THE KÄHLER–EINSTEIN METRIC FOR SOME HARTOGS DOMAINS OVER BOUNDED SYMMETRIC DOMAINS

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Abstract. We study the complete Kähler-Einstein metric of a Hartogs domain $\widetilde{\Omega}$ built on an irreducible bounded symmetric domain Ω, using a power N^μ of the generic norm of Ω. The generating function of the Kähler-Einstein metric satisfies a complex Monge-Ampère equation with boundary condition. The domain $\widetilde{\Omega}$ is in general not homogeneous, but it has a subgroup of automorphisms, the orbits of which are parameterized by $X \in [0,1[$. This allows to reduce the Monge-Ampère equation to an ordinary differential equation with limit condition. This equation can be explicitly solved for a special value μ_0 of μ. We work out the details for the two exceptional symmetric domains. The special value μ_0 seems also to be significant for the properties of other invariant metrics like the Bergman metric; a conjecture is stated, which is proved for the exceptional domains.

Introduction

Let D be a bounded domain in $\mathbb{C}^n$. The *complete (normalized) Kähler-Einstein metric* on D is the Hermitian metric E

$$E_z(u,v) = \partial_u \overline{\partial}_v g\big|_z ,$$

whose *generating function* g is the unique solution of the complex Monge-Ampère equation with boundary condition

$$\det\left(\frac{\partial^2 g}{\partial z^i \partial \overline{z}^j}\right) = \mathrm{e}^{(n+1)g} \qquad (z \in D),$$
$$g(z) \to \infty \qquad\qquad (z \to \partial D)$$

(see [1], [2], [3]).

Let Ω be a bounded irreducible symmetric domain in $V \simeq \mathbb{C}^d$; we will always consider such a domain in its *circled* realization. For a real positive number μ, let $\widetilde{\Omega}$ be the Hartogs type domain defined by

$$\widetilde{\Omega} = \widetilde{\Omega}_k(\mu) = \left\{ (z,Z) \in \Omega \times \mathbb{C}^k \mid \|Z\|^2 < N(z,z)^\mu \right\},$$

where $N(z,z)$ denotes the *generic norm* of Ω (see Appendix A.3). The Bergman kernel of $\widetilde{\Omega}$ has been computed in [4] .

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In this paper, we study the complete Kähler-Einstein metric of $\widetilde{\Omega}$. The domain $\widetilde{\Omega}$ is in general not homogeneous, but it has a subgroup of automorphisms, the orbits of which are parameterized by $X \in [0, 1[$. This allows to reduce the Monge-Ampère equation to an ordinary differential equation, following a method used in [5] when Ω belongs to one of the four series of classical domains. This equation can be explicitly solved for a special value μ_0 of μ. We first study the case $k = 1$, then generalize the results to any integer k. Tables are given in Appendix B, allowing the reader to apply the results to each irreducible bounded symmetric domain. In Section 3, we work out some details for the two exceptional bounded symmetric domains; the construction and the main properties of these domains are recalled in Appendix C. In Section 4, starting from the example of exceptional domains, we state a conjecture which links the critical exponent μ_0 for the Kähler-Einstein metric and the properties of the Bergman kernel of $\widetilde{\Omega}_k(\mu)$.

1. Symmetric domains inflated by discs

This is the case $k = 1$, with

$$\widetilde{\Omega} = \widetilde{\Omega}_1(\mu) = \left\{ (z, w) \in \Omega \times \mathbb{C} \mid |w|^2 < N(z, z)^{\mu} \right\}.$$

1.1. **Automorphisms.** Let Ω be a bounded irreducible symmetric domain, $\mu > 0$ a real number and

$$\widetilde{\Omega} = \widetilde{\Omega}_1(\mu) = \left\{ (z, w) \in \Omega \times \mathbb{C} \mid |w|^2 < N(z, z)^{\mu} \right\}.$$

Let X be the function $X : \widetilde{\Omega} \to [0, 1[$ defined by

$$X(z, w) = \frac{|w|^2}{N(z, z)^{\mu}}. \tag{1.1}$$

Denote by $\operatorname{Aut}' \widetilde{\Omega}$ the subgroup of automorphisms of $\widetilde{\Omega}$ which leave X invariant.

Let $\Phi \in \operatorname{Aut} \Omega$. Denote by $\mathrm{d}\,\Phi(z)$ the differential of Φ at z and by $J\Phi(z) = \det \mathrm{d}\,\Phi(z)$ its Jacobian.

Lemma 1. *Let Ω be a bounded irreducible symmetric domain in $V \simeq \mathbb{C}^d$, with generic norm N and genus γ (see Appendix A.3). Then the function on $\Omega \times \Omega$*

$$\frac{N(z, z)N(t, t)}{|N(z, t)|^2}$$

is invariant by each $\Phi \in \operatorname{Aut} \Omega$ acting diagonally on $\Omega \times \Omega$:

$$\frac{N(\Phi z, \Phi z)N(\Phi t, \Phi t)}{|N(\Phi z, \Phi t)|^2} = \frac{N(z, z)N(t, t)}{|N(z, t)|^2}. \tag{1.2}$$

Proof. If $\Phi \in \operatorname{Aut}_0 \Omega$, the identity component of $\operatorname{Aut} \Omega$, we deduce from (A.1):

$$B(\Phi z, \Phi t) = \mathrm{d}\,\Phi(z) \circ B(z, t) \circ \mathrm{d}\,\Phi(t)^*$$

(see Appendix A.1) and $\det B(z, t) = N(z, t)^{\gamma}$, that

$$N(\Phi z, \Phi t)^{\gamma} = J\Phi(z) N(z, t)^{\gamma} \overline{J\Phi(t)}. \tag{1.3}$$

If $\Phi \in \operatorname{Aut} \Omega$, it can be written $\Phi = \Phi_1 \Phi_2$, where $\Phi_1 \in \operatorname{Aut}_0 \Omega$ and $\Phi_2(0) = 0$; then Φ_2 is linear, unitary (with respect to the Bergman metric at 0) and leaves N invariant, so that (1.3) holds for Φ_2, hence also for every $\Phi \in \operatorname{Aut} \Omega$. The relation (1.2) follows immediately. $\square$

Proposition 1. *The group* $\operatorname{Aut}'\widetilde{\Omega}$ *consists of all* $\Psi = (\Psi_1, \Psi_2)$*:*

$$\Psi_1(z,w) = \Phi(z), \tag{1.4}$$

$$\Psi_2(z,w) = w\psi(z), \tag{1.5}$$

such that $\Phi \in \operatorname{Aut}\Omega$ *and*

$$|\psi(z)|^2 = \left(\frac{N(\Phi z, \Phi z)}{N(z,z)}\right)^{\mu}. \tag{1.6}$$

For $\Phi \in \operatorname{Aut}\Omega$*, let* $z_0 = \Phi^{-1}(0)$*; then the functions* ψ *satisfying (1.6) are the functions*

$$\psi(z) = \mathrm{e}^{\mathrm{i}\,\theta}\,\frac{N\left(z_0, z_0\right)^{\mu/2}}{N\left(z, z_0\right)^{\mu}}. \tag{1.7}$$

The orbits of $\operatorname{Aut}'\widetilde{\Omega}$ *are the level sets*

$$\Sigma_\lambda = \{X = \lambda \mid \lambda \in [0,1[\}.$$

Proof. It is easily checked that the maps Ψ of the form (1.4)-(1.5), satisfying (1.6), are automorphisms of $\widetilde{\Omega}$, leave X invariant and form a subgroup G of automorphisms of $\widetilde{\Omega}$.

Let $z^* = \Phi(z)$. Applying (1.2) to $z_0 = \Phi^{-1}(0)$, we get

$$\frac{N(z^*, z^*)}{N(z,z)} = \frac{N\left(z_0, z_0\right)}{\left|N(z, z_0)\right|^2}, \tag{1.8}$$

as $N(x,0) = 1$ for each x. The relation (1.3) also implies

$$1 = J\Phi(z) N\left(z, z_0\right)^{\gamma}\, \overline{J\Phi\left(z_0\right)};$$

this means that the holomorphic function $z \mapsto N\left(z, z_0\right)$ never vanishes on the convex domain Ω, and we can define the holomorphic function $N\left(z, z_0\right)^{\mu}$ for any real μ assuming it is positive for $z = z_0$. By (1.8), the function

$$\psi_0(z) = \frac{N\left(z_0, z_0\right)^{\mu/2}}{N\left(z, z_0\right)^{\mu}}$$

satisfies (1.6). If ψ is another function satisfying (1.6), then ψ/ψ_0 has constant modulus 1 and $\psi = \mathrm{e}^{\mathrm{i}\,\theta}\,\psi_0$.

Let $\Psi \in \operatorname{Aut}'\widetilde{\Omega}$; as Ψ preserves X, $\Psi(\Omega \times \{0\}) = \Omega \times \{0\}$ and $\Psi(z,0) = (\Phi(z), 0)$ with $\Phi \in \operatorname{Aut}\Omega$. There exists $\Psi^1 \in G$ such that $\Psi^1(z,0) = (\Phi(z),0)$ and $\Theta = \Psi \circ \left(\Psi^1\right)^{-1}$ is an element of $\operatorname{Aut}'\widetilde{\Omega}$, such that $\Theta(z,0) = (z,0)$ for all $z \in \Omega$. In particular, $\Theta(0,0) = (0,0)$; as $\widetilde{\Omega}$ is bounded and circled, it follows from a lemma of H. Cartan (see [6], [7]) that Θ is linear. Then Θ has the form

$$\Theta_1(z,w) = z + wu,$$

$$\Theta_2(z,w) = cw.$$

The invariance of X under Θ implies

$$|c|^2 = \frac{N(z+wu, z+wu)^{\mu}}{N(z,z)^{\mu}}$$

for all $(z,w) \in \widetilde{\Omega}$ and in particular

$$|c|^2 = N\left(wu, wu\right)^{\mu}$$

for all w such that $|w| < 1$. This implies $|c| = 1$ and $u = 0$. Then $\Theta \in G$ and $\Psi = \Theta \circ \Psi^1$ belongs also to G.

If (z_0, w_0) and (z_0', w_0') belong to the same level set Σ_λ, that is $X(z_0, w_0) = X(z_0', w_0') = \lambda$, take $\Phi \in \operatorname{Aut} \Omega$ such that $\Phi(z_0) = z_0'$ and $\psi : \Omega \to \mathbb{C}$ such that

$$|\psi(z)|^2 = \left(\frac{N(\Phi z, \Phi z)}{N(z, z)} \right)^\mu .$$

Then $\Psi \in \operatorname{Aut}' \widetilde{\Omega}$ defined by

$$\begin{aligned} \Psi_1(z, w) &= \Phi(z), \\ \Psi_2(z, w) &= w\psi(z) \end{aligned}$$

maps (z_0, w_0) to $(z_0', w_0\psi(z_0))$; we have $|w_0\psi(z_0)|^2 = |w_0'|^2$, so it suffices to change Ψ_2 in $\alpha\Psi_2$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$, in order to obtain $\Psi(z_0, w_0) = (z_0', w_0')$. So the group $\operatorname{Aut}' \widetilde{\Omega}$ acts transitively on the level sets Σ_λ of X. □

The orbits Σ_λ are real hypersurfaces of $\widetilde{\Omega}$ when $\lambda > 0$; the orbit Σ_0 is $\Omega \times \{0\}$. Note also that $\Psi \in \operatorname{Aut}' \widetilde{\Omega}$ extends continuously to the boundary $\partial\widetilde{\Omega}$ of $\widetilde{\Omega}$, as Φ extends continuously to $\partial\Omega$ and

$$\partial\widetilde{\Omega} = (\partial\Omega \times \{0\}) \cup \left\{ (z, w) \in \Omega \times \mathbb{C} \mid |w|^2 = N(z, z)^\mu \right\}.$$

The part

$$\partial_0\widetilde{\Omega} = \left\{ (z, w) \in \Omega \times \mathbb{C} \mid |w|^2 = N(z, z)^\mu \right\}$$

is an orbit of $\operatorname{Aut}' \widetilde{\Omega}$ (corresponding to $X = 1$), and $\partial\Omega \times \{0\}$ is a finite union of orbits.

In order to compute the differential of ψ and Ψ, we need the following general result in Jordan triple systems. See Appendix A.1 for the notations. In particular, B is the Bergman operator, N the generic norm, m_1 the generic trace, y^x is the quasi-inverse; if V is simple (if Ω is irreducible), the genus is denoted by γ.

Lemma 2. *1) Let V be a Hermitian Jordan triple system. Then*

$$\mathrm{d}_x B(x, y) = -D(\mathrm{d}\, x, y^x) B(x, y), \tag{1.9}$$

$$\mathrm{d}_y B(x, y) = -B(x, y) D(x^y, \mathrm{d}\, y), \tag{1.10}$$

$$\frac{\mathrm{d}_x \det B(x, y)}{\det B(x, y)} = -\operatorname{tr} D(\mathrm{d}\, x, y^x), \tag{1.11}$$

$$\frac{\mathrm{d}_y \det B(x, y)}{\det B(x, y)} = -\operatorname{tr} D(x^y, \mathrm{d}\, y). \tag{1.12}$$

2) If V is simple,

$$\frac{\mathrm{d}_x N(x, y)}{N(x, y)} = -m_1(\mathrm{d}\, x, y^x), \tag{1.13}$$

$$\frac{\mathrm{d}_y N(x, y)}{N(x, y)} = -m_1(x^y, \mathrm{d}\, y), \tag{1.14}$$

$$\partial \left(\frac{\overline{\partial} N(z, z)}{N(z, z)} \right) = -m_1 \left(B(z, z)^{-1} \mathrm{d}\, z, \mathrm{d}\, \overline{z} \right). \tag{1.15}$$

In particular,

$$-\partial\left(\frac{\overline{\partial}N(z,z)}{N(z,z)}\right)\bigg|_{z=0}=m_1\left(\mathrm{d}\,z,\mathrm{d}\,\overline{z}\right). \tag{1.16}$$

Proof. We start from the addition formula for the Bergman operator

$$B(x+z,y)=B(z,y^x)B(x,y)$$

(see [8], p. 469, (J6.4')). Using the definition of B, this identity can be written

$$B(x+z,y)=\left(\mathrm{id}_V-D(z,y^x)+Q(z)Q(y^x)\right)B(x,y).$$

Taking the linear part in z proves (1.9).

