A CHARACTERIZATION OF THE KOSTRIKIN RADICAL OF A LIE ALGEBRA

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Abstract. In this paper we study if the Kostrikin radical of a Lie algebra is the intersection of all its strongly prime ideals, and prove that this result is true for Lie algebras over fields of characteristic zero, for Lie algebras arising from associative algebras over rings of scalars with no 2-torsion, for Artinian Lie algebras over arbitrary rings of scalars, and for some others. In all these cases, this implies that nondegenerate Lie algebras are subdirect products of strongly prime Lie algebras, providing a structure theory for Lie algebras without any restriction on their dimension.

The theory of radicals constitutes an important tool in the study of rings. This notion appears firstly in the context of non-associative rings: in a work of E. Cartan about finite dimensional Lie algebras A over $\mathbb{C}$, he defined the maximal solvable ideal of A as the sum of all solvable ideals of A and proved that A is semisimple (direct sum of simple ideals) if and only if its radical is zero.

For an associative ring R, the Baer radical $r(R)$ is defined as the intersection of all prime ideals of R, so $R/r(R)$ is a subdirect product of prime rings, and $r(R)$ coincides with the smallest ideal of R such that $R/r(R)$ is semiprime, see [18]. Similarly, for Jordan systems J one finds the notion of McCrimmon radical $Mc(J)$, which is the least ideal of J such that $J/Mc(J)$ is nondegenerate. It coincides with the intersection of all strongly prime ideals of J, $J/Mc(J)$ is a subdirect product of strongly prime Jordan systems, and $Mc(J)$ can be characterized as the set of elements such that any m-sequence starting with any of them has finite length, see [23] and [20].

For a Lie algebra L, the smallest ideal inducing a nondegenerate quotient is the Kostrikin radical $K(L)$. This radical was first studied by Filippov in [11]. We highlight the works of E. Zelmanov [25], [24] where the properties of $K(L)$ were established and used intensively. Among other properties, it is shown that the Kostrikin radical is inherited by subalgebras ($K(A) = A$ for any subalgebra $A \subset K(L)$) and by ideals ($K(I) = I \cap K(L)$ for any ideal I of L).

The goal of this paper is to try to answer the question: *Is the Kostrikin radical of a Lie algebra L the intersection of all strongly prime ideals of L?* A positive answer to this question would imply that any nondegenerate Lie algebra is a subdirect

2000 *Mathematics Subject Classification.* Primary 17B05; Secondary 17B60.

Key words and phrases. Lie algebra, absolute zero divisor, Kostrikin radical, strongly prime ideal, m-sequence.

The first author was partially supported by the MEC and Fondos FEDER MTM2007-62390 and by FMQ 264.

The second author was partially supported by the MEC and Fondos FEDER MTM2007-61978 and by FMQ 264 and FQM 3737.

product of strongly prime Lie algebras, providing a structure theory for Lie algebras without any restriction on their dimension.

In this paper we show that this question has a positive answer for the following types of Lie algebras:

(1) Nondegenerate Lie algebras satisfying a technical condition called $\hat{\mathcal{H}}$ (see 2.3) and such that every nonzero ideal of L contains nonzero Jordan elements, Theorem 2.12. In particular, nondegenerate Lie algebras of the form $L = L_n \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_{-n}$, $L_0 = \sum_{i=1}^{n} [L_i, L_{-i}]$, over a ring of scalars Φ with $\frac{1}{k} \in \Phi$ for every $0 \le k \le 4n$, Corollary 2.13. Furthermore, we relate the Kostrikin radical and the McCrimmon radical when the Lie and the Jordan structures are connected, Corollaries 2.7 and 2.8, and Proposition 2.9.

(2) Lie algebras over fields of characteristic zero, Theorem 3.10.

(3) Lie algebras arising from associative algebras over rings of scalars with no 2-torsion, Theorems 4.3, 4.7 and Remark 4.9. Moreover, in these cases we relate the Kostrikin radical of the Lie algebras with the Baer radical of the associative algebras.

(4) Nondegenerate Lie algebras with chain condition on annihilator ideals over arbitrary rings of scalars, Proposition 5.3; in particular, Artinian Lie algebras, Corollary 5.4.

The key point to prove that the Kostrikin radical is the intersection of all strongly prime ideals is to define m-sequences for Lie algebras (a notion similar to that of Jordan systems), and to characterize the elements of the Kostrikin radical as those for which every m-sequence starting with them has finite length. This characterization is true for Lie algebras of type (1) and (3). For Lie algebras as in (2) the notion of m-sequence needs to be generalized. We define generalized m-sequences for Lie algebras, see 3.5, and prove that for Lie algebras over fields of characteristic zero the Kostrikin radical coincides with the set of elements such that every generalized m-sequence starting with them has finite length, Corollary 3.9.

The paper is organized as follows. Section 1 consists on a preliminary section where we recall several notions and results that will be used in the paper. In order to relate the Kostrikin radical of a Lie algebra L and the McCrimmon radical of some Jordan structures associated to L, in Section 2 we define a technical property called $\hat{\mathcal{H}}$, which is satisfied by large families of Lie algebras such as Lie algebras over fields of characteristic zero, Lie algebras generated as algebras by ad-nilpotent elements of index at most n over a ring of scalars Φ with $\frac{1}{k} \in \Phi$ for $k = 1, 2, \ldots, 2n-2$, and Lie algebras with a short $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ with $L_0 = \sum_{i=1}^{n} [L_i, L_{-i}]$ over a ring of scalars Φ with $\frac{1}{k} \in \Phi$ for $k = 1, 2, \ldots, 4n$. There are different constructions to relate Lie and Jordan structures: associated to any ad-nilpotent element x of index ≤ 3 of a Lie algebra L one can build a Jordan algebra L_x, and the Kostrikin radical of L and the McCrimmon radical of L_x can be compared when L satisfies $\hat{\mathcal{H}}$: $Mc(L_x) = \{\overline{a} \in L_x \mid [x, [x, a]] \in K(L)\}$. Similarly, one has the notion of subquotient of a Lie algebra, which is a Jordan pair: if $V = (M, L/\operatorname{Ker} M)$ is the subquotient, then $Mc(V)^+ = M \cap K(L)$, and $Mc(V)^- = \{a + \operatorname{Ker} M \mid [M, [M, a]] \subset K(L)\}$ when L satisfies $\hat{\mathcal{H}}$. This result generalizes the one given by E. Zelmanov in [24, Lemma 3] where he proved that the McCrimmon radical of the Jordan pair (V^+, V^-) consisting of two abelian inner ideals V^+ and V^- of a Lie algebra L satisfies $[[Mc(V)^\sigma, V^{-\sigma}], V^\sigma] \subset K(L)$ and where he related the Kostrikin radical of a Lie algebra L with a short $\mathbb{Z}$-grading

$L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ and the McCrimmon radical of the Jordan pair $V = (L_{-n}, L_n)$.

Under the technical property $\hat{\mathcal{H}}$ the ad-nilpotent elements of index 3 contained in the Kostrikin radical of L satisfy that any m-sequence starting with them has finite length. This makes possible to prove that Lie algebras with enough ad-nilpotent elements are nondegenerate if and only if they are subdirect product of strongly prime ones. In particular, this result applies to any Lie algebra with a finite $\mathbb{Z}$-grading, $L = \oplus_{i=-n}^{n} L_i$, $L_0 = \sum_{i=1}^{n} [L_i, L_{-i}]$, over a ring of scalars of characteristic bigger than $4n$.

Section 3 of the paper follows a private communication with E. Zelmanov where he dropped the hypothesis of having enough idempotents when dealing with Lie algebras over a field of characteristic zero. Basically Section 3 is [21] with some minors changes made by us. We are grateful to E. Zelmanov for allowing us to include them in the final version of this paper. We highlight the notion of generalized m-sequence, which is the key point for the results contained in this section.

In Section 4 we relate the Baer radical of an associative algebra R and the Kostrikin radical of Lie algebras of the form R^- or $\mathrm{Skew}(R, *)$ when R is an associative algebra with involution over a ring of scalars with no 2-torsion. Roughly speaking, the Kostrikin radical of these algebras coincides with the center of R^- or $\mathrm{Skew}(R, *)$ modulo the Baer radical $r(R)$ of R.

Finally, in Section 5 we study Lie algebras satisfying chain conditions on annihilator ideals and defined over arbitrary rings of scalars; in particular, Artinian Lie algebras and Lie algebras with essential socle.

We remark that each Section 2, 3, 4 and 5 can be read independently.

1. Nondegenerate Radicals

1.1. We will be dealing with Lie algebras L, (linear) Jordan algebras J and (linear) Jordan pairs. As usual, given a Lie algebra L, $[x, y]$ will denote the Lie bracket, with ad_x (sometimes denoted by X) the adjoint map determined by x , Jordan algebras J have bilinear product $a \bullet b$, with quadratic operator $U_a b = 2(a \bullet b) \bullet a - a^2 \bullet b$, and Jordan pairs $V = (V^+, V^-)$ have triple products $\{x, y, z\} \in V^\sigma$, for $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$.

1.2. We recall that a (non-necessarily associative) algebra A is a subdirect product of algebras $\{A_\alpha\}_{\alpha\in\Lambda}$ if there exists a monomorphism $f : A \to \prod_{\alpha\in\Lambda} A_\alpha$ such that for every $\beta \in \Lambda$, $\pi_\beta \circ f : A \to A_\beta$ is onto, where $\pi_\beta : \prod_{\alpha\in\Lambda} A_\alpha \to A_\beta$ denotes the canonical projection. Notice that this is equivalent to the existence of a family of ideals $\{I_\alpha\}_{\alpha\in\Lambda}$ of A such that $\bigcap_{\alpha\in\Lambda} I_\alpha = 0$ and $A_\alpha \cong A/I_\alpha$ for all $\alpha \in \Lambda$. A subdirect product of $\{A_\alpha\}_{\alpha\in\Lambda}$ will be called an essential subdirect product if A contains an essential ideal of the direct product $\prod_{\alpha\in\Lambda} A_\alpha$. Recall that an ideal I of an algebra A is essential if it intersects nontrivially any nonzero ideal K of A, i.e, $I \cap K \neq 0$ for every nonzero ideal K of A.

