from (i).

1. Monomials, Free Systems, and Nilpotency

1.1 Given a set X, $\mathrm{FA}[X]$ will denote the free associative algebra over X. It is a free Φ-module with a basis $W[X]$ consisting of the *associative algebra monomials* or *words* $x_{i_1} \cdots x_{i_n}$, for arbitrary $x_{i_1}, \ldots, x_{i_n} \in X$. When dealing with associative words, X will be called the *alphabet*, and its elements will be called *letters*. The algebra $\mathrm{FA}[X]$ is $\mathbf{Z}$-graded by the *degree* or *length* of words, and also $\mathbf{Z}^X$-graded by the *composition* of words.

1.2 Let $\mathrm{FL}[X]$ be the *free Lie algebra* on a set of variables X (see [21, Section 1.2]). This algebra is spanned over Φ by the set $\mathrm{LM}[X]$ of *Lie algebra monomials* (on X), defined inductively as follows: $X \subseteq \mathrm{LM}[X]$, and, given $a, b \in \mathrm{LM}[X]$, the element $[a, b]$ is also a Lie algebra monomial. We will say that the elements of X are Lie algebra monomials of degree 1, and, in general, we will say that a Lie algebra monomial a has degree $n > 1$ if a is $[b, c]$, where b, c are Lie algebra monomials of degrees k, l, and $k + l = n$.

The Lie algebra FL$[X]$ is $\mathbf{Z}^X$-graded, so every monomial has a unique well-defined degree in each generator $x \in X$, and also a well-defined total degree.

For a positive integer n, let $\mathrm{LM}_n[X]$ denote the set of Lie algebra monomials of degree n in FL$[X]$. Clearly,

(i) $\mathrm{FL}_n[X] := \Phi(\cup_{k\geq n} \mathrm{LM}_k[X])$ is an ideal of FL$[X]$.

For an infinite set X, let $Z[X]$ be any of the following subsets of FL$[X]$:

$$\mathrm{FL}[X], \quad \mathrm{LM}_n[X], \quad \mathrm{FL}_n[X].$$

Let L be a Lie algebra, $S \subseteq L$, and $j : S \longrightarrow L$ the inclusion map. For any map $\sigma_0 : X \longrightarrow S$, there exists a unique Lie algebra evaluation homomorphism $\sigma : \mathrm{FL}[X] \longrightarrow L$ extending the composition $j\,\sigma_0 : X \longrightarrow L$. Considering all possible choices of σ_0, we obtain the evaluation $Z[S]$ of $Z[X]$ on S as the union of all the images of $Z[X]$ under all possible evaluation homomorphisms. Notice that,

(ii) since X is infinite, the evaluations FL$[S]$, $\mathrm{LM}_n[S]$, and $\mathrm{FL}_n[S]$ of FL$[X]$, $\mathrm{LM}_n[X]$, and $\mathrm{FL}_n[X]$ are independent of X.

(iii) FL$[S]$ is the subalgebra of L generated by S, and $\mathrm{FL}_n[S]$ is an ideal of FL$[S]$.

(iv) A Lie algebra L is *nilpotent* if and only if $\mathrm{FL}_n[L] = 0$ for some positive integer n, i.e., every Lie algebra monomial in L of degree greater than or equal to n vanishes (in this case L will be said to be nilpotent of *degree* n). Moreover, if S is a generating subset of L, then nilpotency of degree n of L is equivalent to $\mathrm{FL}_n[S] = 0$.

We will also deal with the version of (**1.2**) for Jordan systems:

1.3 The *free Jordan algebra* FJ$[X]$ on X is spanned over Φ by the set JM$[X]$ of *Jordan algebra monomials* (on X), defined inductively as follows: $X \subseteq \mathrm{JM}[X]$, and, given $a, b, c \in \mathrm{JM}[X]$, the elements a^2, $a \circ b$, $U_a b$ and $U_{a,b}c$ are also Jordan algebra monomials. We will say that the elements in X are Jordan algebra monomials of degree 1, and, in general, we will say that a Jordan algebra monomial a has degree $n > 1$ if one of the following holds for b, c, d Jordan algebra monomials of degrees k, l, m: a is b^2 $(n = 2k)$, $b \circ c$ $(n = k + l)$, $U_b c$ $(n = 2k + l)$, or $U_{b,c}d$ $(n = k + l + m)$.