In $GL(V)$ we have the well-known relation

$$\frac{\mathrm{d}(\det A)}{\det A}=\operatorname{tr}\left(A^{-1}\,\mathrm{d}\,A\right)=\operatorname{tr}\left(\mathrm{d}\,A.A^{-1}\right).$$

This implies, using (1.9),

$$\frac{\mathrm{d}_x\det B(x,y)}{\det B(x,y)}=\operatorname{tr}\left(\mathrm{d}_x\,B(x,y).B(x,y)^{-1}\right)=-\operatorname{tr}D(\mathrm{d}\,x,y^x),$$

which proves (1.11). The "dual" formulas (1.10) and (1.12) are proved in the same way.

If V is simple, $\det B(x,y)=N(x,y)^\gamma$ and $\operatorname{tr}D(x,y)=\gamma m_1(x,y)$; so (1.13)-(1.14) immediately follow from (1.11)-(1.12).

As $N(x,y)$ is holomorphic in x and anti-holomorphic in y, we have

$$\frac{\overline{\partial}N(z,z)}{N(z,z)}=-m_1(z^z,\mathrm{d}\,z)$$

and

$$\partial\left(\frac{\overline{\partial}N(z,z)}{N(z,z)}\right)=-m_1(\partial\left(z^z\right),\mathrm{d}\,z).$$

The differential of the quasi-inverse x^y with respect to x is (see [8], p. 471, relation (D2))

$$\mathrm{d}_x\left(x^y\right)=B(x,y)^{-1}\,\mathrm{d}\,x;$$

as x^y is holomorphic in x and anti-holomorphic in y, this implies

$$\partial\left(z^z\right)=B(z,z)^{-1}\,\mathrm{d}\,z$$

and (1.15). □

Lemma 3. *1) Let $\Phi\in\operatorname{Aut}_0\Omega$, $\Phi(z_0)=0$ and let $\psi:\Omega\to\mathbb{C}$ defined by (1.7):*

$$\psi(z)=\mathrm{e}^{\mathrm{i}\,\theta}\,\frac{N\left(z_0,z_0\right)^{\mu/2}}{N\left(z,z_0\right)^{\mu}}.$$

Then

$$\mathrm{d}\,\psi(z)=\mu\psi(z)m_1\left(\mathrm{d}\,z,{z_0}^z\right). \tag{1.17}$$

2) Let $\Psi=(\Psi_1,\Psi_2)\in\operatorname{Aut}'\widetilde{\Omega}$ be defined by

$$\Psi_1(z,w)=\Phi(z),$$
$$\Psi_2(z,w)=w\psi(z).$$

Then the differential of $\varPsi$ is given by

$$\mathrm{d}\,\varPsi_1(z,w) = \mathrm{d}\,\varPhi(z), \tag{1.18}$$

$$\mathrm{d}\,\varPsi_2(z,w) = \mu w \psi(z) m_1\left(\mathrm{d}\,z, z_0{}^z\right) + \psi(z)\,\mathrm{d}\,w. \tag{1.19}$$

The Jacobian of $\varPsi$ is

$$J\varPsi(z,w) = \psi(z) J\varPhi(z) \tag{1.20}$$

and satisfies

$$\left|J\varPsi(z,w)\right|^2 = \left(\frac{N(\varPhi z, \varPhi z)}{N(z,z)}\right)^{\gamma+\mu}. \tag{1.21}$$

The relation (1.17) follows immediately from (1.13), and implies (1.19). The triangular form of (1.18)-(1.19) yields (1.20). The relation (1.21) follows then from (1.3) (for $z = t$) and (1.6).

Remark 1. The relation (1.15) expresses the well-known fact that

$$-\partial\left(\frac{\overline{\partial} N(z,z)}{N(z,z)}\right)$$

is equal, up to the factor $\frac{1}{\gamma}$, to the Bergman metric of $\varOmega$ at z. This could also be established directly, as the Bergman kernel of $\varOmega$ is

$$\mathcal{K}(z) = \frac{1}{\operatorname{vol} \varOmega} N(z,z)^{-\gamma}.$$

In this paper, we will only make use of the special case (1.16) at $z = 0$.

1.2. **Reduction of the Monge-Ampère equation.**

1.2.1. Let $\varOmega$ be a bounded irreducible symmetric domain, $\mu > 0$ a real number and

$$\widetilde{\varOmega} = \widetilde{\varOmega}_1(\mu) = \left\{(z,w) \in \varOmega \times \mathbb{C} \mid |w|^2 < N(z,z)^\mu\right\}.$$

We denote by d the complex dimension of $\varOmega$ and by $n = d+1$ the dimension of $\widetilde{\varOmega}$.

Let g be a C^2 function in $\widetilde{\varOmega}$, which is a solution of the Monge-Ampère equation

$$\det\left(\frac{\partial^2 g}{\partial z^i \partial \overline{z}^j}\right) = \mathrm{e}^{(n+1)g} \tag{1.22}$$

and which generates an invariant form $\partial\overline{\partial} g$. For $\varPsi \in \operatorname{Aut}' \widetilde{\varOmega}$, the invariance of the metric and the Monge-Ampère equation (1.22) imply

$$\mathrm{e}^{(n+1)g(z,w)} = \left|J\varPsi(z,w)\right|^2 \mathrm{e}^{(n+1)g(\varPsi(z,w))}$$

and, using (1.21),

$$\mathrm{e}^{(n+1)g(z,w)}\, N(z,z)^{\gamma+\mu} = \mathrm{e}^{(n+1)g(\varPsi(z,w))}\, N(\varPhi z, \varPhi z)^{\gamma+\mu}.$$

In other words, *the function*

$$g(z,w) + \frac{\gamma+\mu}{n+1} \log N(z,z) \tag{1.23}$$

is constant on the orbits of $\operatorname{Aut}' \widetilde{\Omega}$. Recall that these orbits are also parameterized by $X \in [0,1[$ and *define* $h(X)$ as the value of the function (1.23) on the orbit $X = |w|^2 N(z,z)^{-\mu}$. The function g can then be written

$$g(z,w) = -\frac{\gamma+\mu}{n+1}\log N(z,z) + h\left(\frac{|w|^2}{N(z,z)^{\mu}}\right); \tag{1.24}$$

as $N(0,0)=1$, the function h can be obtained from g by

$$h\left(|w|^2\right) = g(0,w).$$

We will show that the Monge-Ampère equation (1.22) is equivalent to an ordinary differential equation for the function h.

1.2.2. Let $\left(z^1,\dots,z^d,z^{d+1}=w\right)$ be linear coordinates on $\Omega\times\mathbb{C}$ and let

$$\omega = \mathrm{d}\,z^1\wedge\mathrm{d}\,\overline{z}^1\wedge\cdots\wedge\mathrm{d}\,z^d\wedge\mathrm{d}\,\overline{z}^d\wedge\mathrm{d}\,w\wedge\mathrm{d}\,\overline{w}.$$

We choose for $\left(z^1,\dots,z^d\right)$ *orthonormal coordinates w.r.t. to the Hermitian metric* m_1 *relative to* Ω. By Lemma 2, we have

$$-\partial\left(\frac{\overline{\partial}N(z,z)}{N(z,z)}\right)\Bigg|_{z=0} = m_1\left(\mathrm{d}\,z,\mathrm{d}\,\overline{z}\right)$$

and

$$\frac{\left(-\partial\overline{\partial}\log N(z,z)\right)^d}{d!}\Bigg|_{z=0} = \mathrm{d}\,z^1\wedge\mathrm{d}\,\overline{z}^1\wedge\cdots\wedge\mathrm{d}\,z^d\wedge\mathrm{d}\,\overline{z}^d. \tag{1.25}$$

For any C^2 function f on $\widetilde{\Omega}$, we have

$$\frac{1}{(d+1)!}\left(\partial\overline{\partial}f\right)^{d+1} = \det\left(\frac{\partial^2 f}{\partial z^i\partial\overline{z}^j}\right)\omega.$$

Lemma 4. *Let* $g:\widetilde{\Omega}\to\mathbb{R}$ *be a* C^2 *function and let* $h:[0,1[\to\mathbb{R}$ *be related to* g *by (1.24). Then, for* $X=|w|^2>0$,

$$\frac{\left(\partial\overline{\partial}g\right)^{d+1}}{(d+1)!}(0,w) = \left(\mu X h'(X) + \frac{\gamma+\mu}{d+2}\right)^d \left(Xh'(X)\right)'\omega. \tag{1.26}$$

Proof. Let $X=|w|^2 N(z,z)^{-\mu}$. Then

$$\frac{\partial X}{X} = \frac{\mathrm{d}\,w}{w} - \mu\frac{\partial N}{N},$$
$$\frac{\overline{\partial}X}{X} = \frac{\mathrm{d}\,\overline{w}}{\overline{w}} - \mu\frac{\overline{\partial}N}{N}.$$

We have

$$\overline{\partial}\left(h(X)\right) = h'(X)\overline{\partial}X = Xh'(X)\left(\frac{\mathrm{d}\,\overline{w}}{\overline{w}} - \mu\frac{\overline{\partial}N}{N}\right)$$

and

$$\begin{aligned}\partial\overline{\partial}\left(h(X)\right) = {}&\left(Xh'(X)\right)' X\left(\frac{\mathrm{d}\,w}{w} - \mu\frac{\partial N}{N}\right)\wedge\left(\frac{\mathrm{d}\,\overline{w}}{\overline{w}} - \mu\frac{\overline{\partial}N}{N}\right)\\ &- \mu Xh'(X)\partial\left(\frac{\overline{\partial}N}{N}\right).\end{aligned}$$

We conclude that

$$\begin{aligned}\partial\overline{\partial} g = &-\left(\mu X h'(X) + \frac{\gamma+\mu}{d+2}\right)\partial\overline{\partial}\log N(z,z)\\ &+ \left(Xh'(X)\right)' X\left(\frac{\mathrm{d}\, w}{w} - \mu\frac{\partial N}{N}\right)\wedge\left(\frac{\mathrm{d}\,\overline{w}}{\overline{w}} - \mu\frac{\overline{\partial} N}{N}\right)\end{aligned}$$

and

$$\begin{aligned}\left(\partial\overline{\partial} g\right)^{d+1} = &(d+1)\left(\mu X h'(X) + \frac{\gamma+\mu}{d+2}\right)^d \left(Xh'(X)\right)'\left(-\partial\overline{\partial}\log N(z,z)\right)^d\\ &\wedge\frac{X}{|w|^2}\,\mathrm{d}\, w\wedge\mathrm{d}\,\overline{w}.\end{aligned}$$

At $z = 0$, we have $X = |w|^2$ and (1.25):

$$\left.\frac{\left(-\partial\overline{\partial}\log N(z,z)\right)^d}{d!}\right|_{z=0} = \mathrm{d}\, z^1\wedge\mathrm{d}\,\overline{z}^1\wedge\cdots\wedge\mathrm{d}\, z^d\wedge\mathrm{d}\,\overline{z}^d,$$

which implies the result (1.26). □

Lemma 5. *Let g be a C^2 function on $\widetilde{\Omega}$, which is a solution of the Monge-Ampère equation*

$$\det\left(\frac{\partial^2 g}{\partial z^i\partial\overline{z}^j}\right) = \mathrm{e}^{(n+1)g}$$

and which generates an invariant form $\partial\overline{\partial} g$. Let

$$g(z,w) = -\frac{\gamma+\mu}{d+2}\log N(z,z) + h\left(X\right),$$

$$X = \frac{|w|^2}{N(z,z)^\mu}.$$

Then h satisfies on $]0,1[$ the differential equation

$$\left(\mu X h'(X) + \frac{\gamma+\mu}{d+2}\right)^d \left(Xh'(X)\right)' = \mathrm{e}^{(d+2)h}, \tag{1.27}$$

$$Xh'(X)\to 0 \qquad (X\to 0). \tag{1.28}$$

The boundary condition

$$g(z)\to\infty \qquad (z\to\partial\widetilde{\Omega})$$

implies

$$h(X)\to\infty \qquad (X\to 1). \tag{1.29}$$

Proof. The differential equation (1.27) results directly from (1.26). The limit condition (1.29) results from the boundary condition on g, as

$$h\left(|w|^2\right) = g(0,w)$$

and $(0,1)\in\partial\widetilde{\Omega}$. From

$$\overline{w}h'\left(|w|^2\right) = \frac{\partial g}{\partial w}(0,w),$$

we deduce that $X^{1/2}h'(X)\to 0$ as $X\to 0$. □

1.2.3. Let

(1.30) $\beta = \dfrac{\gamma + \mu}{\mu\,(d+2)}$,

(1.31) $Y = Xh'(X) + \beta.$

It results from the previous lemmas that if g is a solution of the Monge-Ampère equation

$$\det\left(\frac{\partial^2 g}{\partial z^i \partial \overline{z}^j}\right) = \mathrm{e}^{(d+2)g}$$

with the boundary condition

$$g(z) \to \infty \qquad (z \to \partial\widetilde{\Omega}),$$

the function Y defined by (1.31) satisfies the differential equation

(1.32) $(\mu Y)^d\, Y' = \mathrm{e}^{(d+2)h}$

with the initial condition $Y(0) = \beta$.

This shows that the derivative of Y^{d+1} is positive and tends to ∞ when $X \to 1$, as $h(X) \to \infty$ when $X \to 1$. So the function Y is strictly increasing and maps $[0,1[$ onto $[\beta, +\infty[$.

