JORDAN QUADRUPLE SYSTEMS

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To our friend and colleague Luiz Antonio Peresi on his retirement

ABSTRACT. We define Jordan quadruple systems by the polynomial identities of degrees 4 and 7 satisfied by the Jordan tetrad $\{a, b, c, d\} = abcd + dcba$ as a quadrilinear operation on associative algebras. We find further identities in degree 10 which are not consequences of the defining identities. We introduce four infinite families of finite dimensional Jordan quadruple systems, and construct the universal associative envelope for a small system in each family. We obtain analogous results for the anti-tetrad $[a, b, c, d] = abcd - dcba$. Our methods rely on computer algebra, especially linear algebra on large matrices, the LLL algorithm for lattice basis reduction, representation theory of the symmetric group, noncommutative Gröbner bases, and Wedderburn decompositions of associative algebras.

1. INTRODUCTION

In this paper we study the quadrilinear operations $\{a, b, c, d\} = abcd + dcba$ and $[a, b, c, d] = abcd - dcba$ in associative algebras. The first is the Jordan tetrad which plays an important role in the structure theory of Jordan algebras [28, 31]. The second is the anti-tetrad, which seems not to have been studied until now.

1.1. **Motivation.** In an associative algebra for $n \geq 2$ we define the n-tad to be this n-ary multilinear operation: $\{a_1, \ldots, a_n\} = a_1 \cdots a_n + a_n \cdots a_1$. For $n = 2$ we obtain the Jordan product $\{a, b\} = ab + ba$ satisfying commutativity and the Jordan identity:

$$\{a, b\} \equiv \{b, a\}, \qquad \{\{\{a, a\}, b\}, a\} \equiv \{\{a, a\}, \{b, a\}\}.$$

There are further "special" identities satisfied by the Jordan product in every associative algebra which do not follow from the defining identities; the simplest occur in degrees 8 and 9 and are called the Glennie identities [17, 18]. A Jordan algebra is "special" if it can be represented as a subspace of an associative algebra closed under the Jordan product; otherwise, it is "exceptional". If a special Jordan algebra is finite dimensional then its universal associative enveloping algebra is also finite dimensional. A survey of the role of identities in Jordan theory has been given by McCrimmon [26]. For the structure and representation theory of finite dimensional Jordan algebras, see Jacobson [21]. For the modern theory including infinite dimensional algebras, see McCrimmon [27].

For $n = 3$ we obtain the Jordan triple product $abc + cba$; in every associative algebra, this operation satisfies identities which define Jordan triple systems (JTS):

$$\{a, b, c\} \equiv \{c, b, a\}, \qquad \{\{a, b, c\}, d, e\} \equiv \{\{a, d, e\}, b, c\} - \{a, \{b, e, d\}, c\} + \{a, b, \{c, d, e\}\}.$$

In contrast to the Jordan identity these identities are multilinear. There are special identities in higher degree: identities satisfied by the Jordan triple product in every associative algebra but which do not follow from the defining identities [24, 25]. For the classification of finite dimensional JTS, see [22, 29, 30] and for their universal associative envelopes, see [23].

Closely related to Jordan triple systems are the anti-Jordan triple systems (AJTS), see [16]. Finite dimensional simple AJTS have been classified [1]. These systems are defined by identities satisfied by the anti-Jordan triple product $abc - cba$ in every associative algebra:

$$\{a, b, c\} + \{c, b, a\} \equiv 0, \qquad \{\{a, b, c\}, d, e\} \equiv \{\{a, d, e\}, b, c\} + \{a, \{b, e, d\}, c\} + \{a, b, \{c, d, e\}\}.$$

Universal associative envelopes for one infinite family of simple AJTS have been constructed [15].

