Speciality of Lie-Jordan algebras

A. N. Grishkov
Instituto de Matemática e Estatística,
Universidade de São Paulo,
São Paulo, Brasil

Ivan P. Shestakov*
Sobolev Institute of Mathematics, Novosibirsk, Russia

Abstract

The class of so called *Lie-Jordan algebras* is introduced, which have one binary (Lie) operation $[x, y]$ and one ternary (Jordan) operation $\{x, y, z\}$, that satisfy the identities (1)-(5) below. It is proved that any such an algebra is *special*, that is, isomorphic to a subalgebra of a Lie-Jordan algebra of the type $A^{\pm}$, obtained from an associative algebra A via the operations $[x, y] = xy - yx,\ \{x, y, z\} = xyz + zyx$. As an application, we prove the conjecture about associativity of a certain loop constructed in [2].

1 Introduction

It is well known that with every associative algebra A one can relate a Lie algebra $A^- = \langle A, +, [,]\rangle$, with the multiplication $[x, y] = xy - yx$, and a Jordan algebra $A^+ = \langle A, +, \circ\rangle$, with the multiplication $x\circ y = \frac{1}{2}(xy+yx)$. Thus, we have the functors $(.)^-$ and $(.)^+$ from the category $\underline{Ass}$ of associative algebras into the categories $\underline{Lie}$ and $\underline{Jord}$ of Lie and Jordan algebras, respectively. It is easy to see that both functors have left adjoint functors $U : \underline{Lie} \longrightarrow \underline{Ass}$

*Supported by the FAPESP, Proc. 98/2162-6.

and $S : \underline{Jord} \longrightarrow \underline{Ass}$, that is, for any $L \in \underline{Lie},\ J \in \underline{Jord},\ A \in \underline{Ass}$ there are bijections

$$Hom_{\underline{Lie}}(L, A^-) \longrightarrow Hom_{\underline{Ass}}(U(L), A),$$
$$Hom_{\underline{Jord}}(J, A^+) \longrightarrow Hom_{\underline{Ass}}(S(J), A),$$

which are functorial in the variables $L,\ J,\ A$. The algebras $U(L)$ and $S(J)$ are called *associative universal enveloping algebras* of L and J, respectively. Furthermore, there are morphisms $\alpha_L^- : L \longrightarrow U(L)^-$ and $\alpha_J^+ : J \longrightarrow S(J)^+$ with the universal properties (these morphisms correspond to the identical morphisms of the algebras $A = U(L),\ A = S(J)$ in the bijections above).

But at this point the Lie and Jordan cases diverge: the mapping α_L^- is always injective when L is a free module over the ring of scalars, due to the celebrated Poincare—Birkhoff—Witt theorem, while the mapping α_J^+ could have nonzero kernel even when J is a finite dimensional algebra over a field (see [4, 7]). A Jordan algebra J for which the universal mapping α_J^+ is injective is called *special*. The condition of speciality plays an important role in the theory of Jordan algebras; there are many results and open problems concerning the speciality of particular Jordan algebras and classes of algebras (see [4, 7]).

One may consider other functors that conserve the additive structure of an algebra, and introduce for them a notion of speciality (see [6]). In this paper we consider the functor $(.)^{\pm}$ that corresponds to an associative algebra A the algebra $A^{\pm}$, which has the same additive structure and two new multiplications: a Lie one $[x, y] = xy - yx$ and a triple Jordan one $\{x, y, z\} = xyz + zyx$. It is easy to check that the operations $[,]$ and $\{,,\}$ satisfy the identities

$$[x, y] = -[y, x], \tag{1}$$
$$\{x, y, z\} = \{z, y, x\}, \tag{2}$$
$$[[x, y], z] = \{x, y, z\} - \{y, x, z\}, \tag{3}$$
$$[\{x, y, z\}, t] = \{[x, t], y, z\} + \{x, [y, t], z\} + \{x, y, [z, t]\}, \tag{4}$$
$$\{\{x, y, z\}, t, v\} = \{\{x, t, v\}, y, z\} - \{x, \{y, v, t\}, z\} + \{x, y, \{z, t, v\}\}. \tag{5}$$

Observe that the Jacobi identity follows from (2) and (3).

Definition 1 *An algebra with two operations* $[,]$ *and* $\{,,\}$ *is called a* Lie-Jordan *algebra (an LJ-algebra for short) if identities (1)-(5) hold.*

Note that an LJ-algebra is a Lie algebra relatively to the binary operation and a triple Jordan system relatively to the ternary operation.

Definition 2 *An LJ-algebra L is called* special *if there exists an associative algebra A such that L is isomorphic to a subalgebra of $A^{\pm}$.*

The main result of the present article is the following

Theorem 1 *Every LJ-algebra over a field of characteristic $\neq 2$ is special.*

In the proof we follow the method developed in [5, 6] and construct the universal enveloping algebra $U(L)$ for a Lie-Jordan algebra L as a deformation of its associated graded algebra $gr\, U(L)$.

In the last section we apply this result to prove the conjecture from [2] that a certain loop constructed there is a group.

All the algebras considered in the paper are assumed to be over a field F of characteristic $\neq 2$.