1.3. A (non-necessarily associative) algebra A is semiprime if for every nonzero ideal I of A, $I^2 := \{xy \mid x, y \in I\} \neq 0$, and it is prime if $IJ := \{yx \mid y \in I, x \in J\} \neq 0$ for every nonzero ideals I, J of A. Moreover, an ideal I of A is semiprime (prime) if the quotient algebra A/I is semiprime (prime). It is well known that every semiprime ideal I of an algebra A is the intersection of all prime ideals of A which contain I, see [4, 18]. This result implies that the Baer or semiprime radical $r(A)$ of an algebra A is the intersection of all prime ideals of A and therefore that semiprime algebras are exactly subdirect products of prime ones.

1.4. An important characterization of primeness and semiprimeness in the associative setting appears in [18]: A ring R is prime if and only if $aRb \neq 0$ for arbitrary nonzero elements $a, b \in R$ and it is semiprime if and only if $aRa \neq 0$ for every nonzero element $a \in R$. Unfortunately (or fortunately) in a general non-associative setting, due to the difficulty of building ideals, these characterizations do not hold. Nevertheless, the above characterizations give rise to new concepts in the Lie and Jordan settings, nondegeneracy and strong primeness (these notions have not been defined in a general nonassociative context): An absolute zero divisor in a Jordan algebra J is an element $x \in J$ such that the quadratic operator $U_x = 0$. A Jordan algebra J is called nondegenerate if it has no nonzero absolute zero divisors and it is strongly prime if J is nondegenerate and prime. An element x in a Lie algebra L is ad-nilpotent of index $k \in \mathbb{N}$ if $\mathrm{ad}_x^k L = 0$ but $\mathrm{ad}_x^{k-1} L \neq 0$. An absolute zero divisor of L is an ad-nilpotent element of index ≤ 2. A Lie algebra L is nondegenerate if it has no nonzero absolute zero divisors and it is strongly prime if L is nondegenerate and prime. Note that if a Lie or Jordan algebra is nondegenerate, then it is semiprime.

1.5. Let L be a Lie algebra. By a nondegenerate (strongly prime) ideal of L we mean an ideal I of L such that the quotient algebra L/I is nondegenerate (strongly prime). The Kostrikin radical $K(L)$ of L is the smallest ideal of L whose associated quotient algebra $L/K(L)$ is nondegenerate. It is radical in the sense of Amitsur-Kurosh, see [11], and can be constructed in the following way: $K_0(L) = 0$ and let $K_1(L)$ be the ideal of L generated by all absolute zero divisors of L; using transfinite induction we define a chain of ideals $K_\alpha(L)$ by $K_\alpha(L) = \bigcup_{\beta<\alpha} K_\beta(L)$ for a limit ordinal α, and $K_\alpha(L)/K_{\alpha-1}(L) = K_1(L/K_{\alpha-1}(L))$ otherwise. The Kostrikin radical of L is defined as $K(L) = \bigcup_\alpha K_\alpha(L)$. By construction, $K(L)$ is the smallest nondegenerate ideal of L, see [24].

1.6. Let J be a Jordan algebra. We will say that an ideal I of J is a nondegenerate (strongly prime) ideal of J if the quotient algebra J/I is nondegenerate (strongly prime). The McCrimmon radical or small radical $Mc(J)$ of a Jordan algebra J is the smallest ideal of J whose associated quotient algebra $J/Mc(J)$ is nondegenerate. It is radical in the sense of Amitsur-Kurosh, see [20, Theorem 4], and can be constructed in the following way: $Mc_0(J) = 0$ and let $Mc_1(J)$ be the subalgebra of J generated by all absolute zero divisors of J ($Mc_1(J)$ is an ideal of J, see [19, Theorem 9]); using transfinite induction we define a chain of ideals $Mc_\alpha(J)$ by $Mc_\alpha(J) = \bigcup_{\beta<\alpha} Mc_\beta(J)$ for a limit ordinal α, and $Mc_\alpha(J)/Mc_{\alpha-1}(J) = Mc_1(J/Mc_{\alpha-1}(J))$ otherwise. Then the McCrimmon radical of J is defined as $Mc(J) = \bigcup_\alpha Mc_\alpha(J)$. Note that $Mc(J)$, by construction, is a nondegenerate ideal and is contained in any nondegenerate ideal of J, see [19], [16].

1.7. For any Jordan system J one has the notion of m-sequence: It is a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $a_{n+1} = U_{a_n} b$ for some $b \in J$. We will say that an m-sequence of J has length k if $a_k \neq 0$ and $a_{k+1} = 0$. There is a beautiful characterization of the elements of the McCrimmon radical in terms of m-sequences: An element $x \in J$ is contained in $Mc(J)$ if and only if any m-sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_1 = x$ has finite length, i.e., there exists $k \in \mathbb{N}$ such that $a_k = 0$. From this property it is shown that the McCrimmon radical of a Jordan algebra J coincides with the intersection of all strongly prime ideals of J or, equivalently, that every nondegenerate ideal I of a Jordan algebra J is the intersection of all strongly prime ideals of J containing I, see [23] for the linear case and [20] for its quadratic extension.

1.8. Following the notion of m-sequence introduced in the previous paragraph for Jordan algebras, we define an analogous concept in the context of Lie algebras: Let L be a Lie algebra. An m-sequence is a set $\{a_n\}_{n\in\mathbb{N}}$ such that $a_{n+1} = [a_n, [a_n, b_n]]$ for some $b_n \in L$. We will say that a m-sequence of L has length k if $a_k \neq 0$ and $a_{k+1} = 0$. Note that, if $x \in L$ satisfies that $[x, [x, L]] \subset K(L)$, then $x \in K(L)$ (because $\overline{x} = x + K(L)$ is an absolute zero divisor in the nondegenerate Lie algebra $L/K(L)$). So if any m-sequence of L starting with x has finite length, then $x \in K(L)$.

2. Lie algebras with enough ad-nilpotent elements

2.1. Let L be a Lie algebra over a ring of scalars Φ such that $\frac{1}{2}, \frac{1}{3} \in \Phi$. We say that an element x in L is a Jordan element if x is ad-nilpotent of index ≤ 3, i.e., if $\mathrm{ad}_x^3 = 0$. Every Jordan element gives rise to a Jordan algebra, called the Jordan algebra of L at x, see [8]: Let L be a Lie algebra and let $x \in L$ be a Jordan element. Then L with the new product given by $a \bullet b := \frac{1}{2}[[a, x], b]$ is an algebra such that

$$\ker(x) := \{z \in L \mid [x, [x, z]] = 0\}$$

is an ideal of $(L, \bullet)$. Moreover, $L_x := (L/\ker(x), \bullet)$ is a Jordan algebra. In this Jordan algebra the U-operator has this very nice expression:

$$U_{\overline{a}}\overline{b} = \frac{1}{8}\overline{\mathrm{ad}_a^2\,\mathrm{ad}_x^2\, b}, \quad \text{for all } a, b \in L, \quad \text{and}$$

$$\{\overline{a}, \overline{b}, \overline{c}\} = -\frac{1}{4}\overline{[a, [\mathrm{ad}_x^2\, b, c]]} \quad \text{for all } a, b, c \in L$$

A Lie algebra is nondegenerate if and only if L_x is nonzero for every Jordan element $x \in L$. Moreover, L_x inherits nondegeneracy from L [8, 2.15(i)].

An inner ideal of L is a subspace M of L such that $[M, [M, L]] \subset M$. It is an abelian inner ideal if it is also an abelian subalgebra, i.e., $[M, M] = 0$. The kernel of M is the set $\mathrm{Ker}_L\, M = \{x \in L : [M, [M, x]] = 0\}$. If M is abelian, then $\mathrm{Ker}_L\, M = \{x \in L : [m, [m, x]] = 0 \text{ for every } m \in M\}$. For any abelian inner ideal M of L, the pair $V = (M, L/\mathrm{Ker}_L\, M)$ with the triple products given by

$$\{m, \overline{a}, n\} := [[m, a], n] \quad \text{for every } m, n \in M \text{ and } a \in L$$

$$\{\overline{a}, m, \overline{b}\} := \overline{[[a, m], b]} \quad \text{for every } m \in M \text{ and } a, b \in L,$$

where $\overline{x}$ denotes the coset of x relative to the submodule $\mathrm{Ker}_L\, M$, is a Jordan pair called the subquotient of L with respect to M. When L is nondegenerate, the notion of subquotient generalizes that of Jordan algebra of a Lie algebra: if x is a Jordan element, M is the abelian inner ideal generated by x, and we consider the subquotient $V = (M, L/\mathrm{Ker}\, M)$ defined by M, then the Jordan homotope algebra $V^{(x)}$ coincides with the Jordan algebra L_x of L at x, cf. [9, §3].

Proposition 2.2. *Let L be a Lie algebra over a ring of scalars Φ such that $\frac{1}{2}, \frac{1}{3} \in \Phi$ and let $x \in L$ be a Jordan element. Then for every $a \in L$ every m-sequence of L of length k starting with $[x, [x, a]]$ gives rise to an m-sequence of L_x starting with $\overline{a}$ with the same length, and vice versa.*

Proof. Let $\{\overline{c_n}\}$ be an m-sequence in L_x. Let us prove that $\{a_n\}$, with $a_n := [x,[x,c_n]]$ is an m-sequence of L with the same number of nonzero terms: we know that for every $n \in \mathbb{N}$ there exists $\overline{b_n} \in L_x$ such that $\overline{c_{n+1}} = U_{\overline{c_n}}\overline{b_n} = \overline{\mathrm{ad}^2_{c_n}\mathrm{ad}^2_x b_n}$. So

$$\mathrm{ad}_{a_n} b_n = \mathrm{ad}^2_x \mathrm{ad}^2_{c_n} \mathrm{ad}^2_x b_n = \mathrm{ad}^2_x c_{n+1} = a_{n+1}.$$

Moreover, by construction, $\overline{a_n} \neq \overline{0}$ if and only if $[x,[x,a_n]] \neq 0$.

