The Splittest Kac Superalgebra K_{10}

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Abstract

We present an isotope of the usual version of the Kac 10-dimensional Jordan superalgebra K_{10} over a general ring of scalars Φ (isomorphic to the original version when $i, \frac{1}{\sqrt{2}} \in \Phi$, but not in characteristic 2), which we take as the "correct" split model for the simple superalgebra in all characteristics. This $J = A \oplus M$ has unit the sum of three reduced orthogonal idempotents. We exhibit a "quaternionic" model $J \subseteq (H \otimes H \boxplus \Phi f) \oplus H$ of the bimodule structure for this model and the original one, as well as an "exterior" model $J \cong \Lambda^2(M) \oplus M$ for both the bimodule structure and the odd product. We give a reference table for all quadratic and triple products, and use this to explicitly describe all inner super-derivations. In a subsequent article we will use this table to investigate the structure of the Grassmann envelope.[1]

Our version $sK_{10} = K_{10}^s$ of the Kac 10-dimensional quadratic Jordan superalgebra $K_{10}(\Phi)$ over a general ring of scalars Φ will be split even further than that of Dan King [2]. The Kac superalgebra consists of an *even* Jordan algebra $A = Jord(Q, e) \boxplus \Phi f$ which is the direct sum of a 5-dimensional algebra $Jord(Q, e)$ of a nondegenerate quadratic form and a 1-dimensional ideal Φf, together with a 4-dimensional *odd* bimodule M having odd products into A. The algebra was called "split" in [2] because the quadratic form has maximal Witt index: in the linear case $Jord(Q, e) = \Phi e \oplus V$ where the form is thought of as residing on V and is there a direct sum of two hyperbolic planes; in the quadratic case the form resides on the entire 5-dimensional space including the basepoint e, and there it is a direct sum of two hyperbolic planes and a 1-dimensional "split" line $Q(e) = 1$. However, in characteristic 2 this Q is traceless (hence in the terminology of Loos [4] *totally ramified*), with the property that $x^2 = -Q(x)e$ for all x, so there are no proper idempotents. In the structure theory for quadratic Jordan algebras this is considered an aberrant case: the "standard" degree-2 algebra has unit a sum of two reduced orthogonal idempotents, $A = \Phi e_1 \oplus \Phi e_2 \oplus V$, and the traceless form arises as a (non-isomorphic) isotope of this standard form. In Jordan theory there is a hierarchy: "reduced" means "has enough idempotents", while "split" means reduced and the coordinate algebra splits. Thus we will refer to our version $J = \Phi e_1 \oplus \Phi e_2 \oplus \Phi f \oplus V \oplus M$ as (intrinsically) *split*, and demote the version [2] to (merely) *standard* (it is *extrinsically split* if $\frac{1}{2} \in \Phi$).

Throughout, we consider unital Jordan superalgebras, $\mathbb{Z}_2$-graded algebras $J = J_0 \oplus J_1 = A \oplus M$ over an arbitrary ring of scalars Φ (possibly of characteristic 2) with graded bilinear and trilinear products $\langle x, y\rangle = V_x(y)$, $\langle x, y, z\rangle = V_{x,y}(z)$ and even products $U_a x, a^2$ quadratic in a and linear in x, such that $\langle a, y, b\rangle = U_{a,b}y = (U_{a+b} - U_a - U_b)y$ is the linearization of the U-operator, and similarly $\langle a, b\rangle = \langle a, 1, b\rangle = (a + b)^2 - a^2 - b^2$ is the linearization of the square. We define $U_{m,p}n := \langle m, n, p\rangle$, even though there is no odd U-operator U_m which gives rise to this.[2] In the absence of a scalar $\frac{1}{2}$, the bilinear products are not sufficient to determine the quadratic products, so we will devote much effort to describing the quadratic products both in the usual and the split version of K_{10}.

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[2]Note that this flouts the tradition that $U_{x,y}$ is symmetric in x, y as the linearization of a quadratic operator U_x.

To avoid subscripts (of which we will have more than enough already), we follow the Racine-Zelmanov convention [7] and distinguish even from odd by using letters a, b, c, d, e, f, g (but u, v for the vector part of $Jord(Q, e)$ in K_{10}) to denote even elements of $J_0 = A$, and letters m, n, p to denote odd elements of $J_1 = M$; general homogeneous elements of J of will be denoted by x, y, z (of degree deg(x) etc.). We denote Jordan bilinear and trilinear products by braces $\{a, ..\}$, Lie products by brackets $[x, ..]$, and androgynous superproducts by $\langle x, ..\rangle$. By abuse of notation we will write $(-1)^x$ for $(-1)^{deg(x)}$, $(-1)^{xy}$ for $(-1)^{deg(x)deg(y)}$ [-1 if both x, y are odd, $+1$ otherwise], and $(-1)^{xyz}$ for $(-1)^{deg(x)deg(y)+deg(y)deg(z)+deg(z)deg(x)}$ ["majority rule': -1 if the majority are odd, $+1$ if the majority are even].

The super-Jordan axioms are that the Grassmann envelope $\Gamma(J)$ becomes a unital quadratic Jordan algebra under "natural" quadratic product. The (as yet not fully listed) quadratic super-identities $F(a_1, \dots, a_r, m_1, \dots, m_s) = 0$ (homogeneous of degree 1 in each m_i) are determined by *Grassmann detour* from quadratic Jordan identities $F(1 \otimes a_1, \dots, 1 \otimes a_r, \gamma_1 \otimes m_1, \dots, \gamma_s \otimes m_r) = 0$ in the Grassmann envelope for independent Grassmann variables $\gamma_i \in \Gamma_1$. For later reference we recall certain of these basic identities for Jordan superalgebra J: M is a Jordan bimodule for the quadratic Jordan algebra A and for $m, n, p \in M, a, b \in A$, homogeneous $x, y, z \in J$

(0.1.1) Switching Rule $\langle x, y, z\rangle + (-1)^{xy}\langle y, x, z\rangle = \langle\langle x, y\rangle, z\rangle$,
SuperSymmetry $\langle x, y\rangle = (-1)^{xy}\langle y, x\rangle$, $\langle x, y, z\rangle = (-1)^{xyz}\langle z, y, x\rangle$,
(0.1.2) Even Symmetry $\langle a, m\rangle = \langle m, a\rangle$, $\langle a, m, b\rangle = \langle b, m, a\rangle$, $\langle a, b, m\rangle = \langle m, b, a\rangle$,
Odd Alternation $\langle m, m\rangle = \langle m, n, m\rangle = 0$, $\langle m, n\rangle = -\langle n, m\rangle$, $\langle m, x, n\rangle = -\langle n, x, m\rangle$.

If $1 = \sum_{i=1}^n e_i$ is a supplementary sum of orthogonal idempotents, the Peirce decomposition of J is $J = \bigoplus_{i \leq j} J_{ij}$ with Peirce projections $E_{ii} = U_{e_i}$, $E_{ij} = U_{e_i, e_j}$ on $J_{ij} = J_{ji}$. In the case of a single idempotent e, we denote these by J_i, E_i ($E_2 = U_e, E_1 = U_{e,1-e}, E_0 = U_{1-e}$ (in our unital case $1 - e$ exists in J, but in general it exists in the unital hull). They satisfy the standard rules

(0.2.1) Peirce Orthogonality $\langle J_{ij}, J_{k\ell}\rangle = \langle J_{ij}, J_{k\ell}, J_{mn}\rangle = 0$ unless indices can be linked,
(0.2.2) $J_{ii}^2 \subseteq J_{ii}$, $\langle J_{ij}, J_{ij}\rangle \subseteq J_{ij}^2 \subseteq J_{ii} + J_{jj}$, $\langle J_{ij}, J_{jk}\rangle \subseteq J_{ik}$ $(k \neq i)$, $\langle J_{ij}, J_{jk}, J_{k\ell}\rangle \subseteq J_{i\ell}$,
(0.2.3) $U_{A_{ij}} J_{ii} \subseteq J_{jj}$, $U_{A_{ij}} J_{ij} \subseteq J_{ij}$, $U_{A_{ij}} J_{k\ell} = 0$ $((k\ell) \neq (i,i), (ij), (jj))$,
(0.2.4) Triple Reduction Formulas $\langle a, a, m\rangle = \langle a^2, m\rangle$, $\langle m, m, x\rangle = \langle m, \langle m, x\rangle\rangle$,
If $x_i, y_i \in J_i(e)$ (i=2, 0, j=3-i), $z_1, w_1 \in J_1(e)$ then $\langle z_1, w_1, x_i\rangle = E_{ii}\langle z_1, \langle w_1, x_i\rangle\rangle$,
$\langle x_i, y_i, z_1\rangle = \langle x_i, \langle y_i, z_1\rangle\rangle$, $\langle z_1, y_i, w_1\rangle = E_{jj}\langle z_1, \langle y_i, w_1\rangle\rangle = E_{jj}\langle\langle z_1, y_i\rangle, w_1\rangle$.

These formulas show that in Jordan superalgebras many of the trilinear products are determined by bilinear products together with the Peirce decomposition; in the split case we will see that *all* trilinear products are so determined.

1 Bases for the Kac superalgebra

The standard version of the quadratic Kac superalgebra $K_{10}(\Phi) = A \oplus M = (B \boxplus \Phi f) \oplus M$ is a free Φ-module of dimension 10 over Φ with 6-dimensional even space A the direct sum of $B = Jord(Q, e)$ (the 5-dimensional Jordan algebra of a quadratic form Q on $\Phi e \oplus V$ with basepoint e) and a 1-dimensional Φf, and with 4-dimensional odd space M, which is a Jordan A-bimodule M with bilinear and trilinear products $\langle \cdot, \cdot\rangle : M \times M \to A$, $\langle \cdot, \cdot, \cdot\rangle : M \times M \times M \to A$. The *King basis* [2, p.31][3, p.391-2], which was adapted from Kac's corrected characteristic zero model to work for arbitrary scalars, consists of 10 elements $x_0, y_0, \tilde{x}_0, \tilde{y}_0, e, f, x_1, y_1, \tilde{x}_1, \tilde{y}_1$ which we shall relabel as $v_1, v_2, v_3, v_4, e, f, m_1, m_2, m_3, m_4$ (King used subscripts to denote parity J_0, J_1, whereas we will use A, M for that purpose, leaving subscripts free to label items in a list). Here e, f are orthogonal idempotents, e the unit of $Jord(Q, e)$, the quadratic form is

(1.1) $$Q(b) = \beta^2 - \beta_1\beta_2 - \beta_3\beta_4, \quad T(b) = 2\beta \quad \textit{for} \quad b = \beta e + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \beta_4 v_4,$$

so the commutative circle products on V are $\{v_1, v_2\} = \{v_3, v_4\} = e$: the multiplication in the direct sum $A = B \boxplus \Phi f$ is given by that in the separate sumands, where for $b = \beta e + v$ in any

$B = Jord(Q,e) = \Phi e \oplus V$ we have quadratic products

$$U_b b' = Q(b,\overline{b'})b - Q(b)\overline{b'}, \quad b^2 = T(b)b - Q(b)e \quad (T(b) = 2\beta,\ Q(b) = \beta^2 + Q(v),\ \bar{b} = T(b)e - b).$$

For our particular V with $Q(v_i) = 0$, $Q(v_i, v_j) = -\delta_{ji'}$, $\overline{v_i} = -v_i$ we get [introducing the convention that $1' = 2$, $2' = 1$, $3' = 4$, $4' = 3$ for switching among the "paired" indices $1,2$ and $3,4$]

$$\begin{array}{c} U_e b = b, \quad \{e,e,b\} = \{e,b,e\} = 2b, \quad U_{v_i} e = v_i^2 = 0, \quad U_{v_i} v_j = \delta_{ji'} v_i, \quad \{v_i, v_i, B\} = 0, \\ \{v_i, v_j, v_k\} = \delta_{ji'} v_k + \delta_{jk'} v_i - \delta_{ki'} v_j, \quad \{v_i, v_j, e\} = \{v_i, e, v_j\} = \{v_i, v_j\} = \delta_{ji'} e, \\ \{v_i, v_{i'}, v_i\} = 2v_i, \quad \{v_i, v_{i'}, v_j\} = v_j, \quad \{v_i, v_j, v_{i'}\} = -v_j \quad (j \neq i, i'). \end{array} \tag{1.2}$$

The Peirce decomposition (0.2) of J relative to e (equivalently, $1 = e + f$) is $J = A_2 \oplus A_0 \oplus M_1$ for $A_2 = \Phi e + V = B,\ A_0 = \Phi f,\ M_1 = M$. Thus King's action of A on M is given by

(1.3) Bimodule Product $\langle A, M \rangle$

$\langle\,,\rangle$	x_1	y_1	$\tilde{x}_1$	$\tilde{y}_1$	$\langle\,,\rangle$	m_1	m_2	m_3	m_4
x_0	0	$\tilde{y}_1$	x_1	0	v_1	0	m_4	m_1	0
y_0	$\tilde{x}_1$	0	0	y_1	v_2	m_3	0	0	m_2
$\tilde{x}_0$	0	$-\tilde{x}_1$	0	x_1	v_3	0	$-m_3$	0	m_1
$\tilde{y}_0$	$\tilde{y}_1$	0	$-y_1$	0	v_4	m_4	0	$-m_2$	0
e, f	x_1	y_1	$\tilde{x}_1$	$\tilde{y}_1$	e, f	m_1	m_2	m_3	m_4

Bimodule Structure

From the Peirce rules (0.2) for the bilinear actions of $B = A_2$ and $\Phi f = A_0$ on $M = M_1$ we immediately get rules for the trilinear actions on M:

$$\begin{array}{c} \langle e,m\rangle = \langle f,m\rangle = \langle e,m,f\rangle = \langle e,e,m\rangle = \langle f,f,m\rangle = m, \quad \langle e,b,m\rangle = \langle b,e,m\rangle = \langle b,m,f\rangle = \langle b,m\rangle, \\ U_e M = U_B M = U_{B,B} M = U_f M = \langle f, B, M\rangle = \langle B, f, M\rangle = \langle v_i, v_i, M\rangle = 0, \\ \langle b, b', m\rangle = \langle b, \langle b', m\rangle\rangle, \quad \langle f, m, b\rangle = \langle b, e, m\rangle = \langle e, b, m\rangle = \langle b, m\rangle. \end{array} \tag{1.4}$$

These rules allow us to give us a complete description of $J = A \oplus M$ as bimodule, equivalently, as a split null extension (before we introduce a nontrivial product on the odd space). We have general trilinear actions $V_{v_i,v_k} = V_{v_i} V_{v_k},\ V_{v_i,v_i} = 0,\ V_{v_i,v_{i'}} + V_{v_{i'},v_i} = \mathbf{1}_M,\ V_{v_i,v_j} = -V_{v_j,v_i}\ (j \neq i, i')$, and particular actions: V_{v_2} kills m_2, m_3 and sends $m_1, m_4 \longrightarrow m_3, m_2 \xrightarrow{V_{v_1}} m_1, m_4$ for V_{v_1,v_2}; similarly V_3 kills m_1, m_3 and sends $m_2, m_4 \longrightarrow -m_3, m_1$, which is sent $\xrightarrow{V_{v_1}} -m_1, 0$ for V_{v_1,v_3}, and sent $\xrightarrow{V_{v_2}} 0, -m_3$ for V_{v_2,v_3}; likewise V_4 kills m_2, m_4 and sends $m_1, m_3 \longrightarrow m_4, -m_2$, which is sent $\xrightarrow{V_{v_1}} 0, -m_4$ for V_{v_1,v_4}, sent $\xrightarrow{V_{v_2}} m_2, 0$ for V_{v_2,v_4}, and sent $\xrightarrow{V_{v_3}} m_1, m_3$ for V_{v_3,v_4}. We can summarize these together with (1.2), (1.4) in the table[3]

[3] Another way to derive the table is to notice that $V_{u,v} = V_u V_v$ where relative to the ordered basis m_1, m_2, m_3, m_4 for M the matrices are $V_{v_1} \cong E_{42} + E_{13}$, $V_{v_2} \cong E_{31} + E_{24}$, $V_{v_3} \cong -E_{32} + E_{14}$, $V_{v_4} \cong E_{41} - E_{23}$.

(1.5.1) V-Operators V_A, $V_{A,A}$

$V_a, \quad V_a(x) = \langle a, x\rangle$	v_1	v_2	v_3	v_4	e	f	m_1	m_2	m_3	m_4
$V_{v_1} = V_{v_1,e} = V_{e,v_1}$	0	e	0	0	$2v_1$	0	0	m_4	m_1	0
$V_{v_2} = V_{v_2,e} = V_{e,v_2}$	e	0	0	0	$2v_2$	0	m_3	0	0	m_2
$V_{v_3} = V_{v_3,e} = V_{e,v_3}$	0	0	0	e	$2v_3$	0	0	$-m_3$	0	m_1
$V_{v_4} = V_{v_4,e} = V_{e,v_4}$	0	0	e	0	$2v_4$	0	m_4	0	$-m_2$	0
$V_e = V_{e,e}$	$2v_1$	$2v_2$	$2v_3$	$2v_4$	$2e$	0	m_1	m_2	m_3	m_4
$V_f = V_{f,f}$	0	0	0	0	0	$2f$	m_1	m_2	m_3	m_4
V_{v_1,v_2}	$2v_1$	0	v_3	v_4	e	0	m_1	0	0	m_4
V_{v_2,v_1}	0	$2v_2$	v_3	v_4	e	0	0	m_2	m_3	0
V_{v_3,v_4}	v_1	v_2	$2v_3$	0	e	0	m_1	0	m_3	0
V_{v_4,v_3}	v_1	v_2	0	$2v_4$	e	0	0	m_2	0	m_4
$V_{v_1,v_3} = -V_{v_3,v_1}$	0	$-v_3$	0	v_1	0	0	0	$-m_1$	0	0
$V_{v_2,v_3} = -V_{v_3,v_2}$	$-v_3$	0	0	v_2	0	0	0	0	0	m_3
$V_{v_1,v_4} = -V_{v_4,v_1}$	0	$-v_4$	v_1	0	0	0	0	0	$-m_4$	0
$V_{v_2,v_4} = -V_{v_4,v_2}$	$-v_4$	0	v_2	0	0	0	m_2	0	0	0
	$V_{v_i,v_i} = V_{f,B} = V_{B,f} = 0$						$V_{b,b'} = V_b V_{b'}$ on M			

(1.5.2) U-Operators U_A, $U_{A,A}$

U_{v_1}	0	v_1	0	0	0	0
U_{v_2}	v_2	0	0	0	0	0
U_{v_3}	0	0	0	v_3	0	0
U_{v_4}	0	0	v_4	0	0	0
U_e	v_1	v_2	v_3	v_4	e	0
U_f	0	0	0	0	0	f
$U_{v_1,v_2} = U_{v_2,v_1}$	0	0	$-v_3$	$-v_4$	e	0
$U_{v_1,v_3} = U_{v_3,v_1}$	0	v_3	0	v_1	0	0
$U_{v_1,v_4} = U_{v_4,v_1}$	0	v_4	v_1	0	0	0
$U_{v_2,v_3} = U_{v_3,v_2}$	v_3	0	0	v_2	0	0
$U_{v_2,v_4} = U_{v_4,v_2}$	v_4	0	v_2	0	0	0
$U_{v_3,v_4} = U_{v_4,v_3}$	$-v_1$	$-v_2$	0	0	e	0
$a^2 = U_a 1$	0	0	0	0	e	f
$U_{b,b} = 2U_b,\ U_{e,b} = V_b,\ U_{f,b} = 0$ on A,						
$U_{f,b} = V_b,\ U_f = U_{e,b} = U_b = U_{b,b'} = 0$ on M						

Odd Products

King *defines* [3, p.392] the alternating odd bilinear product on M by a basis-free recipe involving an alternating bilinear form σ on M and an alternating product $\star$ from $M \times M \to V$ by

$$(1.6) \qquad \langle m, n\rangle := \sigma(m,n)g + 2m \star n \quad (g := e - 3f), \qquad m \star n := \textstyle\sum_{i=1}^4 \sigma(\langle v_i, m\rangle, n)v_i'$$

[where $\sigma(m_1, m_2) = \sigma(m_3, m_4) = 1$ and v' denotes the anti-isometric involution $\sigma(v', w') = \sigma(w, v)$ on V determined by $v_i' = v_{i'}$ for $1' = 2, 3' = 4$ as in (1.2)] as described by the table

(1.7) Products $\star$ and σ on M

$\star$	m_1	m_2	m_3	m_4	σ	m_1	m_2	m_3	m_4
m_1	0	0	$-v_3$	v_1	m_1	0	1	0	0
m_2	0	0	$-v_2$	$-v_4$	m_2	-1	0	0	0
m_3	v_3	v_2	0	0	m_3	0	0	0	1
m_4	$-v_1$	v_4	0	0	m_4	0	0	-1	0

Note for future reference that from (1.7), (1.3) and some calculation we see

(1.8)
$$\begin{array}{c} \textit{for } j \neq i, i' \textit{ we have } \langle V, (\Phi m_i + \Phi m_{i'})\rangle \subseteq \Phi m_j + \Phi m_{j'}, \\ \boxed{\begin{array}{ll} \sigma(\langle v, m_i\rangle, m_i) = 0, & m_i \star m_i = m_i \star m_{i'} = \langle (m_i \star m_j), m_i\rangle = 0, \\ \sigma(m_i, m_{i'}) = (-1)^{i'}, & \langle (m_i \star m_j), m_{i'}\rangle = (-1)^{i'} m_j, \\ \sigma(m_i, m_j) = 0, & \langle (m_i \star m_j), m_{i'}\rangle + \langle (m_{i'} \star m_j), m_i\rangle = 0. \end{array}} \end{array}$$

We can summarize the odd product by the table

(1.9) Odd Product $\langle M, M\rangle$

$\langle\cdot,\cdot\rangle$	x_1	y_1	$\tilde{x}_1$	$\tilde{y}_1$	$\langle\cdot,\cdot\rangle$	m_1	m_2	m_3	m_4
x_1	0	g	$-2\tilde{x}_0$	$2m_0$	m_1	0	g	$-2v_3$	$2v_1$
y_1	$-g$	0	$-2y_0$	$-2\tilde{y}_0$	m_2	$-g$	0	$-2v_2$	$-2v_4$
$\tilde{x}_1$	$2\tilde{x}_0$	$2y_0$	0	g	m_3	$2v_3$	$2v_2$	0	g
$\tilde{y}_1$	$-2m_0$	$2\tilde{y}_0$	$-g$	0	m_4	$-2v_1$	$2v_4$	$-g$	0

The definition of the odd product seems quite mysterious at this point. Notice that odd products $\langle m_i, m_j\rangle$ for $j \neq i'$ produces vectors $v \in V$ "orthogonal" to m_i, m_j:

(1.10)
$$\langle\langle m_i, m_j\rangle, m_i\rangle = \langle\langle m_i, m_j\rangle, m_j\rangle = 0 \qquad (j \neq i').$$

This will become clearer using the Shestakov basis below and the exterior representation in the next section.