2 Universal enveloping algebra and associated graded algebra

Let L be an LJ-algebra; then an associative algebra $U(L)$ is said to be a *universal enveloping algebra* for L if there exists a homomorphism $\alpha_L : L \longrightarrow U(L)^{\pm}$ such that for any associative algebra A and a homomorphism $\beta : L \longrightarrow A^{\pm}$ there exists a homomorphism π of associative algebras $\pi : U(L) \longrightarrow A$ such that $\beta = \alpha_L \circ \pi$. In other words, there is a bijection

$$Hom_{\underline{Lie-Jord}}(L, A^{\pm}) \longrightarrow Hom_{\underline{Ass}}(U(L), A),$$

which is functorial on the variables L and A.

The existence of a universal enveloping algebra $U(L)$ for a given LJ-algebra L is obvious. It is isomorphic to the quotient algebra of the tensor algebra $T(L)$ by the ideal I generated by all the elements

$$a \otimes b - b \otimes a - [a, b],\ a \otimes b \otimes c + c \otimes b \otimes a - \{a, b, c\},\ a, b, c \in L;$$

with the universal homomorphism $\alpha_L : a \longrightarrow a + I$. Theorem 1 is equivalent to the fact that $\ker \alpha = 0$.

The subspace $\alpha(L) \subseteq U(L)$ generates $U(L)$, and so it defines in $U(L)$ an ascending filtration $U_1 = \alpha(L) \subseteq U_2 \subseteq \cdots$, where $U_i = U_{i-1} + U_1^i$. Observe that for any $a, b, c \in L$ we have

$$a \otimes b \otimes c - 1/2\left(\{a, b, c\} + [b, c] \otimes a + b \otimes [a, c] + [a, b] \otimes c\right) \in I,$$

which implies that $U_2 = U_3 = U(L)$. Consider the associated graded algebra $gr\,U = (gr\,U)_1 \oplus (gr\,U)_2$, where $(gr\,U)_1 = U_1,\ (gr\,U)_2 = U(L)/U_1$; then it is easy to see that it is commutative and nilpotent of degree 3: $(gr\,U)^3 = 0$. For more properties of this algebra we need the following

Lemma 1 *In $U(L)^{\pm}$ hold the inclusions*

$$\begin{aligned} [U_i, U_j] &\subseteq U_{i+j-1}, \\ \{U_i, U_j, U_k\} &\subseteq U_{i+j+k-2}. \end{aligned}$$

Proof. Since $U_2 = U(L)$, both inclusions are evident if at least one of the indices is equal to 2. Furthermore, $U_1 = \alpha(L)$, and for any $a, b, c \in L$ we have

$$[\alpha(a), \alpha(b)] = \alpha([a, b]),\ \{\alpha(a), \alpha(b), \alpha(c)\} = \alpha(\{a, b, c\}),$$

which proves the inclusions for $i = j = k = 1$. □

Let $\overline{c_i} = c_i + U_{i-1} \in (gr\,U)_i$, for some $c_i \in U_i$. It follows from Lemma 1 that the following operations are correctly defined in $gr\,U$:

$$\begin{aligned} [\overline{c}_i, \overline{c}_j] &= [c_i, c_j] + U_{i+j-2}, \\ \{\overline{c}_i, \overline{c}_j, \overline{c}_k\} &= \{c_i, c_j, c_k\} + U_{i+j+k-3}. \end{aligned}$$

Since these operations are induced by those in the LJ-algebra $U(L)^{\pm}$, they satisfy identities (1) – (5); that is, the space $gr\,U$ forms an LJ-algebra with respect to the operations $[,],\ \{,,\}$. Moreover, one can easily check that the following identities hold in $U(L)$:

$$\begin{aligned} [xy, z] &= x[y, z] + [x, z]y, \\ \{x \circ y, z, t\} &= \{x, z, t\} \circ y + 1/2([[t, z], y] \circ x + [[t \circ z, y], x]), \end{aligned}$$

where $x \circ y = xy + yx$. The proof is straightforward. As a corollary, we obtain that the associated graded algebra $gr\,U$ satisfies the identities

$$[xy, z] = x[y, z] + [x, z]y, \tag{6}$$

$$\{xy, z, t\} = \{x, z, t\}y + 1/2([[t, z], y]x + [[tz, y], x]). \tag{7}$$

Identity (6) shows that $gr\,U$ is a *Poisson algebra*. By this reason, we will call a commutative associative algebra A with additional operations $[,]$ and $\{,,\}$ a *Poisson LJ-algebra* if $\langle A, +, [,], \{,,\}\rangle$ is an LJ-algebra, $A^3 = 0$, and identities (6),(7) are satisfied in A. As we just have seen, for every LJ-algebra L, the algebra $gr\,U(L)$ is a Poisson LJ-algebra.

Lemma 2 *Every Poisson LJ-algebra A satisfies the identities*

$$\{x,z,t\}y - \{y,z,t\}x - h(x,y,z,t) = 0, \tag{8}$$

$$[xy, zt] = \{xy, z, tw\} = \{xy, zt, w\} = 0, \tag{9}$$

where $h(x,y,z,t) = 1/2([z,[x,y]]t + [t,[x,y]]z + [[t,z],x]y - [[t,z],y]x)$.

Proof. Interchanging x and y in (7) and subtracting the new identity from the former one, by the Jacobi identity and (6), we obtain (8). Furthermore, since $A^3 = 0$, identities (9) follow from (7), (2), (3), and (6). □

Notice that $\alpha(L) = U_1 = (gr\,U)_1$, hence the problem of injectivity of α is reduced to the structure of the graded algebra $gr\,U(L)$. In the next section we will try to construct this algebra starting with the algebra L.