Conversely, let $\{a_n\}_{n\in\mathbb{N}}$ be an m-sequence of L with $a_1 = [x,[x,a]]$ and let us consider $b_n \in L$ such that $a_{n+1} = [a_n,[a_n,b_n]]$ for every $n \in \mathbb{N}$. Let us prove that for every n there exists $c_n \in L$ such that $a_n = [x,[x,c_n]]$: The case $n = 1$ holds by hypothesis. So let us suppose that there exists $c_n \in L$ such that $a_n = [x,[x,c_n]]$. Then

$$a_{n+1} = [a_n,[a_n,b_n]] = \mathrm{ad}^2_{\mathrm{ad}^2_x c_n} b_n = \mathrm{ad}^2_x \mathrm{ad}^2_{c_n} \mathrm{ad}^2_x b_n \tag{1}$$

Now, formula (1) implies that $\{\overline{c}_n\}_{n\geq 2}$ is an m-sequence of L_x because

$$U_{\overline{c}_n}\overline{b_n} = \overline{\mathrm{ad}^2_{c_n}\mathrm{ad}^2_x b_n} = \overline{c_{n+1}}$$

with $\overline{c}_n \neq \overline{0}$ if $a_n \neq 0$. $\square$

2.3. Let α be an ordinal. We say that a Lie algebra L satisfies the property $\mathcal{H}_\alpha$ if for every ordinal β with $\beta \leq \alpha$ which is not a limit ordinal we have that every submodule of $L_\beta := K_\beta(L)/K_{\beta-1}(L)$ which is invariant under inner automorphisms of L_β is indeed an ideal of L_β. Notice that the property $\mathcal{H}_\alpha$ just means that the Lie algebra L_β satisfies the property $\mathcal{H}_1$.

We say that L satisfies the property $\mathcal{H}$ if L satisfies the property $\mathcal{H}_\alpha$ for every ordinal α.

2.4. Examples: Although the property $\mathcal{H}$ might seems somehow "technical", it is satisfied by many Lie algebras. For example:
(i) Every Lie algebra L over a ring of scalars Φ with no torsion which is generated as an algebra by ad-nilpotent elements satisfies the property $\mathcal{H}$. Also if L is generated by ad-nilpotent elements of index at most m and $\frac{1}{k} \in \Phi$ for $1 \leq k \leq 2m-2$, L satisfies the property $\mathcal{H}$. Indeed, using a Vandermonde argument, it is easy to see that every submodule of such L which is invariant under inner automorphisms is an ideal of L.
(ii) Every $\mathbb{Z}$-graded Lie algebra $L = \bigoplus_{i=-n}^{n} L_i$ with $L_0 = \sum_{i=1}^{n}[L_i, L_{-i}]$, defined over a ring of scalars Φ with $\frac{1}{k} \in \Phi$ for $1 \leq k \leq 4n$, satisfies the property $\mathcal{H}$ (just notice that L is generated by ad-nilpotent elements of index at most $2n+1$).
(iii) For every Lie algebra L over a field of zero characteristic one has that for every ordinal α and for every $\beta \leq \alpha$ which is not a limit ordinal $L_\beta = K_\beta(L)/K_{\beta-1}(L) = K_1(L/K_{\beta-1}(L))$ consists on ad-nilpotent elements [24, Lemma 8], so every submodule of L_β which is invariant under inner automorphisms is an ideal by the argument given in (i). Therefore, Lie algebras over fields of characteristic zero satisfy the property $\mathcal{H}$.

Proposition 2.5. *Let L be a Lie algebra over over a ring of scalars Φ and let $x_1, x_2, \ldots, x_n$ be absolute zero divisors of L such that $x = x_1 + \cdots + x_n \in C_1$ is a Jordan element. If L satisfies the property $\mathcal{H}_1$ then $Mc(L_x) = L_x$.*

Proof. First notice that C_1 is an ideal of $K_1(L)$ and $\overline{C_1} \subset Mc(L_x)$ since every absolute zero divisor z of L gives rise to an absolute zero divisor $\overline{z}$ of L_x: $U_{\overline{z}}\overline{a} = \overline{\mathrm{ad}^2_z \mathrm{ad}^2_x a} = \overline{0}$.

Then, for any $a, b \in L$,

$$U_{\overline{a}}\overline{b} = \overline{\operatorname{ad}_a^2 \operatorname{ad}_x^2 b} \subset \overline{\operatorname{ad}_a^2 C_1} \tag{1}$$

since $\operatorname{ad}_x^2 b \in [x, K_1(L)] \subset C_1$, and for every absolute zero divisor $z \in L$,

$$U_{\overline{\operatorname{ad}_a^2 z}}\overline{b} = \overline{[\operatorname{ad}_a^2 z, [\operatorname{ad}_a^2 z, \operatorname{ad}_x^2 b]]} \subset \overline{[\operatorname{ad}_a^2 z, [\operatorname{ad}_a^2 z, C_1]]} \subset \overline{C_1} \subset Mc(L_x)$$

for every $\overline{b} \in L_x$ and therefore $\overline{\operatorname{ad}_a^2 z} \in Mc(L_x)$, which implies that $\overline{\operatorname{ad}_a^2 C_1} \subset Mc(L_x)$ and therefore, by (1), $U_{\overline{a}}\overline{b} \subset Mc(L_x)$ for every $\overline{b} \in L_x$, so $\overline{a} \in Mc(L_x)$, i.e., $Mc(L_x) = L_x$. □

Theorem 2.6. *Let L be a Lie algebra that satisfies $\mathcal{H}_1$ (resp. $\mathcal{H}$) and let $x \in K_1(L)$ (resp. $x \in K(L)$) be a Jordan element. Then every m-sequence of L which starts with x has finite length and $Mc(L_x) = L_x$.*

Proof. Using transfinite induction, we define the following ascending chain of ideals of $K_1(L)$: let $C_1(L)$ be the submodule of $K_1(L)$ generated by all absolute zero divisors of L, which is an ideal of $K_1(L)$ since it is invariant under (inner) automorphisms (of $K_1(L)$), and define $C_\alpha(L) = \bigcup_{\beta<\alpha} C_\beta(L)$ for a limit ordinal α, and

$$C_\alpha(L)/C_{\alpha-1}(L) = C_1(K_1(L)/C_{\alpha-1}(L))$$

otherwise.

By construction every $C_\alpha(L) \subset K_1(L)$ and $K_1(L)/\bigcup C_\alpha(L)$ is a nondegenerate Lie algebra, so $K_1(L) = \bigcup C_\alpha(L)$. Now, if $x \in K_1(L)$ is a Jordan element there exists an ordinal α such that $x \in C_\alpha(L)$ and $x \notin C_\beta(L)$ for $\beta < \alpha$. Hence if $\{x_i\}_{i\in\mathbb{N}}$ is an m-sequence which starts with x, by Proposition 2.5 there exists $n \in \mathbb{N}$ such that $x_n \in C_{\alpha-1}(L)$, which implies by transfinite induction that $\{x_i\}$ has finite length and, therefore, $Mc(L_x) = L_x$ by Proposition 2.2.

If $x \in K(L)$ is a Jordan element, there exists an ordinal α which is not a limit ordinal such that $x \in K_\alpha(L)$, so $\bar{x} = x + K_{\alpha-1}(L) \in K_\alpha(L)/K_{\alpha-1}(L) = K_1(L/K_{\alpha-1}(L))$. By the above, every m-sequence starting with $\bar{x}$ ends in $K_{\alpha-1}(L)$ in a finite number of steps and, by transfinite induction, every m-sequence starting with x has finite length. Moreover, $Mc(L_x) = L_x$ by Proposition 2.2. □

Corollary 2.7. *Let L be a Lie algebra that satisfies $\mathcal{H}$ and let $x \in L$ be a Jordan element. Then $Mc(L_x) = \{\overline{a} \in L_x \mid [x, [x, a]] \in K(L)\}$.*

Proof. If $\overline{a} \in Mc(L_x)$, then every m-sequence of L_x starting with $\overline{a}$ has finite length. Therefore, by Proposition 2.2, every m-sequence of L starting with $[x, [x, a]]$ has finite length and therefore, by 1.8, $[x, [x, a]] \in K(L)$. Conversely, since x is a Jordan element, for every $a \in L$, $\operatorname{ad}_x^2 a$ is a Jordan element. So, if $[x, [x, a]] \in K(L)$, by Proposition 2.6 every m-sequence starting with $[x, [x, a]]$ has finite length in L, so the m-sequences of L_x starting with $\overline{a}$ have finite length by Proposition 2.2, which implies that $\overline{a} \in Mc(L_x)$. □

Corollary 2.8. *Let L be a Lie algebra that satisfies $\mathcal{H}$, M an abelian inner ideal of L and consider the subquotient $V = (M, L/KerM)$. Then*

$$Mc(V)^+ = M \cap K(L), \text{ and}$$
$$Mc(V)^- = \{a + \operatorname{Ker} M \mid [M, [M, a]] \subset K(L)\}.$$

Proof. Notice that $Mc(V)^+$ consists on the elements of M for which every m-sequence has finite length, so $Mc(V)^+ \subset K(L)$. Conversely, since every element of M is a Jordan element, if $x \in K(L) \cap M$ then it satisfies the m-sequence condition by Theorem 2.6, so it belongs to $Mc(V)$.

For the second equality, if $a + \operatorname{Ker} M$ belongs to $Mc(V)^-$, then $[m_1, [m_2, a]] = -\{m_1, a, m_2\} \in Mc(V)^+ \subset K(L)$ for every $m_1, m_2 \in M$. Conversely, if $a \in L$ has $[M, [M, a]] \in K(L)$ then $\{M, (a + \operatorname{Ker} L), M\} \subset K(L) \cap M = Mc(V)^+$, but this implies $a + \operatorname{Ker} M \in Mc(V)^-$ by [1, 3.4]. $\square$

The next result can be found in [25]. We give here an alternative proof.

Proposition 2.9. *Let V be a Jordan pair over over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$ and consider the Lie algebra $L = \mathrm{TKK}(V)$. Then, the Kostrikin radical $K(L)$ of L is a 3-graded ideal with $\pi_{\sigma 1}(K(L)) = Mc(V)^\sigma$, $\sigma = \pm$, where $\pi_{\sigma 1}$ denotes the canonical projection of L onto $L_{\sigma 1}$, and is isomorphic to the center of $L/\operatorname{id}_L(Mc(V)^+ \oplus Mc(V)^-)$.*

Proof. Notice that under these conditions, L satisfies $\mathcal{H}$ by 2.4. We will show that

$$K(L) = Mc(V)^+ \oplus (K(L) \cap [V^+, V^-]) \oplus Mc(V)^-. \tag{1}$$

Clearly, $Mc(V)^+ \oplus Mc(V)^+ \subset K(L)$ since all m-sequences starting with these elements have finite length. Conversely, let $y = y_1 + y_0 + y_{-1} \in K(L)$. If $[V^+, [y, V^+]] = [V^+, [y_{-1}, V^+]] \not\subset Mc(V)^+$, then we would have Jordan elements in $K(L) \cap V^+$ which do not belong to the McCrimmon radical of V, a contradiction with Theorem 2.6. Therefore, $[V^+, [y, V^+]] = \{V^+, y_{-1}, V^+\} \subset Mc(V)^+$, so $y_{-1} \in Mc(V)^-$ by [2, 3.4]. Similarly, $y_1 \in Mc(V)^+$.