It can be shown that FJ$[X]$ is $\mathbf{Z}^X$-graded, so that every monomial has a unique, well-defined degree.

1.4 The *free Jordan triple system* FT$[X]$ on X is spanned over Φ by the set TM$[X]$ of *Jordan triple system monomials* (on X), defined inductively as follows: $X \subseteq \mathrm{TM}[X]$, and, given $a, b, c \in \mathrm{TM}[X]$, the elements $P_a b$ and $P_{a,b}c$ are also Jordan triple monomials. We will say that the elements in X are Jordan triple monomials of degree 1, and, in general, we will say that a Jordan triple monomial a has degree $n > 1$ if one of the following holds for Jordan triple monomials b, c, d of degrees k, l, m: a is $P_b c$ $(n = 2k + l)$, or $P_{b,c}d$ $(n = k + l + m)$.

1.5 We will need to consider, inside $\mathrm{FT}[X]$ the span $\mathrm{FT}\{X\}$ of the set $\mathrm{TM}\{X\}$ of all *Jordan triple* $\{\}$*-monomials*, i.e., those Jordan monomials built exclusively out of the multilinear triple product as follows: $X \subseteq \mathrm{TM}\{X\}$, and, given $a, b, c \in \mathrm{TM}\{X\}$, the element $\{a, b, c\} = P_{a,c}b$ is also a Jordan triple $\{\}$-monomial.

1.6 Let X be a set of variables. For a positive integer n, denote by $\mathrm{JM}_n[X]$ the set of Jordan algebra monomials of degree n in $\mathrm{FJ}[X]$, by $\mathrm{TM}_n[X]$ the set of Jordan triple monomials of degree n in $\mathrm{FT}[X]$, and by $\mathrm{TM}_n\{X\}$ the set of Jordan triple $\{\}$-monomials of degree n. Now, it is immediate that we have ideals

$$\mathrm{FJ}_n[X] := \Phi(\cup_{k\geq n} \mathrm{JM}_k[X]) \triangleleft \mathrm{FJ}[X], \qquad \mathrm{FT}_n[X] := \Phi(\cup_{k\geq n} \mathrm{TM}_k[X]) \triangleleft \mathrm{FT}[X].$$

1.7 As in (**1.2**), the ideals defined in (**1.6**) are used to define nilpotency of Jordan algebras and triple systems.

(i) For an infinite set X, the evaluations of $\mathrm{FJ}[S]$, $\mathrm{JM}_n[S]$, and $\mathrm{FJ}_n[S]$ of $\mathrm{FJ}[X]$, $\mathrm{JM}_n[X]$, and $\mathrm{FJ}_n[X]$, respectively, on a subset S of a Jordan algebra J are independent of X. Moreover,

(ii) $\mathrm{FJ}[S]$ is the subalgebra of J generated by S, and $\mathrm{FJ}_n[S]$ is an ideal of $\mathrm{FJ}[S]$.

(iii) The triple system versions of (i) and (ii) are also true, and we can also consider the evaluations $\mathrm{TM}_n\{S\}$ of $\mathrm{TM}_n\{X\}$, and $\mathrm{TM}\{S\}$ of $\mathrm{TM}\{X\}$, which are also independent of X if X is infinite.

(iv) Given a positive integer n, a Jordan algebra J (resp. triple system T) is said to be *nilpotent of degree* n if the evaluation $\mathrm{FJ}_n[J] = 0$ (resp. $\mathrm{FT}_n[T] = 0$). If S is a generating subset of J or T, then nilpotency of degree n is equivalent to $\mathrm{FJ}_n[S] = 0$ (resp. $\mathrm{FT}_n[S] = 0$).