Comparison with Racine-Zel'manov

The classification paper [7] of Racine and Zel'manov uses a slightly different basis $e, f, u_1, u_2, u_3, u_4, x_1, y_1, x_2, y_2$ (changing their v_i to u_i to avoid conflict with our v_i) with prescribed dot products. To describe the products $\langle x, y\rangle$ in the quadratic case we must double all the dot products $x \cdot y$ in the RZ-list. We introduce $v_i = \frac{1}{2}u_i$ so that $\{v_i, v_j\} = \frac{1}{2}u_i \cdot u_j,\ u_i = 2v_i,\ \{v_1, v_2\} = \{v_3, v_4\} = e$ and $\langle v_i, m\rangle = u_i \cdot m$, but $\langle m, m'\rangle = 2m \cdot m'$. If we further introduce temporary $w_1 := -x_1$, $w_2 := -y_1$, $w_3 := y_2$, $w_4 := -x_2$ and $n_i := \frac{1}{\sqrt{2}} w_i$,[4] then the bilinear action of A on M is given by

[4]Since we are interested in finding a form of the Kac algebra over an algebraically closed field of characteristic $\neq 2$ which will serve as a model for characteristic 2 and all rings of scalars, we have no compunctions about using $\frac{1}{\sqrt{2}}$ here to get rid of a common factor 2.

(1.11) RZ-Bimodule Product $\langle A, M\rangle$

$\cdot$	x_1	y_1	x_2	y_2
u_1	0	x_2	0	$-x_1$
u_2	$-y_2$	0	y_1	0
u_3	0	y_1	x_1	0
u_4	x_2	0	0	y_1
$\langle\,,\rangle$	$-x_1$	$-y_1$	y_2	$-x_2$
$v_1{=}\frac{1}{2}u_1$	0	$-x_2$	$-x_1$	0
$v_2{=}\frac{1}{2}u_2$	y_2	0	0	$-y_1$
$v_3{=}\frac{1}{2}u_3$	0	$-y_2$	0	$-x_1$
$v_4{=}\frac{1}{2}u_4$	$-x_2$	0	y_1	0
$\langle\cdot,\cdot\rangle$	w_1	w_2	w_3	w_4
v_1	0	w_4	w_1	0
v_2	w_3	0	0	w_2
v_3	0	$-w_3$	0	w_1
v_4	w_4	0	$-w_2$	0
$\langle\cdot,\cdot\rangle$	n_1	n_2	n_3	n_4
v_1	0	n_4	n_1	0
v_2	n_3	0	0	n_2
v_3	0	$-n_3$	0	n_1
v_4	n_4	0	$-n_2$	0
e, f	n_1	n_2	n_3	n_4

RZ-Odd Product

$\cdot$	x_1	y_1	x_2	y_2
x_1	0	g	u_1	u_3
y_1	$-g$	0	$-u_4$	u_2
x_2	$-u_1$	u_4	0	g
y_2	$-u_3$	$-u_2$	$-g$	0
$\langle\,,\rangle$	$-x_1$	$-y_1$	y_2	$-x_2$
x_1	0	$-2g$	$2u_3$	$-2u_1$
y_1	$2g$	0	$2u_2$	$2u_4$
y_2	$2u_3$	$2u_2$	0	$2g$
x_2	$2u_1$	$-2u_4$	$-2g$	0
$\langle\cdot,\cdot\rangle$	w_1	w_2	w_3	w_4
$w_1{=}-x_1$	0	$2g$	$-4v_3$	$4v_1$
$w_2{=}-y_1$	$-2g$	0	$-4v_2$	$-4v_4$
$w_3{=}y_2$	$4v_3$	$4v_2$	0	$2g$
$w_4{=}-x_2$	$-4v_1$	$4v_4$	$-2g$	0
$\langle\cdot,\cdot\rangle$	n_1	n_2	n_3	n_4
$n_1{=}\frac{1}{\sqrt{2}}w_1$	0	g	$-2v_3$	$2v_1$
$n_2{=}\frac{1}{\sqrt{2}}w_2$	$-g$	0	$-2v_2$	$-2v_4$
$n_3{=}\frac{1}{\sqrt{2}}w_3$	$2v_3$	$2v_2$	0	g
$n_4{=}\frac{1}{\sqrt{2}}w_4$	$-2v_1$	$2v_4$	$-g$	0

which are clearly the same as tables (1.3),(1.9) with m_i replaced by n_i.

Comparison with Shestakov

The most illuminating basis for K_{10}, organizing the elements with an easy-to-remember multipication table which clearly explains which bimodule products are zero, is due to Ivan Shestakov. His approach was described at the 1996 Oberwolfach Tagung on Jordan Algebras, and was meant to appear in a definitive book on Jordan superalgebras which regretfully was never written.[5] The Shestakov basis uses the 4 odd elements x, y, u, v (our m_1, m_2, m_3, m_4) to parameterize the even variables: A is spanned by e, f, ux, uy, vx, vy where $uy := u \cdot y =: -yu$, etc. Thus the alternating basic odd products $m \cdot n$ are trivial to remember, except that instead of two more basic elements xy, uv we have $g := e - 3f$ $(x \cdot y = -y \cdot x = u \cdot v = -v \cdot u = g)$. The rules for the even-odd products are that e, f act identically ($\langle e, m\rangle = \langle f, m\rangle = m$) and $uy \in A$ kills its parent elements $u, y \in M$, while for non-parents there must be a linked pair x, y or u, v (corresponding to $m_1' = m_2, m_3' = m_4$), in which case the product in order gives $(yu) \cdot v = -y,\ (uy) \cdot x = u$ [the pair elements cancel each other out, leaving the remaining element with $+$ if the order is reversed (y, x) and $-$ for the usual order (u, v)]. Thus the even element uy can only take on values u, y when multiplied by M.

To adjust the products to work in the quadratic case we introduce odd $m_1 := \frac{1}{\sqrt{2}}x,\ m_2 := \frac{1}{\sqrt{2}}y,\ m_3 := \frac{1}{\sqrt{2}}u,\ m_4 := \frac{1}{\sqrt{2}}v$ and even $v_{31} := -v_{13} := \frac{1}{2}ux,\ v_{32} := -v_{23} := \frac{1}{2}uy,\ v_{41} := -v_{14} := \frac{1}{2}vx,\ v_{42} := -v_{24} := \frac{1}{2}vy$, so that $\langle m_1, m_2\rangle = \langle m_3, m_1\rangle = 2m_3 \cdot m_1 = u \cdot x = ux = 2v_{31}$ and $\langle v_{31}, m_2\rangle = 2(\frac{1}{2}ux) \cdot \frac{1}{\sqrt{2}}y = -\frac{1}{\sqrt{2}}u = -m_3$, etc. With this notation the bilinear products become

$$\text{(1.12)}\qquad \boxed{\begin{array}{c} \text{for } i \neq j, j'\ \langle v_{ij}, m_j\rangle = \langle v_{ij}, m_i\rangle = 0, \quad \langle v_{ij}, m_{j'}\rangle = (-1)^j m_i = -\sigma(m_j, m_{j'})m_i, \\ \langle m_i, m_j\rangle = v_{ij} = -v_{ji} \quad (v_{ii} := 0), \quad \langle m_i, m_{i'}\rangle = (-1)^{i'} g. \end{array}}$$

The complete table of bilinear products is given by

[5] The lecture also revealed intriguing connections with the Jordan superalgebras $D_4(1, -3)$ and K_3, and revealed that K_{10} could be generated by a single nonhomogeneous (or two homogeneous) elements, yet was i-exceptional, destroying all hopes for a Shirshov-Cohn theorem for superalgebras.

(1.13) S-Bimodule Product $\langle A, M\rangle$

$\cdot$	x	y	u	v
ux	0	$-u$	0	x
uy	u	0	0	y
vx	0	$-v$	$-x$	0
vy	v	0	$-y$	0
$\langle\,,\rangle$	m_1	m_2	m_3	m_4
v_{31}	0	$-m_3$	0	m_1
v_{32}	m_3	0	0	m_2
v_{41}	0	$-m_4$	$-m_1$	0
v_{42}	m_4	0	$-m_2$	0
$\langle\cdot,\cdot\rangle$	m_1	m_2	m_3	m_4
v_{14}	0	m_4	m_1	0
v_{32}	m_3	0	0	m_2
v_{31}	0	$-m_3$	0	m_1
v_{42}	m_4	0	$-m_2$	0
e, f	m_1	m_2	m_3	m_4

S-Odd Product

$\cdot$	x	y	u	v
x	0	g	$-ux$	$-vx$
y	$-g$	0	$-uy$	$-vy$
u	ux	uy	0	g
v	vx	vy	$-g$	0
$\langle\,,\rangle$	m_1	m_2	m_3	m_4
m_1	0	$-g$	$-2v_{31}$	$2v_{14}$
m_2	g	0	$-2v_{32}$	$-2v_{42}$
m_3	$2v_{31}$	$2v_{32}$	0	g
m_4	$-2v_{14}$	$2v_{42}$	$-g$	0

which are clearly the same as tables (1.3),(1.9) with v_1, v_2, v_3, v_4 replaced by $v_{14}, v_{32}, v_{31}, v_{42}$. Note that the subspaces $\mathcal{M}_1 := \text{Span}(m_1, v_{14}, v_{31}) = \text{Span}(x, vx, ux) = \mathcal{X},\ \mathcal{M}_2 := \text{Span}(m_2, v_{32}, v_{42}) = \text{Span}(y, uy, vy) = \mathcal{Y},\ \mathcal{M}_3 := \text{Span}(m_3, v_{32}, v_{31}) = \text{Span}(u, ux, uy) = \mathcal{U},\ \mathcal{M}_4 := \text{Span}(m_4, v_{14}, v_{42}) = \text{Span}(v, vx, vy) = \mathcal{V}$ all have $\mathcal{M}_i^2 = 0$ (explaining all the zero products).

Comparison with Involution Basis

We will later pass to an isotope determined by

$$(1.14)\qquad \begin{aligned} &s := u + f := v_1 + v_2 + f, \quad s^2 = 1, \ {}^* := U_s \textit{ is an involutive automorphism of } J \textit{ with} \\ &e^* = e,\ f^* = f,\ v_1^* = v_2,\ v_3^* = -v_3,\ v_4^* = -v_4, \quad m^* = \langle u, m\rangle,\ m_1^* = m_3,\ m_2^* = m_4. \end{aligned}$$

In terms of this involution we obtain an *involution basis* $e,\ f,\ b := v_1,\ b^* := v_2,\ c := v_3,\ d := v_4,\ m_1, m_2,\ m_1^* := m_3,\ m_2^* := m_4$. In terms of this basis the bilinear products become

(1.15) $*$-Bimodule Products

$\langle\,,\rangle$	m_1	m_2	m_1^*	m_2^*	$\langle\cdot,\cdot\rangle$	m_1	m_2	m_1^*	m_2^*
b	0	m_2^*	m_1	0	m_1	0	g	$-2c$	$2b$
b^*	m_1^*	0	0	m_2	m_2	$-g$	0	$-2b^*$	$-2d$
c	0	$-m_1^*$	0	m_1	m_1^*	$2c$	$2b^*$	0	g
d	m_2^*	0	$-m_2$	0	m_2^*	$-2b$	$2d$	$-g$	0

Notice that in terms of the involution basis the third and fourth columns of the table are redundant (as the subdiagonal of the original odd table is redundant by skew-symmetry), since by the involution once you know the actions $\langle p, x\rangle$ of all p on x you know all actions $\langle p, x^*\rangle = \langle p^*, x\rangle^*$ on x^*.

2 The Quaternion Model

We can model the split null extension structure of J (M as bimodule for A) using quaternions. We can identify M with a copy of a split quaternion algebra $H = M_2(\Phi)$ via $m_1, m_2, m_3, m_4 \xrightarrow{\varphi} e_{11}, e_{22}, e_{21}, e_{12}$, so M becomes a regular bimodule for H. Since V is the direct sum of two hyperbolic planes, the Clifford algebra of Q on V is the graded tensor product $H\widehat{\otimes}H$ of two split quaternion algebras, i.e., the product of two split quaternion subalgebras H', H'' graded in the natural diagonal/off-diagonal way where H'_0 commutes with H'' and H''_0 commutes with H', but H'_1 anti-commutes with H''_1.

Here $H' := L_H = \Phi[L_{e_{12}}, L_{e_{21}}]$ is isomorphic to H via the left-regular representation. Despite the twist, $H'' := \Phi[SR_{\overline{e_{21}}}, SR_{\overline{e_{12}}}]$ for $S := L_{e_{22}-e_{11}}$ is also isomorphic to H: H is isomorphic to right multiplications $R_H^{op} \cong R_{\overline{H}}$ under the standard quaterion involution $a \mapsto \bar{a}$, and the twist due to S doesn't change a split quaternion algebra, since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longrightarrow \left(\begin{smallmatrix} a & \sigma b \\ \sigma c & d \end{smallmatrix}\right)$ is an automorphism of $M_2(\Phi)$ [indeed, $a + m \to a + \sigma m$ is an automorphism of any $\mathbb{Z}_2$-graded algebra when $\sigma^2 = 1$]. Here the $R_{e_{ii}}$ spanning the even H''_0 commute with all of $L_H = H'$, and the $L_{e_{ii}}$ spanning the even H'_0 commute with all of $R_{\overline{H}}$ and S, hence with H'', but the odd H'_1 anti-commute with H''_1 (indeed, with all SR_H) since $(SR_a)L_b = SL_bR_a = (SL_aS^{-1})SR_a = (-1)^b L_b(SR_a)$ where $SL_{e_{ij}}S^{-1} = L_{e_{22}-e_{11}}L_{e_{ij}}L_{e_{22}-e_{11}} = (-1)^i L_{e_{ii}}L_{e_{ij}}(-1)^j L_{e_{jj}} = (-1)^{i+j}L_{e_{ij}}$ is the grading automorphism on $M_2(\Phi)$. Thus the multiplication algebra $L_HR_{\overline{H}}$ of H is the graded tensor product of two split quaternion subalgebras H', H''; it is generated by operators $V_1 := L_{e_{21}}$, $V_2 := L_{e_{12}}$, $V_3 := SR_{\overline{e_{21}}} = L_{e_{11}-e_{22}}R_{e_{21}}$, $V_4 := SR_{\overline{e_{12}}} = L_{e_{11}-e_{22}}R_{e_{12}}$ with action

(2.1) Regular Quaternion Action $A \times H$

Action of V on:	e_{11}	e_{22}	e_{21}	e_{12}
$V_1 = L_{e_{12}}$	0	e_{12}	e_{11}	0
$V_2 = L_{e_{21}}$	e_{21}	0	0	e_{22}
$V_3 = L_{e_{11}-e_{22}}R_{e_{21}}$	0	$-e_{21}$	0	e_{11}
$V_4 = L_{e_{11}-e_{22}}R_{e_{12}}$	e_{12}	0	$-e_{22}$	0
$V_e = V_f = I$	e_{11}	e_{22}	e_{21}	e_{12}

which is clearly equivalent to our action of V on M in (1.3).

Moreover, the operators V_i generate a Jordan subalgebra $\Phi\mathbf{1}_H \oplus \sum_{i=1}^4 \Phi V_i \subseteq L_HR_{\overline{H}} \cong H \otimes H$ isomorphic to $B = Jord(Q, e)$,

$$V_i^2 = \{V_i, V_j\} = 0 \ (j \neq i'), \quad V_e = \{V_i, V_{i'}\} = \mathbf{1}_H. \tag{2.2}$$

Indeed, all $V_i^2 = 0$ since for $j \neq i$ we have $L_{e_{ij}}^2 = L_{e_{ij}^2} = 0$, $R_{e_{ij}}^2 = R_{e_{ij}^2} = 0$,, while $\{V_1, V_2\} = \{L_{e_{12}}, L_{e_{21}}\} = \{L_{e_{12}, e_{21}}\} = L_{e_{11}+e_{22}} = \mathbf{1}_H$, similarly $\{V_3, V_4\} = L_{e_{11}-e_{22}}^2 R_{\{e_{12}\}, e_{21}} = L_{e_{11}+e_{22}}R_{e_{11}+e_{22}} = \mathbf{1}_H$, and for $r = 1, 2$, $s = 3, 4$ we have $\{V_r, V_j\} = L_{\{e_{ij}, e_{11}-e_{22}\}}R_{e_{kl}} = 0$. Thus $J = B \boxplus \Phi f \oplus M$ imbeds as split null extension in $(H \otimes H \boxplus \Phi f) \oplus H$. However, the excrescence Φf is hard to explain, and the form the odd product takes on $M \cong H \longrightarrow H \otimes H \boxplus \Phi f$ is unilluminating.

3 The Exterior Model

A better way to view the bimodule and odd product, suggested by the Shestakov basis, is through the exterior algebra $\Lambda(M)$. Since $M = \Lambda^1(M)$ is a free Φ-module with ordered basis m_1, m_2, m_3, m_4, the exterior product $\Lambda^2(M)$ is free of rank 6 with basis $\Lambda_1^2 := m_1 \wedge m_4$, $\Lambda_2^2 := m_3 \wedge m_2$, $\Lambda_3^2 := m_3 \wedge m_1$, $\Lambda_4^2 := m_4 \wedge m_2$, $E := \Lambda_5^2 := m_1 \wedge m_2$, $F := \Lambda_6^2 := m_3 \wedge m_4$, and $\Lambda^3(M)$ is free of rank 4 with basis $\Lambda_i^3 := m_i \wedge m_4 \wedge m_3 = -m_i \wedge m_3 \wedge m_4$ $(i = 1, 2)$, $\Lambda_j^3 := m_j \wedge m_2 \wedge m_1 = -m_j \wedge m_1 \wedge m_2$ $(j = 3, 4)$, and $\Lambda^4(M)$ is free of rank 1 with basis $\Lambda_1^4 := E \wedge F = \Lambda_5^2 \wedge \Lambda_6^2 = m_1 \wedge m_2 \wedge m_3 \wedge m_4$. Denote the subspace of $\Lambda^2(M)$ spanned by Λ_i^2, $1 \leq i \leq 4$ by S, so $\Lambda^2(M) = S \oplus \Phi E \oplus \Phi F$.