Taking logarithmic derivatives of both sides of (1.32), we get

(1.33) $\dfrac{\left(Y^d Y'\right)'}{Y^d Y'} = (d+2)h'$

and, using the definition of Y,

(1.34) $\dfrac{\left(Y^d Y'\right)'}{Y^d Y'} = (d+2)\dfrac{Y-\beta}{X}.$

This can be written

$$X\left(Y^d Y'\right)' = (d+2)Y^d\,(Y-\beta)\,Y'$$

or

$$\begin{aligned}\left(XY^d Y'\right)' &= Y^d Y' + (d+2)Y^d\,(Y-\beta)\,Y' \\ &= (d+2)Y^{d+1}Y' - \frac{\gamma}{\mu}Y^d Y',\end{aligned}$$

using the definition (1.30) of β. From (1.28), we deduce that $Y(0) = \beta$; integrating with this initial condition yields

(1.35) $XY^d Y' = Y^{d+2} - \beta^{d+2} - \dfrac{\gamma}{\mu\,(d+1)}\left(Y^{d+1} - \beta^{d+1}\right).$

Let us denote by P the polynomial

(1.36) $P(Y) = Y^{d+2} - \beta^{d+2} - \dfrac{\gamma}{\mu\,(d+1)}\left(Y^{d+1} - \beta^{d+1}\right).$

By construction,

$$P'(Y) = Y^d + (d+2)Y^d\,(Y-\beta)\,,$$

which shows that $P'(t) > 0$ for $t > \beta$ and consequently $P(t) > 0$ for $t > \beta$. Let R be the polynomial defined by

$$P(Y) = (Y-\beta)\,R(Y);$$

then $R(\beta) = \beta^d$ and R is strictly positive on $[\beta, \infty[$.

1.2.4. Now we prove that the resolution of the ordinary differential equation (1.35) allows to construct the generating function g for the Kähler-Einstein metric of $\widetilde{\Omega} = \widetilde{\Omega}_1(\mu)$.

Lemma 6. *The differential equation*

$$XY^dY' = P(Y), \tag{1.37}$$

$$Y \to \infty \qquad\qquad (X \to 1), \tag{1.38}$$

where the polynomial P is defined by (1.36), has a unique solution

$$Y : [0, 1[\to [\beta, +\infty[.$$

This solution is C^∞ at 0.

Proof. Let $Y :]c, 1[\to [\beta, \infty[$ satisfy (1.37)-(1.38). As P is positive on $]\beta, \infty[$, the function Y is monotone and its inverse function satisfies the differential equation

$$\frac{1}{X}\frac{\mathrm{d}\, X}{\mathrm{d}\, Y} = \frac{Y^d}{P(Y)},$$
$$X \to 1 \qquad\qquad (Y \to \infty).$$

The solution of this equation is given by

$$-\log X = \int_Y^\infty \frac{y^d \,\mathrm{d}\, y}{P(y)}. \tag{1.39}$$

This gives X as a function of Y, and Y as an implicit function of X. It is defined on $]\beta, \infty[$ and maps $]\beta, \infty[$ on $[0, 1[$, as $\int_\beta^\infty \frac{y^d \,\mathrm{d}\, y}{P(y)} = +\infty$. So the maximal solution of (1.37)-(1.38) is defined on $]0, 1[$; it is C^∞ on $]0, 1[$ and extends continuously to $[0, 1[$, with $Y(0) = \beta$.

The relation (1.39) can be written

$$-\log X = -\log(Y - \beta) + \log \beta + \int_Y^{2\beta} \frac{\left(y^d - R(y)\right) \mathrm{d}\, y}{(y - \beta)R(y)} + \int_{2\beta}^\infty \frac{y^d \,\mathrm{d}\, y}{P(y)}. \tag{1.40}$$

The polynomial R is positive on $[\beta, \infty[$ and $R(\beta) = P'(\beta) = \beta^d$. Let S be defined by

$$y^d - R(y) = (y - \beta)S(y).$$

Then (1.40) may be written, for $Y > \beta$,

$$-\log X = -\log(Y - \beta) + C_0 - \int_\beta^Y \frac{S(y) \,\mathrm{d}\, y}{R(y)},$$

with

$$C_0 = \log \beta + \int_\beta^{2\beta} \frac{\left(y^d - R(y)\right) \mathrm{d}\, y}{(y - \beta)R(y)} + \int_{2\beta}^\infty \frac{y^d \,\mathrm{d}\, y}{P(y)}.$$

For $Y > \beta$, we have

$$X = \mathrm{e}^{-C_0}(Y-\beta)\exp\int_\beta^Y \frac{S(y)\,\mathrm{d}\,y}{R(y)},$$

which shows that X is a C^∞ invertible function of $Y \in [\beta,\infty[$, and Y a C^∞ function of $X \in [0,1[$. □

Theorem 1. *The generating function g for the Kähler-Einstein metric of*

$$\widetilde{\Omega} = \widetilde{\Omega}(\mu) = \left\{(z,w) \in \Omega \times \mathbb{C} \mid |w|^2 < N(z,z)^\mu\right\}$$

is given by

$$g(z,w) = -\frac{\gamma+\mu}{d+2}\log N(z,z) + h\left(\frac{|w|^2}{N(z,z)^\mu}\right), \tag{1.41}$$

where

$$\mathrm{e}^{(d+2)h} = (\mu Y)^d\, Y' \tag{1.42}$$

and the function $Y : [0,1[\to [\beta,+\infty[$ is the solution of (1.37)-(1.38).

Proof. We have already seen that, if g is the generating function for the Kähler-Einstein metric of $\widetilde{\Omega}(\mu)$, the functions h and Y satisfy the conditions of the theorem.

Let now $Y : [0,1[\to [\beta,\infty[$ be the solution of (1.37)-(1.38).and let h and g be defined from Y by (1.42) and (1.41). As Y verifies (1.37), which is equivalent to (1.35), this implies (1.34). Comparing with the logarithmic derivative of (1.42), we get for $X > 0$

$$\frac{Y-\beta}{X} = h'(X).$$

This can be written

$$h'(X) = \int_0^1 Y'(tX)\,\mathrm{d}\,t \qquad (X > 0),$$

which implies

$$h''(X) = \int_0^1 XY''(tX)\,\mathrm{d}\,t \qquad (X > 0)$$

and the existence of $h'(0) = Y'(0)$ and $h''(0) = \frac{1}{2}Y''(0)$ follows. So h is C^2 on $[0,1[$ and g is C^2 on $\widetilde{\Omega}$.

For $X \in [0,1[$, we have then $Y = Xh'(X) + \beta$ and h satisfies the differential equation (1.27)

$$\left(\mu X h'(X) + \frac{\gamma+\mu}{d+2}\right)^d (Xh'(X))' = \mathrm{e}^{(d+2)h}.$$

Using Lemma 4, we obtain

$$\det\left(\frac{\partial^2 g}{\partial z^i \partial \overline{z}^j}\right)(0,w) = \mathrm{e}^{(d+2)h\left(|w|^2\right)} = \mathrm{e}^{(d+2)g(0,w)}, \tag{1.43}$$

which means that g satisfies the Monge-Ampère equation at the points $(0, w)$. Let $(z, w) \in \widetilde{\Omega}$ and $\Psi \in \operatorname{Aut}' \widetilde{\Omega}$ such that $\Psi(z, w) = (0, w')$. Then

$$\begin{aligned}
\det\left(\frac{\partial^2 g}{\partial z^i \partial \overline{z}^j}\right)(z, w) &= \det\left(\frac{\partial^2 g}{\partial z^i \partial \overline{z}^j}\right)(0, w')\, |J\Psi(z, w)|^2 \\
&= \det\left(\frac{\partial^2 g}{\partial z^i \partial \overline{z}^j}\right)(0, w') \frac{1}{N(z, z)^{\gamma+\mu}}, \\
|w'|^2 &= \frac{|w|^2}{N(z, z)^{\mu}}, \\
g(z, w) &= -\frac{\gamma+\mu}{d+2} \log N(z, z) + g(0, w'),
\end{aligned}$$

so that (1.43) implies that g satisfies the Monge-Ampère equation on $\widetilde{\Omega}$.

It remains to prove that $g(z, w) \to \infty$ as $(z, w) \to (z_0, w_0) \in \partial\widetilde{\Omega}$. From $Y = Xh'(X) + \beta$ and $Y \to \infty$ as $X \to 1$, we see that $h'(X) > 0$, $h'(X) \to \infty$ and $h(X) \to \infty$ as $X \to 1$. The boundary points (z_0, w_0) of $\widetilde{\Omega}$ are of two different types:

- $z_0 \in \Omega$, $|w_0|^2 = N(z_0, z_0)^{\mu}$. Then $X(z_0, w_0) = 1$ and
$$g(z, w) = -\frac{\gamma+\mu}{d+2} \log N(z, z) + h(X(z, w)) \to \infty$$
as $(z, w) \to (z_0, w_0)$.
- $z_0 \in \partial\Omega$, $w_0 = 0$. In this case, $N(z, z) \to 0$ and $h(X) \geq h(0)$, which shows again that $g(z, w) \to \infty$ as $(z, w) \to (z_0, w_0)$.

□

Remark 2. It is easy to check directly that $\partial\overline{\partial} g$ defines a Kähler metric. We have

$$\begin{aligned}
\partial\overline{\partial} g = &- \left(\mu X h'(X) + \frac{\gamma+\mu}{d+2}\right) \partial\overline{\partial} \log N(z, z) \\
&+ (Xh'(X))' X \left(\frac{\mathrm{d}\, w}{w} - \mu \frac{\partial N}{N}\right) \wedge \left(\frac{\mathrm{d}\, \overline{w}}{\overline{w}} - \mu \frac{\overline{\partial} N}{N}\right) \\
= &- \mu Y \partial\overline{\partial} \log N(z, z) + XY' \left(\frac{\mathrm{d}\, w}{w} - \mu \frac{\partial N}{N}\right) \wedge \left(\frac{\mathrm{d}\, \overline{w}}{\overline{w}} - \mu \frac{\overline{\partial} N}{N}\right).
\end{aligned}$$

The associated Hermitian form is

$$\begin{aligned}
H(\zeta, \omega) = &- \mu Y \sum \frac{\partial^2}{\partial z^j \partial \overline{z}^k} \log N(z, z) \zeta^j \overline{\zeta}^k \\
&+ XY' \left| \frac{\omega}{w} - \frac{\mu}{N(z, z)} \sum \frac{\partial}{\partial z^j} N(z, z) \zeta^j \right|^2.
\end{aligned}$$

The term

$$B(\zeta) = -\sum \frac{\partial^2}{\partial z^j \partial \overline{z}^k} \log N(z, z) \zeta^j \overline{\zeta}^k$$

is, up to a constant factor, the Bergman metric of Ω at z. Hence $H(\zeta, \omega) \geq 0$, as $Y > 0$ and $Y' > 0$. If $H(\zeta, \omega) = 0$, then $B(\zeta) = 0$, which implies $\zeta = 0$ and then $\omega = 0$.

1.3. **The critical exponent.** If $\mu = \mu_0$, we have $C = 0$ and (1.39) has a very simple form. We call

$$\mu_0 = \frac{\gamma}{d+1}$$

the *critical exponent* for the bounded symmetric domain Ω.

If $\mu = \mu_0$, we have $C = 0$ and (1.39) is

$$-\log X = \int_Y^\infty \frac{\mathrm{d}\,y}{y^2 - y} = \log \frac{y-1}{y}\bigg|_Y^\infty = -\log \frac{Y-1}{Y},$$

which gives $X = \frac{Y-1}{Y}$ or

$$Y = \frac{1}{1-X}.$$

From $\mathrm{e}^{(d+2)h} = (\mu Y)^d\, Y'$, we obtain

$$\mathrm{e}^{(d+2)h} = (\mu_0)^d\, \frac{1}{(1-X)^{d+2}},$$

that is

$$\begin{aligned} h &= \frac{d}{d+2} \log \mu_0 + \log \left(\frac{1}{1-X} \right), \\ &= \frac{d}{d+2} \log \mu_0 + \log \left(\frac{N(z,z)^{\mu_0}}{N(z,z)^{\mu_0} - |w|^2} \right). \end{aligned}$$

Here

$$\frac{\gamma + \mu_0}{d+2} = \mu_0;$$

applying (1.41), we have

$$\begin{aligned} g(z,w) &= -\mu_0 \log N(z,z) + \frac{d}{d+2} \log \mu_0 + \log \left(\frac{N(z,z)^{\mu_0}}{N(z,z)^{\mu_0} - |w|^2} \right) \\ &= .\frac{d}{d+2} \log \mu_0 + \log \left(\frac{1}{N(z,z)^{\mu_0} - |w|^2} \right). \end{aligned}$$

The Kähler-Einstein metric of $\widetilde{\Omega}_1(\mu_0)$ is associated to the Kähler form

$$\partial\overline{\partial} g = -\partial\overline{\partial} \log \left(N(z,z)^{\mu_0} - |w|^2 \right).$$

2. Inflation by Hermitian balls

The above results can be extended to the domain

$$\widetilde{\Omega} = \widetilde{\Omega}_k(\mu) = \left\{ (z,Z) \in \Omega \times \mathbb{C}^k \mid \|Z\|^2 < N(z,z)^\mu \right\},$$

where $\mathbb{C}^k$ is endowed with the standard Hermitian norm

$$\|Z\|^2 = \sum_{j=1}^k \left| Z^j \right|^2.$$

We outline the results, omitting the proofs when they are entirely analogous to the case $k = 1$.

2.1. **Automorphisms.** Let X be the function $X : \widetilde{\Omega} \to [0,1[$ defined by

$$X(z,Z) = \frac{\|Z\|^2}{N(z,z)^\mu}.$$

Denote by $\operatorname{Aut}' \widetilde{\Omega}$ the subgroup of automorphisms of $\widetilde{\Omega}$ which leave X invariant.

Proposition 2. *The group* $\operatorname{Aut}' \widetilde{\Omega}$ *consists of all* $\Psi = (\Psi_1, \Psi_2)$*:*

$$\begin{aligned} \Psi_1(z,Z) &= \Phi(z), \\ \Psi_2(z,Z) &= \psi(z)U(Z), \end{aligned}$$

where $\Phi \in \operatorname{Aut} \Omega$*,* $U : \mathbb{C}^k \to \mathbb{C}^k$ *is special unitary and* ψ *satisfies*

$$|\psi(z)|^2 = \left(\frac{N(\Phi z, \Phi z)}{N(z,z)}\right)^\mu.$$

The orbits of $\operatorname{Aut}' \widetilde{\Omega}$ *are the level sets*

$$\Sigma_\lambda = \{X = \lambda \mid \lambda \in [0,1[\}.$$

The construction of the functions ψ is given in Proposition 1.