Suppose that now that $y_0 \neq 0$. Then at least $[y_0, V^+] \neq 0$ or $[y_0, V^-] \neq 0$. Suppose $[y_0, V^+] \neq 0$. Since $[V^-, [V^-, [y, V^+]]] = [V^-, [V^-, [y_0, V^+]]] \subset I \cap V^- \subset Mc(V)^-$, then the Jordan triple product

$$\{V^-, [y_0, V^+], V^-\} \subset Mc(V)^-,$$

so $[y_0, V^+] \subset Mc(V)^+$ by [2, 3.4]. Therefore, $[y_0, [y_0, V^+]] \subset Mc(V)^+ \subset K(L)$, and similarly $[y_0, [y_0, V^-]] \subset K(L)$. Therefore $[y_0, [y_0, L]] \subset K(L)$, so also $y_0 \in K(L)$.

From (1), every $x \in K(L)$ satisfies $[x, L] \in \operatorname{id}_L(Mc(V)^+ \oplus Mc(V)^-)$, so $x + \operatorname{id}_L(Mc(V)) \in Z(L/\operatorname{id}_L(Mc(V)))$. Conversely, if $x \in L$ satisfies

$$[x, L] \in \operatorname{id}_L(Mc(V)^+ \oplus Mc(V)^-),$$

then $[x, [x, L]] \in \operatorname{id}_L(Mc(V)) \subset K(L)$, so $x \in K(L)$ by 1.8. $\square$

2.10. We will say that a Lie algebra satisfies the property $\hat{\mathcal{H}}$ if any epimorphic image of L satisfies $\mathcal{H}$. Note that all the examples given 2.4 satisfy this property.

Proposition 2.11. *Let L be a Lie algebra that satisfies $\hat{\mathcal{H}}$ and let M be an m-system of L of nonzero Jordan elements. Then every maximal ideal P of L with respect to the property $P \cap M = \emptyset$ is nondegenerate. Moreover, if M is an m-sequence of L, then P is strongly prime.*

Proof. Let P be a maximal ideal with respect to the property $P \cap M = \emptyset$.

Let us prove that P is nondegenerate: consider the canonical projection $\pi : L \to L/P$. Notice that L/P satisfies the property $\mathcal{H}$. Let us suppose that L/P is degenerate and let $K := \pi^{-1}(K(L/P))$ where $K(L/P)$ is the Kostrikin radical of L/P. By construction, since P is maximal, there exists $x \in M \cap K$ and therefore an m-sequence $\{x_i\}$ which starts with x contained in M. But $\{\overline{x}_i\}$ is an infinite m-sequence in L/P. So, by Theorem 2.6, $\overline{x} \notin K(L/P) \supseteq L/P$, a contradiction.

Now, let us suppose that $M = \{a_n\}_{n\in\mathbb{N}}$ is an m-sequence and let I, J be two ideals of L with $P \subsetneq I$ and $P \subsetneq J$. Then, since P is maximal with respect to $P \cap M = \emptyset$, there exists $i, j \in \mathbb{N}$ such that $a_i \in I$ and $a_j \in J$. Moreover, if $k \geq \max\{i, j\}$, $a_k \in I \cap J$ and $0 \neq a_{k+1} = [a_k, [a_k, b_k]] \in [I, J]$ with $a_{k+1} \notin P$ which proves that P is a strongly prime ideal of L. □

In the following results we will require that every nonzero ideal of L contains nonzero Jordan elements. If the ring of scalars Φ has $\frac{1}{k} \in \Phi$ for every $0 \leq k \leq r$, this hypothesis can be achieved as soon as every ideal of L contains nonzero ad-nilpotent elements of index at most n for $n + [\frac{n}{2}] - 1 \leq r$, see [17, Lemma 1.1, p. 31].

Theorem 2.12. *Let L be a nondegenerate Lie algebra that satisfies the property $\hat{\mathcal{H}}$ and such that every nonzero ideal of L contains nonzero Jordan elements. Then, the intersection of all strongly prime ideals of L is zero. Consequently, L is nondegenerate if and only if it is a subdirect product of strongly prime Lie algebras.*

Proof. We will show that for any nonzero element x of L we can always find a strongly prime ideal of L that does not contain x. Let $I := \mathrm{id}_L(x)$ be the ideal of L generated by x. By hypothesis there is a nonzero Jordan element y of L contained in I. Now, we can construct the following m-sequence of L of infinite length $N = \{a_i\}_{i\in\mathbb{N}}$: $a_1 = y$, and given any $a_i \neq 0$ define $a_{i+1} = [a_i, [a_i, x_i]]$ for any $x_i \in L$ such that $0 \neq [a_i, [a_i, x_i]]$ (there exists such x_i because L is nondegenerate). By Zorn Lemma there exists a maximal ideal in $\{J \triangleleft L \mid J \cap N = \emptyset\}$, which is strongly prime ideal of L by Proposition 2.11 and, by construction, it does not contain y and therefore it does not contain x. □

As mentioned in 2.10, all Lie algebras listed in 2.4 satisfy the property $\hat{\mathcal{H}}$ and, therefore, as soon as all they are nondegenerate and all their nonzero ideals contain nonzero Jordan elements, the intersection of all their strongly prime ideals is zero and they are a subdirect product of strongly prime Lie algebras. Furthermore, all nonzero ideals of a nondegenerate Lie algebra with a finite $\mathbb{Z}$-grading of the form $L = L_n \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_{-n}$, $L_0 = \sum_{i=1}^{n}[L_i, L_{-i}]$, and $\frac{1}{k} \in \Phi$ for every $0 \leq k \leq 4n$, always contain nonzero Jordan elements, and therefore, for such Lie algebras Theorem 2.12 reads as follows:

Corollary 2.13. *Let $L = L_n \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_{-n}$, $L_0 = \sum_{i=1}^{n}[L_i, L_{-i}]$, be a nondegenerate Lie algebra with a finite $\mathbb{Z}$-grading over a ring of scalars Φ with $\frac{1}{k} \in \Phi$ for every $0 \leq k \leq 4n$. Then, the intersection of all strongly prime ideals of L is zero. Consequently, L is nondegenerate if and only if it is a subdirect product of strongly prime Lie algebras.*

Proof. It is enough to prove that nonzero ideals of L have nonzero Jordan elements: let I be a nonzero ideal of L and consider the biggest natural $k \in \mathbb{N}$ such that $\pi_s(I) = 0$ for all $|s| > k$. Then, by nondegeneracy of the ideal $\pi(I) = \pi_k(I) \oplus \cdots \oplus \pi_0(I) \oplus \cdots \oplus \pi_{-k}(I)$ of L, $0 \neq [\pi_k(I), [\pi_k(I), \pi(I)]] = [\pi_k(I), [\pi_k(I), \pi_{-k}(I)]] = [\pi_k(I), [\pi_k(I), I]] \subset I \cap \pi_k(I)$ consists of Jordan elements, and thus the claim follows by Theorem 2.12. □

3. Lie algebras over fields of characteristic zero

Section 2 deals with Lie algebras with enough ad-nilpotent elements. In particular, Theorem 2.12 applies for nondegenerate Lie algebras over fields of characteristic zero such that every nonzero ideal contains at least a nonzero ad-nilpotent element. The hypothesis of having enough ad-nilpotent elements will be removed in this section, where we will prove that a similar theorem holds in general. The results contained in this section were outlined by E. Zelmanov in a private communication [21] to the authors. We are grateful to him for allowing us to include them in the final version of this paper.

Lemma 3.1. *Given a Lie algebra L over a field of characteristic zero and $0 \neq a \in L$ an ad-nilpotent element of index $s > 3$, there exists $a_1 \in L$ such that $[a, a_1] \neq 0$ is ad-nilpotent of index at most 3.*

Proof. In characteristic zero every element of the form $\mathrm{ad}_a^{s-1} x \in [a, L]$ is ad-nilpotent of index at most 3 for any $x \in L$ [13]. □

Lemma 9 of [24] gives conditions that guarantee that an element of L belong to the Kostrikin radical of L. In the next proposition we weaken these conditions.

Proposition 3.2. *Given a Lie algebra L over a field of characteristic zero, if $a \in L$ is such that there exists $q \in \mathbb{N}$ with*

$$\mathrm{ad}_a^q x_0 = \mathrm{ad}_{[a,x_1]}^q x_0 = \mathrm{ad}_{[[a,x_1],x_2]}^q x_0 = 0, \quad \textit{for all } x_0, x_1, x_2 \in L$$

then $a \in K(L)$.

Proof. We can work in $L/K(L)$, assume that L is nondegenerate, and show that $a = 0$. Suppose that $a \neq 0$ and let s be the index of ad-nilpotency of a. If $s > 3$, take $a_1 \in L$ given by Lemma 3.1, and let $b = [a, a_1] \neq 0$, which is ad-nilpotent of index 3; if $s \leq 3$, let $b = a$. Since by hypothesis $[x, b]$ is ad-nilpotent of index at most q for all $x \in L$, every element $\bar{x}$ of the Jordan algebra L_b, see 2.1, is nilpotent of index at most $q + 1$. Indeed, since $\bar{x}^{(n,b)} = \bar{x} \bullet \bar{x}^{(n-1,b)}$, one readily has that

$$\bar{x}^{(2,b)} = \frac{1}{2}\,\overline{[[x, b], x]}, \quad \bar{x}^{(3,b)} = \frac{1}{4}\,\overline{[[x, b], [[x, b], x]]}, \ldots \quad \bar{x}^{(n,b)} = \frac{1}{2^{n-1}}\,\overline{\mathrm{ad}_{[x,b]}^{n-1} x}.$$

Therefore L_b is radical in the sense of McCrimmon, see [22, Lemma 17, pag 849]. But the Jordan algebras of nondegenerate Lie algebras are nondegenerate, see 2.1, so $L_b = Mc(L_b) = 0$, which implies that $\mathrm{Ker}\, b = L$, so $[b, [b, L]] = 0$, i.e., b is an absolute zero divisor, hence $b = 0$, a contradiction. □

3.3. Given $n \in \mathbb{N}$ and a Lie algebra L, let

$$B_n(L) = \{\sum_{i=1}^{n} [[[a_i, b_{i_1}], \ldots, b_{i_{k_i}}]] \mid 0 \leq k_i \leq n,\ b_{i_j} \in L,\ \mathrm{ad}_{a_i}^2 = 0\}$$

be the sums of n monomials in L whose distance to an absolute zero divisor of L is less than or equal to n. Notice that $B_1 \subset B_2 \subset \cdots \subset B_n$ and $K_1(L) = \bigcup_n B_n$.