We obtain an identification isomorphism $\varphi^{(2)} : \Lambda^2(M) = S \oplus \Phi E \oplus \Phi F \longrightarrow A = V \oplus \Phi e \oplus \Phi f$ and *contraction* isomorphisms $\varphi^{(3)} : \Lambda^3(M) \longrightarrow \Lambda^1(M) = M$, $\varphi^{(4)} : \Lambda^4(M) \longrightarrow \Phi$, and a fake copy $\sim : \Lambda^2(M) = S \oplus \Phi E \oplus \Phi F \longrightarrow \widetilde{\Lambda}^2(M) := S \oplus \Phi 1^0 \subseteq \Lambda^2(M) \oplus \Lambda^0(M)$ defined on these bases via

(3.1.1) $\varphi^{(2)}(\Lambda_i^2) := v_i \ (1 \le i \le 4),\ \varphi^{(2)}(\Lambda_5^2) = \varphi^{(2)}(E) := e,\ \varphi^{(2)}(\Lambda_6^2) = \varphi^{(2)}(F) := f,$

i.e., $m_1 \wedge m_4,\ m_3 \wedge m_2,\ m_3 \wedge m_1,\ m_4 \wedge m_2,\ m_1 \wedge m_2,\ m_3 \wedge m_4 \xrightarrow{\varphi^{(2)}} v_1, v_2, v_3, v_4, e, f,$

(3.1.2) $\widetilde{s} = s \ (s \in S), \quad \widetilde{\Lambda}_5^2 =: \widetilde{\Lambda}_6^2 =: 1^0,$

(3.1.3) $\varphi^{(3)}(\Lambda_i^3) := \varphi^{(3)}(m_i \wedge m_4 \wedge m_3) := m_i \ (i{=}1,2),\ \varphi^{(3)}(m_j \wedge m_2 \wedge m_1) := m_j \ (j{=}3,4),$

(3.1.4) $\varphi^{(4)}(\Lambda_1^4) := \varphi^{(4)}(m_1 \wedge m_2 \wedge m_3 \wedge m_4) := 1 = -\varphi^{(4)}(m_3 \wedge m_1 \wedge m_4 \wedge m_2),$

(3.1.5) $V_{\Lambda_i^2} := \varphi \circ L_{\Lambda_i^2} : M{=}\Lambda^1(M) \to \Lambda^3(M) \to M \ (1 \le i \le 4),\ V_{\Lambda_k^2} := L_{\widetilde{\Lambda}_k^2}{=}L_{1^0}{=}\mathbf{1}_M \ (k{=}5,6).$

Abbreviating Λ_i^2, $\varphi^{(3)}$ by Λ_i, φ, we obtain an action table

(3.2) Exterior Bimodule Action $\langle \ell, m\rangle := \varphi^{(3)}(\widetilde{\ell} \wedge m)$

V_ℓ	m_1	m_2	m_3	m_4
$V_{\Lambda_1} =$ $\varphi \circ L_{m_1 \wedge m_4}$	$\varphi(m_1 \wedge m_4 \wedge m_1) =$ 0	$\varphi(m_1 \wedge m_4 \wedge m_2) =$ $\varphi(m_4 \wedge m_2 \wedge m_1) =$ m_4	$\varphi(m_1 \wedge m_4 \wedge m_3) =$ m_1	$\varphi(m_1 \wedge m_4 \wedge m_4) =$ 0
$V_{\Lambda_2} =$ $\varphi \circ L_{m_3 \wedge m_2}$	$\varphi(m_3 \wedge m_2 \wedge m_1) =$ m_3	$\varphi(m_3 \wedge m_2 \wedge m_2) =$ 0	$\varphi(m_3 \wedge m_2 \wedge m_3) =$ 0	$\varphi(m_3 \wedge m_2 \wedge m_4) =$ $\varphi(m_2 \wedge m_4 \wedge m_3) =$ m_2
$V_{\Lambda_3} =$ $\varphi \circ L_{m_3 \wedge m_1}$	$\varphi(m_3 \wedge m_1 \wedge m_1) =$ 0	$\varphi(m_3 \wedge m_1 \wedge m_2) =$ $\varphi(-m_3 \wedge m_2 \wedge m_1) =$ $-m_3$	$\varphi(m_3 \wedge m_1 \wedge m_3) =$ 0	$\varphi(m_3 \wedge m_1 \wedge m_4) =$ $\varphi(m_1 \wedge m_4 \wedge m_3) =$ m_1
$V_{\Lambda_4} =$ $\varphi \circ L_{m_4 \wedge m_2}$	$\varphi(m_4 \wedge m_2 \wedge m_1) =$ m_4	$\varphi(m_4 \wedge m_2 \wedge m_2) =$ 0	$\varphi(m_4 \wedge m_2 \wedge m_3) =$ $\varphi(-m_2 \wedge m_4 \wedge m_3) =$ $-m_2$	$\varphi(m_4 \wedge m_2 \wedge m_4) =$ 0
$V_{\Lambda_k} = L_{\widetilde{\Lambda}_k}$ $=L_{1^0}\ (k{=}5,6)$	m_1	m_2	m_3	m_4

Clearly this coincides with (1.3), and exhibits the bimodule action of V on M as "contracted" multiplication of $\Lambda^2(M)$ on $\Lambda^1(M)$ in the exterior algebra (though with a somewhat artificial replacement of exterior multiplication by $E = \Lambda_5, F = \Lambda_6 \in \Lambda^2(M)$ representing e, f by multiplication by $1 \in \Lambda^0(M)$; our notation E, F indicates that these are only "honorary" members of $\Lambda^2(M)$).

We can also represent the odd multiplication in a natural way through the exterior algebra: there is a natural exterior product of M into $\Lambda^2(M)$, which we map to $S \oplus \Phi G \approx V \oplus \Phi g \subset A$ via

$$\psi(\Lambda_i) := 2\Lambda_i \ (1 \le i \le 4), \quad \psi(\Lambda_5) := \psi(\Lambda_6) := G = E - 3F.$$

(3.3) Exterior Odd Product $\langle m, n\rangle = \psi(m \wedge n)$

$\langle m, n\rangle$	m_1	m_2	m_3	m_4
m_1	$\psi(m_1 \wedge m_1) = 0$	$\psi(m_1 \wedge m_2) = G$	$\psi(m_1 \wedge m_3) = -2\Lambda_3$	$\psi(m_1 \wedge m_4) = 2\Lambda_1$
m_2	$\psi(m_2 \wedge m_1) = -G$	$\psi(m_2 \wedge m_2) = 0$	$\psi(m_2 \wedge m_3) = -2\Lambda_2$	$\psi(m_2 \wedge m_4) = -2\Lambda_4$
m_3	$\psi(m_3 \wedge m_1) = 2\Lambda_3$	$\psi(m_3 \wedge m_2) = 2\Lambda_2$	$\psi(m_3 \wedge m_3) = 0$	$\psi(m_3 \wedge m_4) = G$
m_4	$\psi(m_4 \wedge m_1) = -2\Lambda_1$	$\psi(m_4 \wedge m_2) = 2\Lambda_4$	$\psi(m_4 \wedge m_3) = -G$	$\psi(m_4 \wedge m_4) = 0$

This is just (1.9) in disguise. Tables (3.2-3) are essentially the Shestakov Product Table (1.13).

The exterior viewpoint also allows us to express the symmetric bilinear form $Q(v, w) = Q(\varphi^{(2)}(s), \varphi^{(2)}(t))$ on $V = \varphi^{(2)}(S)$ as

$$(3.4) \qquad Q(\varphi^{(2)}(s), \varphi^{(2)}(t)) = -\varphi^{(4)}(s \wedge t),\ \textit{for}\ s \wedge t = t \wedge s = \sigma(s,t) m_1 \wedge m_2 \wedge m_3 \wedge m_4$$

since from (3.1) $\Lambda_i \wedge \Lambda_j = 0$ due to repeated wedge factors m_k except for the disjoint $\Lambda_1 \wedge \Lambda_2 = m_1 \wedge m_4 \wedge m_3 \wedge m_2 = -m_1 \wedge m_2 \wedge m_3 \wedge m_4$ and $\Lambda_3 \wedge \Lambda_4 = m_3 \wedge m_1 \wedge m_4 \wedge m_2 = -m_1 \wedge m_2 \wedge m_3 \wedge m_4$. We can also explain the "complementary" vector $v_i' = v_{i'}$ as that corresponding to the complementary subset of the index set $\{1,2,3,4\}$,

$$(3.5) \qquad (\Lambda_I^2)' = \Lambda^2_{\{1,2,3,4\}\setminus I}$$

where we parameterize $\Lambda_i^2 = m_j \wedge m_k$ ($m_j < m_k$ in the ordering $m_3 < m_1 < m_4 < m_2$) by the subset $I = \{j,k\}$

4 A Compendium of Triple Products

For quadratic Jordan algebras or superalgebras when $\frac{1}{2} \notin \Phi$, the bilinear products do not determine the quadratic and triple products by the usual rules $2U_a a' = \{a,\{a,a'\}\} - \{a^2,a'\}$ and $2\langle x_i,y_j,z_k\rangle = \langle\langle x_i,y_j\rangle,z_k\rangle + \langle x_i,\langle y_j,z_k\rangle\rangle - (-1)^{ij+jk+ki}\langle y_j,\langle x_i,z_{jk}\rangle\rangle$. We will see below that in the Kac superalgebra scheme K_{10} the bilinear and Peirce structure determines everything but the odd triple products, and only four values $\langle m_i,m_j,m_k\rangle$ ($1 \le i < j < k \le 4$) need to be determined: by Odd Alternation and Switching (0.1.1-2) any product with a repeated variable is determined, $\langle m,n,m\rangle = 0$, $\langle m,n,p\rangle = -\langle p,m,n\rangle$, $\langle m,m,p\rangle = \langle m,\langle m,p\rangle\rangle$, so $\langle m,n,p\rangle + \langle n,m,p\rangle \equiv 0$ modulo bilinear products, and in a triple of distinct variables any one order determines the others, $\langle p,n,m\rangle = -\langle m,n,p\rangle \equiv +\langle n,m,p\rangle = -\langle p,m,n\rangle \equiv +\langle m,p,n\rangle = -\langle n,p,m\rangle$. We will see that in the *split* Kac superalgebra scheme sK_{10} the quadratic structure is *completely determined* by the bilinear structure plus the Peirce decomposition (and King showed [2] that the odd product is determined up to a scalar by the bimodule structure). Thus two forms of sK_{10} which have the same bilinear and Peirce structure *must* have the same quadratic and trilinear structure.

The quadratic operators U_a are determined by the Peirce relations, the quadratic form Q, and the bilinear products: $U_a = U_{\alpha e+v+\beta f} = \alpha^2 U_e + U_v + \beta^2 U_f + \alpha\beta U_{e,f} + \alpha U_{e,v} + \beta U_{f,v}$ where $U_e = E_2, U_f = E_0, U_{e,f} = E_1$ are the Peirce projections on $J_2(e) = \Phi e + V = J_0(f), J_0(e) = \Phi f = J_2(f), J_1(e) = M = J_1(f)$, while by Peirce relations (0.2.4) $U_v f = U_v M = 0$ with $U_v b = Q(v,\bar{b})v - Q(v)\bar{b}$ [as in (1.2)]; $U_{e,v} f = U_{e,v} M = 0$, $U_{e,v} b = V_v b$; and $U_{f,v} f = U_{f,v} B = 0$, $U_{f,v} m = V_v m$.

We will compile a complete list of all possible 10^3 triple products of basis elements m,n,p from M and a,b,c from A (luckily symmetry and Peirce relations reduce this to a manageable collection). The trilinear products with no factors from M are just linearizations $U_{a,a'}a''$ of the quadratic product U_a on A as above. These are just the familiar ones in the direct sum $A = B \boxplus \Phi f$, with those in $B = Jord(Q,e) = \Phi e + V$ being given by (1.2), so we turn to the triple products with a single term from M.

Remark 4.1 *The triple products with only one odd term are completely determined by the Peirce decomposition and the bilinear products as given in* Table (1.5). The outer quadratic products $U_a m$ have $U_B M = U_{e_3} M = 0$, so only triple products $\langle B, M, e_3\rangle$ survive, where $U_{b,e_3} m = V_b m$ reduces to a bilinear product as in (1.5.2). By Even Symmetry (0.1.2) the left multiplications $\langle m,a',a\rangle = \langle a,a',m\rangle$ reduce similarly to repeated bilinear products since $V_{B,f} = V_{f,B} = 0$, $V_{f,f} m = V_f m = m$, and $V_{b,b'} m = V_b V_{b'} m$ by Peirce Orthogonality (0.2.1) and Triple Reduction (0.2.4), which can be read off from Table (1.5.1). ■

We next consider triple products with two or more factors m,n,p from M and a from A.

Remark 4.2 *The triple products with two odd terms are also completely determined by the Perice decomposition and the bilinear products in* Tables (1.3), (1.9)*, since by Triple Reduction* (0.2.4)

$$(4.2.1) \qquad \begin{aligned} \langle m,a_j,n\rangle = E_i\big(\langle m,\langle a_j,n\rangle\rangle\big) = E_i\big(\langle\langle m,a_j\rangle,n\rangle\big), \\ \langle m,n,a_j\rangle = E_j(\langle m,\langle a_j,n\rangle\rangle) \quad (a_j \in A_j(e),\ j=0,2,\ i=2-j). \end{aligned}$$

So far the triple products have all been determined by the bilinear products and the Peirce relations. This is not quite true of the triple products with all odd entries, though in the next section we will see that when the Kac algebra is split further the more refined Peirce decomposition does indeed determine the triples. ■

For the time being, the odd triple product is *defined* in terms of the Peirce relations, *and the alternating bilinear form* σ of (1.7) according to King's explicit formula [3, p.393]

$$(4.3) \qquad \langle m,n,p\rangle = \langle [m\star n-\sigma(m,n)e],p\rangle - \langle [p\star m-\sigma(p,m)e],n\rangle + \langle [n\star p-\sigma(n,p)e],m\rangle,$$

and (4.2.1) can also be formulated as

$$(4.3.1)\ \ \langle m,b,n\rangle = -3\sigma(\langle m,b\rangle,n)f = -3\sigma(m,\langle b,n\rangle)f\ (b\in B), \quad \langle m,f,n\rangle = \sigma(m,n)e+2m\star n$$

since $E_0\langle m,n\rangle = -3\sigma(m,n)f$, $E_2(\langle m,n\rangle) = \sigma(m,n)e+2m\star n$ for $\langle m,n\rangle = \sigma(m,n)g+2m\star n$ [by (1.6)].

From Alternation (0.1.2) we have general relations $\langle m,n,m\rangle = 0$, $\langle m,n,p\rangle = -\langle p,n,m\rangle$, and (letting $1'=2, 3'=4, 4'=3$ as in (1.2),(1.6)) we derive specific relations

$$(4.4) \qquad \begin{array}{ll} \langle m_i,m_k,m_i\rangle = 0, & \langle m_i,m_k,m_j\rangle + \langle m_j,m_k,m_i\rangle = 0, \\ \langle m_{i'},m_i,m_i\rangle = (-1)^{i'}2m_i, & \langle m_j,m_i,m_i\rangle = 0 \quad (j\neq i') \\ \langle m_{i'},m_i,m_j\rangle = (-1)^{i'}m_j, & \langle m_i,m_j,m_{i'}\rangle = (-1)^{i'}m_j \quad (j\neq i,i') \end{array}$$

since from (4.3), (1.8)

$$\begin{aligned} \langle m_{i'},m_i,m_i\rangle &= \langle [0-(-1)^i e],m_i\rangle - \langle [0-(-1)^{i'}e],m_i\rangle + \langle [0-0],m_{i'}\rangle = 2(-1)^{i'}m_i \\ \langle m_j,m_i,m_i\rangle &= \langle [m_j\star m_i-0],m_i\rangle - \langle [m_i\star m_j-0],m_i\rangle + \langle [0-0],m_j\rangle = 0 \\ \langle m_{i'},m_i,m_j\rangle &= \langle [0-(-1)^i e],m_j\rangle - \langle [m_j\star m_{i'}-0],m_i\rangle + \langle [m_i\star m_j-0],m_{i'}\rangle \\ &= [(-1)^{i'}+(-1)^i+(-1)^{i'}]m_j = (-1)^{i'}m_j \\ \langle m_i,m_j,m_{i'}\rangle &= \langle [m_i\star m_j-0],m_{i'}\rangle - \langle [0-(-1)^i e],m_j\rangle + \langle [m_j\star m_{i'}-0],m_i\rangle \\ &= [(-1)^{i'}+(-1)^i-(-1)^i]m_j = (-1)^{i'}m_j. \end{aligned}$$

We quickly arrive at a table of outer odd multiplications, but less quickly at a table of left odd multiplications.

(4.5) Two- or Three Odd Products $\langle M,J,M\rangle, \langle M,M,J\rangle$

$U_{m,n}$	v_1	v_2	v_3	v_4	e	f	m_1	m_2	m_3	m_4
$U_{m_1,m_2}=-U_{m_2,m_1}$	0	0	0	0	$-3f$	e	$-2m_1$	$-2m_2$	m_3	m_4
$U_{m_1,m_3}=-U_{m_3,m_1}$	0	0	0	$3f$	0	$-2v_3$	0	$-m_3$	0	m_1
$U_{m_1,m_4}=-U_{m_4,m_1}$	0	$-3f$	0	0	0	$2v_1$	0	$-m_4$	$-m_1$	0
$U_{m_2,m_3}=-U_{m_3,m_2}$	$3f$	0	0	0	0	$-2v_2$	m_3	0	0	m_2
$U_{m_2,m_4}=-U_{m_4,m_2}$	0	0	$3f$	0	0	$-2v_4$	m_4	0	$-m_2$	0
$U_{m_3,m_4}=-U_{m_4,m_3}$	0	0	0	0	$-3f$	e	m_1	m_2	$-2m_3$	$-2m_4$
$V_{m,n}$	$V_{m,n}a_j = E_jV_mV_na_j$									
$V_{m_3,m_1}=-V_{m_1,m_3}=V_{v_3}$	0	0	0	e	$2v_3$	0	0	$-m_3$	0	m_1
$V_{m_1,m_4}=-V_{m_4,m_1}=V_{v_1}$	0	e	0	0	$2v_1$	0	0	m_4	m_1	0
$V_{m_3,m_2}=-V_{m_2,m_3}=V_{v_2}$	e	0	0	0	$2v_2$	0	m_3	0	0	m_2
$V_{m_4,m_2}=-V_{m_2,m_4}=V_{v_4}$	0	0	e	0	$2v_4$	0	m_4	0	$-m_2$	0
$V_{m_1,m_1}=2V_{v_1,v_3}$	0	$-2v_3$	0	$2v_1$	0	0	0	$-2m_1$	0	0
$V_{m_2,m_2}=2V_{v_2,v_4}$	$-2v_4$	0	$2v_2$	0	0	0	$2m_2$	0	0	0
$V_{m_3,m_3}=2V_{v_3,v_2}$	$2v_3$	0	0	$-2v_2$	0	0	0	0	0	$-2m_3$
$V_{m_4,m_4}=2V_{v_4,v_1}$	0	$2v_4$	$-2v_1$	0	0	0	0	0	$2m_4$	0
V_{m_1,m_2}	$2v_1$	0	$2v_3$	0	e	$-3f$	0	$-2m_2$	$-m_3$	$-m_4$
V_{m_2,m_1}	0	$-2v_2$	0	$-2v_4$	$-e$	$3f$	$2m_1$	0	m_3	m_4
V_{m_3,m_4}	0	$2v_2$	$2v_3$	0	e	$-3f$	$-m_1$	$-m_2$	0	$-2m_4$
V_{m_4,m_3}	$-2v_1$	0	0	$-2v_4$	$-e$	$3f$	m_1	m_2	$2m_3$	0
$U_{m_i,m_i}=0, \quad R_{m_j,m_i}(a) := \langle a,m_j,m_i\rangle = -\langle m_i,m_j,a\rangle = -V_{m_i,m_j}(a)$										

PROOF: The action of the outer multiplications $U_{m,n}$ on the odd m's follows from the general recipes (4.4). For the action on the v's, (4.2.1) shows that $U_{m_i,m_j}v$ vanishes if $\langle m_i, v\rangle$ or $\langle v, m_j\rangle$ vanishes, therefore Table (1.3) shows that for $v_1 \perp m_1, m_4$ only $U_{m_2,m_3}v_1$ survives [with $E_0V_{m_2}V_{m_3}v_1 = E_0V_{m_2}m_1 = E_0(-g) = 3f$], simlarly for $v_2 \perp m_2, m_3$ only U_{m_1,m_4} [with $E_0V_{m_1}V_{m_4}v_2 = E_0V_{m_1}m_2 = E_0(g) = -3f$], for $v_3 \perp m_1, m_3$ only U_{m_2,m_4} [with $E_0V_{m_2}V_{m_4}v_3 = E_0V_{m_2}m_1 = 3f$ again], and for $v_4 \perp m_2, m_4$ only U_{m_1,m_3} [with $E_0V_{m_1}V_{m_3}v_4 = -E_0V_{m_1}m_2 = -E_0(g) = 3f$]. $U_{m,n}e = E_0(\langle m, n\rangle)$ and $U_{m,n}f = E_2(\langle m, n\rangle)$ are read directly from Table (1.9). This completes the table of U-operators.

The left multiplications acting on M can be read from the U-table, $V_{m_i,m_j}(m_k) = U_{m_i,m_k}(m_j)$, or from the general recipes (4.4): V_{m_i,m_i} sends $m_i, m_{i'}, m_j \longrightarrow 0, -(-1)^{i'}2m_i = 2(-1)^i m_i, 0$, while $V_{m_i,m_{i'}}$ sends $m_i, m_{i'}, m_j \longrightarrow 0, (-1)^i m_{i'}, (-1)^i m_j$, giving immediately the last 8 rows on M.

For the last 8 rows acting on A, reading Table (1.3) by columns and then reading Table (1.9) by rows shows that the ordered basis v_1, v_2, v_3, v_4, e, f for A is sent by V_mV_n as follows: V_{m_1} sends $v_1, v_2, v_3, v_4, e, f \longrightarrow 0, m_3, 0, m_4, m_1, m_1$ which is then sent by $E_jV_{m_1}$ to $0, -2v_3, 0, 2v_1, 0, 0$ for V_{m_1,m_1}, and sent by $E_jV_{m_2}$ to $0, -2v_2, 0, -2v_4, -e, 3f$ for V_{m_2,m_1}.

Similarly V_{m_2} sends $v_1, v_2, v_3, v_4, e, f \longrightarrow m_4, 0, -, m_3, 0, m_2, m_2$ which is then sent by $E_jV_{m_1}$ to $2v_1, 0, 2v_3, 0, e, -3f$ for V_{m_1,m_2}, and by $E_jV_{m_2}$ to $-2v_4, 0, 2v_2, 0, 0, 0$ for V_{m_2,m_2}.

Likewise, V_{m_3} sends $v_1, v_2, v_3, v_4, e, f \longrightarrow m_1, 0, 0, -m_2, m_3, m_3$ which is then sent by $E_jV_{m_3}$ to $2v_3, 0, 0, -2v_2, 0, 0$ for V_{m_3,m_3}, and by $E_jV_{m_4}$ to $-2v_1, 0, 0, -2v_4, -e, 3f$ for V_{m_4,m_3}.