3 The Poisson LJ-algebra $\overline{V(L)}$

Let L be an LJ-algebra. In this section we will construct a Poisson LJ-algebra $\overline{V(L)}$ which eventually proves to be isomorphic to the algebra $gr\,U(L)$.

As the first approximation, we consider the space $V(L) = L \oplus S^2(L)$. Here $S^2(L)$ stands for the space of symmetric 2-degree tensors over L, which is spanned by the elements $a \bullet b = 1/2(a \otimes b + b \otimes a),\ a, b \in L$. Define a multiplication $\cdot$ on $V(L)$ by setting for $a, b, c, d \in L$

$$a \cdot b = a \bullet b,\ a \cdot (b \bullet c) = (b \bullet c) \cdot a = (a \bullet b) \cdot (c \bullet d) = 0;$$

then $V(L)$ becomes a commutative associative algebra with $V(L)^3 = 0$. If no ambiguity occurs, we will write ab instead of $a \cdot b$.

Extend now the operations $[,]$ and $\{,,\}$ from L to $V(L)$ by setting

$$
\begin{aligned}
[x \bullet x, y] &= -[y, x \bullet x] = 2x[x,y], \\
\{x \bullet x, y, z\} &= \{z, y, x \bullet x\} = \{x,y,z\}x + 1/2([[z,y],x]x + [[zy,x],x]), \\
\{x, y \bullet y, z\} &= \{y \bullet y, x, z\} + [[x, y^2], z], \\
\{a, b, r\} &= \{a, r, b\} = \{r, a, b\} = [a,b] = 0,
\end{aligned}
$$

for every $x, y, z \in L$, $a, b, \in S^2(L)$, $r \in V(L)$. Since the space $S^2(L)$ is spanned by the elements $x \bullet x, x \in L$, the operations are correctly defined.

Note that L is an LJ-subalgebra of $V(L)$ though $V(L)$ itself, in general, is not an LJ-algebra with respect to the defined operations.

By construction, we have a homomorphism of graded algebras

$$\rho : \langle V(L), +, \cdot, [,], \{,,\} \rangle \longrightarrow gr\, U$$

such that $\rho(a) = \alpha(a), a \in L$.

Obviously, the equality that we need, $\ker \alpha = 0$, is equivalent to the equality $(\ker \rho) \cap L = 0$. Since $\ker \rho$ is a graded ideal, this equality is equivalent to the inclusion

$$\ker \rho \subseteq S^2(L). \tag{10}$$

Therefore, we need an information on the structure of $\ker \rho$.

Notice that, by (8), for every elements $x, y, z, t \in L$ we have

$$j(x,y,z,t) := \{x,z,t\}y - \{y,z,t\}x - h(x,y,z,t) \in \ker \rho. \tag{11}$$

Let $J = J(L) = \mathrm{vect}\,\{j(x,y,z,t) | \, x,y,z,t \in L\}$ be the vector space generated by all the elements of this kind; then evidently $J \subseteq S^2(L) \cap \ker \rho$. We will eventually show that $J = \ker \rho$; this will imply inclusion (10) and Theorem 1.

Our nearest objective is to prove that certain elements from $\ker \rho$ lie in J.

Let $\tilde{L} = LJ[X]$ and $A = A[X]$ be free Lie-Jordan and free associative algebras with free generators $X = \{x_1, x_2, \ldots\}$. Denote by $\widetilde{SL} = SLJ[X]$ a *free special LJ-algebra*, that is, the LJ-subalgebra of $A^{\pm}$ generated by X. Evidently, there exists an epimorphism $\varphi : \tilde{L} \longrightarrow \widetilde{SL}$, identical on X. By definition of a universal enveloping algebra, φ extends uniquely to a homomorphism of associative algebras $\tilde{\varphi} : U(\tilde{L}) \longrightarrow A$, $\tilde{\varphi}(x) = x, x \in X$. But A is a free algebra, hence $\tilde{\varphi}$ is an isomorphism. Thus, we have a homomorphism

$$\rho : \langle V(\tilde{L}), +, \cdot, [,], \{,,\} \rangle \longrightarrow gr\, A = \widetilde{SL} \oplus A/\widetilde{SL}.$$

such that $\rho(a) = \alpha(a), a \in \tilde{L}$. Notice that the algebras $V(\tilde{L})$ and $gr\, A$ have natural gradings by degrees with respect to X, and the homomorphism ρ respects this grading.

The following proposition is crucial for our construction.

Proposition 1 *In the above notations, every multilinear element of degree* ≤ 6 *on* X *from* $\ker \rho$ *lies in* $J = J(\tilde{L})$.

Proof. We will prove that J contains the multilinear elements of degree 6 from $\ker \rho$. For the elements of smaller degree the proof is similar but more easy.