Lemma 7. *Let* $\Psi = (\Psi_1, \Psi_2) \in \operatorname{Aut}' \widetilde{\Omega}$ *be defined as above by*

$$\begin{aligned} \Psi_1(z,Z) &= \Phi(z), \\ \Psi_2(z,Z) &= \psi(z)U(Z), \end{aligned}$$

where $U : \mathbb{C}^k \to \mathbb{C}^k$ *is special unitary. The Jacobian of* Ψ *is*

$$J\Psi(z,Z) = \psi^k(z) J\Phi(z)$$

and satisfies

$$|J\Psi(z,Z)|^2 = \left(\frac{N(\Phi z, \Phi z)}{N(z,z)}\right)^{\gamma + k\mu}. \tag{2.1}$$

2.2. **Reduction of the Monge-Ampère equation.** Let Ω be a bounded irreducible symmetric domain, $\mu > 0$ a real number and

$$\widetilde{\Omega} = \widetilde{\Omega}_k(\mu) = \left\{(z,Z) \in \Omega \times \mathbb{C}^k \mid \|Z\|^2 < N(z,z)^\mu\right\}.$$

We denote by d the complex dimension of Ω and by $n = d + k$ the dimension of $\widetilde{\Omega}$.

Let g be a C^2 function in $\widetilde{\Omega}$, which is a solution of the Monge-Ampère equation

$$\det\left(\frac{\partial^2 g}{\partial z^i \partial \overline{z}^j}\right)_{1 \le i,j \le d+k} = \mathrm{e}^{(n+1)g}$$

and which generates an invariant form $\partial\overline{\partial} g$. For $\Psi \in \operatorname{Aut}' \widetilde{\Omega}$, the invariance of the metric and the Monge-Ampère equation imply

$$\mathrm{e}^{(n+1)g(z,Z)} = |J\Psi(z,Z)|^2 \, \mathrm{e}^{(n+1)g(\Psi(z,Z))}$$

and, using (2.1),

$$\mathrm{e}^{(n+1)g(z,Z)} \, N(z,z)^{\gamma + k\mu} = \mathrm{e}^{(n+1)g(\Psi(z,Z))} \, N(\Phi z, \Phi z)^{\gamma + k\mu}.$$

The function

$$g(z,Z) + \frac{\gamma + k\mu}{d+k+1} \log N(z,z)$$

is then constant on the orbits of $\operatorname{Aut}' \widetilde{\Omega}$. For $X \in [0,1[$, we define $h(X)$ as the value of this function on the orbit $X = \|Z\|^2 N(z,z)^{-\mu}$. The function g can then be written

$$g(z,Z) = -\frac{\gamma + k\mu}{d+k+1} \log N(z,z) + h\left(\frac{\|Z\|^2}{N(z,z)^\mu}\right); \tag{2.2}$$

the function h can be obtained from g by

$$h\left(\|Z\|^2\right) = g(0,Z)$$

or

$$h(X) = g\left(0, \left(X^{1/2}, 0, \ldots, 0\right)\right).$$

Let $(z^1, \ldots, z^d)$ be coordinates on $V \supset \Omega$, which are orthonormal w.r. to the Hermitian metric m_1 relative to Ω and let $\left(Z^1, \ldots, Z^k\right)$ be orthonormal coordinates for the Hermitian space $\mathbb{C}^k$. Let

$$\left(z^1, \ldots, z^{d+k}\right) = \left(z^1, \ldots, z^d, Z^1, \ldots, Z^k\right)$$

and

$$\begin{aligned}\omega(z,Z) &= \omega_d(z) \wedge \omega_k(Z) \\ &= \mathrm{d}\, z^1 \wedge \mathrm{d}\, \overline{z}^1 \wedge \cdots \wedge \mathrm{d}\, z^d \wedge \mathrm{d}\, \overline{z}^d \wedge \mathrm{d}\, Z^1 \wedge \mathrm{d}\, \overline{Z}^1 \wedge \cdots \wedge \mathrm{d}\, Z^k \wedge \mathrm{d}\, \overline{Z}^k.\end{aligned}$$

For any C^2 function f on $\widetilde{\Omega}$, we have

$$\frac{1}{(d+k)!}\left(\partial\overline{\partial} f\right)^{d+k} = \det\left(\frac{\partial^2 f}{\partial z^i \partial \overline{z}^j}\right)\omega.$$

Lemma 8. *Let $g : \widetilde{\Omega} \to \mathbb{R}$ and $h : [0,1[\to \mathbb{R}$ be C^2 functions related by (2.2). Then, for $\|Z\|^2 = X > 0$,*

$$\frac{\left(\partial\overline{\partial} g\right)^{d+k}}{(d+k)!}(0,Z) = \left(h'(X)\right)^{k-1}\left(\mu X h'(X) + \frac{\gamma + k\mu}{d+k+1}\right)^d \left(Xh'(X)\right)' \omega. \tag{2.3}$$

Proof. Let $X = \|Z\|^2 N(z,z)^{-\mu}$. Then

$$\begin{aligned}\frac{\partial X}{X} &= \frac{\partial \|Z\|^2}{\|Z\|^2} - \mu \frac{\partial N}{N}, \\ \frac{\overline{\partial} X}{X} &= \frac{\overline{\partial} \|Z\|^2}{\|Z\|^2} - \mu \frac{\overline{\partial} N}{N}.\end{aligned}$$

We have

$$\overline{\partial}\left(h(X)\right) = h'(X)\overline{\partial} X = Xh'(X)\left(\frac{\overline{\partial}\|Z\|^2}{\|Z\|^2} - \mu\frac{\overline{\partial} N}{N}\right)$$

and

$$\begin{aligned}\partial\overline{\partial}\left(h(X)\right) = {} & \left(Xh'(X)\right)' X \left(\frac{\partial\|Z\|^2}{\|Z\|^2} - \mu\frac{\partial N}{N}\right) \wedge \left(\frac{\overline{\partial}\|Z\|^2}{\|Z\|^2} - \mu\frac{\overline{\partial} N}{N}\right) \\ & + Xh'(X)\partial\left(\frac{\overline{\partial}\|Z\|^2}{\|Z\|^2}\right) - \mu X h'(X)\partial\left(\frac{\overline{\partial} N}{N}\right).\end{aligned} \tag{2.4}$$

As

$$g(z,Z)=-\frac{\gamma+k\mu}{d+k+1}\log N(z,z)+h\left(X\right);$$

we have

$$\begin{aligned}\partial\overline{\partial}g=&Xh'(X)\partial\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}\right)-\left(\mu Xh'(X)+\frac{\gamma+k\mu}{d+k+1}\right)\partial\overline{\partial}\log N(z,z)\\&+XY'\left(\frac{\partial\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}-\mu\frac{\partial N}{N}\right)\wedge\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}-\mu\frac{\overline{\partial}N}{N}\right).\end{aligned}$$

Let

$$\begin{aligned}\beta&=\frac{\gamma+k\mu}{\mu\left(d+k+1\right)},\\Y_0&=Xh'(X),\\Y&=Xh'(X)+\frac{\gamma+k\mu}{\mu\left(d+k+1\right)}=Y_0+\beta.\end{aligned}$$

Then

$$\begin{aligned}\partial\overline{\partial}g=&\;Y_0\partial\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}\right)-\mu Y\partial\overline{\partial}\log N(z,z)\\&+X\,Y'\left(\frac{\partial\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}-\mu\frac{\partial N}{N}\right)\wedge\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}-\mu\frac{\overline{\partial}N}{N}\right)\end{aligned}$$

and

$$\begin{aligned}\left(\partial\overline{\partial}g\right)^{d+k}=&\;\left(Y_0\partial\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}\right)-\mu Y\partial\overline{\partial}\log N(z,z)\right)^{d+k}\\&+(d+k)\left(Y_0\partial\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}\right)-\mu Y\partial\overline{\partial}\log N(z,z)\right)^{d+k-1}\\&\wedge XY'\left(\frac{\partial\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}-\mu\frac{\partial N}{N}\right)\wedge\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}-\mu\frac{\overline{\partial}N}{N}\right)\\=&\binom{d+k}{k}\mu^{d}Y_0^{k}Y^{d}\left(-\partial\overline{\partial}\log N(z,z)\right)^{d}\wedge\left(\partial\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}\right)\right)^{k}\\&+(d+k)\binom{d+k-1}{k-1}\mu^{d}XY'Y_0^{k-1}Y^{d}\\&\left(-\partial\overline{\partial}\log N(z,z)\right)^{d}\wedge\left(\partial\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}\right)\right)^{k-1}\wedge\frac{\partial\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}\wedge\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}\\&+(d+k)\binom{d+k-1}{k}\mu^{d+2}XY'Y_0^{k}Y^{d-1}\\&\left(-\partial\overline{\partial}\log N(z,z)\right)^{d-1}\wedge\frac{\partial N}{N}\wedge\frac{\overline{\partial}N}{N}\wedge\left(\partial\left(\frac{\overline{\partial}\left\|Z\right\|^{2}}{\left\|Z\right\|^{2}}\right)\right)^{k}.\end{aligned}$$

One checks easily that

$$\left(\partial\left(\frac{\overline{\partial}\,\|Z\|^2}{\|Z\|^2}\right)\right)^k = 0$$

and

$$\left(\partial\left(\frac{\overline{\partial}\,\|Z\|^2}{\|Z\|^2}\right)\right)^{k-1} \wedge \frac{\partial\,\|Z\|^2}{\|Z\|^2} \wedge \frac{\overline{\partial}\,\|Z\|^2}{\|Z\|^2} = \frac{(k-1)!\omega_k(Z)}{\|Z\|^{2k}}.$$

So the expression of $\left(\partial\overline{\partial}g\right)^{d+k}$ is reduced to

$$\left(\partial\overline{\partial}g\right)^{d+k} = \frac{(d+k)!}{d!}\mu^d XY'Y_0^{k-1}Y^d\left(-\partial\overline{\partial}\log N(z,z)\right)^d \wedge \frac{\omega_k(Z)}{\|Z\|^{2k}}.$$

At $z=0$, we have $X = \|Z\|^2$ and (1.25):

$$\left.\frac{\left(-\partial\overline{\partial}\log N(z,z)\right)^d}{d!}\right|_{z=0} = \omega_d(z).$$

Finally, we obtain

$$\frac{1}{(d+k)!}\left(\partial\overline{\partial}g\right)^{d+k}(0,Z) = \mu^d X^{1-k}Y'Y_0^{k-1}Y^d\omega,$$

which is equivalent to (2.3). □

Lemma 9. *Let g be a C^2 function in $\widetilde{\Omega}$, which is a solution of the Monge-Ampère equation*

$$\det\left(\frac{\partial^2 g}{\partial z^i\partial\overline{z}^j}\right) = \mathrm{e}^{(n+1)g}$$

and which generates an invariant form $\partial\overline{\partial}g$. Let

$$g(z,Z) = -\frac{\gamma+k\mu}{d+k+1}\log N(z,z) + h\left(X\right),$$
$$X = \frac{\|Z\|^2}{N(z,z)^\mu}.$$

Then h satisfies on $]0,1[$ the differential equation

$$\left(h'(X)\right)^{k-1}\left(\mu Xh'(X) + \frac{\gamma+k\mu}{d+k+1}\right)^d\left(Xh'(X)\right)' = \mathrm{e}^{(d+k+1)h},$$
$$Xh'(X) \to 0 \qquad (X\to 0).$$

The boundary condition

$$g(z)\to\infty \qquad (z\to\partial\widetilde{\Omega})$$

implies

$$h(X)\to\infty \qquad (X\to 1).$$

Let

$$\beta = \frac{\gamma + k\mu}{\mu\,(d+k+1)}, \tag{2.5}$$

$$Y = Y_0 + \beta = Xh'(X) + \frac{\gamma + k\mu}{\mu\,(d+k+1)}. \tag{2.6}$$

It results from the previous lemmas that if g is a solution of the Monge-Ampère equation

$$\det\left(\frac{\partial^2 g}{\partial z^i \partial \overline{z}^j}\right) = \mathrm{e}^{(d+k+1)g}$$

on $\widetilde{\Omega}$, with the boundary condition

$$g(z) \to \infty \qquad (z \to \partial\widetilde{\Omega}),$$

then the function Y defined by (2.6) satisfies the differential equation

$$\left(\frac{Y-\beta}{X}\right)^{k-1} (\mu Y)^d\, Y' = \mathrm{e}^{(d+k+1)h} \tag{2.7}$$

with the initial condition $Y(0) = \beta$.

Writing (2.7) as

$$\mu^d Y_0^{k-1} \left(Y_0 + \beta\right)^d Y_0' = X^{k-1}\, \mathrm{e}^{(d+k+1)h} \tag{2.8}$$

shows that there exists a polynomial R of degree $d+k$, with non negative coefficients, such that the derivative of $R(Y_0)$ is positive on $]0,1[$ and tends to ∞ when $X \to 1$. So the function Y_0 is strictly increasing and maps $[0,1[$ onto $[0,+\infty[$ and the function Y is strictly increasing and maps $[0,1[$ onto $[\beta,+\infty[$.