Lemma 3.4. *For each $n, r \in \mathbb{N}$ there exists $f(n, r) \in \mathbb{N}$ with $f(n, r) \geq 3$ such that for every Lie algebra L over a field of characteristic zero and for every $a \in B_n(L)$*

$$\mathrm{ad}_{[[a,b_1],\ldots,b_k]}^{f(n,r)} = 0 \textit{ for every } b_1, \ldots, b_k \in L,\ 0 \leq k \leq r.$$

Proof. This proof is inspired by [24, Lemma 8]. Let

$$X := \{x_0\} \cup \{x_i \mid i \in \mathbb{N}\} \cup \{x_{ij} \mid i, j \in \mathbb{N}\} \cup \{y_i \mid i \in \mathbb{N}\}$$

and consider the free Lie algebra $\mathcal{L}[X]$. Let $\bar{\mathcal{L}}[X] = \mathcal{L}[X]/\operatorname{Id}_{\mathcal{L}[X]}(\operatorname{ad}^2_{x_i} \mathcal{L}[X] \mid i \in \mathbb{N})$, in which every $\bar{x}_i$ is an absolute zero divisor. For every $n, r \in \mathbb{N}$, define

$$A_{n,r} := \{\sum_{i=1}^{n} [[[[\bar{x}_i, \bar{x}_{i1}], \dots, \bar{x}_{ik_i}], \bar{y}_1], \dots, \bar{y}_k] \mid 0 \le k_i \le n,\ 0 \le k \le r\}.$$

Notice that $A_{n,r} \subset K_1(\bar{\mathcal{L}}[X])$, and it is a finite set, hence also the set $A_{n,r} \cup [A_{n,r}, x_0] \subset K_1(\bar{\mathcal{L}}[X])$ has of a finite number of elements. For fixed $n, r \in \mathbb{N}$, the set $D_{n,r} = \operatorname{Subalg}_{\bar{\mathcal{L}}[X]}(A_{n,r} \cup [A_{n,r}, x_0])$ is nilpotent by a result of Grishkov [14], so there exists $f(n,r) \ge 3$ such that $D_{n,r}^{f(n,r)} = 0$.

Let now L be a Lie algebra, let $a \in B_n(L)$, fix $r \in \mathbb{N}$ and let $b_1, \dots, b_k$ be arbitrary elements of L, $1 \le k \le r$, and $c \in L$. We want to show that $\operatorname{ad}^{f(n,r)}_{[[a,b_1],\dots,b_k]} c = 0$. Since $a \in B_n(L)$, $a = \sum_{i=1}^{n} [[[a_i, b_{i_1}], \dots, b_{i_{k_i}}]]$ for certain absolute zero divisors $a_i \in L$, and certain $b_{ij} \in L$, $i = 1, \dots, n$, $j = 1, \dots, k_i$, $0 \le k_i \le n$. There exists a unique homomorphism of Lie algebras $\varphi : \mathcal{L}[X] \to L$ such that $\varphi(x_0) = c$; $\varphi(x_i) = a_i$ if $1 \le i \le n$ and $\varphi(x_i) = 0$ otherwise; $\varphi(x_{ij}) = b_{ij}$ if $1 \le i \le n$, $1 \le j \le k_i$, and $\varphi(x_{ij}) = 0$ otherwise; $\varphi(y_i) = b_i$ if $1 \le i \le k$ and $\varphi(y_i) = 0$ otherwise. Moreover, since

$$\varphi(\operatorname{Id}_{\mathcal{L}[X]}(\operatorname{ad}^2_{x_i} \mathcal{L}[X] \mid i = 1, \dots, n)) \subset \operatorname{Id}_L(\operatorname{ad}^2_{a_i} L \mid i = 1, \dots, n) = 0,$$

φ gives rise to a unique homomorphism of Lie algebras $\bar{\varphi} : \bar{\mathcal{L}}[X] \to L$ such that $\bar{\varphi}(\bar{x}_0) = c$, $\bar{\varphi}(\bar{x}_i) = a_i$, $1 \le i \le n$, $\bar{\varphi}(\bar{x}_{ij}) = b_{ij}$, $1 \le i \le n$, $1 \le j \le k_i$, and $\bar{\varphi}(\bar{y}_i) = b_i$, $1 \le i \le k$. Finally,

$$\begin{aligned}
&\operatorname{ad}^{f(n,r)}_{[[a,b_1],\dots,b_k]} c = \bar{\varphi}(\operatorname{ad}^{f(n,r)}_{[[\sum_{i=1}^{n}[[\bar{x}_i,\bar{x}_{i1}],\dots,\bar{x}_{ik_i}],\bar{y}_1],\dots,\bar{y}_k]} \bar{x}_0) \\
&= \bar{\varphi}(\operatorname{ad}^{f(n,r)}_{\sum_{i=1}^{n}[[[[\bar{x}_i,\bar{x}_{i1}],\dots,\bar{x}_{ik_i}],\bar{y}_1],\dots,\bar{y}_k]} \bar{x}_0) \in \bar{\varphi}(\operatorname{ad}^{f(n,r)-1}_{A_{n,r}}[A_{n,r}, x_0]) \\
&\subset \bar{\varphi}(D_{n,r}^{f(n,r)}) = 0.
\end{aligned}$$

□

3.5. Given a Lie algebra L over a field of characteristic zero, we say that the sequence $\{c_i\}_{i\in\mathbb{N}}$ is a *generalized m-sequence* of L if $c_1 \in L$ and each c_{i+1}, $i \ge 1$, is an element of the form

$$\operatorname{ad}^{q_i}_{c_i} x_0,\ \operatorname{ad}^{q_i}_{[c_i,x_1]} x_0,\ \text{or}\ \operatorname{ad}^{q_i}_{[[c_i,x_1],x_2]} x_0$$

for some $x_0, x_1, x_2 \in L$ and $q_i = f(i, 3i + 2)$.

Notice that for every i, since $q_i \ge 3$,

$$\begin{aligned}
&\operatorname{ad}^{q_i}_{c_i} x_0 \in [c_i, [c_i, [c_i, L]] \subset [[[c_i, L], L], L] \\
&\operatorname{ad}^{q_i}_{[c_i,x_1]} x_0 \in [[c_i, x_1], [[c_i, x_1], L]] \subset [[[c_i, L], L], L] \\
&\operatorname{ad}^{q_i}_{[[c_i,x_1],x_2]} x_0 \in [[[c_i, x_1], x_2], L] \subset [[[c_i, L], L], L]
\end{aligned}$$

so in each step $c_{i+1} \in [[[c_i, L], L], L]$.

Proposition 3.6. *If a generalized m-sequence $\{c_i\}_{i\in\mathbb{N}}$ in a Lie algebra L over a field of characteristic zero contains an element c_i in $K(L)$, the sequence has finite length.*

Proof. Suppose first that $c_i \in K_1(L) = \bigcup_m B_m$, so c_i belongs to certain B_n (it can be assumed that $n \geq i$). Let us show that $c_{n+1} = 0$: Since c_{i+1} is an element of the form $\mathrm{ad}_{c_i}^{q_i} x_0$, $\mathrm{ad}_{[c_i,x_1]}^{q_i} x_0$, or $\mathrm{ad}_{[[c_i,x_1],x_2]}^{q_i} x_0$ for some $x_0, x_1, x_2 \in L$, it can be expressed as an element $c_{i+1} \in [[[c_i, \underbrace{L], L], L}_{3}]$ by 3.5. Similarly,

$$c_{i+2} \in [[[c_{i+1}, \underbrace{L], L], L}_{3}] \subset [[[c_i, \underbrace{L], \ldots, L}_{3\cdot 2}]].$$

Finally, $c_n \in [[[c_i, \underbrace{L], \ldots, L}_{3(n-i)}]]$. Since $q_n = f(n, 3n+2)$

$$\mathrm{ad}_{c_n}^{q_n} x_0 = 0, \ \mathrm{ad}_{[c_n,x_1]}^{q_n} x_0 = 0, \ \text{and} \ \mathrm{ad}_{[[c_n,x_1],x_2]}^{q_n} x_0 = 0$$

for all $x_0, x_1, x_2 \in L$, so $c_{n+1} = 0$.

We will show by transfinite induction that if $c_i \in K_\alpha(L)$, then the generalized m-sequence $\{c_i\}_{i\in\mathbb{N}}$ has finite length. We have already shown the case $\alpha = 1$. Now assume that our assertion is true for every $\beta < \alpha$.

If α is a limit ordinal, $c_i \in \bigcup_{\beta<\alpha} K_\beta(L)$ so there exists some $\beta < \alpha$ such that $c_i \in K_\beta(L)$ and the sequence has finite length by the induction hypothesis. Otherwise, $\alpha = \beta + 1$ for some β and we can consider the corresponding generalized m-sequence in $L/K_\beta(L)$, $\{c_j + K_\beta(L)\}_{j\in\mathbb{N}}$ for which $c_i + K_\beta(L) \in K_1(L/K_\beta(L))$. By the case $\alpha = 1$ this sequence has finite length and there exists $c_k + K_\beta(L) = \bar{0}$, so $c_k \in K_\beta(L)$ and the result follows by induction. □

Proposition 3.7. *Let L be a Lie algebra over a field of characteristic zero, let $\{c_i\}_{i\in\mathbb{N}}$ be a generalized m-sequence of L, and let P be an ideal of L which is maximal among those ideals of L not containing any element of $\{c_i\}_{i\in\mathbb{N}}$. Then P is a strongly prime ideal of L, i.e., L/P is a strongly prime Lie algebra.*

Proof. To see that L/P is prime, if A/P and B/P are two nonzero ideals of L/P, there exist some $c_j \in A$, some $c_k \in B$, so $c_l \in A \cap B$ for every $l \geq j, k$. Then, $c_{\max(j,k)+1} \in [A, B]$ so $[A/P, B/P] \neq \bar{0}$.