Denote by P the subspace of multilinear elements of degree 6 in $V(\tilde{L})$. A subset W of a vector space V we will call a *pre-basis* of V if $\mathrm{vect}\,\{W\} = V$. We have a decomposition $P = P_1 \oplus P_2$, where $P_1 = P \cap \tilde{L}$ and $P_2 = P \cap S^2(\tilde{L})$. Let us find a pre-basis of the space $(P_2 + J)/J$. Let $L[X]$ be a Lie subalgebra in $\tilde{L}$ generated by X and B be a homogeneous basis of the space of multilinear elements of $L[X]$. We have $B = \cup_{i=1}^{\infty} B_i$, where $B_i = \{b \in B \mid \deg_X b = i\}$. Set

$$\begin{aligned}
Q_1 &= \{a_i b_j \mid a_i \in B_i, b_j \in B_j,\ i+j=6\},\\
Q_2 &= \{a_3\{x,y,z\} \mid a_3 \in B_3,\ x,y,z \in X\},\\
Q_3 &= \{a_2\{b_2,y,z\} \mid a_2, b_2 \in B_2,\ y,z \in X\},\\
Q_4 &= \{\{x,y,z\}\{u,v,w\} \mid x,y,z,u,v,w \in X\}.
\end{aligned}$$

Lemma 3 *The set* $Q = \cup_{i=1}^{4} Q_i$ *is a pre-basis of* $(P_2 + J)/J$.

Proof. Let $v \in P_2$ be a monomial of degree 6. If $v \in L[X] \cdot L[X]$, then $v \in \mathrm{vect}\,\{Q_1\}$.

If the operation $\{,,\}$ appears only once in v, then we can assume, by (3) and (4), that v has one of the forms

$$v_1 = z\{a_3,x,y\}, \quad v_2 = y\{a_2,b_2,x\}, \quad v_3 = a_2\{b_2,x,y\}, \quad v_4 = a_3\{x,y,z\},$$

where $a_i, b_i \in B_i,\ x,y,z \in X$. Now, for any $a,b,c,d \in \tilde{L}$ we have $h(a,b,c,d) \in \mathrm{vect}\,\{Q_1\}$, hence, by (11),

$$\{a,c,d\}b - \{b,c,d\}a = j(a,b,c,d) + h(a,b,c,d) \in J + \mathrm{vect}\,\{Q_1\}. \qquad (12)$$

Therefore, by (3), $v_1, \dots, v_4 \in \text{vect}\,\{Q_1 \cup Q_2 \cup Q_3\} + J$.

Finally, if the operation $\{,,\}$ appears twice in v, then evidently

$$v = \sum_i \{\{x_i, y_i, z_i\}, t_i, u_i\} w_i \quad (mod \quad \text{vect}\,\{Q\}),$$

for some $x_i, \dots, w_i \in X$. By (12), we have

$$\{\{x_i, y_i, z_i\}, t_i, u_i\} w_i \equiv \{x_i, y_i, z_i\}\{w_i, t_i, u_i\} \quad (mod \quad \text{vect}\,\{Q\} + J),$$

so $v \in \text{vect}\,\{Q\} + J$, and Lemma is proved. □

Set $P_2^j = \text{vect}\,\{Q_1 + \dots + Q_j\} + J,\ j = 1, 2, 3, 4$, and $\overline{P}_2^j = P_2^j / P_2^{j-1},\ j = 2, 3, 4$.

Lemma 4 *We have*

$$(i) \quad \dim(P_2^1/J) \leq 274, \qquad (ii) \quad \dim \overline{P}_2^2 \leq 40,$$

$$(iii) \quad \dim \overline{P}_2^3 \leq 45, \qquad (iv) \quad \dim \overline{P}_2^4 \leq 1,$$

$$(v) \quad \dim(P_2 + J/J) \leq 360.$$

Proof. By [1, page 50], the dimension of the space of multilinear elements of degree m in a free n-generated Lie algebra is equal to $B(n, m) = (m-1)! C_m^n$. Therefore, we have

$$\begin{aligned} \dim P_2^1/J &\leq |Q_1| \leq B(6,1)B(5,5) + B(6,2)B(4,4) + B(6,3)B(3,3) \\ &= 144 + 90 + 40 = 274. \end{aligned}$$

Furthermore, by (2) and (3), we have for any $a_i \in L[X],\ i = 1, 2, 3$, and $\sigma \in Sym\{1, 2, 3\}$

$$\{a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}\} \equiv \{a_1, a_2, a_3\} \quad (mod\ L[X]), \tag{13}$$

which implies that $\dim \overline{P}_2^2 \leq B(6,3) = 40$.

To prove (iii), notice that (12) implies

$$a_2\{b_2, x, y\} \equiv b_2\{a_2, x, y\}\ (mod\ L[X] + J),$$

hence $\dim \overline{P}_2^3 \leq 1/2\, B(6,2)B(4,2) = 45$.

For (iv), we need a certain corollary of identity (5). Adding (5) and the identity obtained from it by interchanging y and z, we get by (3)

$$\begin{aligned}
&\{2\{x,y,z\} - [x,[y,z]],t,v\} = 2\{\{x,t,v\},y,z\} - [\{x,t,v\},[y,z]] \\
&\quad -\{x,\{y,v,t\},z\} + \{x,y,\{z,t,v\}\} - \{x,\{z,v,t\},y\} + \{x,z,\{y,t,v\}\} \\
&= 2\{\{x,t,v\},y,z\} - [\{x,t,v\},[y,z]] - \{x,\{y,t,v\} + [y,[v.t]],z\} \\
&\quad +\{x,y,\{z,v,t\} + [z,[t,v]]\} - \{x,\{z,v,t\},y\} + \{x,z,\{y,t,v\}\} \\
&= 2\{\{x,t,v\},y,z\} - [\{x,t,v\},[y,z]] - \{x,[y,[v.t]],z\} + \{x,y,[z,[t,v]]\} \\
&\quad -[x,[\{y,t,v\},z]] + [x,[y,\{z,v,t\}]].
\end{aligned}$$

Since $\tilde{L}$ is an LJ-algebra, this implies that for any $x_i \in X,\ i = 1,\ldots,6$, we have

$$\{\{x_1,x_2,x_3\},x_4,x_5\} = \{\{x_1,x_4,x_5\},x_2,x_3\} + S, \tag{14}$$

where S is a sum of multilinear elements from $\tilde{L}$ on $x_1,\ldots,x_5$, in which the operation $\{,,\}$ appears only once. By the proof of Lemma 3, we have $S \cdot x_6 \in P_2^3$.