Taking logarithmic derivatives of both sides of (2.8), we get

$$\frac{\left(Y_0^{k-1} \left(Y_0+\beta\right)^d Y_0'\right)'}{Y_0^{k-1} \left(Y_0+\beta\right)^d Y_0'} = (d+k+1)\frac{Y_0}{X} + \frac{k-1}{X},$$

which is equivalent to

$$X\left(Y_0^{k-1} \left(Y_0+\beta\right)^d Y_0'\right)' = \left(Y_0^{k-1} \left(Y_0+\beta\right)^d Y_0'\right)\left((d+k+1)Y_0 + k - 1\right)$$

or to

$$\left(XY_0^{k-1} \left(Y_0+\beta\right)^d Y_0'\right)' = \left((d+k+1)Y_0 + k\right) Y_0^{k-1} \left(Y_0+\beta\right)^d Y_0'. \tag{2.9}$$

Let S be the polynomial of degree $d+1$ defined by

$$T^k S(T) = \int_0^T \left((d+k+1)t + k\right) t^{k-1} \left(t+\beta\right)^d \mathrm{d}\,t. \tag{2.10}$$

Integrating (2.9) with the initial condition $Y_0(0) = 0$ yields

$$XY_0^{k-1} \left(Y_0+\beta\right)^d Y_0' = Y_0^k S(Y_0)$$

or

$$X\left(Y_0+\beta\right)^d Y_0' = Y_0 S(Y_0).$$

Integrating by parts in (2.10), we get

$$T^k S(T) = T^k (T+\beta)^{d+1} + k(1-\beta)\int_0^T t^k \left(t+\beta\right)^d \mathrm{d}\,t;$$

let

$$\int_0^T t^k (t+\beta)^d \,\mathrm{d}\, t = T^{k+1} S_1(T). \tag{2.11}$$

Then S_1 is a polynomial of degree d, and

$$(-1)^{k+d} \int_0^{-\beta} t^k (t+\beta)^d \,\mathrm{d}\, t > 0$$

shows that $S_1(-\beta) \neq 0$. Finally,

$$S(T) = (T+\beta)^{d+1} + k(1-\beta) T S_1(T), \quad S_1(-\beta) \neq 0. \tag{2.12}$$

Lemma 10. *The differential equation*

$$\begin{aligned} &X (Y_0+\beta)^d Y_0' = Y_0 S(Y_0), \\ &Y_0 \to \infty \qquad\qquad (X \to 1), \end{aligned}$$

where the polynomial S is defined by (2.10), has a unique solution

$$Y_0 : [0,1[\to [0,+\infty[.$$

This solution is C^∞ at 0.

Theorem 2. *The generating function g for the Kähler-Einstein metric of*

$$\widetilde{\Omega} = \widetilde{\Omega}_k(\mu) = \left\{ (z,Z) \in \Omega \times \mathbb{C}^k \mid \|Z\|^2 < N(z,z)^\mu \right\}$$

is given by

$$g(z,Z) = -\frac{\gamma + k\mu}{d+k+1} \log N(z,z) + h\left(\frac{\|Z\|^2}{N(z,z)^\mu} \right),$$

where

$$\mathrm{e}^{(d+k+1)h} = \mu^d X^{1-k} Y_0' Y_0^{k-1} (Y_0+\beta)^d$$

and the function $Y_0 : [0,1[\to [0,+\infty[$ satisfies

$$X (Y_0+\beta)^d Y_0' = Y_0 S(Y_0), \tag{2.13}$$

$$Y_0 \to \infty \qquad\qquad (X \to 1) \tag{2.14}$$

with

$$T^k S(T) = \int_0^T ((d+k+1)t + k)\, t^{k-1} (t+\beta)^d \,\mathrm{d}\, t.$$

In view of (2.12), the differential equation (2.13) may also be written as

$$X Y_0' = Y_0(Y_0+\beta) + k(1-\beta) \frac{Y_0^2 S_1(Y_0)}{(Y_0+\beta)^d}, \tag{2.15}$$

where S_1 is the polynomial defined by (2.11).

2.3. **The critical exponent.** The expression (2.12) shows that the polynomial S defined by (2.10) is divisible by $(T+\beta)^d$ if and only if $\beta = 1$. In this case, $S = (T+\beta)^{d+1}$. The equation (2.13) is then

$$XY_0' = Y_0\,(Y_0+1)\,.$$

With the limit condition (2.14), it integrates as

$$-\log X = \int_{Y_0}^{\infty} \frac{1}{y(y+1)}\,\mathrm{d}\,y,$$

which gives

$$Y_0 = \frac{X}{1-X}$$

and

$$Y = Y_0 + 1 = \frac{1}{1-X}.$$

As

$$\beta = \frac{\gamma + k\mu}{\mu\,(d+k+1)},$$

the value of μ corresponding to $\beta = 1$ is again

$$\mu_0 = \frac{\gamma}{d+1},$$

that is, the same value as for $k = 1$.

For $\mu = \mu_0$, it is again possible to compute explicitly the Kähler-Einstein metric of $\widetilde{\Omega}_k(\mu)$. Actually,

$$\mathrm{e}^{(d+k+1)h} = \mu^d X^{1-k} Y_0' Y_0^{k-1}\,(Y_0+\beta)^d$$

yields

$$\mathrm{e}^{(d+k+1)h} = \mu_0^d \frac{1}{(1-X)^{d+k+1}},$$

$$h(X) = \frac{d}{d+k+1}\log\mu_0 + \log\frac{1}{1-X}.$$

As $\beta = 1$, we have $\frac{\gamma+k\mu_0}{d+k+1} = \mu_0$,

$$\begin{aligned} g(z,Z) &= -\mu_0 \log N(z,z) + h\left(\frac{\|Z\|^2}{N(z,z)^{\mu_0}}\right) \\ &= \frac{d}{d+k+1}\log\mu_0 + \log\frac{1}{N(z,z)^{\mu_0} - \|Z\|^2} \end{aligned}$$

and finally

$$\partial\overline{\partial} g = -\partial\overline{\partial}\log\left(N(z,z)^{\mu_0} - \|Z\|^2\right).$$

3. The exceptional cases

Here we study the Kähler-Einstein metric and the Bergman metric of $\widetilde{\Omega}_k\,(\mu_0)$, when Ω is one of the two exceptional domains Ω_V and Ω_{VI} (see Appendix C).

3.1. The *Bergman kernel* of $\widetilde{\Omega}_k(\mu)$ has been computed (for all $\mu > 0$) in [4]. If the polynomial $k \mapsto \chi(k\mu)$ (which has degree $d = \dim \Omega$) is decomposed as

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{d} c_{\mu,j} \frac{(k+1)_j}{j!}, \tag{3.1}$$

where $(k+1)_j = \frac{\Gamma(k+j)}{\Gamma(k)}$ denotes the raising factorial, let the function $F_{\chi,\mu}$ be defined by

$$F_{\chi,\mu}(t) = \sum_{j=0}^{d} c_{\mu,j} \left(\frac{1}{1-t}\right)^j. \tag{3.2}$$

Then the Bergman kernel $\widetilde{\mathcal{K}}_k(z,Z)$ of $\widetilde{\Omega}_k(\mu)$ is given by the relations

$$\widetilde{\mathcal{K}}_k(z,Z) = \mathcal{L}_k\left(z, \|Z\|^2\right), \tag{3.3}$$

$$\mathcal{L}_k(z,r) = \frac{1}{k!}\frac{\partial^k}{\partial r^k}\mathcal{L}_0(z,r), \tag{3.4}$$

$$\mathcal{L}_0(z,r) = \mathcal{K}(z) F_{\chi,\mu}\left(\frac{r}{N(z,z)^\mu}\right), \tag{3.5}$$

where

$$\mathcal{K}(z) = \frac{1}{\operatorname{vol}\Omega}\frac{1}{N(z,z)^\gamma}$$

is the Bergman kernel of Ω.

Applying these results, we get

$$\mathcal{L}_0(z,r) = \frac{1}{\operatorname{vol}\Omega}\frac{1}{N(z,z)^\gamma} F_{\chi,\mu}\left(\frac{r}{N(z,z)^\mu}\right),$$
$$\mathcal{L}_k(z,r) = \frac{1}{k!}\frac{1}{\operatorname{vol}\Omega}\frac{1}{N(z,z)^{\gamma+k\mu}} F^{(k)}_{\chi,\mu}\left(\frac{r}{N(z,z)^\mu}\right)$$

and

$$\widetilde{\mathcal{K}}_k(z,Z) = \frac{1}{k!}\frac{1}{\operatorname{vol}\Omega}\frac{1}{N(z,z)^{\gamma+k\mu}} F^{(k)}_{\chi,\mu}(X),$$

where

$$X = \frac{\|Z\|^2}{N(z,z)^\mu}.$$

Hence the *Bergman metric* of $\widetilde{\Omega}_k(\mu)$ is associated to the $(1,1)$ form

$$\phi = -(\gamma + k\mu)\,\partial\overline{\partial}\log N(z,z) + \partial\overline{\partial}\log F^{(k)}_{\chi,\mu}(X).. \tag{3.6}$$

If $\mu = \mu_0$, the critical exponent, then $\gamma + k\mu_0 = \mu_0(d+k+1)$ and the Bergman metric of $\widetilde{\Omega}_k(\mu_0)$ is associated to the $(1,1)$ form

$$\phi_0 = -\mu_0(d+k+1)\partial\overline{\partial}\log N(z,z) + \partial\overline{\partial}\log F^{(k)}_{\chi,\mu_0}(X). \tag{3.7}$$

For the critical exponent $\mu = \mu_0$, the Kähler-Einstein metric of $\widetilde{\Omega}_k(\mu_0)$ is associated to the $(1,1)$ form

$$\Psi_0 = \partial\overline{\partial} g = -\partial\overline{\partial}\log\left(N(z,z)^{\mu_0} - \|Z\|^2\right). \tag{3.8}$$

3.2. **The exceptional case of dimension 16.** See Appendix C.2 for definitions, notations, and basic results.

Let

$$V = \mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}}) = \{(a_2, a_3) \mid a_2, a_3 \in \mathbb{O}_{\mathbb{C}}\}$$

and consider the exceptional symmetric domain of dimension 16

$$\Omega = \Omega_V = \left\{x \in \mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}}) \mid 1 - (x|x) + (x^\sharp \mid x^\sharp) > 0,\ 2 - (x|x) > 0\right\}.$$

Here $d = 16$ and $\gamma = 12$. The *generic norm* is

$$N(x, y) = 1 - (x|y) + (x^\sharp|y^\sharp).$$

The critical exponent is

$$\mu_0 = \frac{12}{17}.$$

The inflated Hartogs domain is $\widetilde{\Omega} = \widetilde{\Omega}_k(\mu_0) \subset \mathbb{C}^{16+k}$, consisting in $(z, Z) \in \mathbb{O}_{\mathbb{C}} \times \mathbb{C}^k$ which verify

$$\begin{aligned}
&2 - (z|z) > 0,\\
&1 - (z|z) + (z^\sharp \mid z^\sharp) > 0,\\
&\|Z\|^2 < \left(1 - (z|z) + (z^\sharp|z^\sharp)\right)^{12/17}.
\end{aligned}$$

The *Kähler-Einstein metric* of $\widetilde{\Omega}_k(\mu_0)$ is associated to the $(1,1)$-form

$$\Psi_0 = -\partial\overline{\partial} \log\left(\left(1 - (z|z) + (z^\sharp|z^\sharp)\right)^{12/17} - \|Z\|^2\right). \tag{3.9}$$

The *Hua polynomial* χ is

$$\chi(s) = (s+1)_8(s+4)_8.$$

For $\mu = \mu_0 = \frac{12}{17}$, the Bergman metric of $\widetilde{\Omega}_k(\mu_0)$ is associated to the $(1,1)$ form

$$\phi_0 = -\mu_0(d+k+1)\partial\overline{\partial} \log N(z, z) + \partial\overline{\partial} \log F^{(k)}_{\chi,\mu_0}(X). \tag{3.10}$$

Recall that

$$F_{\chi,\mu_0}(t) = \sum_{j=0}^{d} c_{\mu_0,j} \left(\frac{1}{1-t}\right)^j,$$

where the $c_{\mu,j}$ are determined by

$$\frac{\chi(k\mu_0)}{\chi(0)} = \sum_{j=0}^{16} c_{\mu_0,j} \frac{(k+1)_j}{j!}.$$

A computation with Maple gives

$$\chi\left(\frac{12}{17}s\right) = \left(\frac{12}{17}\right)^{16} \sum_{j=0}^{16} c_j (s+1)_j,$$

with

$$c_{16} = 1, \qquad c_{15} = 0, \qquad c_{14} = \frac{595}{12}, \qquad c_{13} = \frac{4165}{6}, \qquad c_{12} = \frac{30042145}{3456},$$

$$c_{11} = \frac{14448385}{144}, \qquad c_{10} = \frac{790269316375}{746496}, \qquad c_9 = \frac{1259425781075}{124416},$$

$$c_8 = \frac{12447571001586875}{143327232}, \qquad c_7 = \frac{2957566710311675}{4478976},$$
$$c_6 = \frac{11300125622942496725}{2579890176}, \qquad c_5 = \frac{10677213117341703625}{429981696},$$
$$c_4 = \frac{651907704485453963181 25}{557256278016}, \qquad c_3 = \frac{10209484788366056549125}{23219011584},$$
$$c_2 = \frac{114818904611324955416375}{92876046336}, \qquad c_1 = \frac{35779252854815307462625}{15479341056},$$
$$c_0 = \frac{3336889241222254530312 5}{15479341056}.$$

This shows that *all coefficients* $c_{\mu_0,j}$ $(0 \leq j \leq 16)$ in

$$F_{\chi,\mu_0}(t) = \sum_{j=0}^{16} c_{\mu_0,j} \left(\frac{1}{1-t}\right)^j$$

are strictly positive, except $c_{\mu_0,15} = 0$.

3.3. **The exceptional case of dimension 27.** See Appendix C.1 for definitions, notations, and basic results.