To see that L/P is nondegenerate, suppose on the contrary that $K(L/P) \neq 0$. Consider $\hat{K} = \pi^{-1}(K(L/P))$, where $\pi : L \to L/P$ denotes the canonical projection, which is an ideal of L properly containing P, so there exists some $c_j \in \hat{K}$, hence $c_j + P \in K(L/P)$. By Proposition 3.6 the sequence $\{c_i + P\}_{i\in\mathbb{N}}$ has finite length, so there exists some $c_k + P = \bar{0}$, i.e., $c_k \in P$, a contradiction. □

Proposition 3.8. *Given a Lie algebra L over a field of characteristic zero, if $a \in L$ does not belong to $K(L)$ then there exists an infinite generalized m-sequence starting with a.*

Proof. Consider $\bar{0} \neq \bar{a} = a + K(L) \in L/K(L)$ and let $\bar{c}_0 = \bar{a}$. If $\bar{c}_i \neq \bar{0}$ then there exists $\bar{c}_{i+1} \neq \bar{0}$ since otherwise it would mean that $\mathrm{ad}_{\bar{c}_i}^{q_i} \bar{x}_0 = \mathrm{ad}_{[\bar{c}_i,x_1]}^{q_i} \bar{x}_0 = \mathrm{ad}_{[[\bar{c}_i,\bar{x}_1],\bar{x}_2]}^{q_i} \bar{x}_0 = 0$, for all $\bar{x}_0, \bar{x}_1, \bar{x}_2 \in L/K(L)$, $q_i = f(i, 3i+2)$, but by Proposition 3.2 this implies that $\bar{c}_i \in K(L/K(L)) = \bar{0}$, a contradiction. The infinite generalized m-sequence $\{\bar{c}_i\}$ in $L/K(L)$ induces an infinite generalized m-sequence in L. □

By Lemma 3.4 and Proposition 3.8 one readily has

Corollary 3.9. *Let L be a Lie algebra over a field of characteristic zero, and let $K(L)$ denote its Kostrikin radical. Then*

$$K(L) = \{x \in L \mid \text{ every generalized m-sequence starting with } x \text{ has finite length}\}.$$

Theorem 3.10. *The Kostrikin radical $K(L)$ of a Lie algebra L over a field of characteristic zero is the intersection of all strongly prime ideals of L. Therefore, $L/K(L)$ is isomorphic to a subdirect product of strongly prime Lie algebras.*

Proof. If $\{P_i\}$ denotes the set of all strongly prime ideals of L, it is clear that $K(L) \subset P_i$ for each i since L/P_i is nondegenerate, so $K(L) \subset \bigcap P_i$. Conversely, let $a \in L$ be an element that does not belong to $K(L)$. By Proposition 3.8 there exists an infinite generalized m-sequence starting with a. Let P be an ideal of L maximal among those not containing any element of the m-sequence. By Proposition 3.7 P is an strongly prime ideal of L, and $a \notin P$, so $a \notin \bigcap P_i$. □

4. Lie algebras arising from associative algebras

There are two important ways of producing Lie algebras out of an associative algebra R:

- If R is an associative algebra, R^- with product $[x, y] := xy - yx$ is a Lie algebra.
- If R is an associative algebra with involution $*$, the set of skew elements of R, $\mathrm{Skew}(R, *) = \{x \in R \mid x^* = -x\}$, becomes a Lie subalgebra of R^-.

We begin by studying some relations between the Baer radical $r(R)$ of an associative algebra R and the Kostrikin radical of R^-.

Lemma 4.1. *Let R be an associative algebra and let $x \in r(R)$. Then any m-sequence $\{a_n\}_{n\in\mathbb{N}}$ of R^- with $a_1 = x$ has finite length, i.e., there exists $k \in \mathbb{N}$ such that $a_k = 0$*

Proof. It is well known that the Baer radical of R can be constructed as in 1.5 or 1.6. Moreover, since the (associative) ideal generated by all absolute zero divisors of R coincides with the submodule generated by all absolute zero divisors, we only need to show that the proposition holds when x is a sum of absolute zero divisors of R. Let $x = a_1 + \cdots a_k$ where each a_i is an absolute zero divisor of R, $i = 1, 2, \dots, k$. Then any product of elements of R in which x appear at least $k + 1$ times is zero. Therefore, any m-sequence of R^- which starts with x has at most length n, for $2^n \leq k$. □

Lemma 4.2. *Let R be an associative algebra defined over a ring of scalars Φ with no 2-torsion. If R is semiprime, the Lie algebra $R^-/Z(R)$ is nondegenerate. Furthermore, if R is prime, $R^-/Z(R)$ is strongly prime.*

Proof. We can suppose that R is not commutative, otherwise $R = Z(R)$ and the result is trivial.

Let us first see that $R^-/Z(R)$ is nondegenerate when R is semiprime: Suppose that $x \in R$ satisfies $[x, [x, R]] \in Z(R)$. Given any $a \in R$,

$$0 =[a, [x, [x, xa]]] = [a, [x, x[x, a]]] = [a, x[x, [x, a]]] = [a, x][x, [x, a]]$$

since $[x, [x, a]] \in Z(R)$, which implies $0 = \mathrm{ad}_x([a, x][x, [x, a]]) = -([x, [x, a]])^2$ and, therefore, $[x, [x, a]] = 0$ because R is semiprime and $[x, [x, a]]$ is a nilpotent element of $Z(R)$; now, by [15, Sublemma, p. 5], $[x, [x, R]] = 0$ implies $x \in Z(R)$.

Now suppose that R is prime. By [3, Theorem 3.4] if $I/Z(R)$ is a nonzero ideal of $R/Z(R)$ there exists a nonzero ideal I' of R such that $[I', R] \subset I$. Let us prove that for every nonzero ideal I' of R, $[I', R]$ is not contained in $Z(R)$. Otherwise, $[I', [I', R]] = 0$ which implies $0 \neq I' \subset Z(R)$ (because $R^-/Z(R)$ is nondegenerate) and this is not possible because in a prime noncommutative associative algebra there are no nonzero ideals contained in the center. Finally, if $I_1/Z(R)$ and $I_2/Z(R)$ are two nonzero ideals of $R^-/Z(R)$, there exist two nonzero ideals I_1', I_2' of R with $[I_i', R] \subset I_i$ for $i = 1, 2$. Now, $\overline{0} \neq [(I_1' \cap I_2') + Z(R)/Z(R), R/Z(R)] \subset I_1/Z(R) \cap I_2/Z(R)$, which implies that $R^-/Z(R)$ is prime. □

Theorem 4.3. *Let R be an associative algebra defined over a ring of scalars Φ with no 2-torsion, and denote by $K(R^-)$ the Kostrikin radical of R^-. Then:*

(1) *$K(R^-)$ coincides with the intersection of all strongly prime ideals of R^-.*
(2) *$K(R^-) = \pi^{-1}(Z(R/r(R)))$ where $r(R)$ is the Baer radical of R and $\pi : R \to R/r(R)$ denotes the (associative) canonical projection.*
(3) *$K(R^-) = \{x \in R \mid$ every m-sequence starting with x has finite length$\}$.*

Proof. The intersection of all prime ideals $\{I_i\}_i$ of R coincides with the Baer radical $r(R)$. For every prime ideal I_i of R, R/I_i is a prime algebra, and the maps

$$\Psi_i : R^- \to (R/I_i)/Z(R/I_i)$$

are epimorphisms of Lie algebras, which implies by Lemma 4.2 that $\mathrm{Ker}(\Psi_i)$ is a strongly prime ideal of R^-, and since the Kostrikin radical is contained in every strongly prime ideal of R^-, $K(R^-) \subset \mathrm{Ker}(\Psi_i)$. Now, if $x \in \bigcap \mathrm{Ker}(\Psi_i)$, $x + I_i \in Z(R/I_i)$ for every prime ideal I_i of R and therefore $[x, R] \subset \bigcap I_i = r(R)$. Hence $x \in \pi^{-1}(Z(R/r(R)))$, and if $\{\mathcal{I}_i\}$ denotes the family of all strongly prime ideals of R^-,

$$K(R^-) \subset \bigcap \mathcal{I}_i \subset \bigcap \mathrm{Ker}(\Psi_i) \subset \pi^{-1}(Z(R/r(R))).$$

Finally, if $x \in \pi^{-1}(Z(R/r(R)))$, $[x, [x, a]] \in r(R)$ for every $a \in R$ and therefore, every m-sequence of R^- starting with $[x, [x, a]]$ has finite length by Lemma 4.1, which implies that $x \in K(R^-)$ by 1.8. □

Corollary 4.4. *Let R be a semiprime algebra over a ring of scalars Φ with no 2-torsion, and let us consider the Lie algebra $L = R^-/Z(R)$. Then the intersection of all strongly prime ideals of L is zero.*

Now we turn associative algebras with involution and study the relation between the Kostrikin radical of $\mathrm{Skew}(R, *)$ and the Baer radical of R.

Lemma 4.5. *If Q is a simple Lie algebra with involution $*$ over a ring of scalars Φ with no 2-torsion, $*$ is of the first kind and $\dim_{Z(Q)} Q \leq 4$, then the Lie algebra $L = \mathrm{Skew}(Q, *)$ is either strongly prime or central and, in the second case, L has dimension one over $Z(Q)$.*

Proof. Let $0 \neq t \in L = \mathrm{Skew}(Q, *)$ be an element such that $[t, [t, \mathrm{Skew}(Q, *)]] = 0$. In [3, Theorem 2.10] it is shown that $[t, \mathrm{Skew}(Q, *)] = 0$, and from this we get that t commutes with the subalgebra $\overline{\mathrm{Skew}(Q, *)}$ generated by $\mathrm{Skew}(Q, *)$. But Herstein in [15, Lemma 2.2] showed that either $\overline{\mathrm{Skew}(Q, *)} = Q$, leading to $t \in Z(Q) \cap \mathrm{Skew}(Q, *) = 0$, or L is one-dimensional over its center. Furthermore, if L is nondegenerate, it is prime since $\dim_{Z(Q)} Q \leq 4$ and there cannot exist two nonzero ideals with zero intersection. □

Proposition 4.6. *Let R be a $*$-prime associative algebra with involution $*$ over a ring of scalars Φ with no 2-torsion and let $L = \mathrm{Skew}(R, *)$.*

- *If the involution is of the second kind or the involution is of the first kind and R is not an order in a simple algebra Q of dimension at most 16 over its center, then $L/Z(L)$ is strongly prime. In these cases, $Z(R) \cap L = Z(L)$.*
- *If the involution is of the first kind and R is an order in a simple algebra Q with $\dim_{Z(Q)} Q = 9$ or 16, then $Z(R) \cap L = Z(L) = K(L) = 0$ and the intersection of all strongly prime ideals of $L/Z(L)$ is zero.*
- *If the involution is of the first kind and R is an order in a simple algebra Q with $\dim_{Z(Q)} Q \leq 4$, then either L is abelian or strongly prime.*

Proof. If R is a commutative algebra, all the results are trivial, so we can suppose that R is noncommutative.