At the next step $n = 4$ we obtain the Jordan tetrad $abcd + dcba$, which arises in the study of symmetric elements of associative algebras. Let A_n be the free unital associative algebra on generators $x_1, \ldots, x_n$ over a field F. The involution $*: A_n \to A_n$ defined on monomials by $(x_{i_1} \cdots x_{i_d})^* = x_{i_d} \cdots x_{i_1}$ satisfies $(ab)^* = b^* a^*$ and $(a^*)^* = a$. The subspace $H_n = \{ a \in A_n \mid a^* = a \}$ of symmetric elements is a Jordan algebra under the Jordan product. Let SJ_n be the free special Jordan algebra: the Jordan subalgebra of H_n generated by $x_1, \ldots, x_n$. Cohn [12, 13] has shown that $SJ_n = H_n$ for $n \le 3$; but for $n \ge 4$, $SJ_n \ne H_n$ and H_n is generated as a Jordan algebra by $x_1, \ldots, x_n$ and the tetrads $x_{i_1} x_{i_2} x_{i_3} x_{i_4} + x_{i_4} x_{i_3} x_{i_2} x_{i_1}$ for $1 \le i_1 < i_2 < i_3 < i_4 \le n$. Since A_n is unital, H_n is generated by $x_1, \ldots, x_n$ using the tetrad as a quadrilinear operation. From the tetrad we recover the Jordan product by setting two arguments to 1, and so this operation provides the natural algebraic structure on H_n for $n \ge 4$.

Definition 1.1. The **tetrad** and **anti-tetrad** are respectively these quadrilinear operations on associative algebras:

$$\{a, b, c, d\} = abcd + dcba, \qquad [a, b, c, d] = abcd - dcba.$$

1.2. **Outline.** In §2 we recall basic results from the representation theory of the symmetric group, emphasizing a computational point of view, with a focus on applications to polynomial identities. In §3 we use computer algebra to determine a complete set of generators for the multilinear polynomial identities of degrees 4 and 7 satisfied by the tetrad in every associative algebra. These identities define Jordan quadruple systems (JQS). In §4 we use representation theory to show that there are further "special" identities in degree 10 satisfied by the tetrad in every associative algebra which do not follow from the defining identities for JQS. We use the LLL algorithm for lattice basis reduction to obtain five explicit nonlinear special identities. In §5 we introduce four infinite families of finite dimensional JQS. For one system in each family, we use noncommutative Gröbner bases to construct its universal associative envelope; in each case the envelope is finite dimensional, and we use the Wedderburn decomposition of an associative algebra to classify the finite dimensional irreducible representations. In §§6–8 we describe analogous results for the anti-tetrad. Throughout the paper we suggest a number of open problems as possible directions for further research.

1.3. **Conventions.** Unless otherwise indicated, all computations are performed with the computer algebra system `Maple` using arithmetic over either the ring $\mathbb{Z}$ of integers or the field $\mathbb{Q}$ of rational numbers. To save computer memory, we often use arithmetic over the finite field $\mathbb{F}_p$ for some prime p, followed by rational reconstruction to recover results over $\mathbb{Z}$. We use the symbol $\equiv$ to indicate that an equation holds for all values of the arguments.

2. Computational methods for polynomial identities

In this section we review rather informally the structure theory of the group algebra $\mathbb{Q}S_n$ from an algorithmic point of view, with a focus on applications to polynomial identities. For a more detailed exposition, see [9, §5].

2.1. **Representation matrices.** If $\mathbb{F}$ is a field of characteristic 0 or $p > n$, then the group algebra $\mathbb{F}S_n$ is semisimple, and decomposes as the direct sum of simple two-sided ideals, each isomorphic to a full matrix algebra:

$$R \colon \mathbb{F}S_n \xrightarrow{\ \approx\ } \bigoplus_\lambda M_{d_\lambda}(\mathbb{F})$$

The sum is over all partitions λ of n, and we write $R_\lambda \colon \mathbb{F}S_n \to M_{d_\lambda}(\mathbb{F})$ for the projection onto component λ. The dimension d_λ of the irreducible representation $[\lambda]$ can be computed from the Young diagram of λ using the hook formula. Given any permutation $\sigma \in S_n$, the $d_\lambda \times d_\lambda$ representation matrix $R_\lambda(\sigma)$ in the natural representation has entries in $\{0, 1, -1\}$ and can be efficiently computed using the methods of [11] and [9, Figure 1].