Let

$$V = \mathcal{H}_3(\mathbb{O}_\mathbb{C}),$$

the space of Cayley-Hermitian 3×3 matrices with entries in $\mathbb{O}_\mathbb{C}$ and consider the exceptional symmetric domain $\Omega = \Omega_V$ of dimension 27, defined by the inequalities

$$1 - (z|z) + (z^\sharp|z^\sharp) - |\det z|^2 > 0,$$
$$3 - 2(z|z) + (z^\sharp|z^\sharp) > 0,$$
$$3 - (z|z) > 0.$$

The generic minimal polynomial is

$$m(T, x, y) = T^3 - (x|y)T^2 + (x^\sharp|y^\sharp)T - \det x \det \overline{y}.$$

Here $d = 27$ and $\gamma = 18$. The *generic norm* is

$$N(x, y) = 1 - (x \mid y) + (x^\# \mid y^\#) - \det x \det \overline{y}.$$

The critical exponent is

$$\mu_0 = \frac{9}{14}..$$

The inflated Hartogs domain is $\widetilde{\Omega} = \widetilde{\Omega}_k(\mu_0) \subset \mathbb{C}^{16+k}$, consisting in $(z, Z) \in \mathbb{O}_\mathbb{C} \times \mathbb{C}^k$ which verify

$$3 - (z|z) > 0,$$
$$3 - 2(z|z) + (z^\sharp \mid z^\sharp) > 0,$$
$$1 - (z|z) + (z^\sharp|z^\sharp) - |\det z|^2 > 0,$$
$$\|Z\|^2 < \left(1 - (z|z) + (z^\sharp|z^\sharp) - |\det z|^2\right)^{9/14}.$$

The *Kähler-Einstein metric* of $\widetilde{\Omega}_k(\mu_0)$ is associated to the $(1,1)$-form

$$\Psi_0 = -\partial\overline{\partial} \log \left(\left(1 - (z|z) + (z^\sharp|z^\sharp) - |\det z|^2\right)^{9/14} - \|Z\|^2\right). \tag{3.11}$$

The *Hua polynomial* χ is

$$\chi(s) = (s+1)_9(s+5)_9(s+9)_9.$$

For $\mu = \mu_0 = \frac{9}{14}$, the Bergman metric of $\widetilde{\Omega}_k\,(\mu_0)$ is associated to the $(1,1)$ form

$$\phi_0 = -\mu_0(d+k+1)\partial\overline{\partial}\log N(z,z) + \partial\overline{\partial}\log F^{(k)}_{\chi,\mu_0}(X), \tag{3.12}$$

with

$$F_{\chi,\mu_0}(t) = \sum_{j=0}^{27} c_{\mu_0,j}\left(\frac{1}{1-t}\right)^j,$$

where the $c_{\mu,j}$ are determined by

$$\frac{\chi(k\mu_0)}{\chi(0)} = \sum_{j=0}^{27} c_{\mu_0,j}\frac{(k+1)_j}{j!}.$$

A computation with Maple gives

$$\chi\left(\frac{9}{14}s\right) = \left(\frac{9}{14}\right)^{27}\sum_{j=0}^{27} c_j(s+1)_j,$$

with

$$\begin{aligned}
c_{27} &= 1, \qquad c_{26} = 0, \qquad c_{25} = \frac{2275}{9},\\
c_{24} &= \frac{56875}{9}, \qquad c_{23} = \frac{38591735}{243},\\
c_{21} &= 15425515970150/3^{11},\\
c_{20} &= 37061881356500/3^{9},\\
c_{19} &= 184328710104188650/3^{14},\\
c_{18} &= 3564334218619774600/3^{14},\\
c_{17} &= 58473532468117741975 0/3^{16},\\
c_{16} &= 1002073289416306081975 0/3^{16},\\
c_{15} &= 3520016113512955878642535 00/3^{23},\\
c_{14} &= 586664566244061492395923000/3^{21},\\
c_{13} &= 1988637252859632373297511212000/3^{26},\\
c_{12} &= 2567225171703812439228939630100 0/3^{26},\\
c_{11} &= 823366348706160597280348633164437 5/3^{29},\\
c_{10} &= 8938479344382100037086237438262500 0/3^{29},\\
c_9 &= 192375429310254004220153919832695936687 5/3^{36},\\
c_8 &= 2097789080057125888595916491235338018 75/3^{32},\\
c_7 &= 3991925523773734763185503956827514329756 25/3^{37},\\
c_6 &= 2728484170046421839052459199725228012518 750/3^{37},\\
c_5 &= 4784035119796249263140931690273985222683 1250/3^{38},\\
c_4 &= 2324682577625177531584606418615396267101 25000/3^{38},
\end{aligned}$$

$$c_3 = 73037107041363504672642146434776735686797778125/3^{42},$$
$$c_2 = 23557400955895564936769134062033297681918662500/3^{40},$$
$$c_1 = 409456797752799914624225389536137199476376953125/3^{42},$$
$$c_0 = 394594700340674453245747775040231797415576953125/3^{42}.$$

This shows again that *all coefficients* $c_{\mu_0,j}$ $(0 \le j \le 27)$ in

$$F_{\chi,\mu_0}(t) = \sum_{j=0}^{27} c_{\mu_0,j} \left(\frac{1}{1-t}\right)^j$$

are strictly positive, except $c_{\mu_0,26} = 0$.

4. A conjecture

It has been shown above that for the critical exponent $\mu = \mu_0$, the Kähler-Einstein metric of $\widetilde{\Omega}_k\,(\mu_0)$ is associated to the $(1,1)$ form

$$\Psi_0 = \partial\overline{\partial} g = -\partial\overline{\partial}\log\left(N(z,z)^{\mu_0} - \|Z\|^2\right).$$

On the other hand, the Bergman metric of $\widetilde{\Omega}_k\,(\mu)$ is associated to the $(1,1)$ form

$$\phi_0 = -\mu(d+k+1)\partial\overline{\partial}\log N(z,z) + \partial\overline{\partial}\log F^{(k)}_{\chi,\mu}(X),$$

with

$$F_{\chi,\mu}(t) = \sum_{j=0}^{d} c_{\mu,j} \left(\frac{1}{1-t}\right)^j,$$

where the $c_{\mu,j}$ are determined by

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{d} c_{\mu,j} \frac{(k+1)_j}{j!}. \tag{4.1}$$

Conjecture. *Let Ω be an irreducible circled bounded symmetric domain of genus γ and dimension d, $\mu_0 = \frac{\gamma}{d+1}$ its critical exponent, χ its Hua polynomial. The coefficients $c_{\mu,j}$ in (4.1) are all strictly positive if and only if*

$$\mu < \mu_0.$$

For $\mu = \mu_0$, all coefficients $c_{\mu,j}$ in (4.1) are strictly positive, except $c_{\mu,d-1} = 0$ and except for the rank 1 type $I_{1,n}$ (where $c_{\mu,d} > 0$ and $c_{\mu,j} = 0$ for all $j < d = n$).

The values of the critical exponent are

Type	$I_{m,n}$	II_n	III_n	IV_n	V	VI
μ_0	$\frac{m+n}{mn+1}$	$\frac{2}{n+\frac{2}{n-1}}$	$\frac{2}{n+\frac{2}{n+1}}$	$\frac{n}{n+1}$	$\frac{12}{17}$	$\frac{9}{14}$

We have always $\mu_0 < 1$, except in the rank 1 case $I_{1,n}$. For $\mu = 1$, the signs of $c_{\mu,j}$ are alternating, starting with $c_{\mu,d} > 0$ and ending with $c_{\mu,j} = 0$ for $j < j_0$, where j_0 is a positive integer depending on Ω.

Remark 3. The conjecture has been checked with help of computer algebra software in many significant cases:

- for $\mu = \mu_0$ and the types $I_{3,3}$, IV_3, IV_4, IV_6, V, VI;
- for type V and all values of μ.

Remark 4. As the function $F_{\chi,\mu}$ is related to the Bergman kernel of $\widetilde{\Omega}_k(\mu)$, all derivatives $F^{(k)}_{\chi,\mu}$, $k>0$, of this function are strictly positive on $[0,1[$ for all $\mu>0$.

Remark 5. If the conjecture is true, it would help to compare the Bergman metric and the Kähler-Einstein metric of $\widetilde{\Omega}_k(\mu)$ for $\mu=\mu_0$. The exponent μ_0 seems also to be a limit case for other comparisons: in [12], a comparison theorem is given between the Kähler-Einstein metric and the Kobayashi metric of $\widetilde{\Omega}_1(\mu)$ when Ω is a symmetric domain of type $I_{m,n}$ and $\mu<\mu_0$.

Appendix A. Bounded symmetric domains and Jordan triple systems

Hereunder we give a review of properties of the Jordan triple structure associated to a complex bounded symmetric domain (see [7], [8]).

A.1. **Jordan triple system associated to a bounded symmetric domain.** Let Ω be an irreducible bounded circled homogeneous domain in a complex vector space V. Let K be the identity component of the (compact) Lie group of (linear) automorphisms of Ω leaving 0 fixed. Let ω be a volume form on V, invariant by K and by translations. Let $\mathcal{K}$ be the Bergman kernel of Ω with respect to ω, that is, the reproducing kernel of the Hilbert space $H^2(\Omega,\omega)=\operatorname{Hol}(\Omega)\cap L^2(\Omega,\omega)$. The Bergman metric at $z\in\Omega$ is defined by

$$h_z(u,v)=\partial_u\overline{\partial}_v\log\mathcal{K}(z).$$

The *Jordan triple product* on V is characterized by

$$h_0(\{uvw\},t)=\partial_u\overline{\partial}_v\partial_w\overline{\partial}_t\log\mathcal{K}(z)\,|_{z=0}\,.$$

The triple product $(x,y,z)\mapsto\{xyz\}$ is complex bilinear and symmetric with respect to (x,z), complex antilinear with respect to y. It satisfies the *Jordan identity*

$$\{xy\{uvw\}\}-\{uv\{xyw\}\}=\{\{xyu\}vw\}-\{u\{vxy\}w\}.$$

The space V endowed with the triple product $\{xyz\}$ is called a *(Hermitian) Jordan triple system.* For $x,y,z\in V$, denote by $D(x,y)$ and $Q(x,z)$ the operators defined by

$$\{xyz\}=D(x,y)z=Q(x,z)y.$$

The Bergman metric at 0 is related to D by

$$h_0(u,v)=\operatorname{tr}D(u,v).$$

A Jordan triple system is called *Hermitian positive* if $(u|v)=\operatorname{tr}D(u,v)$ is positive definite. As the Bergman metric of a bounded domain is always definite positive, the Jordan triple system associated to a bounded symmetric domain is Hermitian positive.

The *quadratic representation*

$$Q:V\longrightarrow\operatorname{End}_{\mathbb{R}}(V)$$

is defined by $Q(x)y=\frac{1}{2}\{xyx\}$. The following fundamental identity for the quadratic representation is a consequence of the Jordan identity:

$$Q(Q(x)y)=Q(x)Q(y)Q(x).$$

The *Bergman operator* B is defined by

$$B(x,y)=I-D(x,y)+Q(x)Q(y),$$

where I denotes the identity operator in V. It is also a consequence of the Jordan identity that the following fundamental identity holds for the Bergman operator:

$$Q(B(x,y)z) = B(x,y)Q(z)B(y,x).$$

The Bergman operator gets its name from the following property:

$$h_z\left(B(z,z)u,v\right) = h_0(u,v) \quad (z \in \Omega;\ u,v \in V).$$

If $\Phi \in (\operatorname{Aut}\Omega)_0$, the identity component of the automorphism group of Ω, the relation

$$B(\Phi x, \Phi y) = \mathrm{d}\,\Phi(x) \circ B(x,y) \circ \mathrm{d}\,\Phi(y)^* \tag{A.1}$$

holds for $x, y \in \Omega$, where * denotes the adjoint with respect to the Hermitian metric h_0. As a consequence, the Bergman kernel of Ω is given by

$$\mathcal{K}(z) = \frac{1}{\operatorname{vol}\Omega}\frac{1}{\det B(z,z)}. \tag{A.2}$$

The *quasi-inverse* x^y is defined, for each pair (x,y) such that $B(x,y)$ is invertible, by

$$x^y = B\left(x,y\right)^{-1}\left(x - Q(x)y\right).$$

A.2. **Spectral theory.** An Hermitian positive Jordan triple system is always *semi-simple*, that is, the direct sum of a finite family of simple subsystems, with component-wise triple product.

As the domain Ω is assumed to be irreducible, the associated Jordan triple system V is *simple*, that is V is not the direct sum of two non trivial subsystems.

An *automorphism* $f : V \to V$ of the Jordan triple system V is a complex linear isomorphism preserving the triple product : $f\{u,v,w\} = \{fu, fv, fw\}$. The automorphisms of V form a group, denoted $\operatorname{Aut} V$, which is a compact Lie group; we will denote by K its identity component.

An element $c \in V$ is called *tripotent* if $\{ccc\} = 2c$. If c is a tripotent, the operator $D(c,c)$ annihilates the polynomial $T(T-1)(T-2)$.

Let c be a tripotent. The decomposition $V = V_0(c) \oplus V_1(c) \oplus V_2(c)$, where $V_j(c)$ is the eigenspace $V_j(c) = \{x \in V\ ;\ D(c,c)x = jx\}$, is called the *Peirce decomposition* of V (with respect to the tripotent c).

Two tripotents c_1 and c_2 are called *orthogonal* if $D(c_1,c_2) = 0$. If c_1 and c_2 are orthogonal tripotents, then $D(c_1,c_{1)}$ and $D(c_2,c_2)$ commute and $c_1 + c_2$ is also a tripotent.

A non zero tripotent c is called *primitive* if it is not the sum of non zero orthogonal tripotents. A tripotent c is *maximal* if there is no non zero tripotent orthogonal to c. The set of maximal tripotents is equal to the *Shilov boundary* of the domain Ω.

A *frame* of V is a maximal sequence $(c_1, \ldots, c_r)$ of pairwise orthogonal primitive tripotents. The frames of V form a manifold $\mathcal{F}$, which is called the *Satake-Furstenberg boundary* of Ω.

Let $\mathbf{c} = (c_1, \ldots, c_r)$ be a frame. For $0 \le i \le j \le r$, let

$$V_{ij}(\mathbf{c}) = \left\{x \in V \mid D(c_k,c_k)x = (\delta_i^k + \delta_j^k)x,\ 1 \le k \le r\right\} :$$

the decomposition $V = \bigoplus_{0 \le i \le j \le r} V_{ij}(\mathbf{c})$ is called the *simultaneous Peirce decomposition* with respect to the frame $\mathbf{c}$.

Let V be a simple Hermitian positive Jordan triple system. Then there exist frames for V. All frames have the same number of elements, which is the *rank* r of V. The subspaces $V_{ij} = V_{ij}(\mathbf{c})$ of the simultaneous Peirce decomposition have the following properties: $V_{00} = 0$; $V_{ii} = \mathbb{C}e_i$ $(0 < i)$; all V_{ij}'s $(0 < i < j)$ have the same dimension a; all V_{0i}'s $(0 < i)$ have the same dimension b.