Now, for any $x_i \in X,\ i = 1,\ldots,6$ we have by (12)

$$\{\{x_1,x_2,x_3\},x_4,x_5\}x_6 \equiv \{x_1,x_2,x_3\}\{x_4,x_5,x_6\}\ (mod\ P_2^3),$$

which implies by (14) that

$$\{x_1,x_2,x_3\}\{x_4,x_5,x_6\} \equiv \{x_1,x_4,x_5\}\{x_2,x_3,x_6\}\ (mod\ P_2^3). \tag{15}$$

It follows easily from (13) and (15) that for any $\sigma \in Sym\{1,\ldots,6\}$ holds

$$\{x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}\}\{x_{\sigma(4)},x_{\sigma(5)},x_{\sigma(6)}\} \equiv \{x_1,x_2,x_3\}\{x_4,x_5,x_6\}\ (mod\ P_2^3),$$

which implies (iv).

Finally, we have

$$\dim(P_2+J/J) \leq \dim P_2^1/J + \dim \overline{P}_2^2 + \dim \overline{P}_2^3 + \dim \overline{P}_2^4 = 274+40+45+1 = 360,$$

proving the Lemma. □

Lemma 5 $\dim P_1 \leq 360$.

Proof. Let us write $P_1 = P_1^0 + P_1^1 + P_1^2$, where P_1^i means the space generated by monomials from P_1 in which the operation $\{,,\}$ appears i times. Furthermore, for $i = 1,2$ we set $\overline{P}_1^i = (P_1^0 + \cdots + P_1^i)/(P_1^0 + \cdots + P_1^{i-1})$. Clearly, $\dim P_1^0 \leq B(6,6) = 5! = 120$.

Let B and B_i be as in Lemma 3. By (13), the union of the sets $\{B_4, X, X\}$, $\{B_3, B_2, X\}$, and $\{B_2, B_2, B_2\}$ forms a pre-basis of $\overline{P}_1^1$. Taking in account (13), we obtain

$$\begin{aligned} \dim \operatorname{vect}\{\{B_4, X, X\}\} &\leq B(6,4) = 90, \\ \dim \operatorname{vect}\{\{B_3, B_2, X\}\} &\leq B(6,3)B(3,2) = 120, \\ \dim \operatorname{vect}\{\{B_2, B_2, B_2\}\} &\leq B(6,2)B(4,2)/3! = 15, \end{aligned}$$

which implies that $\dim \overline{P}_1^1 \leq 90 + 120 + 15 = 225$.

Furthermore, it follows from (14) and (13) that the set $\{\{B_2, x_i, x_j\}, x_k, x_l\}$, where $x_s \in X$, $i < j < k < l$, forms a pre-basis of $\overline{P}_1^2$; hence $\dim \overline{P}_1^2 \leq B(6,2) = 15$.

Thus, $\dim P_1 = \dim P_1^0 + \dim \overline{P}_1^1 + \dim \overline{P}_1^2 \leq 120 + 225 + 15 = 360$. □

Return to the proof of Proposition 1. By Lemmas 4 and 5 we have that

$$\dim P/P \cap J = \dim(P + J/J) \leq \dim P_1 + \dim(P_2 + J/J) \leq 360 + 360 = 720.$$

On the other hand, the image $\rho(P)$ coincides with the space $(gr\, A)_{\langle 6 \rangle}$ of multilinear elements of degree 6 on X in $gr\, A$, which has the same dimension as the corresponding space $A_{\langle 6 \rangle}$ in A. Therefore, $\dim \rho(P) = \dim A_{\langle 6 \rangle} = 6! = 720$, and hence $P \cap ker \rho = P \cap J$. □

Corollary 1 *$J = J(\tilde{L})$ is an ideal of the algebra $\langle V(L), +, \cdot, [,], \{,,\} \rangle$.*

Proof. It suffices to prove that

$$[J, \tilde{L}] \subseteq J, \quad \{J, \tilde{L}, \tilde{L}\} \subseteq J.$$

Moreover, the space J is evidently stable with respect to endomorphisms of the space $V(\tilde{L})$ induced by those of the LJ-algebra $\tilde{L}$; hence we need only to prove that for any $x, y, z, t, u, v \in X$ hold the inclusions

$$[j(x, y, z, t), u] \in J, \ \{j(x, y, z, t), u, v\} \in J.$$

Observe that the elements under consideration belong to $\ker \rho$. Since they are multilinear and of degrees 5 and 6, they belong to J as well. □

Corollary 2 *The quotient algebra* $\overline{V(\tilde{L})} = V(\tilde{L})/J(\tilde{L})$ *is a Poisson LJ-algebra.*

Proof. We have to prove that identities (1)—(5) hold in $V(\tilde{L})$ modulo $J(\tilde{L})$. If an identity contains at least two arguments from $S^2(\tilde{L})$, it holds trivially. Consequently, we can assume, as before, that all but one arguments in identities under consideration belong to X, and one argument belongs to $X \cup X \cdot X$. The identities evidently hold modulo $\ker\rho$. Since they are multilinear and of degree ≤ 6 on X, they hold modulo $J(\tilde{L})$ as well. □