(v) A Jordan pair V will be said *nilpotent of degree* n if the Jordan triple system $\mathrm{T}(V)$ is nilpotent of degree n.

2. From Lie Algebras to Jordan Systems

We will use the following results on Lie algebras over arbitrary rings of scalars which can be found in [2, 6, 20].

2.1 LIE SANDWICH THEOREM. *A Lie algebra generated by a finite set of sandwiches is nilpotent. Hence a Lie algebra generated by a set of sandwiches is locally nilpotent.* ■

2.2 First we stress some connections between the products of a Jordan pair V and the products in the Lie algebra $\mathrm{TKK}(V)$ which are particular cases of (**0.5**)(1):

(i) $[x, y] = -[y, x] = \delta(x, y)$, for any $x \in V^+$, $y \in V^-$,

(ii) $[[x, y], z] = \{x, y, z\}$, for any $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$.

2.3 Lemma. *Let T be a Jordan triple system, $S \subseteq T$, and $L = \mathrm{TKK}((\mathrm{V}(T))$. Let $\mathrm{V}(S)$ be the set consisting of two copies of S in $\mathrm{V}(T)^+$ and $\mathrm{V}(T)^-$. Then, for any positive integer k, $\mathrm{TM}_k\{S\} \subseteq \mathrm{LM}_k[\mathrm{V}(S)] \cap L_i$, for $i = \pm 1$.*

Proof: We will prove the result by induction on k. Fix an infinite set X.

Indeed, the assertion is clear for $k = 1$ since $\mathrm{TM}_1\{S\} = S = \mathrm{V}(S) \cap L_i = \mathrm{LM}_1[\mathrm{V}(S)] \cap L_i$, $i = \pm 1$. If we assume that the result is true for all positive integers smaller than k ($k \geq 2$), and $a \in \mathrm{TM}_k\{S\}$, then a is some evaluation on S of a monomial $\tilde{a} \in \mathrm{TM}_k\{X\}$. This means $a = \sigma(\tilde{a})$, for some Jordan triple system homomorphism $\sigma : \mathrm{FT}[X] \longrightarrow T$ extending a map $X \longrightarrow S$. Now, $\tilde{a} = \{\tilde{b}, \tilde{c}, \tilde{d}\}$, where $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ are monomials in $\mathrm{TM}\{X\}$ such that $\deg(\tilde{b}) + \deg(\tilde{c}) + \deg(\tilde{d}) = k$, and thus $a = \sigma(\tilde{a}) = \sigma(\{\tilde{b}, \tilde{c}, \tilde{d}\}) = \{\sigma(\tilde{b}), \sigma(\tilde{c}), \sigma(\tilde{d})\} = \{b, c, d\}$, where $b = \sigma(\tilde{b}), c = \sigma(\tilde{c}), d = \sigma(\tilde{d})$ are the evaluations through σ of $\tilde{b}, \tilde{c}, \tilde{d}$, respectively. By the induction assumption, there exist Lie monomials $\hat{b}, \hat{c}, \hat{d} \in \mathrm{LM}[X]$ with degrees $\deg(\hat{b}) = \deg(\tilde{b})$, $\deg(\hat{c}) = \deg(\tilde{c})$, $\deg(\hat{d}) = \deg(\tilde{d})$, and evaluations $b \in T = V^+ = L_1$, $c \in T = V^- = L_{-1}$, $d \in T = V^+ = L_1$, respectively. Being more precise, this means that there are three Lie algebra homomorphisms $\tau_1, \tau_2, \tau_3 : \mathrm{FL}[X] \longrightarrow L$, extending three maps $X \longrightarrow \mathrm{V}(S)$ with $b = \tau_1(\hat{b})$, $c = \tau_2(\hat{c})$, $d = \tau_3(\hat{d})$. Using that X is infinite, we can assume that $\hat{b}, \hat{c}, \hat{d}$ do not have common variables, which allows us to find a single Lie algebra homomorphism $\tau : \mathrm{FL}[X] \longrightarrow L$, extending a map $X \longrightarrow \mathrm{V}(S)$, such that $\tau(\hat{b}) = b \in T = V^+ = L_1, \tau(\hat{c}) = c \in T = V^- = L_{-1}, \tau(\hat{d}) = d \in T = V^+ = L_1$. Now $a = \{b, c, d\} =_{(\mathbf{2.2})(\mathrm{ii})} [[b, c], d] = [[\tau(\hat{b}), \tau(\hat{c})], \tau(\hat{d})] = \tau([[\hat{b}, \hat{c}], \hat{d}]) \in \mathrm{LM}_k[\mathrm{V}(S)] \cap L_1$. Similarly, we can show that $a \in \mathrm{LM}_k[\mathrm{V}(S)] \cap L_{-1}$. ∎