The *numerical invariants* of V (or of Ω) are the rank r and the two integers

$$\begin{aligned} a &= \dim V_{ij} \quad (0 < i < j), \\ b &= \dim V_{0i} \quad (0 < i). \end{aligned}$$

The *genus* of V is the number γ defined by

$$\gamma = 2 + a(r-1) + b;$$

the genus is generally denoted by g or p, but we denote it here by γ as g stands for the generating function of the Kähler-Einstein metric. The HPJTS V and the domain Ω are said to be of *tube type* if $b = 0$.

Let V be a simple Hermitian positive Jordan triple system. Then any $x \in V$ can be written in a unique way

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p, \tag{A.3}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$ and $c_1, c_2 \ldots, c_p$ are pairwise orthogonal tripotents. The element x is *regular* iff $p = r$; then $(c_1, c_2, \ldots, c_r)$ is a frame of V. The decomposition (A.3) is called the *spectral decomposition* of x.

A.3. **The generic minimal polynomial.** Let V be a Jordan triple system of rank r. There exist polynomials $m_1, \ldots, m_r$ on $V \times \overline{V}$, homogeneous of respective bidegrees $(1,1), \ldots, (r,r)$, such that for each regular $x \in V$, the polynomial

$$m(T, x, y) = T^r - m_1(x,y)T^{r-1} + \cdots + (-1)^r m_r(x,y)$$

satisfies

$$m(T, x, x) = \prod_{i=1}^{r} (T - \lambda_i^2),$$

where $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r$ is the spectral decomposition of x. Here $\overline{V}$ denotes the space V with the conjugate complex structure. The polynomial

$$m(T, x, y) = T^r - m_1(x,y)T^{r-1} + \cdots + (-1)^r m_r(x,y)$$

is called the *generic minimal polynomial* of V (at (x,y)). The (inhomogeneous) polynomial $N : V \times \overline{V} \to \mathbb{C}$ defined by

$$N(x,y) = m(1,x,y)$$

is called the *generic norm.* The following identities hold:

$$\begin{aligned} \det B(x,y) &= N(x,y)^{\gamma}, \\ \operatorname{tr} D(x,y) &= \gamma\, m_1(x,y). \end{aligned} \tag{A.4}$$

A.4. **The spectral norm.** Let V be an HPJTS. The map $x \mapsto \lambda_1$, where $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p$ is the spectral decomposition of x ($\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$) is a norm on V, called the *spectral norm.* The *bounded symmetric domain* Ω is the unit ball of V for the *spectral norm.* It is also characterized by the set of polynomial inequalities

$$\left.\frac{\partial^j}{\partial T^j} m(T,x,x)\right|_{T=1} > 0, \qquad 0 \leq j \leq r-1.$$

A.5. **The Hua integral.** Let Ω be an irreducible bounded circled homogeneous domain, N its generic norm. The *Hua integral* relative to Ω is

$$\int_\Omega N(x,x)^s \alpha^n,$$

where $\alpha = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial} m_1$. It converges for $s > -1$ and has be computed for the four classical series of symmetric domains by Hua Lookeng in [9].

For a general irreducible bounded circled homogeneous domain with numerical invariants a, b and rank r, the Hua integral is given by

$$\int_\Omega N(x,x)^s \alpha^n = \frac{\chi(0)}{\chi(s)} \int_\Omega \alpha^n,$$

where χ is the *Hua polynomial*

$$\chi(s) = \prod_{j=1}^{r} \left(s + 1 + (j-1)\frac{a}{2}\right)_{1+b+(r-j)a}$$

(see [4], Theorem 2.3). See the tables below for the expression of this polynomial for the six cases of bounded irreducible symmetric domains.

Appendix B. Tables for bounded symmetric domains

The following examples exhaust the list of simple Hermitian positive Jordan triple systems (see [10]). The HPJTS occurring in the four infinite series $I_{p,q}$, II_n, III_n, IV_n are called *classical*; the two HPJTS of type V and VI are called *exceptional.* There is some overlapping between the classical series, due to a finite number of isomorphisms in low dimension. We give hereunder for each type:

- the definition of the space V, its Jordan triple product, and the corresponding bounded circled homogeneous domain;
- the generic norm;
- the numerical invariants $r, a, b, \gamma = 2 + a(r-1) + b$;
- the Hua polynomial

$$\chi(s) = \prod_{j=1}^{r} \left(s + 1 + (j-1)\frac{a}{2}\right)_{1+b+(r-j)a}.$$

B.1. **Type $\mathbf{I}_{m,n}$ ($1 \leq m \leq n$).** $V = \mathcal{M}_{m,n}(\mathbb{C})$ (space of $m \times n$ matrices with complex entries), endowed with the triple product

$$\{xyz\} = x^t\overline{y}z + z^t\overline{y}x.$$

The domain Ω is the set of $m \times n$ matrices x such that $I_m - x^t\overline{x}$ is definite positive. The generic minimal polynomial is

$$m(T,x,y) = \mathrm{Det}(TI_m - x^t\overline{y}),$$

where Det is the usual determinant of square matrices. The numerical invariants are $r = m$, $a = 2$, $b = n - m$, $\gamma = m + n$. These HPJTS are of tube type only for $m = n$.

The polynomial χ is

$$\chi(s) = \prod_{j=1}^{m} (s + j)_{m+n+1-2j} = \prod_{j=1}^{m} (s + j)_n.$$

B.2. **Type II**$_n$ $(n \geq 2)$**.** $V = \mathcal{A}_n(\mathbb{C})$ (space of $n \times n$ alternating matrices) with the same triple product as for Type I. The domain Ω is the set of $n \times n$ alternating matrices x such that $I_n + x\overline{x}$ is definite positive.

B.2.1. *Type* II_{2p} *($n = 2p$ even).* The generic minimal polynomial is here given by

$$m(T, x, y)^2 = \operatorname{Det}(TI_n + x\overline{y}).$$

The numerical invariants are $r = \frac{n}{2} = p$, $a = 4$, $b = 0$, $\gamma = 2\,(n - 1)$; these HPJTS are of tube type.

The polynomial χ is

$$\chi(s) = \prod_{j=1}^{p} (s + 2j - 1)_{1+4(p-j)} = \prod_{j=1}^{p} (s + 2j - 1)_{2p-1}.$$

B.2.2. *Type* II_{2p+1} *($n = 2p + 1$ odd).* The generic minimal polynomial is given by

$$Tm(T, x, y)^2 = \operatorname{Det}(TI_n + x\overline{y}).$$

The numerical invariants are $r = \left[\frac{n}{2}\right] = p$, $a = 4$, $b = 2$, $\gamma = 2(n-1)$; these HPJTS are not of tube type.

The polynomial χ is

$$\chi(s) = \prod_{j=1}^{p} (s + 2j - 1)_{3+4(p-j)} = \prod_{j=1}^{p} (s + 2j - 1)_{2p+1}.$$

B.3. **Type III**$_n$ $(n \geq 1)$**.** $V = \mathcal{S}_n(\mathbb{C})$ (space of $n \times n$ symmetric matrices) with the same triple product as for Type I. The domain Ω is the set of $n \times n$ symmetric matrices x such that $I_n - x\overline{x}$ is definite positive. The generic minimal polynomial is

$$m(T, x, y) = \operatorname{Det}(TI_n - x\overline{y}).$$

The numerical invariants are $r = n$, $a = 1$, $b = 0$, $\gamma = n + 1$. These HPJTS are of tube type.

The polynomial χ is

$$\chi(s) = \prod_{j=1}^{n} \left(s + \frac{j+1}{2}\right)_{1+n-j}.$$

B.4. **Type IV$_n$** $(n \neq 2)$**.** $V = \mathbb{C}^n$ with the quadratic operator defined by

$$Q(x)y = q(x, \overline{y})x - q(x)\overline{y},$$

where $q(x) = \sum x_i^2$, $q(x,y) = 2\sum x_i y_i$. The domain Ω is the set of points $x \in \mathbb{C}^n$ such that

$$1 - q(x, \overline{x}) + |q(x)|^2 > 0, \qquad 2 - q(x, \overline{x}) > 0.$$

The generic minimal polynomial is

$$m(T, x, y) = T^2 - q(x, \overline{y}) + q(x)q(\overline{y}).$$

The numerical invariants are $r = 2$, $a = n - 2$, $b = 0$, $\gamma = n$. These HPJTS are of tube type.

The polynomial χ is

$$\chi(s) = (s+1)_{n-1}\left(s + \frac{n}{2}\right).$$

B.5. **Type V.** $V = \mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$, the subspace of $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ consisting in matrices of the form

$$\begin{pmatrix} 0 & a_3 & \tilde{a}_2 \\ \tilde{a}_3 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}$$

with the same quadratic operator as for type VI (see below). Here $\tilde{a}$ denotes the Cayley conjugate of $a \in \mathbb{O}_{\mathbb{C}}$. The generic minimal polynomial is

$$m(T, x, y) = T^2 - (x|y)T + (x^\sharp|y^\sharp).$$

The domain Ω is the "exceptional domain of dimension 16" defined by

$$1 - (x|x) + (x^\sharp \mid x^\sharp) > 0, \qquad 2 - (x|x) > 0.$$

The numerical invariants are $r = 2$, $a = 6$, $b = 4$, $\gamma = 12$. This HPJTS is not of tube type.

The polynomial χ is

$$\chi(s) = (s+1)_{11}\,(s+4)_5\,;$$

it can also be written

$$\chi(s) = (s+1)_8(s+4)_8.$$

B.6. **Type VI.** $V = \mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$, the space of 3×3 matrices with entries in the space $\mathbb{O}_{\mathbb{C}}$ of octonions over $\mathbb{C}$, which are Hermitian with respect to the Cayley conjugation; the quadratic operator is defined by

$$Q(x)y = (x|y)x - x^\sharp \times \overline{y},$$

where $\times$ denotes the Freudenthal product, $x^\sharp$ the adjoint matrix in $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ and $(x|y)$ the standard Hermitian product in $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ (see Appendix C.1). The domain Ω is the "exceptional domain of dimension 27" defined by

$$1 - (x|x) + (x^\sharp|x^\sharp) - |\det x|^2 > 0,$$
$$3 - 2(x|x) + (x^\sharp|x^\sharp) > 0,$$
$$3 - (x|x) > 0.$$

The generic minimal polynomial is

$$m(T, x, y) = T^3 - (x|y)T^2 + (x^\sharp|y^\sharp)T - \det x \det \overline{y},$$

where det denotes the determinant in $\mathcal{H}_3(\mathbb{O}_\mathbb{C})$. The numerical invariants are $r = 3$, $a = 8$, $b = 0$, $\gamma = 18$. This HPJTS is of tube type.

The polynomial χ is

$$\begin{aligned}\chi(s) &= (s+1)_{17}(s+5)_9(s+9)\\ &= (s+1)_9(s+5)_9(s+9)_9.\end{aligned}$$

Appendix C. The exceptional bounded symmetric domains

In this appendix, we recall without proofs the construction and the main results about the two exceptional bounded symmetric domains. For details and proofs, see [11].

C.1. **The exceptional Jordan triple system $H_3(\mathbb{O}_\mathbb{C})$ and the exceptional bounded symmetric domain of dimension 27.** Let $\mathbb{O}_\mathbb{C}$ denote the 8-dimensional algebra of complex octonions, with Cayley conjugation $a \mapsto \tilde{a}$, trace $t(a) = a + \tilde{a}$, Cayley norm $n(a) = a\tilde{a}$. If $\mathbb{O}$ denotes the real Cayley division algebra, we consider $\mathbb{O}_\mathbb{C}$ as its complexification $\mathbb{O}_\mathbb{C} = \mathbb{C}\otimes_\mathbb{R}\mathbb{O}$; the complex conjugate of $a = \lambda\alpha$ ($\lambda \in \mathbb{C}$, $\alpha \in \mathbb{O}$) is $\overline{a} = \overline{\lambda}\alpha$. (Recall that the algebras $\mathbb{O}$ and $\mathbb{O}_\mathbb{C}$ are neither commutative nor associative). The (complex bilinear) scalar product of $a, b \in \mathbb{O}_\mathbb{C}$ is defined by

$$(a : b) = a\widetilde{b} + \widetilde{a}b = \widetilde{b}a + b\widetilde{a}; \tag{C.1}$$

the Hermitian scalar product of $a, b \in \mathbb{O}_\mathbb{C}$ is defined by

$$(a \mid b) = \left(a : \overline{b}\right) = a\widetilde{\overline{b}} + \widetilde{a}\overline{b} = \widetilde{\overline{b}}a + \overline{b}\widetilde{a}. \tag{C.2}$$

We denote by $H_3(\mathbb{O}_\mathbb{C})$ the $\mathbb{C}$-vector space (with the natural operations) of 3×3 matrices with entries in $\mathbb{O}_\mathbb{C}$, which are *Hermitian with respect to the Cayley conjugation in $O_\mathbb{C}$*. An element $a \in H_3(\mathbb{O}_\mathbb{C})$ will be written

$$a = \begin{pmatrix} \alpha_1 & a_3 & \widetilde{a_2} \\ \widetilde{a_3} & \alpha_2 & a_1 \\ a_2 & \widetilde{a_1} & \alpha_3 \end{pmatrix}, \tag{C.3}$$

with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ and $a_1, a_2, a_3 \in \mathbb{O}_\mathbb{C}$. Instead of (C.3), we will also write

$$a = \sum_{j=1}^{3} \alpha_j e_j + \sum_{j=1}^{3} F_j(a_j), \tag{C.4}$$

with the obvious definitions for e_j and $F_j(a_j)$. The vector space $H_3(\mathbb{O}_\mathbb{C})$ decomposes into the direct sum

$$H_3(\mathbb{O}_C) = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3, \tag{C.5}$$

where $\mathcal{F}_j = \{F_j(a); a \in \mathbb{O}_\mathbb{C}\}$. The subspaces $\mathcal{F}_j$ are 8-dimensional and

$$\dim_\mathbb{C} H_3(\mathbb{O}_C) = 27.$$

On $H_3(\mathbb{O}_\mathbb{C})$, define a bilinear form by

$$(a : b) = \sum_{j=1}^{3} \alpha_j\beta_j + \sum_{j=1}^{3} (a_j : b_j) \tag{C.6}$$

for $a = \sum_{j=1}^3 \alpha_j e_j + \sum_{j=1}^3 F_j(a_j)$, $b = \sum_{j=1}^3 \beta_j e_j + \sum_{j=1}^3 F_j(b_j)$. Here $(a_j : b_j)$ denotes the scalar product (C.16) in $\mathbb{O}_\mathbb{C}$. The form defined by (C.6) is clearly non-singular and the decomposition (C.5) is orthogonal with respect to it. We will refer to $(a : b)$ as the (complex bilinear) *scalar product* of a and b in $H_3(\mathbb{O}_\mathbb{C})$.