Finally, V_{m_4} sends $v_1, v_2, v_3, v_4, e, f \longrightarrow m_2, m_1, m_4, m_4$ which is sent by $E_jV_{m_3}$ to $0, 2v_2, 2v_3, 0, e, -3f$ for V_{m_3,m_4}, and by $E_jV_{m_4}$ to $0, 2v_4, -2v_1, 0, 0, 0$ for V_{m_4,m_4}.[6] Comparison of this with (1.5) shows (for no apparent reason)

$$(4.6) \qquad V_{m_1,m_1} = 2V_{v_1,v_3},\ V_{m_2,m_2} = 2V_{v_2,v_4},\ V_{m_3,m_3} = 2V_{v_3,v_2},\ V_{m_4,m_4} = 2V_{v_4,v_1}.$$

The remaining first 4 rows of left multiplications $V_{m,n}$ reduce to operators V_b which can be read off from Table (1.5): we claim that $V_{m_3,m_1} = -V_{m_1,m_3} = V_{v_3}$, $V_{m_1,m_4} = -V_{m_4,m_1} = V_{v_1}$, $V_{m_3,m_2} = -V_{m_2,m_3} = V_{v_2}$, $V_{m_4,m_2} = -V_{m_2,m_4} = V_{v_4}$ as operators on all of J, since by (1.7) $m_3 \star m_1 = v_3$, $m_1 \star m_4 = v_1$, $m_3 \star m_2 = v_2$, $m_4 \star m_2 = v_4$, where in general

$$(4.7) \qquad \textit{for } j \neq i, i' \quad \textit{we have} \quad V_{m_i,m_j} = V_{m_i \star m_j} \in V_B.$$

This holds on M since by (1.8) $V_{m_i \star m_j}$ sends $m_i, m_{i'}, m_j, m_{j'}$ to 0, $(-1)^{i'}m_j$, 0, $(-1)^j m_i$, while by (4.4) $\langle m_i, m_j, m_i\rangle = 0$, $\langle m_i, m_j, m_{i'}\rangle = (-1)^{i'}m_j$, $\langle m_i, m_j, m_j\rangle = 0$, $\langle m_i, m_j, m_{j'}\rangle = -\langle m_{j'}, m_j, m_i\rangle = -(-1)^{j'}m_i = (-1)^j m_i$. To see it also holds on A, we check that $\langle m_i, m_j, f\rangle = E_0\big(\langle m_i, m_j\rangle\big) = -3\sigma(m_i, m_j)f = 0 = V_{m_i \star m_j}(f) \in V_Bf$. Similarly $\langle m_i, m_j, e\rangle = E_2\big(\langle m_i, m_j\rangle\big) = E_2\big(\sigma(m_i, m_j)g + 2m_i \star m_j\big) = 0 + 2m_i \star m_j = V_{m_i \star m_j}(e)$. Finally, $\langle m_i, m_j, v_k\rangle = E_2\big(\langle m_i, \langle m_j, v_k\rangle\rangle\big)$ [by Triple Reduiction (0.2.4)] $= E_2\big(\sigma(m_i, \langle m_j, v_k\rangle)g + 2m_i \star \langle m_j, v_k\rangle\big)$ [by (1.6)] $= \sigma(m_i, \langle m_j, v_k\rangle)e$ [by (1.8) $m_i \star \langle V, m_j\rangle \in m_i \star (\Phi m_i + \Phi m_{i'}) = 0$], while $V_{m_i \star m_j}(v_k) = -V_{m_j \star m_i}(v_k)$ [by skewness of $\star$] $= -\big(\sum_\ell \langle \sigma(\langle v_\ell, m_j\rangle, m_i)v_{\ell'}\big),\ v_k\rangle = -\sum_\ell \sigma(\langle v_\ell, m_j\rangle, m_i)\delta_{\ell k}e = -\sigma(\langle v_k, m_j\rangle, m_i) = +\sigma(m_i, \langle v_k, m_j\rangle)$ [by skewness of σ]. This establishes the last cases of (4.5). ■

Notice that most of the $V_{m,n}$ reduce rather mysteriously to $V_{a,b}$'s. We now turn to the mixed left multiplications $V_{a,m}$, $V_{m,a}$; they too reduce surprisingly to $V_{m,e}, V_{m,f}$:

$$(4.8) \qquad V_{m,b} = V_{\langle b,m\rangle,e}, \quad V_{b,m} = V_{e,\langle b,m\rangle}, \quad V_{m,f} = V_{e,m}, \quad V_{f,m} = V_{m,e}$$

because in the Peirce decomposition relative to e we have by Triple Reduction (0.2.4) that $\langle m, b, f\rangle = 0 = \langle\langle b, m\rangle, e, f\rangle$, $\langle m, b, c\rangle = \langle\langle b, m\rangle, c\rangle = \langle\langle\langle b, m\rangle, e\rangle, c\rangle = \langle\langle b, m\rangle, e, c\rangle$, and $\langle m, b, n\rangle = E_0\big(\langle\langle b, m\rangle, n\rangle\big) =$

[6] This can also be calculated from the matrices of $A \overset{V'_x}{\longleftarrow} M \overset{V''_x}{\longleftarrow} A$ of V_x relative to the ordered bases $\{v_1, v_2, v_3, v_4, e, f\}$ for A and $\{m_1, m_2, m_3, m_4\}$ for M, since $V'_{m_1} \cong E_{52} - 3E_{62} - 2E_{33} + 2E_{14}$, $V'_{m_2} \cong -E_{51} + 3E_{61} - 2E_{23} - 2E_{44}$, $V'_{m_3} \cong E_{54} - 3E_{61} + 2E_{31} + 2E_{22}$, $V'_{m_4} \cong -E_{53} + 3E_{63} - 2E_{11} + 2E_{42}$, $V''_{m_1} \cong E_{32} + E_{44} + E_{15} - E_{16}$, $V''_{m_2} \cong E_{41} - E_{33} + E_{25} + E_{26}$, $V''_{m_2} \cong E_{11} - E_{24} + E_{35} + E_{36}$, $V''_{m_4} \cong E_{22} + E_{13} + E_{45} + E_{46}$.

$\langle\langle b,m\rangle,e,n\rangle$; similarly $\langle b,m,f\rangle = \langle\langle b,m\rangle,f\rangle = \langle b,m\rangle = \langle e,\langle b,m\rangle,f\rangle$, $\langle b,m,c\rangle = 0 = \langle e,\langle b,m\rangle,c\rangle$, and $\langle b,m,n\rangle = E_2\big(\langle\langle b,m\rangle,n\rangle\big) = \langle e,\langle b,m\rangle,n\rangle$; by Switching (0.1.1) $V_{m,f} = V_{\langle m,f\rangle} - V_{f,m} = V_m - V_{f,m} = V_{1,m} - V_{f,m} = V_{e,m}$, dually $V_{f,m} = V_m - V_{m,f} = V_{m,1-f} = V_{m,e}$. Thus

$$V_{A,M} + V_{M,A} = V_{M,e} + V_{e,M} = V_{M,\Phi e+\Phi f}. \tag{4.9}$$

Combining this with (4.8), the 32 odd left multiplications reduce to 8:

(4.10)

$V_{m_1,e} = V_{m_3,v_1} = V_{m_4,v_3} = V_{f,m_1}$,	$V_{e,m_1} = V_{v_1,m_3} = V_{v_3,m_4} = V_{m_1,f}$,
$V_{m_2,e} = -V_{m_3,v_4} = V_{m_4,v_2} = V_{f,m_2}$,	$V_{e,m_2} = -V_{v_4,m_3} = V_{v_2,m_4} = V_{m_2,f}$,
$V_{m_3,e} = V_{m_1,v_2} = -V_{m_2,v_3} = V_{f,m_3}$,	$V_{e,m_3} = V_{v_2,m_1} = -V_{v_3,m_2} = V_{m_3,f}$,
$V_{m_4,e} = V_{m_1,v_4} = V_{m_2,v_1} = V_{f,m_4}$,	$V_{e,m_4} = V_{v_4,m_1} = V_{v_1,m_2} = V_{m_4,f}$.

leading to the following brief table of values of these odd left multiplications.

(4.11) Odd Left Multiplications $\langle M,A,J\rangle, \langle A,M,J\rangle$

	$V_{M,A} = V_{M,e}$						$V_{A,M} = V_{e,M}$			
$V_{m,a}(x)$ for $x =$	v_1	v_2	v_3	v_4	e	f	m_1	m_2	m_3	m_4
$V_{m_1,e} = V_{f,m_1}$	0	m_3	0	m_4	m_1	0	0	$-3f$	0	0
$V_{m_2,e} = V_{f,m_2}$	m_4	0	$-m_3$	0	m_2	0	$3f$	0	0	0
$V_{m_3,e} = V_{f,m_3}$	m_1	0	0	$-m_2$	m_3	0	0	0	0	$-3f$
$V_{m_4,e} = V_{f,m_4}$	0	m_2	m_1	0	m_4	0	0	0	$3f$	0
$V_{e,m_1} = V_{m_1,f}$	0	0	0	0	0	m_1	0	e	$-2v_3$	$2v_1$
$V_{e,m_2} = V_{m_2,f}$	0	0	0	0	0	m_2	$-e$	0	$-2v_2$	$-2v_4$
$V_{e,m_3} = V_{m_3,f}$	0	0	0	0	0	m_3	$2v_3$	$2v_2$	0	e
$V_{e,m_4} = V_{m_4,f}$	0	0	0	0	0	m_4	$-2v_1$	$2v_4$	$-e$	0

PROOF: The table for $V_{m,e}a = V_m a$ can be read off vertically from the rows of (1.3) [note $V_{M,f}B = 0, V_{m,f}f = m$] and $V_{m,a}n = U_{m,n}a$ from the f-column of (4.5) [or from (1.9)]. ■

5 The Split Kac Superalgebra $sK_{10}(\Phi)$

In this section we introduce the isotope $sK_{10}(\Phi) := K_{10}(\Phi)^{split} := K_{10}(\Phi)^{(s)}$ $(s = v_1 + v_2 + f)$ of the standard Kac superalgebra scheme which provides 3 reduced idempotents over an arbitrary ring Φ of scalars. We find a "split basis" and compute all bilinear and trilinear products in this isotope. Later we will show that when $i, \frac{1}{\sqrt{2}} \in \Phi$ this isotope is isomorphic to the standard superalgebra.

We call this isotope the "split K_{10} scheme". Using $x^* = U_s x$ as promised in (1.12), its operations are[7]

$$U_a^{(s)}(x_i) := U_a U_s x_i = U_a x_i^*, \quad a^{(2,s)} := U_a s,, \quad 1^{(s)} := s = u + f,$$

$$\langle x_i, y_j\rangle^{(s)} := \langle x_i, s, y_j\rangle, \quad \langle x_i, y_j, z_k\rangle^{(s)} := \langle x_i, U_s y_j, z_k\rangle = \langle x_i, y_j^*, z_k\rangle.$$

Here A remains a subalgebra in the isotope, $A^{(s)} = B^{(u)} \boxplus (\Phi f)^{(f)} = Jord(-Q,u) \boxplus \Phi f$ [since in general $Jord(Q,e)^{(u)} = Jord(Q(u)Q, u^{-1})$ where here $Q(u) = -1$, $1^{(s)} = u^{-1} = u$. In particular, $f^{(2,s)} = U_f f = f$, $b^{(2,s)} = U_b u$ so $e^{(2,s)} = U_e u = u$, $v_i^{(2,s)} = U_{v_i}(v_1 + v_2) = U_{v_i} v_{i'} = v_i$ $(i = 1,2)$, $v_j^{(2,s)} = 0$ $(j = 3,4)$, and for the bilinear products we have the following table. For Peirce reasons which will become clear shortly, we re-order our basis for M as m_1, m_4, m_3, m_2 (interchanging the second and fourth members).

[7] Again we use a Grassmann detour to make sure isotopy works for quadratic superalgebras; the quadratic Jordan algebra $\Gamma(J)^{(1\otimes s)}$ has operations $\widetilde{U}_{\beta\otimes a} = \widetilde{U}_{\beta\otimes a}\widetilde{U}_{1\otimes s} = \beta^2 \otimes U_a x$, $(\beta\otimes a)^2 = \widetilde{U}_{\beta\otimes a}\tilde{1} = \beta^2 \otimes U_a s$, $\{\delta_i\otimes x_i, \delta_j\otimes y_j\} = \{\delta_i\otimes x_i, 1\otimes s, \delta_j\otimes y_j\} = \delta_i\delta_j\langle x_i, s, y_j\rangle$ and analogously $\{\delta_i\otimes x_i, \delta_j\otimes y_j, \delta_k\otimes z_k\} = \{\delta_i\otimes x_i, \widetilde{U}_{1\otimes s}(\delta_j\otimes y_j), \delta_k\otimes z_k\} = \delta_i\delta_j\delta_k\langle x_i, U_s y_j, z_k\rangle$.

(5.1) sK_{10} Split Bimodule Products

$\langle a,a'\rangle^{(s)}$	v_1	v_2	v_3	v_4	e	f	$\langle a,m\rangle^{(s)}$	m_1	m_4	m_3	m_2
v_1	$2v_1$	0	v_3	v_4	e	0	v_1	m_1	m_4	0	0
v_2	0	$2v_2$	v_3	v_4	e	0	v_2	0	0	m_3	m_2
v_3	v_3	v_3	0	$-u$	0	0	v_3	0	$-m_3$	0	m_1
v_4	v_4	v_4	$-u$	0	0	0	v_4	$-m_2$	0	m_4	0
e	e	e	0	0	$2u$	0	e	m_3	m_2	m_1	m_4
f	0	0	0	0	0	$2f$	f	m_1	m_4	m_3	m_2
$a^{(2,s)}$	v_1	v_2	0	0	u	f					

PROOF: The algebra products with f follow from $\{f,b\}^{(s)} = \{f,s,b\} = 0$, while $\{b,b'\}^{(s)} = \{b,u,b'\}$ can be read directly from Table (1.5) by adding the v_1- and v_2-columns for the U-operators.

As for the bimodule products, these can be read from the V-operators of Table (1.5) [$\langle f,m\rangle^{(s)} = \langle f,s,m\rangle = \langle f,f,m\rangle = m,\ \langle b,m\rangle^{(s)} = \langle b,u,m\rangle = V_{b,v_1+v_2}(m)$], or easily using $\langle b,u,m\rangle = \langle b,\langle u,m\rangle\rangle = \langle b,m^*\rangle$ [recall (1.12)] so that the new product of b on m_1,m_4,m_3,m_2 is just the old product of b on m_3,m_2,m_1,m_4 [hence obtained from transposing the m_1 and m_3 columns in Table (1.3)]. ■

Born Again

In particular, the elements v_1,v_2,f now become supplementary orthogonal idempotents and v_3,v_4,e,m_1,m_4 span the Peirce 1-space of v_1, while v_3,v_4,e,m_3,m_2 span the Peirce 1-space of v_2. *We make the further replacement of v_3 by $-v_3$*, so that $\langle -v_3,v_4\rangle^{(s)} = u$ in $B^{(u)}$ (as $\{v_3,v_4\} = e$ in B). To indicate this Peirce structure we hereby rechristen our basis to indicate their Peirce space J_{ij}. When we use this new labelling it is clear that we are working in the isotope, so we drop the superscripts $\langle\ldots\rangle^{(s)}$ for isotope-products and use the usual notation $\langle\ldots\rangle$, U_a for superalgebra products. Dressed in its new clothes

Old	v_1	v_2	$-v_3$	v_4	e	f	$v_2{+}v_2$	m_1	m_4	m_3	m_2	$\langle\ldots\rangle^{(s)}$	$U_a^{(s)}$
New	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	u	m_{13}	n_{13}	m_{23}	n_{23}	$\langle\ldots\rangle$	U_a

the new quadratic form becomes (compare with (1.1))

$$(5.2) \qquad Q(b) = \beta_1\beta_2 - \beta_3\beta_4 - \beta_5^2, \quad T(b) = \beta_1 + \beta_2 \quad \textit{for} \quad b = \beta_1 e_1 + \beta_2 e_2 + \beta_3 c_{12} + \beta_4 d_{12} + \beta_5 q_{12}$$

and the bimodule structure becomes

(5.3) sK_{10} Bimodule Product

$\circ$	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	m_{13}	n_{13}	m_{23}	n_{23}
e_1	$2e_1$	0	c_{12}	d_{12}	q_{12}	0	m_{13}	n_{13}	0	0
e_2	0	$2e_2$	c_{12}	d_{12}	q_{12}	0	0	0	m_{23}	n_{23}
c_{12}	c_{12}	c_{12}	0	u	0	0	0	m_{23}	0	$-m_{13}$
d_{12}	d_{12}	d_{12}	u	0	0	0	$-n_{23}$	0	n_{13}	0
q_{12}	q_{12}	q_{12}	0	0	$2u$	0	m_{23}	n_{23}	m_{13}	n_{13}
e_3	0	0	0	0	0	$2e_3$	m_{13}	n_{13}	m_{23}	n_{23}
a^2	e_1	e_2	0	0	u	e_3				

We can also give a closed-form expression for the action of B_{12} on M_{i3}:

$$(5.4) \qquad \begin{array}{|lll|} \hline \langle q_{12},m_{i3}\rangle = m_{j3}, & \langle q_{12},n_{i3}\rangle = n_{j3}, & \langle c_{12},m_{i3}\rangle = \langle d_{12},n_{i3}\rangle = 0, \\ \langle c_{12},n_{i3}\rangle = (-1)^j m_{j3}, & \langle d_{12},m_{i3}\rangle = (-1)^i n_{j3}, & (j = 3-i). \\ \hline \end{array}$$

We can translate Table (1.5) directly into a bimodule table for the isotope.

(5.5) Split Bimodule Structure

$V,\quad \langle a,x\rangle = V_a(x)$	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	m_{13}	n_{13}	m_{23}	n_{23}
$V_{e_1,q_{12}} = V_{q_{12},e_2}$	0	q_{12}	0	0	$2e_1$	0	0	0	m_{13}	n_{13}
$V_{e_2,q_{12}} = V_{q_{12},e_1}$	q_{12}	0	0	0	$2e_2$	0	m_{23}	n_{23}	0	0
$V_{q_{12},c_{12}} = -V_{c_{12},q_{12}}$	0	0	0	q_{12}	$-2c_{12}$	0	0	m_{13}	0	$-m_{23}$
$V_{d_{12},q_{12}} = -V_{q_{12},d_{12}}$	0	0	$-q_{12}$	0	$2d_{12}$	0	n_{13}	0	$-n_{23}$	0
$V_{q_{12},q_{12}}$	$2e_1$	$2e_2$	$2c_{12}$	$2d_{12}$	$2q_{12}$	0	m_{13}	n_{13}	m_{23}	n_{23}
$V_f = V_{f,f}$	0	0	0	0	0	$2f$	m_{13}	n_{13}	m_{23}	n_{23}
$V_{c_{12}}$	c_{12}	c_{12}	0	u	0	0	0	m_{23}	0	$-m_{13}$
$V_{d_{12}}$	d_{12}	d_{12}	u	0	0	0	$-n_{23}$	0	n_{13}	0
$V_{q_{12}}$	q_{12}	q_{12}	0	0	$2u$	0	m_{23}	n_{23}	m_{13}	n_{13}
$V_{e_1} = V_{e_1,e_1}$	$2e_1$	0	c_{12}	d_{12}	q_{12}	0	m_{13}	n_{13}	0	0
$V_{e_2} = V_{e_2,e_2}$	0	$2e_2$	c_{12}	d_{12}	q_{12}	0	0	0	m_{23}	n_{23}
$V_{c_{12},d_{12}}$	e_1	e_2	$2c_{12}$	0	q_{12}	0	m_{13}	0	m_{23}	0
$V_{d_{12},c_{12}}$	e_1	e_2	0	$2d_{12}$	q_{12}	0	0	n_{13}	0	n_{23}
$V_{e_1,c_{12}} = V_{c_{12},e_2}$	0	c_{12}	0	e_1	0	0	0	0	0	$-m_{13}$
$V_{e_2,c_{12}} = V_{c_{12},e_1}$	c_{12}	0	0	e_2	0	0	0	m_{23}	0	0
$V_{e_1,d_{12}} = V_{d_{12},e_2}$	0	d_{12}	e_1	0	0	0	0	0	n_{13}	0
$V_{e_2,d_{12}} = V_{d_{12},e_1}$	d_{12}	0	e_2	0	0	0	$-n_{23}$	0	0	0
$a^2 = U_a u$	e_1	e_2	0	0	u	f	$U_{f,b} = V_b$ on M			
$V_{e_1,e_2} = V_{e_2,e_1} = 0,\qquad V_{c_{12},c_{12}} = V_{d_{12},d_{12}} = V_{f,B} = V_{B,f} = 0,\qquad V_{b',b} = V_{b'}V_{b^*}$ on M										
$U_{b,b} = 2U_b,\quad U_{u,b} = V_b,\qquad U_f = U_b = U_{b,b'} = 0$ on M										

U	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3
U_{e_1}	e_1	0	0	0	0	0
U_{e_2}	0	e_2	0	0	0	0
$U_{c_{12}}$	0	0	0	c_{12}	0	0
$U_{d_{12}}$	0	0	d_{12}	0	0	0
$U_{q_{12}}$	e_2	e_1	$-c_{12}$	$-d_{12}$	q_{12}	0
U_f	0	0	0	0	0	f
$U_{e_1,e_2} = U_{e_2,e_1}$	0	0	c_{12}	d_{12}	q_{12}	0
$U_{e_1,c_{12}} = U_{c_{12},e_1}$	c_{12}	0	0	e_1	0	0
$U_{e_1,d_{12}} = U_{d_{12},e_1}$	d_{12}	0	e_1	0	0	0
$U_{e_1,q_{12}} = U_{q_{12},e_1}$	q_{12}	0	0	0	$2e_1$	0
$U_{e_2,c_{12}} = U_{c_{12},e_2}$	0	c_{12}	0	e_2	0	0
$U_{e_2,d_{12}} = U_{d_{12},e_2}$	0	d_{12}	e_2	0	0	0
$U_{e_2,q_{12}} = U_{q_{12},e_2}$	0	q_{12}	0	0	$2e_2$	0
$U_{c_{12},d_{12}} = U_{d_{12},c_{12}}$	e_2	e_1	0	0	$-q_{12}$	0
$U_{q_{12},c_{12}} = U_{c_{12},q_{12}}$	0	0	0	q_{12}	$2c_{12}$	0
$U_{q_{12},d_{12}} = U_{d_{12},q_{12}}$	0	0	q_{12}	0	$2d_{12}$	0

PROOF: Beginning with the V-operators $V^{(s)}_{a,b} = V_{a,b^*}$, we set $i = 1,2$, $j = 3-i$, $k = 3,4$. For the first two lines we have $V^{(s)}_{e_i,q_{12}} = V_{v_i,e^*} = V_{v_i,e} = V_{v_i} = V_{e,v_i} = V_{e,v_j^*} = V^{(s)}_{q_{12},e_j}$. Then $V_{v_3,e} = V_{e,v_3}$ becomes $V^{(s)}_{-c_{12},q_{12}} = V^{(s)}_{q_{12},c_{12}}$ and $V_{v_4,e} = V_{e,v_4}$ becomes $V^{(s)}_{d_{12},q_{12}} = V^{(s)}_{q_{12},-d_{12}}$; $V_{e,e}$ becomes $V^{(s)}_{q_{12},q_{12}}$, $V_{f,f}$ remains $V^{(s)}_{f,f}$; V_{v_i,v_j} becomes $V^{(s)}_{e_i,e_i}$; V_{v_3,v_4} becomes $V^{(s)}_{-c_{12},-d_{12}}$, and dually for V_{v_4,v_3}; $V_{v_i,v_3} = -V_{v_3,v_i}$ becomes $V^{(s)}_{e_i,c_{12}} = -V^{(s)}_{-c_{12},e_j}$; $V_{v_i,v_4} = -V_{v_4,v_i}$ become $V^{(s)}_{e_i,-d_{12}} = -V^{(s)}_{d_{12},e_j}$ [so we negate the row in (1.5)]; $V_{b',b} = V_{b'}V_b$ becomes $V^{(s)}_{b',b^*} = V^{(s)}_{b'}V^{(s)}_{b^*}$ [by Peirce relations, not translation]. $V^{(s)}_b = V_{b,u} = V_{b,v_1} + V_{b,v_2} = V^{(s)}_{b,e_2} + V^{(s)}_{b,e_1}$ is obtained by adding the first two columns of V_b in (1.5). $V_{c_{12}}, V_{d_{12}}, V_{q_{12}}$ are most easily read directly from (5.4) [alternately, $V^{(s)}_b = V_{b,u} = V_{b,u^*} = V^{(s)}_{b,u}$ is the sum $V^{(s)}_{b,e_1}+V^{(s)}_{b,e_2}$ in (5.4)].