2.4 Lemma. *Let V be a Jordan pair, and $a \in V^+$ be an absolute zero divisor of V. Then, for any $d \in \mathrm{Der}(V)$, and any $y, y' \in V^-$,*

(i) $D_{a,y} d^+(a) = 0$,

(ii) $[\delta(a, y), \delta(a, y')] = 0$.

Proof: (i) Using that d is a derivation,

$$D_{a,y} d^+(a) = \{a, y, d^+(a)\} = d^+(Q_a y) - Q_a d^-(y) = 0$$

since $Q_a = 0$.

(ii) Also $[\delta(a, y), \delta(a, y')] = ([D_{a,y}, D_{a,y'}], [D_{y,a}, D_{y',a}]) = 0$ since, for any $b, b' \in V^-$, using [8, JP13] and [8, JP9], respectively,

$$D_{a,b} D_{a,b'} = D_{Q_a b, b'} + Q_a Q_{b,b'} = 0, \qquad D_{b,a} D_{b',a} = Q_{b,b'} Q_a + D_{b, Q_a b'} = 0. \ ∎$$

We want to relate Jordan absolute zero divisors to Lie sandwiches. One direction is easy.

2.5 Main Lemma. *If V is a Jordan pair, and $a \in V^\sigma$, $\sigma \in \{+, -\}$, is an absolute zero divisor of V, then a is a sandwich of $\mathrm{TKK}(V)$.*

Proof: We will prove the result for $a \in V^+$ since the case $\sigma = -$ follows by passing to the opposite pair $V^{\mathrm{op}} = (V^-, V^+)$ (see [8, 1.5]).

For any $x, x' \in V^+$, $d, d' \in \mathrm{InDer}(V)$, $y, y' \in V^-$, we can use (**0.5**)(1) to obtain

$$\begin{aligned}\big[[x \oplus d \oplus y, a], a\big] =& [d^+(a) \oplus -\delta(a,y) \oplus 0, a] \\ =& -\delta(a,y)^+(a) = -D_{a,y}a = -\{a,y,a\} = -2Q_a y = 0\end{aligned} \tag{1}$$

since $Q_a = 0$, and

$$\begin{aligned}&\Big[\big[[x \oplus d \oplus y, a], x' \oplus d' \oplus y'\big], a\Big] \\ =& \Big[\big[[x \oplus d \oplus y, a], a\big], x' \oplus d' \oplus y'\Big] + \big[[x \oplus d \oplus y, a], [x' \oplus d' \oplus y', a]\big] \\ &\qquad \text{(since } Ad(a) \text{ is a derivation in } \mathrm{TKK}(V)) \\ =& \big[[x \oplus d \oplus y, a], [x' \oplus d' \oplus y', a]\big] \qquad \text{(as in (1))} \\ =& [d^+(a) \oplus -\delta(a,y) \oplus 0, d'^+(a) \oplus -\delta(a,y') \oplus 0] \qquad \text{(by } (\mathbf{0.5})(1)) \\ =& \big(-\delta(a,y)^+ d'^+(a) + \delta(a,y')^+ d^+(a)\big) \oplus [\delta(a,y), \delta(a,y')] \oplus 0 \qquad \text{(by } (\mathbf{0.5})(1)) \\ =& \big(-D_{a,y} d'^+(a) + D_{a,y'} d^+(a)\big) \oplus [\delta(a,y), \delta(a,y')] \oplus 0 = 0\end{aligned} \tag{2}$$