First, let us suppose that the involution is of the second kind: Let us consider the Lie algebra $L' := L/(Z(R) \cap L)$ and let $\bar{t}$ an absolute zero divisor of L'. If $[t, [t, L]] = 0$, then by [3, Theorem 2.13] (which also holds for $*$-prime algebras), $t \in Z(R)$, so $\bar{t} = \bar{0}$ in L'. If $0 \neq [t, [t, L]] \subset Z(R)$, there exists $x \in L$ such that $0 \neq [t, [t, x]] = \alpha$. Since $\alpha \in Z(R)$, $\alpha[t, [t, H(R, *)]] = [t, [t, \alpha H(R, *)]] \subset [t, [t, L]] \subset Z(R)$, but then also $[t, [t, H(R, *)]] \subset Z(R)$ since R is $*$-prime (notice that in any $*$-prime R, $0 \neq \alpha \in Z(R)$ and $r \in R$ with $\alpha r \in Z(R)$ implies $r \in Z(R)$). Therefore, $[t, [t, R]] \subset Z(R)$ and we get that $t \in Z(R)$ by Lemma 4.2 , i.e, L' is nondegenerate. Therefore $K(L') = 0$, so $K(L) = Z(R) \cap L$, which implies, in particular, that $Z(L) = Z(R) \cap L$.

Now, let us suppose that the involution is of the first kind and R is not an order in a simple algebra of dimension less than 9 over its center. Then, by [3, Theorem 2.10] (notice that the proof of this result also works in the $*$-prime setting) the Lie algebra L is nondegenerate. So $K(L) = 0$ which implies that $Z(R) \cap L = Z(L) = 0$.

Suppose that either the involution is of the second kind, or it is of the first kind but R is not an order in a simple algebra Q of dimension at most 16 over its center. To show that $L/(Z(R) \cap L)$ is strongly prime, assume firstly that R is prime. Then, by [6, Theorem 1 (a), p. 525] if $*$ is of the second kind, or by [6, Corollary, p. 533] if $*$ is of the first kind and R is not an order in a simple algebra Q which is at most 16-dimensional over its center, given a nonzero ideal $I'/(Z(R) \cap L)$ of $L/(Z(R) \cap L)$, there exists a nonzero $*$-ideal I of R such that $[I \cap \mathrm{Skew}(R, *), \mathrm{Skew}(R, *)] \subset I'$. Let us show that $[I \cap \mathrm{Skew}(R, *), \mathrm{Skew}(R, *)] \neq 0$. Otherwise, $I \cap \mathrm{Skew}(R, *)$ can be regarded as a nilpotent ideal of the nondegenerate Lie algebra L, so it is zero modulo $Z(R)$, in which case:

(I) If $I \cap \mathrm{Skew}(R, *) = 0$, then for every $y \in I$, $y - y^* \in \mathrm{Skew}(R, *) \cap I = 0$, so $y = y^*$ for every $y \in I$, and given $r, s \in R$,

$$yrs = (yrs)^* = s^* r^* y; \quad yrs = (yr)^* s = r^* y s = r^* (ys)^* = r^* s^* y,$$

hence $(s^* r^* - r^* s^*) y = 0$, and since R is prime, $(rs)^* = (sr)^*$ for every $r, s \in R$, which implies R is commutative, a contradiction.

(II) If $0 \neq I \cap \mathrm{Skew}(R, *) \subset Z(R)$, then there exists $\alpha \in I \cap \mathrm{Skew}(R, *) \cap Z(R)$. Since $I = I \cap \mathrm{Skew}(R, *) \oplus I \cap H(R, *)$, we have that $I \subset Z(R)$ because also $I \cap H(R, *) \subset Z(R)$ since $\alpha(I \cap H(R, *)) \subset I \cap \mathrm{Skew}(R, *) \subset Z(R)$. But a noncommutative prime R cannot have nonzero $*$-ideals I contained in $Z(R)$, a contradiction.

Thus if $I/(Z(R) \cap \mathrm{Skew}(R,*))$ and $J/(Z(R) \cap \mathrm{Skew}(R,*))$ are ideals of $L/(Z(R) \cap L)$, there exist ideals I', J' of R such that

$$0 \neq [I' \cap J' \cap \mathrm{Skew}(R,*), \mathrm{Skew}(R,*)] \subset I \cap J,$$

so $L/(Z(R) \cap L)$ is a prime nondegenerate algebra, i.e, it is strongly prime.

If R is $*$-prime but not prime, there exists a prime ideal I of R such that $I \cap I^* = 0$. The map $f : R \to R/I \times R/I^*$ is a $*$-monomorphism of algebras with exchange involution

$$* : R/I \times R/I^* \to R/I \times R/I^*$$

given by $(x,y)^* = (y^*, x^*)$. Now, $f(I \oplus I^*)$ is an essential ideal of $R/I \times R/I^*$ and

$$I \cong \mathrm{Skew}(f(I \oplus I^*)) \triangleleft \mathrm{Skew}(R/I \times R/I^*) \cong R/I^*,$$

which implies that $I/(Z(R) \cap I) \cong \mathrm{Skew}(f(I \oplus I^*))/(\mathrm{Skew}(f(I \oplus I^*)) \cap Z(f(R)))$ is a strongly prime algebra and, since it is essential in L, $L/(Z(R) \cap L)$ is strongly prime.

Suppose that R is $*$-prime with involution of the first kind and R is an order in a simple algebra Q of dimension at most 16 over its center. Since Q is simple and finite dimensional, Q is a PI algebra, so R is a PI algebra and it is a central order in Q: for every $q \in Q$ there exists $\alpha \in Z(R)$ and $x \in R$ such that $q = \alpha^{-1}x$. Now, we can extend the involution to Q and since the center of Q is the extended centroid of R, we have that the involution on Q is of the first kind. If $\dim_{Z(Q)} Q = 16$ or 9, by [3, Theorem 2.10], $\mathrm{Skew}(Q,*)$ is nondegenerate, and if $\dim_{Z(Q)} Q = 4$ or 1, by Lemma 4.5, $\mathrm{Skew}(Q,*)$ is either central or strongly prime. In any case, L is abelian if $\mathrm{Skew}(Q,*)$ is abelian and L is strongly prime (nondegenerate) if $\mathrm{Skew}(Q,*)$ is so: Let us show that L is strongly prime when $\mathrm{Skew}(Q,*)$ is strongly prime (the inheritance of nondegeneracy follows analogously). Given $x, y \in L$ such that $[x,[y,L]] = 0$ we have that for every $q \in \mathrm{Skew}(Q,*)$ there exists $\alpha \in Z(R)$ and $z \in R$ such that $q = \alpha^{-1}z$ and therefore $[x,[y,q]] = [x,[y,\alpha z]] = \alpha[x,[y,z]] = 0$ which implies that $x = 0$ or $y = 0$ and L is strongly prime, see [12, Theorem 1.6]. Finally, if $\dim_{Z(Q)} Q = 16$ or 9, $Z(R) \cap L \subset Z(L) \subset K(L) = 0$ and by Corollary 5.4 the intersection of all strongly prime ideals of L is zero. □

Theorem 4.7. *Let R be an associative algebra with involution $*$ over a ring of scalars Φ with no 2-torsion, let $L = \mathrm{Skew}(R,*)$, and denote by $K(L)$ its Kostrikin radical. Then:*

1. *$K(L)$ coincides with the intersection of all strongly prime ideals of L.*
2. *$K(L) = \pi^{-1}(Z(L/(r(R) \cap L)))$ where $r(R)$ is the Baer radical of R and $\pi : L \to L/(r(R) \cap L)$ denotes the canonical projection.*
3. *$K(L) = \{x \in L \mid$ every m-sequence starting with x has finite length$\}$.*

Proof. The intersection of all $*$-prime ideals of R, $\{I_i\}_{i \in \Delta}$, is equal to the Baer radical $r(R)$. Now, for every $*$-prime ideal I_i of R, let us consider the epimorphism of Lie algebras

$$\Psi_i : \mathrm{Skew}(R,*) \to \mathrm{Skew}(R/I_i,*)/Z(\mathrm{Skew}(R/I_i,*)).$$

By Proposition 4.6, $\mathrm{Ker}\,\Psi_i$ is either a strongly prime ideal of L, or it is the intersection of strongly prime Lie algebras, or it is the whole algebra L. Therefore, if $x \in \bigcap \mathrm{Ker}\,\Psi_i$ (which is an intersection of strongly prime ideals of L) and $a \in L$, we have that $[x,a] \in I_i$ for all $i \in \Delta$ and therefore, $[x,a] \in r(R)$, which implies that $x \in \pi^{-1}(Z(L/(r(R) \cap L)))$. So if $\{\mathcal{I}_i\}$ denotes the family of all strongly prime ideals of L,

$$K(L) \subset \bigcap \mathcal{I}_i \subset \pi^{-1}(Z(L/(r(R) \cap L))).$$

Finally, if $x \in \pi^{-1}(Z(L/(r(R) \cap L)))$, $[x,[x,a]] \in r(R)$ for every element $a \in L$ and therefore, every m-sequence of L starting with $[x,[x,a]]$ has finite length, Lemma 4.1, which implies that $x \in K(L)$ by 1.8. $\square$

Corollary 4.8. *Let R be a semiprime associative algebra with involution $*$ over a ring of scalars Φ with no 2-torsion and let $L = \mathrm{Skew}(R,*)$. Then $K(L) = Z(L)$ and it coincides with the intersection of all strongly prime ideals of L.*

4.9. Remark. Since the Kostrikin radical of any ideal I of a Lie algebra L coincides with $K(L) \cap I$, see [24, Corollary 1, pg 543], Theorems 4.3, 4.7 and Corollaries 4.4, 4.8 also hold for ideals of the Lie algebras mentioned there. In particular, they hold for the Lie algebras $[R,R]$ and $[R,R]/(Z(R) \cap [R,R])$, and for $[\mathrm{Skew}(R,*), \mathrm{Skew}(R,*)]$, $[\mathrm{Skew}(R,*), \mathrm{Skew}(R,*)]/(Z(R) \cap [\mathrm{Skew}(R,*), \mathrm{Skew}(R,*)])$.