The U-operators are $U^{(s)}_{a,b}(x)=U_{a,b}(x^*)$, so we read their values directly from (1.5) with rows with c_{12} negated (due to $c_{12}=-v_3$), columns v_1,v_2 interchanged, column v_4 negated (but not column $c_{12}=-v_3$ since it is negated twice), omitting the m_i since $U^{(s)}_B M=0$. Here $U^{(s)}_{q_{12},e_i}a=U_{e,v_i}a^*=V_{v_i}a^*=\{v_i,a^*\}$, $U^{(s)}_{q_{12},c_{12}}a=-U_{e,v_3}a^*=-V_{v_3}a^*=-\{v_3a^*\}$, dually $U^{(s)}_{q_{12},d_{12}}=\{v_4,a^*\}$ are read from (1.5). ■

Odd Products

Turning to the odd products, we introduce the abbreviation

$$g_i := 2v_i - 3f = 2e_i - 3e_3 \quad (i=1,2).$$

Again $\langle m,n\rangle^{(s)}=\langle m,s,n\rangle$ can be read off Tables (1.3), (1.9) using $\langle m,s,n\rangle=\langle m,u+f,n\rangle=E_0\big(\langle m,\langle u,m\rangle\rangle\big)+E_2\big(\langle m,\langle f,n\rangle\rangle\big)=E_0\big(\langle m,n^*\rangle\big)+E_0\big(\langle m,n\rangle\big)$, or by adding the v_1,v_2,f-columns of Table (4.5). This leads to $\langle m_1,m_2\rangle^{(s)}=0+0+e=q_{12}$ and $\langle m_3,m_4\rangle^{(s)}=0+0+e=q_{12}$, $\langle m_1,m_3\rangle^{(s)}=0+0-2v_3=2c_{12}$, $\langle m_1,m_4\rangle^{(s)}=0-3f+2v_1=g_1$ and $\langle m_2,m_3\rangle^{(s)}=3f+0-2v_2=-g_2$, $\langle m_2,m_4\rangle^{(s)}=0+0-2v_4=-2d_{12}$, thus

(5.6)
$$\boxed{\begin{array}{lll}\langle m_{i3},n_{j3}\rangle=q_{12}, & \langle m_{i3},m_{j3}\rangle=(-1)^j2c_{12} & (j{=}3{-}i),\\ \langle m_{i3},n_{i3}\rangle=g_i, & \langle n_{i3},n_{j3}\rangle=(-1)^j2d_{12}, & \end{array}}$$

(5.7) Isotope Odd Product / Split Odd Product

$\langle\cdot,\cdot\rangle^{(s)}$	m_1	m_4	m_3	m_2	$\langle\cdot,\cdot\rangle$	m_{13}	n_{13}	m_{23}	n_{23}
m_1	0	g_1	$-2v_3$	e	m_{13}	0	g_1	$2c_{12}$	q_{12}
m_4	$-g_1$	0	$-e$	$2v_4$	n_{13}	$-g_1$	0	$-q_{12}$	$2d_{12}$
m_3	$2v_3$	e	0	g_2	m_{23}	$-2c_{12}$	q_{12}	0	g_2
m_2	$-e$	$-2v_4$	$-g_2$	0	n_{23}	$-q_{12}$	$-2d_{12}$	$-g_2$	0

Following the Shestakov model, we could write these in the more mnemonic form (for $i\neq j\in\{1,2\}$)

$$\boxed{\begin{array}{c} c_{12}=:b^{(m)}_{12},\ d_{12}=:b^{(n)}_{12},\ q_{12}=:b^{(m,n)}_{12}\ \textit{with}\ \langle m_{13},m_{23}\rangle=2b^{(m)}_{12},\ \langle n_{13},n_{23}\rangle=2b^{(n)}_{12},\\ \langle m_{i3},n_{j3}\rangle=b^{(m,n)}_{12},\quad \langle b^{(m)}_{12},m_{i3}\rangle=\langle b^{(n)}_{12},n_{i3}\rangle=0,\\ \langle b^{(m)}_{12},n_{i3}\rangle=(-1)^jm_{j3},\ \langle b^{(n)}_{12},m_{i3}\rangle=(-1)^in_{j3},\ \langle b^{(m,n)}_{12},m_{i3}\rangle=m_{j3},\ \langle b^{(m,n)}_{12},n_{i3}\rangle=n_{j3}.\end{array}}$$

Quaternion Representation

The quaternionic structure for the isotope is still easy to describe in terms of the split basis. Under the isomorphism φ of §2 our newly-ordered basis for M becomes $m_{13},n_{13},m_{23},n_{23}=m_1,m_4,m_3,m_2\stackrel{\varphi}{\longrightarrow}e_{11},e_{12},e_{21},e_{22}$ and the action (2.1) takes the form

(5.8) Split Quaternion Action $A\times H$

Action of V on:	e_{11}	e_{12}	e_{21}	e_{22}
$V_{e_1}=L_{e_{11}}$	e_{11}	e_{12}	0	0
$V_{e_2}=L_{e_{22}}$	0	0	e_{21}	e_{22}
$V_{c_{12}}=L_{e_{21}-e_{12}}R_{e_{21}}$,	0	e_{21}	0	$-e_{11}$
$V_{d_{12}}=L_{e_{12}-e_{21}}R_{e_{12}}$	$-e_{22}$	0	e_{12}	0
$V_{q_{12}}=L_{e_{12}+e_{21}}$	e_{21}	e_{22}	e_{11}	e_{12}
$V_{e_3}=\mathbf{1}_M$	e_{11}	e_{12}	e_{21}	e_{22}

PROOF: In the isotope the actions are $V^{(s)}_a=V_{a,s}$, $V^{(s)}_b=V_{b,u}=V_bV_u$ where $V_u=V_{e_1+e_2}=L_{e_{12}+e_{21}}$, so recalling the actions (2.1) we see

$$V_{e_1}^{(s)} = V_{v_1}V_u = L_{e_{12}}L_{e_{12}+e_{21}} = L_{e_{11}}, \quad V_{e_2}^{(s)} = V_{v_2}V_u = L_{e_{21}}L_{e_{12}+e_{21}} = L_{e_{22}},$$
$$V_{c_{12}}^{(s)} = -V_{v_3}V_u = L_{e_{22}-e_{11}}R_{e_{21}}L_{e_{12}+e_{21}} = L_{e_{21}-e_{12}}R_{e_{21}},$$
$$V_{d_{12}}^{(s)} = V_{v_4}V_u = L_{e_{11}-e_{22}}R_{e_{12}}L_{e_{12}+e_{21}} = L_{e_{12}-e_{21}}R_{e_{12}},$$
$$V_{q_{12}}^{(s)} = V_eV_u = V_u = L_{e_{12}+e_{21}}, \quad V_{e_3}^{(s)} = V_{f,s} = V_{f,f} = V_f = \mathbf{1}_M.$$

Thus the regular representation of the $V_a^{(s)}$ as quaternion multiplications on H is precisely the action of Table (5.3). ■

6 Split Triple Products

We now translate our tables for triple products in the standard K_{10} into tables for the split sK_{10}. We will see that *all* triple products are determined by bilinear products and the Peirce decomposition. We noted in Remark 4.1 that the triple products with only one odd term are completely determined by the Peirce decomposition and the bilinear products, of the form $V_{a',a} = V_{a'}V_a$ or $U_b = U_{b',b} = U_f = 0$ and $U_{b,f} = V_b$ on M, which can all be read off from Table (5.3). Alternately, $U_a^{(s)} = U_aU_s = U_s*$, $V_{a',a}^{(s)} = V_{a',U_sa} = V_{a',a^*}$ can be read off directly from Bimodule Table (5.5). By Remark 4.2, triple products with two odd terms are outer $U_{m,n}a_j = E_i\big(\langle\langle m, a_j\rangle, n\rangle\big) = E_i\big(\langle m, \langle a_j, n\rangle\rangle\big)$ or left $V_{m,n}a_j = E_j\big(\langle m, \langle a_j, n\rangle\rangle\big)$.

(6.1) Split Two- or Three-Odd Multiplication $U_{M,M}, V_{M,M}$

$U_{m,n}p, V_{m,n}p$ for $p =$	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	m_{13}	n_{13}	m_{23}	n_{23}
$[U_{m,n}p^*,\ V_{m,n^*}p]$ for $p^* =]$	$[v_2]$	$[v_1]$	$[v_3]$	$[-v_4]$	$[e]$	$[f]$	$[m_3]$	$[m_2]$	$[m_1]$	$[m_4]$
$U_{m_{13},n_{23}} = -U_{n_{23},m_{13}}$ $[=U_{m_1,m_2}*]$	0	0	0	0	$-3e_3$	q_{12}	m_{23}	$-2n_{23}$	$-2m_{13}$	n_{13}
$U_{m_{13},m_{23}} = -U_{m_{23},m_{13}}$ $[=U_{m_1,m_3}*]$	0	0	0	$-3e_3$	0	$2c_{12}$	0	$-m_{23}$	0	m_{13}
$U_{m_{13},n_{13}} = -U_{n_{13},m_{13}}$ $[=U_{m_1,m_4}*]$	$-3e_3$	0	0	0	0	$2e_1$	$-m_{13}$	$-n_{13}$	0	0
$U_{n_{23},m_{23}} = -U_{m_{23},n_{23}}$ $[=U_{m_2,m_3}*]$	0	$3e_3$	0	0	0	$-2e_2$	0	0	m_{23}	n_{23}
$U_{n_{23},n_{13}} = -U_{n_{13},n_{23}}$ $[=U_{m_2,m_4}*]$	0	0	$3e_3$	0	0	$-2d_{12}$	$-n_{23}$	0	n_{13}	0
$U_{m_{23},n_{13}} = -U_{n_{13},m_{23}}$ $[=U_{m_3,m_4}*]$	0	0	0	0	$-3e_3$	q_{12}	$-2m_{23}$	n_{23}	m_{13}	$-2n_{13}$
$U_{m,m} = 0, \quad U_{n,m} = -U_{m,n}, \quad V_{m,n}a_i = E_jV_mV_na_j$										
$V_{m_{23},m_{23}} = V_{q_{12},c_{12}}$ $= -V_{m_{13},m_{13}}$ $[=V_{m_3,m_1}]$	0	0	0	q_{12}	$-2c_{12}$	0	0	m_{13}	0	$-m_{23}$
$V_{m_{13},n_{23}} = V_{q_{12},e_2}$ $= -V_{n_{13},m_{23}}$ $[=V_{m_1,m_4}]$	0	q_{12}	0	0	$2e_1$	0	0	0	m_{13}	n_{13}
$V_{m_{23},n_{13}} = V_{q_{12},e_1}$ $= -V_{n_{23},m_{13}}$ $[=V_{m_3,m_2}]$	q_{12}	0	0	0	$2e_2$	0	m_{23}	n_{23}	0	0
$V_{n_{13},n_{13}} = -V_{q_{12},d_{12}}$ $= -V_{n_{23},n_{23}}$ $[=V_{m_4,m_2}]$	0	0	$-q_{12}$	0	$2d_{12}$	0	n_{13}	0	$-n_{23}$	0
$V_{m_{13},m_{23}} = 2V_{e_1,c_{12}}$ $[=V_{m_1,m_1}]$	0	$2c_{12}$	0	$2e_1$	0	0	0	0	0	$-2m_{13}$
$V_{n_{23},n_{13}} = -2V_{e_2,d_{12}}$ $[=V_{m_2,m_2}]$	$-2d_{12}$	0	$-2e_2$	0	0	0	$2n_{23}$	0	0	0
$V_{m_{23},m_{13}} = -2V_{c_{12},e_1}$ $[=V_{m_3,m_3}]$	$-2c_{12}$	0	0	$-2e_2$	0	0	0	$-2m_{23}$	0	0
$V_{n_{13},n_{23}} = 2V_{d_{12},e_2}$ $[=V_{m_4,m_4}]$	0	$2d_{12}$	$2e_1$	0	0	0	0	0	$2n_{13}$	0
$V_{m_{13},n_{13}}$ $[=V_{m_1,m_2}]$	$2e_1$	0	$2c_{12}$	0	q_{12}	$-3e_3$	0	$-n_{13}$	$-m_{23}$	$-2n_{23}$
$V_{n_{23},m_{23}}$ $[=V_{m_2,m_1}]$	0	$-2e_2$	0	$-2d_{12}$	$-q_{12}$	$3e_3$	$2m_{13}$	n_{13}	m_{23}	0
$V_{m_{23},n_{23}}$ $[=V_{m_3,m_4}]$	0	$2e_2$	$2c_{12}$	0	q_{12}	$-3e_3$	$-m_{13}$	$-2n_{13}$	0	$-n_{23}$
$V_{n_{13},m_{13}}$ $[=V_{m_4,m_3}]$	$-2e_1$	0	0	$-2d_{12}$	$-q_{12}$	$3e_3$	m_{13}	0	$2m_{23}$	n_{23}

Proof: This can be computed by brute force directly from Tables (5.4), (5.6).[8] More elegantly, since $U^{(s)}_{m,n}(a) = U_{m,n}(a^*)$, $\langle m,p,n\rangle^{(s)} = \langle m, U_s p, n\rangle = \langle m, p^*, n\rangle$, and the action of $U^{(s)}_{m,n}$ on m_1, m_4, m_3, m_2 is just that of $U_{m,n}$ on $m_1^*, m_4^*, m_3^*, m_2^* = m_3, m_2, m_1, m_4$, the U-table follows immediately from Table (4.5) by switching columns v_1, v_2 (from $v_1^* = v_2$) and columns m_1, m_3 and negating column v_4 (from $v_j^* = -v_j$ for $j = 3, 4$), and remembering that $c_{12}^* = (-v_3)^* = v_3$.

We can similarly read off the V-operators directly from Table (4.5) via $V^{(s)}_{m,n} = V_{m,n^*}$ [switching columns m_1, m_3, recalling that $m_1^* = m_3, m_2^* = m_4$] so that $V^{(s)}_{m_{13},m_{13}} = V_{m_1,m_3} = -V_{m_3,m_1} = -V^{(s)}_{m_{23},m_{23}}$; $V^{(s)}_{m_{13},n_{23}} = V_{m_1,m_4} = -V_{m_4,m_1} = -V^{(s)}_{n_{13},m_{23}}$; $V^{(s)}_{n_{23},m_{13}} = V_{m_2,m_3} = -V_{m_3,m_2} = -V^{(s)}_{m_{23},n_{13}}$; $V^{(s)}_{n_{23},n_{23}} = V_{m_2,m_4} = -V_{m_4,m_2} = -V^{(s)}_{n_{13},n_{13}}$; $V^{(s)}_{m_{13},m_{23}} = V_{m_1,m_1}$; $V^{(s)}_{n_{23},n_{13}} = V_{m_2,m_2}$; $V^{(s)}_{m_{23},m_{13}} = V_{m_3,m_3}$; $V^{(s)}_{n_{13},n_{23}} = V_{m_4,m_4}$; $V^{(s)}_{m_{13},n_{13}} = V_{m_1,m_2}$; $V^{(s)}_{n_{23},m_{23}} = V_{m_2,m_1}$; $V^{(s)}_{m_{23},n_{23}} = V_{m_3,m_4}$; $V^{(s)}_{n_{13},m_{13}} = V_{m_4,m_3}$. ■

Remark 6.2 *Because the Peirce spaces M_{i3} are only 2-dimensional, all the odd triple products in the split Kac superalgebra are completely determined by Peirce orthogonality relations and Reductions from bilinear products.*

Indeed, every triple $\langle x_{i3}, y_{k3}, z_{\ell 3}\rangle$ for $i, k, \ell = 1, 2$ will have a repeated index, hence by alternation (0.1.2) is of the form $\langle x_{i3}, y_{j3}, z_{i3}\rangle$ or $\langle x_{i3}, y_{i3}, z_{i3}\rangle$ or $\pm\langle x_{i3}, y_{i3}, z_{j3}\rangle$ for $i = 1, 2$, $j = 3 - i$. But $\langle x_{i3}, y_{j3}, z_{i3}\rangle = 0$ by Peirce Orthogonality (0.2.1), $\langle x_{i3}, z_{i3}, y_{j3}\rangle = \langle x_{i3}, \langle z_{i3}, y_{j3}\rangle\rangle$, while $\langle x_{i3}, y_{i3}, z_{i3}\rangle$ for basis vectors from M_{i3} *must have a repetition since* $\dim(M_{i3}) = 2$, where from Reduction (0.2.4) we have $\langle m, n, m\rangle = 0$, $\langle m, m, n\rangle = \langle m, \langle m, n\rangle\rangle = -\langle n, m, m\rangle$. This leads to the following reduction formulas for all odd triple products:

$$
(6.3)\qquad
\begin{array}{llll}
\langle m_{i3}, m_{i3}, n_{i3}\rangle & = \langle m_{i3}, \langle m_{i3}, n_{i3}\rangle\rangle & \langle n_{i3}, n_{i3}, m_{i3}\rangle & = \langle n_{i3}, \langle n_{i3}, m_{i3}\rangle\rangle \\
\langle m_{i3}, m_{i3}, m_{j3}\rangle & = \langle m_{i3}, \langle m_{i3}, m_{j3}\rangle\rangle & \langle n_{i3}, n_{i3}, n_{j3}\rangle & = \langle n_{i3}, \langle n_{i3}, n_{j3}\rangle\rangle \\
\langle m_{i3}, m_{i3}, n_{j3}\rangle & = \langle m_{i3}, \langle m_{i3}, n_{j3}\rangle\rangle & \langle n_{i3}, n_{i3}, m_{j3}\rangle & = \langle n_{i3}, \langle n_{i3}, m_{j3}\rangle\rangle \\
\langle m_{i3}, n_{i3}, n_{j3}\rangle & = \langle m_{i3}, \langle n_{i3}, n_{j3}\rangle\rangle & \langle n_{i3}, m_{i3}, m_{j3}\rangle & = \langle n_{i3}, \langle m_{i3}, m_{j3}\rangle\rangle \\
\langle n_{i3}, m_{i3}, n_{j3}\rangle & = \langle n_{i3}, \langle m_{i3}, n_{j3}\rangle\rangle & \langle m_{i3}, n_{i3}, m_{j3}\rangle & = \langle m_{i3}, \langle n_{i3}, m_{j3}\rangle\rangle
\end{array}
\qquad ■
$$

We can translate Table (4.11) into a table of odd left multiplications for the split algebra, where the identity e for B becomes $u = e_1 + e_2$. (4.9) shows again that $V_{A,M} + V_{M,A} = V_{M,\Phi u + \Phi f}$, where $V_{M_{i3},u} = V_{M_{i3},e_i}$. Since in the isotope $v_1, v_2, v_3, v_4, e, f, m_1, m_2, m_3, m_4 \longrightarrow e_1, e_2, c_{12}, d_{12}, q_{12}, e_3, m_{13}, n_{23}, m_{23}, n_{13}$ and $V^{(s)}_{m_i,b} = V_{m_i,b^*}$ with $e_1, e_2, c_{12}, d_{12}, q_{12}, e_3, m_{13}, n_{13}, m_{23}, n_{23} \stackrel{*}{\longrightarrow} e_2, e_1, -c_{12}, -d_{12}, q_{12}, e_3, m_{23}, n_{23}, m_{13}, n_{13}$, the 8 odd left multiplications reduce to:

$$
(6.4)\qquad
\boxed{
\begin{array}{ll}
V_{m_{13},q_{12}} = V_{m_{23},e_2} = V_{n_{13},c_{12}} = V_{e_3,m_{23}}, & V_{q_{12},m_{23}} = V_{e_1,m_{13}} = -V_{c_{12},n_{23}} = V_{m_{13},e_3}, \\
V_{n_{23},q_{12}} = V_{m_{23},d_{12}} = V_{n_{13},e_1} = V_{e_3,n_{13}}, & V_{q_{12},n_{13}} = -V_{d_{12},m_{13}} = V_{e_2,n_{23}} = V_{n_{23},e_3}, \\
V_{m_{23},q_{12}} = V_{m_{13},e_1} = -V_{n_{23},c_{12}} = V_{e_3,m_{13}}, & V_{q_{12},m_{13}} = V_{e_2,m_{23}} = V_{c_{12},n_{13}} = V_{m_{23},e_3}, \\
V_{n_{13},q_{12}} = -V_{m_{13},d_{12}} = V_{n_{23},e_2} = V_{e_3,n_{23}}, & V_{q_{12},n_{23}} = V_{d_{12},m_{23}} = V_{e_1,n_{13}} = V_{n_{13},e_3},
\end{array}
}
$$

leading to the following brief table of values of these odd left multiplications.