by (**2.4**). But (1) and (2) are exactly (**0.9**)(i)(ii) for a. ∎

2.6 Theorem. *Any Jordan pair or Jordan triple system generated by a finite collection of absolute zero divisors is nilpotent.*

Proof: First let T be a Jordan triple system generated by a finite set $S \subseteq T$ of absolute zero divisors of T. Fix an infinite set X.

We may assume without loss of generality that $0 \in S$, so that always $0 \in \mathrm{TM}_n\{S\}$ and $0 \in \mathrm{TM}_n[S]$.

We will show, by induction on k, that,

(1) for any positive integer k, $\mathrm{TM}_k\{S\} = \mathrm{TM}_k[S]$, and any element in $\mathrm{TM}_k[S]$ is an absolute zero divisor of T.

Indeed, the assertion is obvious for $k = 1$ since $\mathrm{TM}_1\{S\} = \mathrm{TM}_1[S] = S$.

Assume that (1) is true for all positive integers smaller than k ($k \geq 2$). We will show that $\mathrm{TM}_k\{S\} \supseteq \mathrm{TM}_k[S]$, which obviously yields the desired equality $\mathrm{TM}_k\{S\} = \mathrm{TM}_k[S]$. If $a \in \mathrm{TM}_k[S]$, then a is some evaluation on S of a monomial $\tilde a \in \mathrm{TM}_k[X]$, i.e., $a = \sigma(\tilde a)$, for some Jordan triple system homomorphism $\sigma : \mathrm{FT}[X] \longrightarrow T$ extending a map $X \longrightarrow S$. Now, either (i) $\tilde a = P_{\tilde b}\tilde c$, or (ii) $\tilde a = \{\tilde b, \tilde c, \tilde d\}$, where $\tilde b$, $\tilde c$, $\tilde d$ are monomials in $\mathrm{TM}[X]$ of degree smaller than k, so by the induction assumption $b := \sigma(\tilde b)$, $c := \sigma(\tilde c)$, $d := \sigma(\tilde d)$ are absolute zero divisors of T. In (i), $a = \sigma(\tilde a) = \sigma(P_{\tilde b}\tilde c) = P_{\sigma(\tilde b)}\sigma(\tilde c) = P_b c = 0$ (which is certainly an absolute zero divisor of T!), and $a = 0 \in \mathrm{TM}_k\{S\}$. In (ii), $a = \sigma(\tilde a) = \sigma(\{\tilde b, \tilde c, \tilde d\}) = \{\sigma(\tilde b), \sigma(\tilde c), \sigma(\tilde d)\} = \{b, c, d\}$

which is an absolute zero divisor of T by (**0.7**). Moreover, again by the induction assumption, we can find Jordan monomials $\hat{b}, \hat{c}, \hat{d} \in \mathrm{TM}\{X\}$ of the same degrees as $\tilde{b}, \tilde{c}, \tilde{d}$, respectively, having b, c, d as certain evaluations on S. More precisely, there exist three Jordan triple system homomorphisms $\tau_1, \tau_2, \tau_3 : \mathrm{FT}[X] \longrightarrow T$ extending three maps $X \longrightarrow S$, such that $b = \tau_1(\hat{b})$, $c = \tau_2(\hat{c})$, $d = \tau_3(\hat{d})$. Using that X is infinite, we can assume that $\hat{b}, \hat{c}, \hat{d}$ do not have common variables, which allows us to find a single Jordan triple system homomorphism $\tau : FJT[X] \longrightarrow T$ extending a map $X \longrightarrow S$ such that $b = \tau(\hat{b})$, $c = \tau(\hat{c})$, $d = \tau(\hat{d})$. Now $a = \{b, c, d\} = \{\tau(\hat{a}), \tau(\hat{b}), \tau(\hat{c})\} = \tau(\{\hat{a}, \hat{b}, \hat{c}\}) \in \mathrm{TM}_k\{S\}$.