We study multilinear polynomial identities for nonassociative quadrilinear operations. Monomials in such an operation have degrees $n \equiv 1 \pmod 3$. By an association type in degree n we mean a placement of operation symbols in a sequence of n arguments (without specifying the arguments). By the identity monomial for an association type in degree n we mean the monomial with the identity permutation $a_1 \cdots a_n$ of the arguments. In general, a monomial in degree n consists of an association type applied to a permutation of the arguments. If there are $t = t(n)$ distinct association types in degree n, totally ordered in some way, then any multilinear polynomial I of degree n can be written as a sum of t components $I_1 + \cdots + I_t$; in each component, the terms differ only by the permutation of the arguments. We can therefore regard I as an element of the direct sum of t copies of the group algebra, $(\mathbb{F}S_n)^t$, on which S_n acts by left multiplication. For each partition λ, we apply the projection R_λ to each component $I_1, \ldots, I_t$ to obtain a sequence of t matrices of size $d_\lambda \times d_\lambda$ which we combine horizontally into a matrix of size $d_\lambda \times t d_\lambda$. This is the representation matrix for the component of I in partition λ. The row canonical form (RCF) of this matrix is the normal form of I for partition λ. (We assume that zero rows have been removed from the RCF, so that a matrix in RCF always has full rank.) Each row of the RCF generates a submodule of $(\mathbb{F}S_n)^t$ isomorphic to $[\lambda]$. Hence the rank of this matrix is the multiplicity of $[\lambda]$ in the submodule of $(\mathbb{F}S_n)^t$ generated by I.

More generally, we consider a sequence $I^{(1)}, \ldots, I^{(s)}$ of s multilinear identities in degree n and the $s d_\lambda \times t d_\lambda$ matrix in which the (i, j) block is the image under R_λ of the terms of the i-th identity in the j-th association type. The rows of the RCF of this matrix provide a canonical set of generators for the isotypic component of type $[\lambda]$ generated by $I^{(1)}, \ldots, I^{(s)}$ in $(\mathbb{F}S_n)^t$.

2.2. **Polynomial identities.** The process of finding a complete set of generators for the S_n-module of multilinear identities in degree n satisfied by the tetrad $\{a_1, a_2, a_3, a_4\}$ consists of three steps. (For the anti-tetrad, the process is similar, but we must also keep track of sign changes resulting from reversal.) These computations take place in the multilinear subspaces $\mathrm{Quad}(n)$ of degree n in the free quaternary algebra with one operation satisfying $\{a_1, a_2, a_3, a_4\} - \{a_4, a_3, a_2, a_1\} \equiv 0$. Since $\mathrm{Quad}(n)$ is also a left S_n-module (permutations act on the subscripts), we can regard it as the degree n component of the symmetric operad Quad generated by one quaternary operation for which the action of S_4 is given by the symmetry in degree 4; that is, the generating space $\mathrm{Quad}(4)$ is the quotient of the regular S_4-module $\mathbb{Q}S_4$ by the left ideal generated by $a_1 a_2 a_3 a_4 - a_4 a_3 a_2 a_1$.

Step 1. We distinguish two types of identities satisfied by the tetrad in degree n which are consequences of known identities in lower degrees.

• Type 1: Symmetries. Since the tetrad satisfies symmetry in degree 4, we use this to reduce the number of association types in degrees $n > 4$. For example, in degree 7 we reduce the number of association types from 4 to 2 as follows:

$$\{a_1, a_2, \{a_3, a_4, a_5, a_6\}, a_7\} \longrightarrow \{a_7, \{a_3, a_4, a_5, a_6\}, a_2, a_1\},$$
$$\{a_1, a_2, a_3, \{a_4, a_5, a_6, a_7\}\} \longrightarrow \{\{a_4, a_5, a_6, a_7\}, a_3, a_2, a_1\}.$$

This does not eliminate all the identities in degree n which follow from symmetry in degree 4, since for each association type there remain identities of the form $\iota - \tau \equiv 0$ where ι is the identity monomial and τ is a monomial in the same association type obtained from ι by a single application of symmetry. For example, for one of the association types in degree 10 we have these identities:

$$\{\{a_1, a_2, a_3, a_4\}, a_5, a_6, \{a_7, a_8, a_9, a_{10}\}\} - \{\{a_4, a_3, a_2, a_1\}, a_5, a_6, \{a_7, a_8, a_9, a_{10}\}\} \equiv 0,$$
$$\{\{a_1, a_2, a_3, a_4\}, a_5, a_6, \{a_7, a_8, a_9, a_{10}\}\} - \{\{a_1, a_2, a_3, a_4\}, a_5, a_6, \{a_{10}, a_9, a_8, a_7\}\} \equiv 0,$$
$$\{\{a_1, a_2, a_3, a_4\}, a_5, a_6, \{a_7, a_8, a_9, a_{10}\}\} - \{\{a_7, a_8, a_9, a_{10}\}, a_6, a_5, \{a_1, a_2, a_3, a_4\}\} \equiv 0.$$

Symmetries of the association types generate a submodule $\mathrm{Symm}(n) \subset \mathrm{Quad}(n)$.