[8] The computation uses the formulas $\langle x_{i3}, e_i, y_{i3}\rangle = -3\sigma(x_{i3}, y_{i3})e_3$, $\langle x_{i3}, e_3, y_{i3}\rangle = 2\sigma(x_{i3}, y_{i3})e_i$, $\langle x_{i3}, e_j, y_{i3}\rangle = \langle x_{i3}, A_{12}, y_{i3}\rangle = 0$, $\langle x_{i3}, e_i, y_{j3}\rangle = \langle x_{i3}, e_j, y_{j3}\rangle = 0$, $\langle x_{i3}, e_3, y_{j3}\rangle = \langle x_{i3}, y_{j3}\rangle$, $\langle x_{i3}, a_{12}, y_{j3}\rangle = -3\sigma(\langle x_{i3}, a_{12}\rangle, y_{j3})e_3 = -3\sigma(x_{i3}, \langle a_{12}, y_{j3}\rangle)e_3$ resulting from Peirce Orthogonality and Triple Reduction.

(6.5) Odd Left Multiplications $\langle M, A, J\rangle = \langle A, M, J\rangle$

$V_{m,a}(x)$ for $x =$	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	m_{13}	n_{13}	m_{23}	n_{23}
V_{m_{13},e_1}	m_{13}	0	0	$-n_{23}$	m_{23}	0	0	$-3e_3$	0	0
V_{n_{13},e_1}	n_{13}	0	m_{23}	0	n_{23}	0	$3e_3$	0	0	0
V_{m_{23},e_2}	0	m_{23}	0	n_{13}	m_{13}	0	0	0	0	$-3e_3$
V_{n_{23},e_2}	0	n_{23}	$-m_{13}$	0	n_{13}	0	0	0	$3e_3$	0
V_{m_{13},e_3}	0	0	0	0	0	m_{13}	0	$2e_1$	$2c_{12}$	q_{12}
V_{n_{13},e_3}	0	0	0	0	0	n_{13}	$-2e_1$	0	$-q_{12}$	$2d_{12}$
V_{m_{23},e_3}	0	0	0	0	0	m_{23}	$-2c_{12}$	q_{12}	0	$2e_2$
V_{n_{23},e_3}	0	0	0	0	0	n_{23}	$-q_{12}$	$-2d_{12}$	$-2e_2$	0

PROOF: This is just table (4.11)with the m_2, m_4 columns and m_1, m_3 rows and m_2, m_4 rows switched, and the v_3 column negated. Alternately, the values for $V_{m,e}a = V_m a$ and $V_{m,a}n = U_{m,n}a$ can be read off (vertically) from (1.3), [note $V_{M,f}B = 0, V_{m,f}f = m$] and from the f-column of (4.5) [or from (1.9)]. ■

7 Inner Super-Derivations

We will compile a table of *inner super-derivations* $D = D_0 + D_1 = \alpha\mathbf{1}_{sK} + \sum V_{x_i,y_i}$ with $D(1) = 0$. An analysis of *all* derivations of the Kac and other simple superalgebras has been carried out by Michael Smith [8] in general, and by G. Benkart and A. Elduque [1] for the Kac algebra in characteristic $\neq 2$. Recall that by Grassmann detour $D = D_0 + D_1$ is a super-derivation of a unital quadratic Jordan superalgebra J iff $\widetilde{D} := \gamma_0 \otimes D_0 + \gamma_1 \otimes D_1$ is a derivation of the Grassmann envelope for all $\gamma_i \in \Gamma_i$. Intrinsically, the conditions amount to the following conditions for the homogeneous components D_i on homogeneous elements $x, y, z \in \Gamma(J)$:

$$
\begin{aligned}
&D_i\langle x,y,z\rangle = \langle D_i(x),y,z\rangle + (-1)^{ix}\langle x,D_i(y),z\rangle + (-1)^{ix+iy}\langle x,y,D_i(z)\rangle,\\
&D_i(U_a x) = \langle D_i(a),x,a\rangle + U_a D_i(x), \quad \textit{which imply}\\
(7.1)\quad &\langle D_i, V_{x,y}\rangle = V_{D_i(x),y} + (-1)^{ix}V_{x,D_i(y)}, \quad D_i\langle x,z\rangle = \langle D_i(x),z\rangle + (-1)^{ix}\langle x,D_i(z)\rangle,\\
&D_i(\hat{1}) = 0, \quad D_i(a^2) = \langle D_i(a),a\rangle.
\end{aligned}
$$

By a Grassmann detour,[9] the left multiplications $V_i = V_{s,t}$ $(i = deg(s) + deg(t))$ belong to the structure Lie superalgebra, satisfying

$$
\begin{aligned}
&V_i\langle x,y,z\rangle = \langle V_i(x),y,z\rangle - (-1)^{ix}\langle x,V_i^*(y),z\rangle + (-1)^{ix+iy}\langle x,y,V_i(z)\rangle,\\
(7.2)\quad &V_i(U_a x) = \langle V_i(a),x,a\rangle - (-1)^i U_a V_i^*(x), \quad V_i(a^2) = \langle V_i(a),a\rangle - U_a v_i,\\
&V_i(\langle x,z\rangle) = \langle V_i(x),z\rangle - (-1)^{ix}\langle x,v_i,z\rangle + (-1)^{ix}\langle x,V_i(z)\rangle,\\
&V_i(1) = (-1)^{st}V_i^*(1) = \langle s,t\rangle =: v_i \qquad (V_i^* = V_{t,s}).
\end{aligned}
$$

The map $V_{s,t}$ is itself a superderivation iff $v_i = \langle s,t\rangle = 0$, since a general inner structural map $W_i = \sum V_{s_k,t_k}$ is a superderivation iff $W_i(\hat{1}) = \sum\langle s_k,t_k\rangle = 0$, in which case $W_i^* = -(-1)^i W$. Automatically all $D_{x_i,y_j} := V_{x_i,y_j} - (-1)^{ij}V_{y_j,x_i}$ and all $D_m := V_{m,m}$ for odd m are super-derivations [since $\langle x_i,y_j\rangle = (-1)^{ij}\langle y_j,x_i\rangle$ by supersymmetry (0.1.1) and $\langle m,m\rangle = 0$ by odd alternation (0.1.2)], as are all *Smith derivations* of the form V_x for $2x = 0$ (see [8] for more on this phenomenon), in particular ηV_x if $\eta \in \Phi_{2^\perp} = \{\eta | 2\eta = 0\}$, so in Inder$(J)$ we have the *standard inner super-derivations*

[9] Inner maps which kill 1 are derivations of the superalgebra because their extensions to the Grassmann envelope remain inner maps which kill 1, hence are derivations of the quadratic Jordan algebra. The superskew condition $W_i^* = -(-1)^i W_i$ alone is not enough: together with $W_i + (-1)^i W_i^* = V_{W_i(1)}$ it implies $V_{W(1)} = 0, 2W(1) = 0$, but in general does not imply $W(1) = 0$.

$$(7.3)\qquad \begin{array}{c} D_m := V_{m,m},\ \ D_{m,n} := V_{m,n}+V_{n,m},\ \ D_{a,b} := V_{a,b}-V_{b,a},\ \ S(a') := V_{a'}\ \text{ in Inder}(J)_0, \\ D_{m,a} := V_{m,a} - V_{a,m},\quad S_{m'} := V_{m'}\ \text{ in Inder}(J)_1\quad (\textit{for}\quad 2a' = 2m' = 0). \end{array}$$

In general, the odd $V_{m,n}$ do not contribute many new inner derivations, since they are skew-symmetric by Switching (0.1.1): we have general rules

$$(7.4)\qquad \begin{array}{c} D_{a,a} = 0,\ D_{x,y} = -(-1)^{xy} D_{y,x},\ D_{x,1} = 0,\ V_{m,n} - V_{a,b} \in \text{Inder}(J)\ \ \textit{if}\ \ \langle m,n\rangle = \{a,b\}, \\ \textit{If } a,b \in A \textit{ have } \{a,b\} = 0 \textit{ then } V_{a,b} = -V_{b,a} \in \text{Inder}(J)_0 \textit{ with}\quad D_{a,b} = 2V_{a,b}, \\ V_{m,n} - V_{n,m} = V_{\langle n,m\rangle} \in V_A,\quad 2V_{m,n} = D_{m,n} + V_{\langle m,n\rangle} \in D_{M,M} + V_A, \\ V_{a,b} = V_{b,a} = 0\ \ \textit{if}\ \ a \in A_2(e_i), b \in A_0(e_i)\quad (\ \textit{so}\ \ D_{A,e_3} = D_{e_1,e_2} = 0). \end{array}$$

using Peirce Orthogonality (0.2.1) and noting that $\{a,b\} = 0$ implies $V_{a,b}(1) = \{a,b\} = 0$ and $V_{b,a} = V_{\{a,b\}} - V_{a,b}$ [by Switching (0.1.1)] $= -V_{a,b}$, so $D_{a,b} = 2V_{a,b}$.

In the particular case of the split Kac superalgebra $J = sK_{10}(\Phi)$ the odd standard and even Smith inner super-derivations reduce to

$$(7.5)\qquad D_{m,e_3} = -D_{m,u},\ D_{m,b} = D_{\langle m,b\rangle,u} = \Delta_{\langle m,b\rangle}\ \textit{for}\ \Delta_m := D_{m,u} = V_{m,u_*}\ (u_* := u - e_3),$$

since by (4.8) $D_{m,b} = V_{m,b} - V_{b,m} = V_{\{m,b\},u} - V_{u,\langle b,m\rangle} = D_{\{m,b\},u}$ and $D_{m,e_3} = D_{u,m} = -D_{m,u}$ with $\Delta_m := D_{m,u} = V_{m,u} - V_{u,m} = V_{m,u} - V_{m,e_3} = V_{m,u_*}$ [for reassurance, note $\langle m,u_*,1\rangle = \langle m,u\rangle - \langle m,e_3\rangle = m - m = 0$ so this is indeed a superderivation].

With this notation out of the way, we can describe all the inner super-derivations.

Inner Super-Derivation Theorem 7.6 *The space $\mathcal{I}nder(sK_{10}) = \mathcal{I}_0 \oplus \mathcal{I}_1$ of inner derivations of sK_{10} is $\mathcal{I} \cong \big(osp_{1,2}(\Phi) \otimes \Phi[\mu]\big) \oplus D_0(\Phi_{2\perp})$,*

$$\mathcal{I}_0 = \mathcal{D} \oplus \mathcal{D}' \oplus D_0(\Phi_{2\perp}) \cong sl_2(\Phi) \oplus sl_2(\Phi)\mu \oplus D_0(\Phi_{2\perp}) = \big(sl_2(\Phi) \otimes \Phi[\mu]\big) \oplus D_0(\Phi_{2\perp})$$

$$\mathcal{I}_1 := \mathcal{E} \oplus \mathcal{E}' \cong M_{13} \oplus M_{23} \cong V(\Phi) \oplus V(\Phi) = V(\Phi) \otimes \Phi[\mu]\quad (V(\Phi) := \Phi^2)$$

where μ is a scalar in $\Phi[\mu]$ with $\mu^2 = -1$. Here the even inner derivations are built from[10]

$$(7.6.1)\qquad \begin{array}{l} \mathcal{D} := \bigoplus_{i=1,2,3} \Phi D_i \ \textit{for}\ D_1 := V_{c_{12},q_{12}},\quad D_2 := V_{q_{12},d_{12}},\quad D_3 := D_{c_{12},d_{12}}, \\ \textit{where we have alternate descriptions} \\ \qquad D_3 = 3\big(V_{c_{12},d_{12}} - \mathbf{1}_M\big) + V_{e_1} - V_{m_{13},n_{13}} = 3\big(V_{c_{12},d_{12}} - \mathbf{1}_M\big) + V_{e_2} - V_{m_{23},n_{23}}; \\ \mathcal{D}' := \bigoplus_{i=1,2,3} \Phi D'_i\ \textit{for}\ D'_1 := D_{e_1,c_{12}} = V_{e_1-e_2,c_{12}},\quad D'_2 := D_{e_1,d_{12}} = V_{e_1-e_2,d_{12}}, \\ \qquad D'_3 := -D_{e_1,q_{12}} = V_{q_{12},e_1-e_2}; \\ D_0(\Phi_{2\perp})\ \textit{consists of all}\ D_0(\eta) := S_{\eta e_2} = \eta V_{e_2}\ (\eta \in \Phi_{2\perp}) \\ \qquad (\textit{so}\ D_0 = 0\ \textit{if}\ \Phi\ \textit{has no 2-torsion}). \end{array}$$

The odd superderivations are built from

$$(7.6.2)\qquad \begin{array}{l} \mathcal{E} = \bigoplus_{i=1,2} \Phi\Delta_i\quad \textit{for}\quad \Delta_1 = \Delta_{m_{13}} = V_{m_{13},u_*},\quad \Delta_2 = \Delta_{n_{13}} = V_{n_{13},u_*}, \\ \mathcal{E}' = \bigoplus_{i=1,2} \Phi\Delta'_i\quad \textit{for}\quad \Delta'_1 = \Delta_{m_{23}} = V_{m_{23},u_*},\quad \Delta'_2 = \Delta_{n_{23}} = V_{n_{23},u_*}. \end{array}$$

The action of the inner derivations on the split basis is given by the table

[10] Note the *symmetry* between D'_1 and D'_2, but the *asymmetry* between $D_1 = V_{c_{12},q_{12}}$ and $D_2 = V_{q_{12},d_{12}} = -V_{d_{12},q_{12}}$.

(7.6.3) Even Inner Derivations $\mathcal{I}_0$ $(k := e_1 - e_2)$

D	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	m_{13}	n_{13}	m_{23}	n_{23}
$D_1{=}V_{c_{12},q_{12}}$	0	0	0	$-q_{12}$	$2c_{12}$	0	0	$-m_{13}$	0	m_{23}
$D_2{=}V_{q_{12},d_{12}}$	0	0	q_{12}	0	$-2d_{12}$	0	$-n_{13}$	0	n_{23}	0
$D_3{=}D_{c_{12},d_{12}}$	0	0	$2c_{12}$	$-2d_{12}$	0	0	m_{13}	$-n_{13}$	m_{23}	$-n_{23}$
$D'_1{=}V_{k,c_{12}}$	$-c_{12}$	c_{12}	0	k	0	0	0	$-m_{23}$	0	$-m_{13}$
$D'_2{=}V_{k,d_{12}}$	$-d_{12}$	d_{12}	k	0	0	0	n_{23}	0	n_{13}	0
$D'_3{=}V_{q_{12},k}$	q_{12}	$-q_{12}$	0	0	$-2k$	0	m_{23}	n_{23}	$-m_{13}$	$-n_{13}$
$D_0(\eta){=}\eta V_{e_2}$ $(2\eta = 0)$	0	0	ηc_{12}	ηd_{12}	ηq_{12}	0	0	0	ηm_{23}	ηn_{23}

Odd Inner Super-Derivations $\mathcal{I}_1$ $(h_i := 2e_i + 3e_3)$

D	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	m_{13}	n_{13}	m_{23}	n_{23}
$\Delta_1{=}\Delta_{m_{13}}$	m_{13}	0	0	$-n_{23}$	m_{23}	$-m_{13}$	0	$-h_1$	$-2c_{12}$	$-q_{12}$
$\Delta_2{=}\Delta_{n_{13}}$	n_{13}	0	m_{23}	0	n_{23}	$-n_{13}$	h_1	0	q_{12}	$-2d_{12}$
$\Delta'_1{=}\Delta_{m_{23}}$	0	m_{23}	0	n_{13}	m_{13}	$-m_{23}$	$2c_{12}$	$-q_{12}$	0	$-h_2$
$\Delta'_2{=}\Delta_{n_{23}}$	0	n_{23}	$-m_{13}$	0	n_{13}	$-n_{23}$	c_{12}	$2d_{12}$	h_2	0

To make the multiplication table of Lie super-brackets $[D,E]^s = DE - (-1)^{DE}ED$ *for the Lie superalgebra of inner derivations appear more familiar, we introduce* $E_i := \Delta_i$, $E'_i := \Delta'_i$ *EXCEPT* $E_2 = -\Delta_2$ (!!), *and obtain*

(7.6.4) Lie Superalgebra of Inner Derivations

D	D_1	D_2	D_3	D'_1	D'_2	D'_3	$D_0(\eta)$	E_1	E_2	E'_1	E'_2
D_1	0	D_3	$-2D_1$	0	D'_3	$-2D'_1$	0	0	E_1	0	E'_1
D_2	$-D_3$	0	$2D_2$	$-D'_3$	0	$2D'_2$	0	E_2	0	E'_2	0
D_3	$2D_1$	$-2D_2$	0	$2D'_1$	$-2D'_2$	0	0	E_1	$-E_2$	E'_1	$-E'_2$
D'_1	0	D'_3	$-2D'_1$	0	$-D_3$	$2D_1$	$\eta D'_1$	0	E'_1	0	$-E_1$
D'_2	$-D'_3$	0	$2D'_2$	D_3	0	$-2D_2$	$\eta D'_2$	E'_2	0	$-E_2$	0
D'_3	$2D'_1$	$-2D'_2$	0	$-2D_1$	$2D_2$	0	$\eta D'_3$	E'_1	$-E'_2$	$-E_1$	E_2
$D_0(\eta)$	0	0	0	$\eta D'_1$	$\eta D'_2$	$\eta D'_3$	0	0	0	$\eta E'_1$	$\eta E'_2$
E_1	0	$-E_2$	$-E_1$	0	$-E'_2$	$-E'_1$	0	$-2D_1$	D_3	$-2D'_1$	D'_3
E_2	$-E_1$	0	E_2	$-E'_1$	0	E'_2	0	D_3	$2D_2$	D'_3	$2D'_2$
E'_1	0	$-E'_2$	$-E'_1$	0	E_2	E_1	$\eta E'_1$	$-2D'_1$	D'_3	$2D_1$	$-D_3$
E'_2	$-E'_1$	0	E'_2	E_1	0	$-E_2$	$\eta E'_2$	D'_3	$2D'_2$	$-D_3$	$-2D_2$

The standard inner derivations reduce to 10 *in* D_M, 6 *in* $D_{A,A}$, *and* 4 *in* $D_{M,A}$ *which can be written in terms of the* D_i, Δ_i *by*

$$
\text{(7.6.5)}\quad
\begin{aligned}
&D_{m_{13}} = -D_{m_{23}} = D_1, \qquad D_{n_{23}} = -D_{n_{13}} = D_2,\\
&D_{m_{13},m_{23}} = 2D_{c_{12},e_2} = 2D'_1, \qquad D_{n_{13},n_{23}} = 2D_{d_{12},e_2} = 2D'_2,\\
&D_{m_{23},n_{13}} = -D_{m_{13},n_{23}} = V_{q_{12},e_1-e_2} = D'_3, \quad D_{m_{13},n_{13}} = D_{m_{23},n_{23}} = D_{c_{12},d_{12}} = D_3,\\
&D_{e_1,b_{12}} = -D_{e_2,b_{12}} = V_{e_1-e_2,b_{12}} \in \mathcal{D} \quad (b_{12} = c_{12}, d_{12}, q_{12}),\\
&D_{e_1,e_2} = D_{e_3,A} = 0, \quad D_{c_{12},q_{12}} = 2D_1, \quad D_{q_{12},d_{12}} = 2D_2, \quad D_{c_{12},d_{12}} = D_3,\\
&\Delta_{m_{13}} = \Delta_1, \quad \Delta_{n_{13}} = \Delta_2, \quad \Delta_{m_{23}} = \Delta'_1, \quad \Delta_{n_{23}} = \Delta'_2,\\
&S_{x'} \in \mathrm{Span}(D_i, \Delta_i) : S_{\eta e_2} = D_0(\eta),\ S_{\eta e_3} = S_{\eta u} = \eta D_3,\ S_{\eta e_1} = S_{\eta u} - S_{\eta e_2},\\
&S_{\eta c_{12}} = \eta D'_1,\ S_{\eta d_{12}} = \eta D'_2,\ S_{\eta q_{12}} = \eta D'_3, \quad S_{m'} = \Delta_{m'}.
\end{aligned}
$$

Proof: The table (7.6.3) can be read directly from (5.5) for D_i, D'_i and from (6.5) for Δ_i, Δ'_i [noting $V_{p_{i3},u_*} = V_{p_{i3},e_i-e_3}$]. That the D'_i have an alternate description in (7.6.1) comes from $D_{e_1,b_{12}} = V_{e_1,b_{12}} - V_{b_{12},e_1} = V_{e_1,b_{12}} - V_{e_2,b_{12}} = V_{e_1-e_2,b_{12}}$ for $b_{12} = c_{12}, d_{12}, -q_{12}$. It is more work to show that the three expressions for D_3 in (7.6.1) agree: from (6.1), (5.5) $V_{e_1} - V_{m_{13},n_{13}}$ sends the ordered basis for sK to $\big(2e_1 - 2e_1,\ 0-0,\ c_{12} - 2c_{12},\ d_{12} - 0,\ q_{12} - q_{12},\ 0 + 3e_3,\ m_{13} - 0,\ n_{13} + n_{13},\ 0 + m_{23},\ 0 + 2n_{23}\big) = \big(0, 0, -c_{12}, d_{12}, 0, 3e_3, m_{13}, 2n_{13}, m_{23}, 2n_{23}\big)$, while $V_{e_2} - V_{m_{23},n_{23}}$ also sends the ordered basis to $\big(0-0,\ 2e_2-2e_2,\ c_{12}-2c_{12},\ d_{12}-0,\ q_{12}-q_{12},\ 0+3e_3,\ 0+m_{13},\ 0+2n_{13},\ m_{23}-0,\ n_{23}+n_{23}\big) = \big(0,0,-c_{12},d_{12},0,3e_3,m_{13},2n_{13},m_{23},2n_{23}\big)$, while $3V_{c_{12},d_{12}} - 3\mathbf{1}$ sends the basis to $3\big((1-1)e_1,\ (1-1)e_2,\ (2-1)c_{12},\ (0-1)d_{12},\ (1-1)q_{12},\ (0-1)e_3,\ (1-1)m_{13},\ (0-1)n_{13},\ (1-1)m_{23},\ (3-1)n_{23}\big) = \big(0,0,3c_{12},-3d_{12},0,-3e_3,0,-3n_{13},0,-3n_{23}\big)$. Adding these shows that the second and third versions of D_3 send $\big(e_1,e_2,c_{12},d_{12},q_{12},e_3,m_{13},n_{13},m_{23},n_{23}\big)$ to

$$\big(0,0,2c_{12},-2d_{12},0,0,m_{13},-n_{13},m_{23},-n_{23}\big), \tag{7.6.6}$$

which is precisely where the first version $D_3 = D_{c_{12},d_{12}} = V_{c_{12},d_{12}} - V_{d_{12},c_{12}}$ sends it by subtracting two rows in (5.5).