Notice that the constructions of $V(L)$ and $J(L)$ are functorial; that is, any LJ-algebra homomorphism $\varphi : L_1 \longrightarrow L_2$ can be extended to a homomorphism

$$\tilde{\varphi} : \langle V(L_1), +, \cdot, [,], \{,,\}\rangle \longrightarrow \langle V(L_2), +, \cdot, [,], \{,,\}\rangle,$$

and $\tilde{\varphi}(J(L_1)) = J(\varphi(L_1)) \subseteq J(L_2)$. Moreover, if φ is surjective then so is $\tilde{\varphi}$. Since every LJ-algebra can be represented as a quotient algebra of a free LJ-algebra, corollaries 1 and 2 imply

Proposition 2 *For every LJ-algebra L the space* $J(L)$ *is an ideal of the algebra* $V(L)$ *with respect to the operations* $\cdot, [,], \{,,\}$*, and the quotient algebra* $\overline{V(L)} = V(L)/J(L)$ *is a Poisson LJ-algebra.*

Since $J(L) \subseteq \ker\rho$, the homomorphism $\rho : V(L) \longrightarrow gr\,U(L)$ induces a Poisson LJ-algebra homomorphism $\tilde{\rho} : \overline{V(L)} \longrightarrow gr\,U(L)$, such that $\tilde{\rho}(l) = \alpha(l)$ for any $l \in L$. In the next section we will prove that $\tilde{\rho}$ is an isomorphism, which will imply that $\ker\rho = J(L)$, proving inclusion (10) and Theorem 1.

4 $U(L)$ is a deformation of $\overline{V(L)}$

As we have seen, for the proving of theorem 1 it suffices to prove that $\overline{V(L)}$ is isomorphic to $gr\,U(L)$. We will prove it by constructing an associative multiplication on $\overline{V(L)}$, isomorphic to that in $U(L)$.

Consider the space of polynomials $\overline{V(L)}[t]$ over $\overline{V(L)}$. An associative multiplication $*$ on $\overline{V(L)}[t]$ we will call a *quantization deformation* of $\overline{V(L)}$, if it satisfies the conditions:

$$\begin{array}{rl} (i) & a * a = a^2, \\ (ii) & a * b - b * a = [a,b]t, \\ (iii) & a * b * c + c * b * a = \{a,b,c\}t^2, \\ (iv) & t * r = r * t, \end{array}$$

for any $a, b, c \in L,\ r \in \overline{V(L)}$.

The role of this definition is illustrated by the following

Proposition 3 *Let L be an LJ-algebra. If the Poisson LJ-algebra $\overline{V(L)}$ admits a quantization deformation, then L is special. Moreover, in this case $gr\,U(L)$ is isomorphic to $\overline{V(L)}$ and $U(L)$ is isomorphic to the quotient algebra $\langle \overline{V(L)}[t], +, *\rangle / \overline{V(L)}[t](1-t)$.*

Proof. Assume that $\overline{V(L)} = L \oplus S^2(L)/J(L)$ admits a quantization deformation $*$. Define a new multiplication $a \star b = a * b|_{t=1}$ on $\overline{V(L)}$; then the algebra $B = \langle \overline{V(L)}, +, \star\rangle$ is associative, and, by properties (ii) and (iii), the LJ-algebra L is a subalgebra of the LJ-algebra $B^{\pm}$. Therefore, L is special.

Consequently, there exists an algebra homomorphism $\pi : U(L) \longrightarrow B$ such that $\pi\alpha(l) = l$ for any $l \in L$. The algebra B is evidently generated by L and has a filtration $B_1 = L \subseteq B_2 = B$, where $B_2 = L + L \star L$; and it is easily seen that $\pi(U_k) \subseteq B_k$. Therefore, π induces a homomorphism of associated graded algebras $\tilde{\pi} : gr\,U \longrightarrow gr\,B$, with $\tilde{\pi}\alpha(l) = l$ for any $l \in L$. But $gr\,B$ is easily seen to be isomorphic to $\overline{V(L)}$, and we have seen that there exists a graded algebra homomorphism $\tilde{\rho} : \overline{V(L)} \longrightarrow gr\,U(L)$, such that $\tilde{\rho}(l) = \alpha(l)$ for any $l \in L$. This proves that $gr\,U(L)$ is isomorphic to $\overline{V(L)}$. Observe that $U(L)$ and $gr\,U(L)$ are isomorphic as vector spaces. Choose in $U(L)$ a base $\{u_i\}$ such that the set $\{\bar{u}_i\}$ is a graded base of $gr\,U(L)$; then evidently $\{\pi(u_i)\}$ is a base of the space $\overline{V(L)}$, and so $\ker \pi = 0$ and π is an isomorphism. □

Now, everything is finished by the following

Theorem 2 *For any LJ-algebra L, the Poisson LJ-algebra $\overline{V(L)}$ admits a quantization deformation.*

Proof. We first define a multiplication $*$ on $V(L)[t]$ by setting

$$\begin{aligned} a * b &= ab + 1/2\,[a,b]t, \\ a^2 * b &= 1/2\,([a^2,b]t + \{a,a,b\}t^2), \\ b * a^2 &= 1/2\,([b,a^2]t + \{a,a,b\}t^2), \\ a^2 * b^2 &= 1/2\,(\{a,a,b^2\}t^2 + [\{a,a,b\},b]t^3). \end{aligned}$$