Let $V = \mathrm{V}(T)$. It is clear that V is generated as a Jordan pair by the finite set $\mathrm{V}(S)$ consisting of the copies of the elements of S in both $V^+ = T$ and $V^- = T$. Let $L = \mathrm{TKK}(V)$. We claim that

(2) L is generated by $\mathrm{V}(S)$ as a Lie algebra.

Indeed, by its definition (**0.5**), L is generated by $V^+ \cup V^-$, but

$$V^+ = V^- = T =_{(\mathbf{1.7})(\mathrm{iii})} \mathrm{FT}[S] = \Phi(\cup_{k\in\mathbf{N}} \mathrm{TM}_k[S]) =_{(1)} \Phi(\cup_{k\in\mathbf{N}} \mathrm{TM}_k\{S\}) \\ \subseteq_{(\mathbf{2.3})} \Phi(\cup_{k\in\mathbf{N}} \mathrm{LM}_k[\mathrm{V}(S)]) = \mathrm{FL}[\mathrm{V}(S)],$$

which is the subalgebra of L generated by $\mathrm{V}(S)$ (**1.2**)(iii).

Notice that all the elements in $\mathrm{V}(S)$ are absolute zero divisors of V, hence sandwiches in $L := \mathrm{TKK}(V)$ by (**2.5**). Thus, L is nilpotent by (**2.1**) and (2), hence there exists a positive integer n such that $\mathrm{FL}_n[\mathrm{V}(S)] = 0$ (**1.2**)(iv), i.e.,

(3) $\mathrm{LM}_k[\mathrm{V}(S)] = 0$, for all $k \geq n$.

By (3) and (**2.3**), $\mathrm{TM}_k\{S\} = 0$ for all $k \geq n$, hence $\mathrm{TM}_k[S] = 0$ for all $k \geq n$ using (1), i.e., $\mathrm{FT}_n[S] = 0$, and T is nilpotent (**1.7**)(iv).

Now let V be a Jordan pair generated by a finite collection of absolute zero divisors. The nilpotency of V follows from the nilpotency of $\mathrm{T}(V)$, which is also generated by a finite collection of absolute zero divisors. ∎

2.7 Notice that, in general, if S is a set of generators for a Jordan triple system T, then $\mathrm{V}(S)$ need not generate $L = \mathrm{TKK}(\mathrm{V}(T))$. Take, for example, $\Phi = \mathbf{Z}$ and $T_0 = \mathrm{FA}[X]^{(+)}$, for $X = \{x\}$, and T be the subtriple of T_0 generated by $S = X$, so that T is freely spanned over $\mathbf{Z}$ by $\{x, x^3, x^5, \ldots\}$. However, it can be shown that the subalgebra M of L generated by $\mathrm{V}(S)$ satisfies $M \cap L_1 = M \cap L_{-1} \subseteq \mathbf{Z}(x, 2x^3, 2x^5, \ldots)$, hence $M \neq L$.

The validity of (2) in the proof above is due to the fact that the elements of S are absolute zero divisors, so that (1) is also true.