From this table it is easy to see that the transformations $D_i, D_i', D_0(\Phi), \Delta_i, \Delta_i'$ are independent over Φ [8]: if $D = D_0(\eta) + \sum_{i=1}^{3}\alpha_i D_i + \sum_{j=1}^{3}\alpha_i' D_i' = 0$ then identifying coefficients of c_{12}, d_{12}, q_{12} in $D(e_1) = 0$ gives $\alpha_1' = \alpha_2' = \alpha_3' = 0$, identifying coefficients of q_{12} in $D(c_{12}) = D(d_{12}) = 0$ gives $\alpha_1 = \alpha_2 = 0$, the coefficient of n_{13} in $D(n_{13}) = 0$ gives $\alpha_3 = 0$, and finally the coefficient of c_{12} in $D(c_{12}) = 0$ gives $\eta = 0$, so that D vanishes iff all its coefficients vanish. Similarly, since $\Delta_m(u) = m$ if $\Delta = \sum_{i=1}^{2}\alpha_i\Delta_i + \sum_{j=1}^{2}\Delta_i' = 0$ then $0 = \Delta(u) = \alpha_1 m_{13} + \alpha_2 n_{13} + \alpha_1' m_{23} + \alpha_2' n_{23}$ implies all $\alpha_i, \alpha_i' = 0$.

First we check that the D_i, Δ_i are actually superderivations. This is clear for the standard D_3, D_i' by (7.3), for the Δ_i by (7.5), for $D_0(\eta) = S_{\eta e_2}$ by (7.3), and for D_1, D_2 by (7.4) since $\{c_{12}, q_{12}\} = \{d_{12}, q_{12}\} = 0$.

Next we check that the space $\mathcal{L}$ spanned by these 7 even and 4 odd superderivations span all inner derivations by giving explicit expressions (7.6.5) for the standard inners. From (7.3) we see there are 10 basic even standard inner derivations D_m, $D_{m,n}$. The 4 basic even $D_{m_{i*}} = V_{m_{i*},m_{i*}}$ reduce by (6.1) to $D_{m_{13}} = -D_{m_{23}} = V_{c_{12},q_{12}} = -V_{q_{12},c_{12}} = D_1$, $D_{c_{12},q_{12}} = V_{c_{12},q_{12}} - V_{q_{12},c_{12}} = 2V_{c_{12},q_{12}} = 2D_1$, analogously $D_{n_{23}} = -D_{n_{13}} = V_{q_{12},d_{12}} = -V_{d_{12},q_{12}} = D_2$, $D_{q_{12},d_{12}} = V_{q_{12},d_{12}} - V_{d_{12},q_{12}} = 2V_{q_{12},d_{12}} = 2D_2$. For the other 6 basic $D_{m_{i*},m_{j*}}$ we have by (6.1) that $D_{m_{13},m_{23}} = V_{m_{13},m_{23}} + V_{m_{23},m_{13}} = 2V_{e_1,c_{12}} - 2V_{c_{12},e_1} = 2D_{e_1,c_{12}} = 2D_1'$, $D_{n_{13},n_{23}} = V_{n_{13},n_{23}} + V_{n_{23},n_{13}} = 2V_{d_{12},e_2} - 2V_{e_2,d_{12}} = 2D_{d_{12},e_2} = 2D_2'$, and $D_{m_{13},n_{23}} = V_{m_{13},n_{23}} + V_{n_{23},m_{13}} = V_{q_{12},e_2} - V_{q_{12},e_1} = -V_{q_{12},e_1-e_2} = -D_3'$ which also equals $-V_{n_{13},m_{23}} - V_{m_{23},n_{13}} = -D_{m_{23},n_{13}}$. From adding two rows in (6.1) and subtracting two rows in (5.5) we see that $D_{m_{i3},n_{i3}} = V_{m_{i3},n_{i3}} + V_{n_{i3},n_{i3}}$ for $i = 1,2$ both mysteriously coincide with D_3 as in (7.6.5) since they all send the ordered basis for sK_{10} to $\big(0,0,2c_{12},-2d_{12},0,0,m_{13},-n_{13},m_{23},-n_{23}\big)$.

In view of (7.4) there are 6 basic even D_{a_i,a_j} with $a_i > a_j$ [in the order $e_1 > c_{12} > d_{12} > q_{12}$, since $D_{A,e_3} = D_{e_1,e_2} = 0$ by (7.5), while $D_{e_2,b_{12}} = -D_{e_1,b_{12}}$], and for these we have $D_{e_1,b_{12}} \in \mathcal{L}$ since $V_{e_1,b_{12}} - V_{b_{12},e_1} = V_{e_1,b_{12}} - V_{e_2,b_{12}} = V_{e_1-e_2,b_{12}}$, and by (7.4) we have $D_{c_{12},q_{12}} = 2D_1$, $D_{d_{12},q_{12}} = -2D_2$, $D_{c_{12},d_{12}} = D_3$.

To see (7.6.5) for the even Smith derivations, $a' = \sum\alpha_i e_i + \beta_1 c_{12} + \beta_2 d_{12} + \beta_3 q_{12}$ has $2a' = 0$ iff all $\eta = \alpha_i, \beta_i \in \Phi_{2^\perp}$ [by freeness of A as Φ-module]; here[11]

$$S_{\eta e_3} = S_{\eta u} = \eta D_3 = \mathbf{0}_A \oplus \eta\mathbf{1}_M, \quad S_{\eta b_{12}} = \eta V_{b_{12},k} = \eta V_{k,b_{12}} \quad (k := e_1 - e_2) \tag{7.6.7}$$

since (7.6.3) shows ηD_3 vanishes on A [from $2\eta = 0$] and is $\eta\mathbf{1}$ on M [from $\eta = -\eta$], and the same holds for $S_{\eta e_3}, S_{\eta u}$ since $V_{e_3} = 2\mathbf{1}, V_u = 0$ on Φe_3, $V_{e_3} = 0, V_u = 2\mathbf{1}$ on B, and $V_{e_3} = V_u = \mathbf{1}$ on M. Also $S_{\eta b_{12}} = \eta V_{b_{12},1} = V_{b_{12},e_1+e_2} = \eta V_{b_{12},e_1-e_2}$ [by $2\eta = 0$] $= \eta V_{b_{12},k} = -\eta V_{k,b_{12}}$ [by Switching (0.1.1) since $\{k, b_{12}\} = 0$] $= +\eta V_{k,b_{12}}$ [since $-\eta = \eta$]. This yields the formulas (7.6.5) for $S_{\eta e_2}$, $S_{\eta e_3}$, $S_{\eta u}$, $S_{\eta e_1}$, $S_{\eta b_{12}}$ ($b_{12} = c_{12}, d_{12}, q_{12}$), so each piece of $S_{a'}$ lies in $\mathcal{L}$. The odd Smith derivations are absorbed as in (7.6.5) since $S_{m'} = V_{m'} = V_{m',1} = V_{m',u+e_3} = V_{m',u_*+2e_3} = V_{m',u_*} = \Delta_{m'}$ when $2m' = 0$.

For the odd standard super-derivations, in view of (7.5) $D_{M,A}$ and $S_{M'}$ reduce to Δ_M spanned by the Δ_i, Δ_i'. Thus all standard inner derivations lie in $\mathcal{L} \subseteq \mathrm{Inder}(sK_{10})$ as stated in (7.6.5).

Now we check that space $\mathcal{L}$ contains *all* inner derivations (not just the standard ones). Setting $\widehat{V}_{A,A} := V_{A,A} + \Phi\mathbf{1}_{sK_{10}}$ for convenience,[12] we first note

[11] When Φ has characteristic 2 then $1 \in \Phi_{2^\perp}$, and S_1 is the surprising Smith derivation $0_A \oplus \mathbf{1}_M$.

[12] Note that if $\frac{1}{2} \in \Phi$ this latter term is unnecesaary since $\mathbf{1}_{sK_{10}} = \frac{1}{2}V_{1,1}$.

$$\widehat{V}_{A,A} + V_{M,M} \subseteq \Phi V_{m_{13},n_{13}} + \Phi V_{m_{23},n_{23}} + \widehat{V}_{A,A} \subseteq \Phi D_3 + \widehat{V}_{A,A} \subseteq \mathcal{L} + \widehat{V}_{A,A},$$
$$\widehat{V}_{A,A} \subseteq \Phi V_{e_2} + \Phi\mathbf{1} + V_{e_1,B_{12}} + \Phi V_{c_{12},d_{12}} + \mathcal{L}. \tag{7.6.8}$$

Indeed, Table (4.5) shows that all $V_{m,n}$ lie in $V_{B,B}$ except for $V_{m_{i3},n_{i3}}$ and $V_{n_{i3},m_{i3}} = V_{m_{i3},n_{i3}} - V_{\langle m_{i3},n_{i3}\rangle} = V_{m_{i3},n_{i3}} - V_{g_i} \in V_{m_{i3},n_{i3}} - V_A$, where $V_{m_{i3},n_{i3}} \in \widehat{V}_{A,A} - D_3$ by (7.6.1). All terms of $\widehat{V}_{A,A}$ fall in $\mathcal{L}$ up to the 4 terms indicated in (7.6.8) because

$$V_{e_3,A} = \Phi V_{e_3,e_3} = \Phi V_{e_3},\ V_{e_3} = 2\mathbf{1} - V_{e_1} - V_{e_2},\quad V_{e_1} = -V_{e_2} + V_{c_{12},d_{12}} + V_{d_{12},c_{12}}$$
$$V_{e_2,A} + V_{A,e_2} = \Phi V_{e_2} + V_{e_2,B_{12}} + V_{B_{12},e_2} = \Phi V_{e_2} + V_{B_{12},e_1} + V_{e_1,B_{12}},$$
$$V_{e_1,A} + V_{A,e_1} = \Phi V_{e_1} + V_{e_1,B_{12}} + V_{B_{12},e_1},\ V_{B_{12},e_1} = V_{e_1,B_{12}} - V_{e_1-e_2,B_{12}} = V_{e_1,B_{12}} + \sum_{i=1}^{3} \Phi D'_i,$$
$$V_{B_{12},B_{12}} \subseteq \Phi V_{c_{12},q_{12}} + \Phi V_{d_{12},q_{12}} + \Phi V_{c_{12},d_{12}} + \Phi V_{d_{12},c_{12}} = \sum_{i=1}^{3} \Phi D_i + \Phi V_{c_{12},d_{12}},$$
$$V_{M,A} + V_{A,M} \subseteq V_{M,u} + V_{M,e_3} \subseteq \Delta_M + V_{M,e_3}.$$

Finally, we check that an even inner map $D = \alpha_1\mathbf{1} + \eta V_{v_2} + \alpha_2 V_{e_1,c_{12}} + \alpha_3 V_{e_1,d_{12}} + \alpha_4 V_{e_1,q_{12}} + \alpha_5 V_{c_{12},d_{12}}$ is a derivation iff $\alpha_i = 0$ and $2\eta = 0$: $D(1) = \alpha_1(e_1+e_2+e_3) + 2\eta e_2 + \alpha_2 c_{12} + \alpha_3 d_{12} + \alpha_4 q_{12} + \alpha_5(e_1+e_2)$ vanishes iff $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_1 + \alpha_5 = \alpha_1 + 2\eta + \alpha_5 = 0$ [identifying coefficients of $e_3, c_{12}, d_{12}, q_{12}, e_1, e_2$], which reduces to all $\alpha_i = 0$, $2\eta = 0$, i.e. $D = \eta V_{e_2} \in D_0(\Phi_{2^\perp})$ and hence $D \in \mathcal{L}$. Similarly, an odd inner map $D = V_{m,e_3}$ is a super-derivation, i.e., $D(1) = 0$, iff $m = 0$, since $D(1) = \{m, e_3\} = m$.

The table (7.6.4) of Lie superbrackets results from straightforward calculation using the definition of the D's, the action table (7.6.3), and $[D_i, V_{x,y}]^s = V_{D_i(x),y} + (-1)^{ix} V_{x,D_i(y)}$, $[D_i, V_x]^s = V_{D_i(x)}$, from (7.1). For the even products note that

$$2V_{c_{12},d_{12}} - V_{k,k} = 2V_{c_{12},d_{12}} - V_{q_{12},q_{12}} = D_{c_{12},d_{12}} \tag{7.6.9}$$

since from $k^2 = q^2 = u$ we have $2V_{c,d} - V_{k,k} = 2V_{c,d} - V_u = 2V_{c,d} - V_{\langle c,d\rangle} = V_{c,d} - V_{d,c}$ [by Switching (0.1.1)] $= D_{c,d}$, similarly $2V_{c,d} - V_{q,q} = D_{c,d}$. Note also that $[D, D_0(\eta)] = \eta V_{D(e_2)}$ where $D(e_2) = 0$ for $D = D_i$ and $D(e_2) = b_{12}$ for $D = D'_i$, where by (7.6.5) $\eta V_{b_{12}} = \eta V_{k,b_{12}} = \eta D'_i$.

For the mixed even-odd products, we show that the Δ_m span a 4-dimensional space $\mathcal{I}_1$ naturally isomorphic to M, with the adjoint action of $\mathcal{I}_0$ on $\mathcal{I}_1$ isomorphic to the action of $\mathcal{I}_0$ as linear transformations on M:

$$[D_0, \Delta_m] = \Delta_{D_0(m)} \tag{7.6.10}$$

since $[D_0, V_{m,u_*}] = V_{D_0(m),u_*} + V_{m,D_0(u_*)} = V_{D_0(m),u_*} = \Delta_{D_0(m)}$ [by (7.6.3) $D_0(e_3) = D_0(u) = 0$ so $D_0(u_*) = 0$]. Thus $\mathcal{I}_1 \cong M$ as $\mathcal{I}_0$-modules.

For the odd products, we have

$$\Delta_m^2 = -V_{m,m} = -D_m,\ \langle \Delta_m, \Delta_n\rangle = -D_{m,n} \tag{7.6.11}$$

By a Grassmann detour $V_{m,a}V_{n,a}x = V_{m,U_a n}x + (-1)^x U_{m,n}U_a x$, so by Alternation (0.1.2) and $U_{u_*}m = -m$ we have $\Delta_m^2 x = V_{m,u_*}^2 x = -V_{m,m}x + (-1)^x U_{m,m}U_{u_{ast}}x = -V_{m,m} = -D_m$, hence by linearization $\langle \Delta_m, \Delta_n\rangle = -D_{m,n}$. This, together with (7.6.5) (remembering that $D_{m,n}$ is *symmetric* in m, n), shows that

$$
\begin{array}{llll}
\langle E_1,E_1\rangle = 2E_1^2, & (E_1)^2 = (\Delta_1)^2 = \Delta_{m_{13}}^2 & = -D_{m_{13}} & = -D_1, \\
\langle E_2,E_2\rangle = 2E_2^2, & (E_2)^2 = (-\Delta_2)^2 = \Delta_{n_{13}}^2 & = -D_{n_{13}} & = D_2, \\
\langle E'_1,E'_1\rangle = 2(E'_1)^2, & (E'_1)^2 = (\Delta_3)^2 = \Delta_{m_{23}}^2 & = -D_{m_{23}} & = D_1, \\
\langle E'_2,E'_2\rangle = 2(E'_2)^2, & (E'_2)^2 = (\Delta_4)^2 = \Delta_{n_{23}}^2 & = -D_{n_{23}} & = -D_2, \\
\langle E_1,E_2\rangle = \langle\Delta_1,-\Delta_2\rangle & = \langle\Delta_{m_{13}},-\Delta_{n_{13}}\rangle & = D_{m_{13},n_{13}} & = D_3, \\
\langle E_1,E'_1\rangle = \langle\Delta_1,\Delta'_1\rangle & = \langle\Delta_{m_{13}},\Delta_{m_{23}}\rangle & = -D_{m_{13},m_{23}} & = -2D'_1, \\
\langle E_1,E'_2\rangle = \langle\Delta_1,\Delta'_2\rangle & = \langle\Delta_{m_{13}},\Delta_{n_{23}}\rangle & = -D_{m_{13},n_{23}} & = D'_3, \\
\langle E_2,E'_1\rangle = \langle-\Delta_2,\Delta'_1\rangle & = \langle-\Delta_{n_{13}},\Delta_{m_{23}}\rangle & = D_{n_{13},m_{23}} & = D'_3, \\
\langle E_2,E'_2\rangle = \langle-\Delta_2,\Delta'_2\rangle & = \langle-\Delta_{n_{13}},\Delta_{n_{23}}\rangle & = D_{n_{13},n_{23}} & = 2D'_2, \\
\langle E'_1,E'_2\rangle = \langle\Delta'_1,\Delta'_2\rangle & = \langle\Delta_{m_{23}},\Delta_{n_{23}}\rangle & = -D_{m_{23},n_{23}} & = -D_3.
\end{array} \tag{7.6.12}
$$

Finally, we turn to the isomorphisms mentioned at the beginning of the Theorem. We have $\mathcal{I}_0 = \mathcal{L}_0 = \big(\sum_{i=1}^3 \Phi D_i\big) \oplus \big(\sum_{i=1}^3 \Phi D_i'\big) \oplus D_0(\Phi) = \mathcal{D} \oplus \mathcal{D}' \oplus D_0(\Phi_{2\perp}) \cong sl_2(\Phi) \oplus sl_2(\Phi)\mu \oplus D_0(\Phi_{2\perp}) = \big(sl_2(\Phi) \otimes \Phi[\mu]\big) \oplus D_0(\Phi_{2\perp})$ for $\mu^2 = -1$ because immediately from the table that $[D, C'] = [D', C] = [D, C]'$, $[D', C'] = -[D, C]$ and $\mathcal{D} \cong sl_2(\Phi)$ via $D_1, D_2, D_3 \longrightarrow E_{12}, E_{21}, E_{11} - E_{22}$. In characteristic $\neq 2$ (no 2-torsion) $\Phi_{2\perp} = 0$ and the Lie algebra $\mathcal{I}_0$ is free of rank 6. In characteristic 2 it is free of rank 7 since $\Phi_{2\perp} = \Phi$.[13]

The odd bimodule $\mathcal{I}_1 = \big(\sum_{i=1}^2 \Phi E_i\big) \oplus \big(\sum_{i=1}^2 \Phi E_i'\big) = \mathcal{E} \oplus \mathcal{E}'$ is isomorphic to $V(\Phi) \oplus V(\Phi)\mu = V(\Phi) \otimes \Phi[\mu]$ for $V(\Phi) = \Phi v_1 \oplus \Phi v_2$ the standard bimodule for sl_2 because (once we carefully replace Δ_2 by $-\Delta_2$) we again immediately read off from the table that $[D, E'] = [D, E]'$, $[D', E'] = -[D, E]$ and $\mathcal{E} \cong V(\Phi)$ via $E_1, E_2 \longrightarrow v_1, v_2$. Thus as bimodule we have $\mathcal{I} \cong D_0(\Phi_{2\perp}) \oplus \big(sl_2(\Phi) \oplus V(\Phi)\big)[\mu]$. Here $V(\Phi) \cong \Phi E_{13} \oplus \Phi E_{23} \cong \Phi(E_{13} - E_{32}) \oplus \Phi(E_{23} + E_{31})$, so $sl_2(\Phi) \oplus V(\Phi)$ can be identified with the set of all 3×3 matrices

$$\left[\begin{array}{cc|c} \alpha & \beta & \epsilon \\ \gamma & -\alpha & \delta \\ \hline \delta & -\epsilon & 0 \end{array}\right]$$

which is $osp_{1,2}(\Phi)$ (turned upside down). Under this identification the (symmetric) odd Lie superproducts also correspond: The matrix product $\big(\epsilon(E_{13} - E_{32}) + \delta(E_{23} + E_{31})\big)^2 = -\epsilon^2 E_{12} + \delta^2 E_{21} + \epsilon\delta(E_{11} - E_{33} - E_{22} + E_{33}) = \epsilon\delta(E_{11} - E_{22}) - \epsilon^2 E_{12} + \delta^2 E_{21}$. On the other hand, $(\epsilon E_1 + \delta E_2)^2 = \Delta_m^2$ ($m := \epsilon m_{13} - \delta n_{13}$) [beware the minus] $= -D_m$ [by (7.6.11)] $= -\big(\epsilon^2 D_{m_{13}} - \epsilon\delta D_{m_{13},n_{13}} + \delta^2 D_{n_{13}}\big) = -\epsilon^2 D_1 + \epsilon\delta D_3 - \delta^2(-D_2)$ [by (7.6.5)] $\longrightarrow -\epsilon^2 E_{12} + \epsilon\delta(E_{11} - E_{22}) + \delta^2 E_{21}$. Thus $\mathcal{D} \oplus \mathcal{E} \cong osp_{1,2}(\Phi)$ as Lie superalgebra. ■

Note that this inner derivation superalgebra of sK_{10} is *not* the same as that of the ordinary K_{10} (found elegantly in [1, 2.8, p. 3213]) where $\mu^2 = +1$. Here $\mathrm{Inder}(sK_{10})_0 \cong \mathrm{Inder}(B) = \mathrm{Der}(J(Q, u)) = \{D \in \mathrm{Instrl}(J) \mid D(u) = 0, Q(D(b), b) = 0 \text{ for all } b \in B\}$ is the "isotropy subalgebra" of the inner structure algebra at the point u, while the usual $\mathrm{Inder}(sK_{10})_0$ is the isotropy subalgebra at the point e. Our split superalgebra is an isotope of the standard one, and while in general isotopes share the same inner structure algebra $V_{J,J}^{(s)} = V_{J,U_sJ} = V_{J,J}$, they are sensitive to isotropy: in the standard K_{10} the basepoint e lies in the bilinear radical of Q in characteristic 2 ($Q(e, J) = 0$), whereas in the split algebra the basepoint $u = e_1 + e_2$ does not ($Q(u, e_1) = 1$).