5. Lie algebras with descending chain conditions

5.1. Recall that the annihilator of an ideal I in a Lie algebra L is defined as $\mathrm{Ann}_L(I) = \{x \in L \mid [x,I] = 0\}$. If I is an ideal of L which is nondegenerate as a Lie algebra (in particular if L is nondegenerate), then $\mathrm{Ann}_L(I) = \{x \in L \mid [x,[x,I]] = 0\}$ and $I \cap \mathrm{Ann}_L(I) = 0$. Moreover, if I is nondegenerate and $\mathrm{Ann}_L(I) = 0$, L is a nondegenerate Lie algebra, see [7, 2.5].

If L is nondegenerate, $\mathrm{Ann}_L(I)$ is a nondegenerate ideal of L for every ideal I of L: let $\overline{x} \in L/\mathrm{Ann}_L(I)$ such that $[\overline{x},[\overline{x}, L/\mathrm{Ann}_L(I)]] = \overline{0}$. Then $[x,[x,L]] \subset \mathrm{Ann}_L(I)$, so $[x,[x,I]] \subset I \cap \mathrm{Ann}_L(I) = 0$, hence $x \in \mathrm{Ann}_L(I)$.

5.2. We say that a Lie algebra L satisfies the descending chain condition for annihilator ideals if every descending chain of annihilator ideals $\{\mathrm{Ann}_L(I_i)\}_i$, $\mathrm{Ann}_L(I_i) \supset \mathrm{Ann}_L(I_{i+1})$, reaches zero in a finite number of steps. Since $\mathrm{Ann}_L(\mathrm{Ann}_L(\mathrm{Ann}_L(I))) = \mathrm{Ann}_L(I)$ for every ideal I of L, we have that L satisfies the descending chain condition for annihilator ideals if and only it is satisfies the ascending one.

A nonzero ideal I of L is said to be uniform if for every two nonzero ideals J, J' of L such that $J, J' \subset I$ we have that $J \cap J' \neq 0$. If L is semiprime, by [10, Proposition 3.1 (i)] I is a uniform ideal of L if and only if $\mathrm{Ann}_L(I)$ is maximal among all annihilator ideals of nonzero ideals of L. The next proposition can be deduced from [10, Theorem 4.1].

Proposition 5.3. *If L is nondegenerate and every annihilator ideal of L is contained in a maximal annihilator ideal, then the intersection of all strongly prime ideals of L is zero. Moreover, if L satisfies the chain condition for annihilator ideals, then L is an essential subdirect product of finitely many strongly prime Lie algebras.*

Proof. Let $0 \neq x \in L$, consider the ideal J of L generated by x and its annihilator $\mathrm{Ann}_L(J)$. By hypothesis, there exists a nonzero ideal I of L such that $\mathrm{Ann}_L(I)$ is a maximal annihilator ideal with $\mathrm{Ann}_L(J) \subset \mathrm{Ann}_L(I)$. Now, if $x \in \mathrm{Ann}_L(I)$, $J \oplus \mathrm{Ann}_L(J) \subset \mathrm{Ann}_L(I)$, a contradiction because $J \oplus \mathrm{Ann}_L(J)$ is an essential ideal of L. Therefore, the intersection of all maximal annihilator ideals of L, which are strongly prime ideals of L by 5.1 and [10, Proposition 3.1 (ii)], is zero.

Now suppose that L satisfies the chain condition for annihilator ideals and consider the set of all uniform ideals $\{I_i\}_i$ of L. By 5.1 and [10, Proposition 3.1 (ii)], $L/\operatorname{Ann}_L(I_i)$ is strongly prime. Moreover, since $\bigcap_i \operatorname{Ann}_L(I_i) = \operatorname{Ann}_L(\sum_i I_i)$ and every descending chain of annihilator ideals reaches zero, there exists a finite number of uniforms ideals $\{I_i\}_{i=1}^n$ such that $\bigcap_{i=1}^n \operatorname{Ann}_L(I_i) = 0$. Finally, every $I_i \cong (I_i + \operatorname{Ann}_L(I_i))/\operatorname{Ann}_L(I_i)$ is an essential ideal of the strongly prime Lie algebra $L/\operatorname{Ann}_L(I_i)$, hence L is an essential subdirect product of the strongly prime Lie algebras $\{L_i = L/\operatorname{Ann}_L(I_i)\}_{i=1}^n$. $\square$

The following corollary shows that the characterization of the Kostrikin radical of a Lie algebra L as the intersection of all strongly prime ideals of L holds for Artinian Lie algebras, hence in particular for finite dimensional Lie algebras.

Corollary 5.4. *If L is an Artinian Lie algebra, the Kostrikin radical of L coincides with the intersection of all strongly prime ideals of L, and $L/K(L)$ is an essential subdirect product of finitely many strongly prime Lie algebras.*

Proof. The nondegenerate Lie algebra $L/K(L)$ remains Artinian and satisfies the chain condition for annihilator ideals, so the intersection of all strongly prime ideals of L is $K(L)$ and $L/K(L)$ is an essential subdirect product of finitely many strongly prime Lie algebras. $\square$

5.5. An inner ideal of a Lie algebra L is a Φ-submodule B of L such that $[B,[B,L]] \subset B$. An abelian inner ideal is an inner ideal B which is also an abelian subalgebra, i.e., $[B,B] = 0$. If L is defined over a field of scalars with $1/2, 1/3$ and $1/5$, the socle of a nondegenerate Lie algebra L is an ideal $\operatorname{Soc}(L)$ defined as the sum of all minimal inner ideals of L, and it is a direct sum of simple ideals [5, 2.4, 2.5].

Proposition 5.6. *If L is defined over a field of scalars with $1/2, 1/3$ and $1/5$, and L is nondegenerate and has essential socle, then the intersection of all strongly prime ideals of L is zero and, therefore, L is an essential subdirect product of strongly prime Lie algebras.*

Proof. Let $\operatorname{Soc}(L) = \bigoplus_i L_i$ be the decomposition of the socle of L into simple ideals, see [5, 2.5(i)]. It is easy to see that $\bigcap_i(\operatorname{Ann}_L(L_i)) = \operatorname{Ann}(\bigoplus_i L_i) = \operatorname{Ann}(\operatorname{Soc}(L)) = 0$ because $\operatorname{Soc}(L)$ is essential, so the intersection of all strongly prime ideals of L is zero and L is an essential subdirect product of the strongly prime Lie algebras $\{L/\operatorname{Ann}_L(L_i)\}_i$. $\square$

Acknowledgements. The authors are grateful to Prof. E. Zelmanov for allowing them to include part of his notes in the final version of this paper. They would also like to thank Prof. J. A. Anquela, Prof. T. Cortés and Prof. A. Fernández López for their careful reading of this manuscript and their valuable suggestions and comments.

References

[1] José A. Anquela, Teresa Cortés, Esther García, and Miguel Gómez-Lozano. Polynomial identities and speciality of Martindale-like covers of Jordan algebras. *J. Pure Appl. Algebra*, 202(1-3):1–10, 2005.

[2] José A. Anquela, Teresa Cortés, Esther García, and Miguel Gómez-Lozano. Polynomial identities and speciality of Martindale-like covers of Jordan algebras. *J. Pure Appl. Algebra*, 202(1-3):1–10, 2005.

[3] Georgia Benkart. The Lie inner ideal structure of associative rings. *J. Algebra*, 43(2):561–584, 1976.
[4] Bailey Brown and Neal H. McCoy. Prime ideals in nonassociative rings. *Trans. Amer. Math. Soc.*, 89:245–255, 1958.
[5] Cristina Draper, Antonio Fernández López, Esther García, and Miguel Gómez Lozano. The socle of a nondegenerate Lie algebra. *J. Algebra*, 319(6):2372–2394, 2008.
[6] Theodore S. Erickson. The Lie structure in prime rings with involution. *J. Algebra*, 21:523–534, 1972.
[7] Antonio Fernández López, Esther García, and Miguel Gómez Lozano. The Jordan socle and finitary Lie algebras. *J. Algebra*, 280(2):635–654, 2004.
[8] Antonio Fernández López, Esther García, and Miguel Gómez Lozano. The Jordan algebras of a Lie algebra. *J. Algebra*, 308(1):164–177, 2007.
[9] Antonio Fernández López, Esther García, Miguel Gómez Lozano, and Erhard Neher. A construction of gradings of Lie algebras. *Int. Math. Res. Not. IMRN*, (16):Art. ID rnm051, 34, 2007.
[10] Antonio Fernández López and Eulalia García Rus. Algebraic lattices and nonassociative structures. *Proc. Amer. Math. Soc.*, 126(11):3211–3221, 1998.
[11] V. T. Filippov. Nil-elements of index 2 in Mal′tsev algebras. *Algebra i Logika*, 37(3):358–373, 377, 1998.
[12] Esther García and Miguel Gómez Lozano. An elemental characterization of strong primeness in Lie algebras. *J. Algebra*, 312(1):132–141, 2007.
[13] Esther García and M. Gómez Lozano. A note on a result of kostrikin. *Comm. Algebra*, 37:1–5, 2009.
[14] A. N. Grishkov. Local nilpotency of an ideal of a Lie algebra generated by second-order elements. *Sibirsk. Mat. Zh.*, 23(1):181–182, 222, 1982.
[15] I. N. Herstein. *Topics in ring theory.* The University of Chicago Press, Chicago, Ill.-London, 1969.
[16] Nathan Jacobson. *Structure and representations of Jordan algebras.* American Mathematical Society Colloquium Publications, Vol. XXXIX. American Mathematical Society, Providence, R.I., 1968.
[17] A. I. Kostrikin. *Around Burnside*, volume 20 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by James Wiegold.
[18] Neal H. McCoy. Prime ideals in general rings. *Amer. J. Math.*, 71:823–833, 1949.
[19] Kevin McCrimmon. The radical of a Jordan algebra. *Proc. Nat. Acad. Sci. U.S.A.*, 62:671–678, 1969.
[20] Armin Thedy. Z-closed ideals of quadratic Jordan algebras. *Comm. Algebra*, 13(12):2537–2565, 1985.
[21] E. I. Zelmanov. Some notes on the Kostrikin radical. *Private communication.*
[22] E. I. Zelmanov. Absolute zero-divisors in Jordan pairs and Lie algebras. *Mat. Sb. (N.S.)*, 112(154)(4(8)):611–629, 1980.
[23] E. I. Zelmanov. Characterization of the McCrimmon radical. *Sibirsk. Mat. Zh.*, 25(5):190–192, 1984.
[24] E. I. Zelmanov. Lie algebras with algebraic adjoint representation. *Mat. Sb. (N.S.)*, 49(2):537–552, 1984.
[25] E. I. Zelmanov. Lie algebras with finite grading. *Mat. Sb. (N.S.)*, 124(166)(3):353–392, 1984.

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