2.8 Corollary. *A Jordan algebra generated by a finite collection of absolute zero divisors is nilpotent.*

Proof: If J is a Jordan algebra generated by the collection $A = \{a_1, \ldots, a_n\}$ of absolute zero divisors, then the underlying triple system of J is generated (as a triple system) by the finite set $\widetilde{A} = A \cup \{a^2, a \circ b \mid a, b \in A\}$ [1, 1.4], which also consists of absolute zero divisors of the algebra and the triple system J by (**0.7**). Thus, J is nilpotent as a triple system (**2.6**), i.e., there is a positive integer N such that $\mathrm{FT}_N[J] = 0$. In particular, $\mathrm{FT}_N[\widetilde{A}] = 0$, which implies $\mathrm{FJ}_{2N}[A] = 0$ by [1, 1.9]. Since J is generated by A as a Jordan algebra, this implies that J is nilpotent as a Jordan algebra by (**1.7**)(iv). ■

We can rephrase the above results (**2.6**) and (**2.8**) as follows.

2.9 Corollary. *A Jordan system (algebra, triple system or pair) generated by a set of absolute zero divisors is locally nilpotent.* ■

2.10 We recall that local nilpotency in Jordan systems is a radical property (cf. [12, 0.1]) and, in particular, is a transitive property in the sense that

(i) If I is an ideal of a Jordan system J and both I and J/I are locally nilpotent, then J is locally nilpotent too.

This assertion follows from the following facts:

(ii) Every finitely generated solvable Jordan system is nilpotent.

(iii) If a Jordan system is finitely generated, then its derived ideals are also finitely generated systems.

In [21, Chapter 4], (ii), (iii), and consequently (i) are proved for linear Jordan algebras. For general (quadratic) Jordan algebras, (ii) is given in [11, Albert-Zhevlakov Theorem], and (iii) in [11, Proposition 9]. For Jordan pairs, (ii) and (iii) are established in [16, Theorems 2 and 3], and from them one can easily derive the corresponding results for Jordan triple systems.

2.11 Corollary. *For any Jordan system J, the McCrimmon radical* $\mathrm{Mc}(J)$ *is locally nilpotent.*

Proof: Taking into account (**0.8**), we just need to prove that $M_\alpha(J)$ is locally nilpotent for any ordinal α, which we will do by transfinite induction (on α):

(1) $M_1(J)$ is locally nilpotent by (**2.9**).

(2) Let us assume that $M_\alpha(J)$ is locally nilpotent for any ordinal $\alpha < \beta$, $\beta > 1$.

  (a) If β is a limit ordinal, then $M_\beta(J) = \cup_{\alpha<\beta} M_\alpha(J)$, and any finite set S of elements of $M_\beta(J)$ is contained in $M_\alpha(J)$ for some $\alpha < \beta$, hence the subsystem generated by S is nilpotent by local nilpotency of $M_\alpha(J)$.

  (b) If β is not a limit ordinal, then $M_\beta(J)/M_{\beta-1}(J) = M_1(J/M_{\beta-1}(J))$ is locally nilpotent by (1), and $M_{\beta-1}(J)$ is locally nilpotent since $\beta - 1 < \beta$. Hence $M_\beta(J)$ is locally nilpotent too by (**2.10**)(i). ■

2.12 Corollary. *Simple Jordan systems are always nondegenerate.*

PROOF: Let J be a simple Jordan system. If J is degenerate, then $\mathrm{Mc}(J) \neq 0$, hence $J = \mathrm{Mc}(J)$ by simplicity, which implies that J is locally nilpotent by (**2.11**). But then, we can apply [1, 2.3; 11, Cor. on page 476] to obtain $J = 0$, which is a contradiction. ■

2.13 REMARK: Some authors consider a stronger definition of an absolute zero divisor z of a Jordan algebra J. They require $z^2 = 0$ besides $U_z J = 0$. Obviously, those "strong" absolute zero divisors are absolute zero divisors in our sense (**0.6**), so that (**2.8**) and (**2.9**) remain valid. Using (**0.7**), it is immediate to check that the absence of nonzero "strong" absolute zero divisors is equivalent to the absence of nonzero absolute zero divisors (**0.6**), so that there is only one notion of nondegeneracy for Jordan algebras, hence only one McCrimmon radical.

Acknowledgements: The authors want to thank the referee for his or her valuable comments and suggestions, which have brought substantial improvements to the final version of the paper.

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