The *adjoint* $a^\#$ of an element

$$a = \sum_{j=1}^{3} \alpha_j e_j + \sum_{j=1}^{3} F_j(a_j) \in H_3(\mathbb{O}_\mathbb{C})$$

is defined by

$$\text{(C.7)} \quad a^\# = \sum_i (\alpha_j \alpha_k - n(a_i))\, e_i + \sum_i \widetilde{F}_i\,(a_j a_k - \alpha_i \widetilde{a_i})\,.$$

In (C.7) and below, $\sum_i$ means $\sum_{i=1}^3$ and j, k are defined by (i, j, k) being an *even* permutation of $(1, 2, 3)$; $\widetilde{F}_i(c)$ stands for $F_i(\widetilde{c})$. The symmetric bilinear map, associated to the quadratic map $a \mapsto a^\#$ by

$$a \times b = (a + b)^\# - a^\# - b^\#, \quad a \times a = 2a^\#,$$

is called the *Freudenthal product* in $H_3(\mathbb{O}_\mathbb{C})$. It follows directly from the definitions that

$$\text{(C.8)} \quad a \times b = \sum_i (\alpha_j \beta_k + \alpha_k \beta_j - (a_i : b_i))\, e_i + \sum_i \widetilde{F}_i \left(a_j b_k + b_j a_k - \alpha_i \widetilde{b_i} - \beta_i \widetilde{a_i} \right).$$

In particular, we have

$$\text{(C.9)} \quad \begin{aligned} &e_i \times e_i = 0, \quad e_i \times e_j = e_k, \\ &e_i \times F_i(b) = -F_i(b), \quad e_i \times F_j(b) = 0, \\ &F_i(a) \times F_i(b) = -(a : b)e_i, \quad F_i(a) \times F_j(b) = \widetilde{F_k}(ab). \end{aligned}$$

In these relations, (i, j, k) is always an even permutation of $(1, 2, 3)$. The Freudenthal product verifies

$$\text{(C.10)} \quad (a \times b : c) = (a : b \times c) \qquad (a, b, c \in H_3(\mathbb{O}_\mathbb{C})).$$

Let T denote the trilinear symmetric form on $H_3(\mathbb{O}_\mathbb{C})$ defined by

$$T(a, b, c) = (a \times b : c)\,.$$

The *determinant* in $H_3(\mathbb{O}_\mathbb{C})$ is the associated polynomial of degree 3, defined by

$$\text{(C.11)} \quad \det a = \frac{1}{3!} T(a, a, a) = \frac{1}{3}(a^\# : a).$$

If $a \in H_3(\mathbb{O}_\mathbb{C})$ is written as in (C.4), then

$$\text{(C.12)} \quad \det a = \alpha_1 \alpha_2 \alpha_3 - \sum_i \alpha_i n(a_i) + a_1(a_2 a_3) + (\widetilde{a_3}\widetilde{a_2})\widetilde{a_1}.$$

The following identities hold in $H_3(\mathbb{O}_\mathbb{C})$:

$$\text{(C.13)} \quad (a^\#)^\# = (\det a) a,$$

$$\text{(C.14)} \quad \det(a^\#) = (\det a)^2.$$

The *complex conjugate* of $a = \sum_{j=1}^{3} \alpha_j e_j + \sum_{j=1}^{3} F_j(a_j) \in H_3(\mathbb{O}_{\mathbb{C}})$ is defined as $\overline{a} = \sum_{j=1}^{3} \overline{\alpha_j} e_j + \sum_{j=1}^{3} F_j(\overline{a_j})$. On $H_3(\mathbb{O}_{\mathbb{C}})$, we consider the Hermitian scalar product defined by

$$(a \mid b) = \left(a : \overline{b}\right). \tag{C.15}$$

The *Jordan triple product* on $H_3(\mathbb{O}_{\mathbb{C}})$ is defined by

$$Q(x)y = (x \mid y)x - x^{\#} \times \overline{y}, \tag{C.16}$$

$$D(x,y)z = \{xyz\} = (x \mid y)z + (z \mid y)x - (x \times z) \times \overline{y}. \tag{C.17}$$

With this triple product, $H_3(\mathbb{O}_{\mathbb{C}})$ is an Hermitian positive Jordan triple system. The space $H_3(\mathbb{O}_{\mathbb{C}})$ endowed with the triple product defined by (C.17) will be referred to as the *Hermitian Jordan triple system* $H_3(\mathbb{O}_{\mathbb{C}})$, or the *Hermitian Jordan triple system of type VI*, or the *exceptional Hermitian triple system of dimension* 27.

The *generic minimal polynomial* of the Jordan triple system $H_3(\mathbb{O}_{\mathbb{C}})$ is

$$m(T,x,y) = T^3 - (x \mid y)T^2 + (x^{\#} \mid y^{\#})T - \det x \det \overline{y}; \tag{C.18}$$

the rank of $H_3(\mathbb{O}_{\mathbb{C}})$ is 3. The *generic norm* is

$$N(x,y) = 1 - (x \mid y) + (x^{\#} \mid y^{\#}) - \det x \det \overline{y}$$

The *set $\mathcal{E}$ of tripotents* of $H_3(\mathbb{O})$ is the disjoint union $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$, where

$$\begin{aligned}
&\mathcal{E}_0 = \{0\}, \quad \mathcal{E}_1 = \left\{x; (x \mid x) = 1,\ x^{\#} = 0\right\}, \\
&\mathcal{E}_2 = \left\{x; (x \mid x) = 2,\ (x^{\#} \mid x^{\#}) = 1,\ \det x = 0\right\}, \\
&\mathcal{E}_3 = \left\{x; (x \mid x) = 3,\ (x^{\#} \mid x^{\#}) = 3,\ \left|\det x\right|^2 = 1\right\}.
\end{aligned}$$

The elements e_1, e_2, e_3 belong to $\mathcal{E}_1$ and are therefore minimal tripotents. It is easily checked that the Peirce spaces for e_1 are

$$\begin{aligned}
&V_0(e_1) = \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathcal{F}_1, \\
&V_1(e_1) = \mathcal{F}_2 \oplus \mathcal{F}_3, \quad V_2(e_1) = \mathbb{C}e_1.
\end{aligned}$$

Similar results hold for the Peirce decomposition w.r. to e_2 and e_3. As e_2 and e_3 belong to $V_0(e_1)$, they are orthogonal to e_1; also, e_2 is orthogonal to e_3. So (e_1, e_2, e_3) is a frame for the Jordan triple system $H_3(\mathbb{O}_{\mathbb{C}})$. It is then easily checked that the simultaneous Peirce decomposition w.r. to the frame (e_1, e_2, e_3) is

$$H_3(\mathbb{O}_{\mathbb{C}}) = \bigoplus_{1 \le i \le j \le 3} V_{ij},$$

with $V_{ii} = \mathbb{C}e_i$, $V_{ij} = \mathcal{F}_k$. As all V_{ij} $(1 \le i < j \le 3)$ for this frame are non-zero, the Hermitian Jordan triple system $H_3(\mathbb{O}_{\mathbb{C}})$ is *simple*. Hence the numerical invariants are

$$a = \dim \mathcal{F}_i = 8, \quad b = \dim V_{0i} = 0, \quad r = 3, \quad g = 2 + a(r-1) = 18.$$

In particular, $H_3(\mathbb{O}_{\mathbb{C}})$ is of tube type. In $H_3(\mathbb{O}_{\mathbb{C}})$,

$$\operatorname{tr} D(x,y) = 18(x \mid y), \tag{C.19}$$

$$\det B(x,y)) = \left(1 - (x \mid y) + (x^{\#} \mid y^{\#}) - \det x \det \overline{y}\right)^{18}. \tag{C.20}$$

The *bounded circled symmetric domain* Ω_{VI} corresponding to the Jordan triple system $H_3(\mathbb{O}_{\mathbb{C}})$ is defined by the set of inequalities

$$1-(x|x)+(x^\sharp|x^\sharp)-|\det x|^2>0,$$
$$3-2(x|x)+(x^\sharp|x^\sharp)>0,$$
$$3-(x|x)>0.$$

C.2. **The exceptional Jordan triple and the exceptional domain of dimension 16.** The 16-dimensional vector space

$$W=\mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})=\{(a_2,a_3)\mid a_2,a_3\in\mathbb{O}_{\mathbb{C}}\}$$

is identified with the subspace of $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ consisting in matrices of the form

$$\begin{pmatrix} 0 & a_3 & \tilde a_2\\ \tilde a_3 & 0 & 0\\ a_2 & 0 & 0\end{pmatrix}.$$

This is the Peirce subspace $V_1(e_1)$ of $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ with respect to the tripotent e_1 and hence a Jordan subsystem of the Jordan triple system $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$. The Peirce subspace $V_0(e_1)=\mathbb{C}e_2\oplus\mathbb{C}e_3\oplus\mathcal{F}_1$ will be identified with the space

$$\mathcal{H}_2(\mathbb{O}_{\mathbb{C}})=\left\{\begin{pmatrix}\alpha_2 & \widetilde{a_1}\\ a_1 & \alpha_3\end{pmatrix};\alpha_2,\alpha_3\in\mathbb{C},\ a_1\in\mathbb{O}_{\mathbb{C}}\right\}$$

of 2×2 Cayley-Hermitian matrices with coefficients in $\mathbb{O}_{\mathbb{C}}$; the space $\mathcal{H}_2(\mathbb{O}_{\mathbb{C}})$ is also a Jordan subsystem of the Jordan triple system $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$, and it can be shown it is isomorphic to the *classical* Hermitian Jordan triple of type IV_{10}.

For $x=(x_2,x_3)\in\mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$, the adjoint $x^\sharp\in\mathcal{H}_2(\mathbb{O}_{\mathbb{C}})$ is then defined by

$$x^\sharp=\begin{pmatrix}-n\,(x_2) & \widetilde{x_3}\widetilde{x_2}\\ x_2x_3 & -n\,(x_3)\end{pmatrix}.$$

The *Freudenthal product* of $x=(x_2,x_3)\,,\ y=(y_2,y_3)\in\mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$ is then

$$x\times y=(x+y)^\sharp-x^\sharp-y^\sharp.$$

The spaces $\mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$ and $\mathcal{H}_2(\mathbb{O}_{\mathbb{C}})$ are endowed with the following Hermitian scalar products, inherited from the Hermitian scalar product in $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$:

- if $x=(x_2,x_3)\,,\ y=(y_2,y_3)\in\mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$,

(C.21) $(x|y)=(x_2|y_2)+(x_3|y_3)$;

- if $u=\begin{pmatrix}\lambda_2 & u_1\\ \widetilde{u_1} & \lambda_3\end{pmatrix}$, $v=\begin{pmatrix}\mu_2 & v_1\\ \widetilde{v_1} & \mu_3\end{pmatrix}\in\mathcal{H}_2(\mathbb{O}_{\mathbb{C}})$, $(\lambda_2,\lambda_3,\mu_2,\mu_3\in\mathbb{C}$, $u_1,v_1\in\mathbb{O}_{\mathbb{C}})$,

(C.22) $(u|v)=\lambda_2\overline{\mu_2}+\lambda_3\overline{\mu_3}+(u_1|v_1)$.

The *quadratic operator* of the *Jordan triple system of type V* is defined, for $x=(x_2,x_3)\,,\ y=(y_2,y_3)\in\mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$, by

(C.23) $Q(x)y=\left(x_2\widetilde{\overline{y_2}}x_2+(x_2\overline{y_3})\widetilde{x_3},\widetilde{x_2}(\overline{y_2}x_3)+x_3\widetilde{\overline{y_3}}x_3\right)$.

The *generic minimal polynomial* of W is

$$m(T,x,y)=T^2-(x\mid y)T+(x^{\#}\mid y^{\#});$$

the *generic norm* is

$$N(x,y) = 1 - (x \mid y) + (x^{\#} \mid y^{\#}).$$

The *set of tripotents* of W is $\mathcal{E}' = \mathcal{E}'_0 \cup \mathcal{E}'_1 \cup \mathcal{E}'_2$, with

$$\mathcal{E}'_0 = \{0\}, \quad \mathcal{E}'_1 = \left\{x \in W \mid (x \mid x) = 1,\ x^{\#} = 0\right\},$$
$$\mathcal{E}'_2 = \left\{x \in W \mid (x \mid x) = 2,\ (x^{\#} \mid x^{\#}) = 1\right\}.$$

The triple system W is simple. Its numerical invariants are

$$a = 6,\ b = 4,\ r = 2,\ g = 12.$$

(The numerical invariants and the simplicity of W are obtained by computing the simultaneous Peirce decomposition w.r. to a set of two orthogonal tripotents). In $W = \mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$, we have

$$\operatorname{tr} D(x,y) = 12(x \mid y),$$
$$\det B(x,y) = \left(1 - (x \mid y) + (x^{\#} \mid y^{\#})\right)^{12}.$$

The *exceptional domain of dimension* 16 is

$$\Omega_V = \left\{x \in \mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}}) \mid 1 - (x|x) + (x^{\sharp} \mid x^{\sharp}) > 0,\ 2 - (x|x) > 0\right\}. \tag{C.24}$$

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