Let us check that $J(L)*V(L)+V(L)*J(L)\subseteq J(L)[t]$. Let $x,y,z,t\in L$; consider

$$\begin{aligned} j(x,y,z,t) &= \{x,z,t\}y-\{y,z,t\}x-1/2[z,[x,y]]t \\ &\quad -1/2\,([t,[x,y]]z+[[t,z],x]y-[[t,z],y]x) \\ &= \sum_i j_1^{(i)} j_2^{(i)}. \end{aligned}$$

We claim that for any $a\in V(L)$ holds

$$\sum_i \{j_1^{(i)}, j_2^{(i)}, a\}+\{j_2^{(i)}, j_1^{(i)}, a\}\in J. \qquad (16)$$

In fact, we have

$$\begin{aligned} &\sum_i \{j_1^{(i)}, j_2^{(i)}, a\}+\{j_2^{(i)}, j_1^{(i)}, a\}= \\ &=\{\{x,z,t\},y,a\}+\{y,\{x,z,t\},a\}-\{\{y,z,t\},x,a\} \\ &\quad -\{x,\{y,z,t\},a\}-1/2\{[z,[x,y]],t,a\}-1/2\{t,[z,[x,y]],a\} \\ &\quad -1/2\{[t,[x,y]],z,a\}-1/2\{z,[t,[x,y]],a\}-1/2\{[[t,z],x],y,a\} \\ &\quad -1/2\{y,[[t,z],x],a\}+1/2\{[[t,z],y],x,a\}+1/2\{x,[[t,z],y],a\}. \end{aligned}$$

One can straightforwardly check that this sum is identically zero in the algebra $\widetilde{SL}$. Since it is of degree 5, by Proposition 1 it is an identity in the free LJ-algebra $\tilde{L}$. Consequently, this identity holds in the LJ-algebra $\overline{V(L)}=V(L)/J(L)$, which implies inclusion (16).

Now, by definition of the multiplication $*$, we have

$$j(x,y,z,t)*a=1/2\,[j,a]t+1/4\sum_i(\{j_1^{(i)}, j_2^{(i)}, a\}+\{j_2^{(i)}, j_1^{(i)}, a\})t^2,$$

which lies in $J[t]$ by Corollary 1 and (16). Furthermore, if $a\in S^2(L),\ a=b^2$, then

$$\begin{aligned} j(x,y,z,t)*b^2 &= 1/2\sum_i(\{j_1^{(i)}, j_2^{(i)}, b^2\}+\{j_2^{(i)}, j_1^{(i)}, b^2\})t^2 \\ &\quad +1/2\sum_i[\{j_1^{(i)}, j_2^{(i)}, b\}+\{j_2^{(i)}, j_1^{(i)}, b\}, b]t^3, \end{aligned}$$

which lies in $J[t]$ by the same reasons. Therefore, $V(L)*J(L)\subseteq J(L)[t]$. Similarly, $J(L)*V(L)\subseteq J(L)[t]$.

The inclusions just proved show that $*$ can be defined naturally on the quotient algebra $\overline{V(L)}[t]$. Conditions (i)–(iv) are evidently satisfied by $*$. We have only to prove that $*$ is associative.

We will consider only the most complicated case; the remaining cases are proved similarly.

Compute

$$(a^2 * b^2) * c^2 = 1/2\,(\{a, a, b^2\} * c^2)t^2 + 1/2([\{a, a, b\}, b] * c^2)t^3.$$

One can easily check that in any Poisson LJ-algebra hold the identities

$$\{a, a, b^2\} = \{a, a, b\}b + [a, b]^2 + a[[a, b], b].$$

Hence,

$$\begin{aligned}\{a, a, b^2\} * c^2 \;\; = \;\; & 1/4\,(\{\{\{a, a, b\}, b, c^2\} + \{b, \{a, a, b\}, c^2\})t^2 \\ & +1/4\,(\{a, [[a, b], b], c^2\} + \{[[a, b], b], a, c^2\})t^2 \\ & +1/2\,\{[a, b], [a, b], c^2\}t^2 + 1/2[\{[a, b], [a, b], c\}, c]t^3 \\ & +1/4\,([\{\{a, a, b\}, b, c\}, c] + [\{b, \{a, a, b\}, c\}, c])t^3 \\ & +1/4\,([\{\{a, [[a, b], b], c\}, c] + [\{[[a, b], b], a, c\}, c])t^3.\end{aligned}$$

Therefore, $(a^2 * b^2) * c^2 = At^4 + Bt^5$, where

$$\begin{aligned}A = 1/8\,((\{\{a, a, b\}, b, c^2\} + \{b, \{a, a, b\}, c^2\} \\ + \{a, [[a, b], b], c^2\} + \{[[a, b], b], a, c^2\}, \\ +2\,[[\{a, a, b\}, b], c^2]) + 2\{[a, b], [a, b], c^2\},\end{aligned}$$

and

$$\begin{aligned}B = 1/8\,([\{\{a, a, b\}, b, c\}, c] + [\{b, \{a, a, b\}, c\}, c] \\ + [\{\{a, [[a, b], b], c\}, c] + [\{[[a, b], b], a, c\}, c] \\ +2\,[\{[a, b], [a, b], c\}, c] + 2\{c, c, [\{a, a, b\}, b]\}.\end{aligned}$$