8 Imbedding the Split in the Standard Kac Superalgebra

Finally, we show how over an algebraically closed field $\overline{\Phi}$ of characteristic not 2 the split isotope sK_{10} can be imbedded inside the standard Kac superalgebra $\overline{K}_{10}$. By the usual Grassmann detour, two isotopes $J^{(a)}$, $J^{(b)}$ by even elements $a, b \in A$ are isomorphic if the elements a^{-1}, b^{-1} are conjugate under the inner structure group of J (since $(1 \otimes T)(1 \otimes a^{-1}) = 1 \otimes b^{-1}$ then holds in $\widetilde{J}$ for structural $1 \otimes T$), and in our case $\overline{J}^{(s)} \cong \overline{J}$ since $s \in A$ has a square root t in $\overline{A}$.

Imbedding Theorem 8.1 *If Φ contains τ with $\tau^4 = -\frac{1}{4}$ then $\varphi' := U_t : K_{10} \longleftrightarrow sK_{10}$ is an isomorphism of Jordan superalgebras (in both directions) for*

$$\text{(8.1.1)} \qquad t := v + f, \quad v := \tau(e + iu), \qquad v^2 = u := v_1 + v_2 \qquad (i := -2\tau^2).$$

In this case Φ also contains λ such that

$$\text{(8.1.2)} \qquad \lambda = \frac{1 - i}{2}, \quad i\lambda = \frac{1 + i}{2}, \quad i^2 = -1, \quad 2\lambda^2 i = 1, \quad \lambda^4 = -\frac{1}{4}.$$

[13]Then $sl_2(\Phi)$ is nilpotent, and $\mathrm{Inder}(sK_{10})_0$ is solvable but not nilpotent: $\mathcal{I}_0^{(1)} = [\mathcal{D}, \mathcal{I}_0] = \Phi D_1 + \Phi D_2 + \Phi D_3 + \Phi D_3'$, $\mathcal{I}_0^{(2)} = \Phi D_3$, $\mathcal{I}_0^{(3)} = 0$, but $[D_0(1), \Phi D_1 + \Phi D_2 + \Phi D_3'] = \Phi D_1 + \Phi D_2 + \Phi D_3'$.

PROOF: By choice of τ the element $i := -2\tau^2$ has $i^2 = 4\tau^4 = -1$ and $2\tau^2 i = -i^2 = 1$. Then $v^2 = \tau^2(e + 2iu - e) = 2\tau^2 iu = u$. Thus $U_t(1^{-1}) = t^2 = v^2 + f = u + f = s = s^{-1}$ and at the same time $U_t s^{-1} = U_t(t^2) = (t^2)^2 = s^2 = 1 = 1^{-1}$, and φ' is an isomorphism of superalgebras in both directions. [While φ' is not an involution, $\varphi'^2 = U_s = *$ is an involution on J.] If we define $\lambda := \frac{1-i}{2}$ then $i\lambda = \frac{i+1}{2}$, $\lambda^2 = \frac{1-2i+i^2}{4} = -\frac{i}{2}$, $2i\lambda^2 = -i^2 = 1$, $\lambda^4 = \frac{i^2}{4} = -\frac{1}{4}$ and λ is another fourth-root of $-\frac{1}{4}$ with the same i.

The isomorphism φ' must take the split basis $\{e_1, e_2, c_{12}, d_{12}, q_{12}, e_3, m_{13}, n_{13}, m_{23}, n_{23}\} = \{v_1, v_2, -v_3, v_4, e, f, m_1, m_4, m_3, m_2\}$ to a split basis inside K_{10}. Here $\varphi' = U_{v+f}$ reduces on $B = A_{11+12+22}$ to $U_v = \lambda^2(U_e + iU_{e,u} - U_u) = \lambda^2(\mathbf{1}_B - * + iV_u)$, while on $\Phi e_3 = \Phi f$ it is just U_f, and on $M = M_{13} + M_{23}$ it is $U_{v,f} m = V_v m = \langle v, m\rangle = \lambda(\langle e, m\rangle + i\langle u, m\rangle) = \lambda(m + m^*)$. If we set $k := v_1 - v_2$ (as in (7.6.2)) we get a split basis

$$
\begin{aligned}
e'_1 &:= \varphi'(v_1) = \lambda^2(v_1 - v_2 + ie) = \tfrac{1}{2}(e - ik), & e'_3 &:= \varphi'(f) = U_f f = f,\\
e'_2 &:= \varphi'(v_2) = \lambda^2(v_2 - v_1 + ie) = \tfrac{1}{2}(e + ik), & m'_{13} &:= \varphi'(m_1) = \lambda(m_1 + im_3),\\
c'_{12} &:= \varphi'(-v_3) = \lambda^2(-v_3 - v_3 + i0) = iv_3, & n'_{13} &:= \varphi'(m_4) = \lambda(m_4 + im_2),\\
d'_{12} &:= \varphi'(v_4) = \lambda^2(v_4 + v_4 + i0) = -iv_4, & m'_{23} &:= \varphi'(m_3) = \lambda(m_3 + im_1),\\
q'_{12} &:= \varphi'(e) = \lambda^2(e - e + i2u) = u, & n'_{23} &:= \varphi'(m_2) = \lambda(m_2 + im_4).
\end{aligned}
\tag{8.1.3}
$$

inside $\overline{K}_{10}$. ∎

Remark 8.2 *In characteristic not* 2 *this means the two algebras $sK_{10}(\overline{\Phi})$ and $K_{10}(\overline{\Phi})$ are isomorphic as Jordan superalgebras over $\overline{\Phi}$, and the split scheme sK_{10} is a $\mathbb{Z}$-form of the standard scheme K_{10}. But the split and standard algebras in characteristic* 2 do not *become isomorphic under any scalar extension (they are* not *forms of each other): the condition $b^2 = T(b)b - Q(b)1 = -Q(b)1 \in \Phi 1$ satisfied by the standard K_{10} in characteristic 2 (due to the traceless nature of Q) will persist in all scalar extensions, and K_{10} will never be able to grow 3 reduced supplementary orthogonal idempotents.* ∎

Another imbedding (another Kac basis) creates the splitting idempotents e_1, e_2 more naturally from the element $u = v_1 + v_2$: if $\frac{1}{2} \in \Phi$ then $Jord(Q, e)$ is a degree 2 Jordan algebra whose identity can be decomposed as a sum of two orthogonal idempotents

$$
e''_1 := \tfrac{1}{2}(e + u),\ e''_2 := \tfrac{1}{2}(e - u),\ e''_3 := f, \quad u := v_1 + v_2, \quad u^2 = e = e_1 + e_2. \tag{8.3}
$$

Then e''_1, e''_2, e''_3 are supplementary reduced orthogonal idempotents in $K_{10}(\overline{\Phi})$ and A is a degree 3 Jordan algebra with unit $1 = e''_1 + e''_2 + e''_3$; the Peirce decomposition of $K_{10}(\overline{\Phi})$ is

$$
\begin{gathered}
K_{10}(\overline{\Phi}) = \big(\textstyle\bigoplus_{i=1}^{3} A_{ii} \bigoplus A_{12}\big) \bigoplus \big(M_{13} \oplus M_{23}\big), \quad \textit{where for} \quad w := v_1 - v_2\\
B = Jord(Q, e) = A_{11} \oplus A_{22} \oplus A_{12}, \quad A_{ii} = \Phi e''_i,\ A_{12} = \Phi w \oplus \Phi v_3 \oplus \Phi v_4,\\
Q(a'') = \varepsilon_1\varepsilon_2 + \alpha^2 - \alpha_3\alpha_4, \quad T(a'') = \varepsilon_1 + \varepsilon_2 \quad \textit{for } a'' = \varepsilon_1 e''_1 + \varepsilon_2 e''_2 + \alpha w + \alpha_3 v_3 + \alpha_4 v_4,\\
w^2 = -e, \quad \{w, v_j\} = 0, \quad v_j^2 = 0, \quad \{v_3, v_4\} = e \quad (j = 3, 4)\\
v_1^* = v_2, \quad v_2^* = v_1, \quad e''^*_i = e''_i, \quad v_{12}^* = -v_{12}\ (\textit{for} \quad v = w, v_3, v_4).
\end{gathered}
\tag{8.4}
$$

The original basis $\{v_1, v_2, v_4, v_4, e, f, m_1, m_2, m_3, m_4\}$ for $K_{10}(\overline{\Phi})$ is not adapted to these new idempotents. Over an algebraically closed field of characteristic $\neq 2$ (where we are seeking the "true" split superalgebra; all we need are $i = \sqrt{-1}$ and $\sqrt{2}$) we obtain another split $\mathbb{Z}$-basis for $K_{10}(\overline{\Phi})$ (cf. (5.2)):

$$
\begin{aligned}
e''_1 &:= \tfrac{1}{2}(e + v_1 + v_2) = \tfrac{1}{2}(e + \ell), & e''_3 &:= f,\\
e''_2 &:= \tfrac{1}{2}(e - v_1 - v_2) = \tfrac{1}{2}(e - \ell), & m''_{13} &:= \tfrac{1}{\sqrt{2}}(m_1 + m_3),\\
c''_{12} &:= iv_3, & n''_{13} &:= \tfrac{1}{\sqrt{2}}(m_2 + m_4),\\
d''_{12} &:= -iv_4, & m''_{23} &:= \tfrac{i}{\sqrt{2}}(m_1 - m_3),\\
q''_{12} &:= i(v_1 - v_2) = iw, & n''_{23} &:= \tfrac{-i}{\sqrt{2}}(m_2 - m_4).
\end{aligned}
\tag{8.5}
$$

One can check that the multiplication table for the basis (8.4) is the analogue of (5.4), (5.7) directly from Tables (1.3), (1.9), (1.12); more conceptually, this holds because there is an inner automorphism of $\overline{J}$ with $\psi(x') = x''$ for each element of the split basis.

Theorem 8.6 *If* $i = \sqrt{-1}$, $\sqrt{2} \in \Phi$ *we can define* $\lambda_1, \lambda_2 \in \Phi$ *related to* $\lambda := \frac{1-i}{2}$ *of* (8.1.2) *by*

$$\lambda_1 := \tfrac{1-i}{\sqrt{2}} = \lambda\sqrt{2}, \quad \lambda_2 := \tfrac{1+i}{\sqrt{2}} = i\lambda\sqrt{2}, \quad \textit{which will then satisfy}$$
$$\lambda_1\lambda_2 = 1, \quad \lambda_1^2 = -i, \quad \lambda_2^2 = i, \quad \lambda\lambda_1 = -\tfrac{i}{\sqrt{2}}, \ \lambda\lambda_2 = \tfrac{1}{\sqrt{2}}. \tag{8.6.1}$$

If we set $v := \lambda_1 v_1 + \lambda_2 v_2 + f$, *then the map* $\varphi'' := U_s U_{v+f}$ *is an inner automorphism of* $K_{10}(\Phi)$ *sending the split basis* (8.1.3) *to the split basis* (8.5).

Proof: The formulas (8.6.1) follow by standard calculations with i. Already $*' := U_{v+f}$ for $v := \lambda_1 v_1 + \lambda_2 v_2$ is an involutory inner automorphism since $v^2 = \lambda_1\lambda_2\{v_1, v_2\} = e$ implies $(v+f)^2 = e + f = 1$. Composition with the involution $* = U_s$ then yields an inner automorphism of superalgebras ψ. To see that ψ does transform x' to x'', note that $*' = U_v + U_{v,f} + U_f$ for $U_v = \lambda_1^2 U_{v_1} + \lambda_2^2 U_{v_2} + \lambda_1\lambda_2 U_{v_1,v_2} = -iU_{v_1} + iU_{v_2} + U_{v_1,v_2}$, so that on B ψ sends

$$\begin{aligned}
&e \xrightarrow{*'} -0 + 0 + \{v_1, v_2\} = e \xrightarrow{*} e,\\
&v_1 \xrightarrow{*'} -0 + iv_2 + 0 = iv_2 \xrightarrow{*} iv_1,\\
&v_2 \xrightarrow{*'} -iv_1 + 0 + 0 = -iv_1 \xrightarrow{*} -iv_2,\\
&v_3 \xrightarrow{*'} -0 + 0 - v_3 = -v_3 \xrightarrow{*} v_3,\\
&v_4 \xrightarrow{*'} -0 + 0 - v_4 = -v_4 \xrightarrow{*} v_4,\\
&k = v_1 - v_2 \longrightarrow iv_1 - (-iv_2) = i(v_1 + v_2) = i\ell,\\
&u = v_1 + v_2 \longrightarrow iv_1 + (-iv_2) = i(v_1 - v_2) = iw,
\end{aligned}$$

(note that with respect to e_1', e_2' the element $k = v_1 - v_2$ is "diagonal" and $u = v_1 + v_2$ is "off-diagonal", while with respect to e_1'', e_2'' the element $w = v_1 - v_2$ is "off-diagonal" and $\ell = v_1 + v_2$ is "diagonal," hence their new names), and hence ψ sends $e_1' = \frac{1}{2}(e - ik) \xrightarrow{\psi} \frac{1}{2}(e + \ell) = e_1''$, $e_2' = \frac{1}{2}(e + ik) \xrightarrow{\psi} \frac{1}{2}(e - \ell) = e_2''$, $c_{12}' = iv_3 \xrightarrow{\psi} iv_3 = c_{12}''$, $d_{12}' = -iv_4 \xrightarrow{\psi} -iv_4 = d_{12}''$, $q_{12}' = u \xrightarrow{\psi} iw = q_{12}''$ as claimed. On Φf we have $*' = * = \Psi = U_f$ sending

$$f \xrightarrow{*'} f \xrightarrow{*} f$$

as claimed. Finally, on M the involution $*'$ becomes $U_{v,f} = V_v = \lambda_1 V_{v_1} + \lambda_2 V_{v_2}$, so ψ sends

$$\begin{aligned}
&m_1 \xrightarrow{*'} 0 + \lambda_2 m_3 \xrightarrow{*} \lambda_2 m_1,\\
&m_2 \xrightarrow{*'} \lambda_1 m_4 + 0 \xrightarrow{*} \lambda_1 m_2,\\
&m_3 \xrightarrow{*'} \lambda_1 m_1 + 0 \xrightarrow{*} \lambda_1 m_3,\\
&m_4 \xrightarrow{*'} 0 + \lambda_2 m_2 \xrightarrow{*} \lambda_2 m_4,
\end{aligned}$$

and hence sends $m_{13}' = \lambda(m_1 + im_3) \longrightarrow \lambda\lambda_2 m_1 + i\lambda\lambda_1 m_3 = \frac{1}{\sqrt{2}}(m_1 + m_3) = m_{13}''$, $n_{13}' = \lambda(m_4 + im_2) \longrightarrow \lambda\lambda_2 m_4 + i\lambda\lambda_1 m_2 = \frac{1}{\sqrt{2}}(m_4 + m_2) = n_{13}''$, $m_{23}' = \lambda(m_3 + im_1) \longrightarrow \lambda\lambda_1 m_3 + i\lambda\lambda_2 m_1 = \frac{i}{\sqrt{2}}(m_1 - m_3) = m_{23}''$, $n_{23}' = \lambda(m_2 + im_4) \longrightarrow \lambda\lambda_1 m_2 + i\lambda\lambda_2 m_4 = \frac{-i}{\sqrt{2}}(m_2 - m_4) = n_{23}''$ as claimed. ■

The quaternion action (5.7) of $sK_{10}(\Phi)$ on M can be duplicated in $K_{10}(\Phi)$ using the above basis. The element $\ell = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ has $\ell^2 = 1$ and determines an involutive isomorphism ψ of $H = M_2(\Phi)$ with $\psi(e_{ij}) =: f_{ij}$ another family of supplementary matrix units for H, with $f_{11} =$

$\frac{1}{2}\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$, $f_{22} = \frac{1}{2}\left[\begin{smallmatrix}1&-1\\-1&1\end{smallmatrix}\right]$, $f_{12} = \frac{1}{2}\left[\begin{smallmatrix}1&-1\\1&-1\end{smallmatrix}\right]$, $f_{21} = \frac{1}{2}\left[\begin{smallmatrix}1&1\\-1&-1\end{smallmatrix}\right]$ so that $e_{12}-e_{21} = \frac{1}{2}\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right] = f_{21}-f_{12}$, $e_{11}-e_{22} = \frac{1}{2}\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right] = f_{12}+f_{21}$, $e_{11}+e_{21} = \frac{1}{2}\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right] = f_{11}+f_{12}$, $e_{11}-e_{21} = \frac{1}{2}\left[\begin{smallmatrix}1&0\\-1&0\end{smallmatrix}\right] = f_{22}+f_{21}$, $2e_{11} = f_{11}+f_{12}+f_{21}+f_{22}$, $e_{22}+e_{12} = \frac{1}{2}\left[\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right] = f_{11}-f_{12}$, $e_{22}-e_{12} = \frac{1}{2}\left[\begin{smallmatrix}0&-1\\0&1\end{smallmatrix}\right] = f_{22}-f_{21}$, $2e_{22} = f_{11}-f_{12}-f_{21}+f_{22}$, $e_{12}+e_{21} = \frac{1}{2}\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] = f_{11}-f_{22}$, $2e_{12} = f_{11}-f_{12}+f_{21}-f_{22}$, $2e_{21} = f_{11}+f_{12}-f_{21}-f_{22}$. In these terms the regular quaternion action (2.1) of A'' on M is $V_{e_1''} = \frac{1}{2}(\mathbf{1}+V_1+V_2) = \frac{1}{2}L_{(e_{11}+e_{22})+e_{12}+e_{21}} = \frac{1}{2}L_{(e_{11}+e_{21})+(e_{22}+e_{12})} = \frac{1}{2}L_{(f_{11}+f_{12})+(f_{11}-f_{12})} = L_{f_{11}}$, $V_{e_2''} = \frac{1}{2}(\mathbf{1}-V_1-V_2) = \frac{1}{2}L_{(e_{11}+e_{22})-e_{12}-e_{21}} = \frac{1}{2}L_{(e_{11}-e_{21})+(e_{22}-e_{12})} = \frac{1}{2}L_{(f_{22}+f_{21})+(f_{22}-f_{21})} = L_{f_{22}}$, with more complicated actions $V_{c_{12}''} = iV_3 = iL_{e_{11}-e_{22}}R_{e_{21}} = \frac{1}{2}iL_{f_{12}+f_{21}}R_{f_{11}+f_{12}-f_{21}-f_{22}}$, $V_{d_{12}''} = -iV_4 = -iL_{e_{11}-e_{22}}R_{e_{12}} = -\frac{1}{2}iL_{f_{12}+f_{21}}R_{f_{11}-f_{12}+f_{21}-f_{22}}$, $V_{q_{12}''} = i(V_1-V_2) = i(L_{e_{12}-e_{21}}) = -iL_{f_{12}-f_{21}}$, $V_{e''} = V_{e_3''} = \mathbf{1}_M$.

Under the isomorphism $M \longrightarrow H$ via $m_1, m_2, m_3, m_4 \xrightarrow{\varphi} e_{11}, e_{22}, e_{21}, e_{12}$ the new basis for M is (up to the scalar $\frac{1}{\sqrt{2}}$) m_1+m_3, m_2+m_4, $i(m_1-m_3)$, $-i(m_2-m_4) \xrightarrow{\varphi} f_{11}+f_{12}$, $f_{11}-f_{12}$, $i(f_{21}+f_{22})$, $-i(f_{22}-f_{21}) = i(f_{21}-f_{22})$, and after a routine calculation the action of Table (2.1) takes the split form

(8.7) Quaternion Action $A'' \times H$

Action of V on:	$f_{11}+f_{12}$	$f_{11}-f_{12}$	$i(f_{21}+f_{22})$	$i(f_{21}-f_{22})$
$V_{e_1''} = L_{f_{11}}$	$f_{11}+f_{12}$	$f_{11}-f_{12}$	0	0
$V_{e_2''} = L_{f_{22}}$	0	0	$i(f_{21}+f_{22})$	$i(f_{21}-f_{22})$
$V_{q_{12}''} = iL_{f_{21}-f_{12}}$	$i(f_{21}+f_{22})$	$i(f_{21}-f_{22})$	$f_{11}+f_{12}$	$f_{11}-f_{12}$
$V_{c_{12}''} = iL_{f_{12}+f_{21}}R_{e_{21}}$	0	$i(f_{21}+f_{22})$	0	$-(f_{11}+f_{12})$
$V_{d_{12}''} = -iL_{f_{12}+f_{21}}R_{e_{12}}$	$-i(f_{21}-f_{22})$	0	$f_{11}-f_{12}$	0
$V_{e''} = V_{e_3''} = I$	$f_{11}+f_{12}$	$f_{11}-f_{12}$	$i(f_{21}+f_{22})$	$i(f_{21}-f_{22})$

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