Similarly, $a^2 * (b^2 * c^2) = A_1t^4 + B_1t^5$, where

$$A_1 = 1/4\,(\{a, a, \{b, b, c^2\}\} + [a^2, [\{b, b, c\}, c]]),$$

and

$$B_1 = 1/8\,([\{a,a,b\},\{b,b,c\}] + [\{a,a,\{b,b,c\}\},c]
+ [\{a,a,b\},[[b,c],c]] + [\{a,a,[[b,c],c]\},b]
+2\,[\{a,a,[b,c]\},[b,c]] + 2\,\{a,a,[\{b,b,c\},c]\}).$$

One can straightforwardly check that $A = A_1$ and $B = B_1$ in the algebra $gr\,A[X]$ for any $a,b,c \in X$. Since the elements $A - A_1, B - B_1$ are of degree 6, by Proposition 1 we have $A - A_1, B - B_1 \in J(\tilde{L})$, and hence $A = A_1, B = B_1$ in $\overline{V(L)}$.

This proves the theorem. □

5 Application

In order to formulate an application of our result, we recall some definitions from [2].

Let $A = A[x,y]$ be the free associative algebra over a field F of characteristic $p > 0$ on free generators x, y. For any $f = f(x,y) \in A$ we define a space A_f as a minimal subspace of A with the properties:

(i) $x, y \in A_f$,

(ii) If $a, b \in A_f$, then $f(a,b) \in A_f$.

A series $E(x) = \sum_{i=0}^{\infty} a_i x^i$ is called an *f-exponent* if $a_0 = 1$ and in the natural completion $\overline{A}$ of A we have

$$E(x)E(y) = E(z), \tag{17}$$

where $z = \sum_{i=0}^{\infty} z_i,\ \deg z_i = i,\ z_i \in A_f$.

A series $E(x)$ we will call a *p-exponent* if $E(x)$ is an f-exponent for some polynomial f, and for any other polynomial g we have $A_f \subseteq A_g$.

It was proved in [2] that in the case $p = 0$ a 0-exponent exists and coincides with the classical exponent $\exp x = \sum_{i=0}^{\infty} x^i/i!$. It was also conjectured there that a p-exponent exists (certainly not unique!) and coincides with an f_p-exponent, where the element $f_p \in A$ is defined as follows.

It is well known (see for instance [3, section 5.7]) that for every simple $p > 0$ an equality of the following type holds:

$$(x+y)^p = x^p + y^p + s(x,y) + p\tilde{f}_p(x,y),$$

where $s(x,y)$ and $\tilde{f}_p(x,y)$ are a Lie and associative polynomials on x, y with integer coefficients. Of course, the polynomial $\tilde{f}_p(x,y)$ is not uniquely defined by this equation; but it is unique modulo Lie polynomials.

Now, we fix $\tilde{f}_p$, reduce its coefficients modulo p, and set $f_p(x,y) = [x,y] + \tilde{f}_p(x,y)$.

For $p = 3$, it is easy to see that we can take $\tilde{f}_3 = xyx + yxy$; hence $f_3 = xy - yx + xyx + yxy$, and the subspace A_{f_3} coincides with the free special LJ-algebra $SLJ[x,y]$. Consequently, the elements z_i in equality (17) belong to $SLJ[x,y]$ and can be written as concrete polynomials on x, y in terms of operations $[,]$ and $\{,,\}$. More specifically, it was proved in [2] that the series

$$E(x) = x + \sqrt{1 + x^2} = 1 + x - x^2 + x^4 + \cdots$$

is a 3-exponent and an f_3-exponent; and the element z from (17) is given by

$$z = E^{-1}(E(x)E(y)) = x * y = x + y - [x,y] + \{x,x,y\} + \{x,y,y\} + \cdots. \quad (18)$$

The following Theorem was conjectured in [2].

Theorem 3 *Let $L = LJ[x,y]$ be the free LJ-algebra over a field F of characteristic 3 on generators x, y, and $\overline{L}$ be the completion of L; then the operation $u * v = w$, defined by the right part of (18), introduces on $\overline{L}$ a group structure.*

Proof. By Theorem 1, the LJ-algebra L is special; hence it is isomorphic to $SL = SLJ[x,y]$, and by (17) we have $u * v = E^{-1}(E(u)E(v))$ for any $u, v \in L = SL \subseteq A$. This implies easily the statement of the Theorem. □

References

[1] Yu. A. Bahturin, *Identities of Lie algebras*, Moscow, Nauka, 1985 (in Russian); English translation by VNU Science Press: Utrecht, 1987.

[2] A. Grishkov, An analog of exponential map and Campbell–Hausdorff series in characteristic 3, to appear.

[3] N. Jacobson, *Lie algebras*, Interscience Publishers, 1962.

[4] N. Jacobson, *Structure and representations of Jordan algebras*, AMS, Providence, RI, 1968.

[5] I. P. Shestakov, Every Akivis algebra is linear, Geometriae dedicata, to appear.

[6] I. P. Shestakov, Speciality and Deformations of Algebras, Proceedings of the International Algebraic Conference dedicated to memory of A.G.Kurosh, May 1998, Moscow, to appear.

[7] K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, A. I. Shirshov, *Rings that are nearly associative*, Moscow, Nauka, 1978 (in Russian); English translation by Academic Press: N.Y., 1982.