• Type 2: Liftings. We assume by induction that we have already determined a set of S_{n-3}-module generators for the multilinear identities of degree $n-3$ satisfied by the tetrad. (We do not include the symmetry in degree 4 in this process, so $n - 3 > 4$.) Let $I(a_1, \ldots, a_{n-3}) \equiv 0$ be such an identity. We consider $n-3$ substitutions of a tetrad for an argument of I, and two embeddings of I into a tetrad:

$$I(\{a_1, a_{n-2}, a_{n-1}, a_n\}, a_2, \ldots, a_{n-3}), \quad \ldots, \quad I(a_1, \ldots, \{a_i, a_{n-2}, a_{n-1}, a_n\}, \ldots, a_{n-3}), \quad \ldots,$$
$$I(a_1, a_2, \ldots, \{a_{n-3}, a_{n-2}, a_{n-1}, a_n\}),$$
$$\{I(a_1, a_2, \ldots, a_{n-3}), a_{n-2}, a_{n-1}, a_n\}, \quad \{a_{n-2}, I(a_1, a_2, \ldots, a_{n-3}), a_{n-1}, a_n\}.$$

These multilinear polynomials clearly vanish, and form a set of S_n-module generators for the multilinear identities in degree n which are consequences of I. We repeat this process for every generator in degree $n-3$. Iteration of this process produces a set of generators for the S_n-module of identities for the tetrad in degree n which are consequences of the known identities of lower degree. The liftings of the identities of lower degree generate a submodule $\mathrm{Lift}(n) \subset \mathrm{Quad}(n)$.

Definition 2.1. The submodule $\mathrm{Old}(n) = \mathrm{Symm}(n) + \mathrm{Lift}(n) \subset \mathrm{Quad}(n)$ consists of identities in degree n which are consequences of known identities of lower degree.

We use the methods of §2.1 to find canonical generators for $\mathrm{Old}(n)$. For each λ, we obtain the matrix $\mathrm{RCF}_\lambda(\mathrm{Old}(n))$ whose rows are a set of independent generators for the simple summands $[\lambda]$ in the isotypic component of type λ in $\mathrm{Old}(n)$.

Step 2. We determine all the multilinear identities satisfied by the tetrad in degree n. By §2.1, these identities are elements of $\mathrm{Quad}(n) = (\mathbb{F}S_n)^t$, where $t = t(n)$ is the number of association types in degree n. Each monomial in degree n can be expanded by applying the definition $\{a, b, c, d\} = abcd + dcba$ to each occurrence of the operation symbol; the result is a multilinear associative polynomial of degree n: an element of $\mathbb{F}S_n$. This process extends linearly to $\mathrm{Quad}(n)$, giving the expansion map $E_n \colon \mathrm{Quad}(n) \to \mathbb{F}S_n$, which is an S_n-module morphism. The kernel of E_n is the submodule of $\mathrm{Quad}(n)$ consisting of those multilinear polynomials which vanish after expansion into the free associative algebra.

Definition 2.2. The submodule $\mathrm{All}(n) \subset \mathrm{Quad}(n)$ is the kernel of $E_n \colon \mathrm{Quad}(n) \to \mathbb{F}S_n$: all the multilinear identities in degree n satisfied by the tetrad.

We use the methods of §2.1 to find canonical generators for $\mathrm{All}(n)$. For each λ, we calculate the matrix $\mathrm{RCF}_\lambda(\mathrm{All}(n))$ whose rows are a set of independent generators for distinct simple summands of type $[\lambda]$ in $\mathrm{All}(n)$. We restrict E_n to the corresponding isotypic component of its domain and codomain, obtaining the map $E_n^\lambda \colon M_{d_\lambda}(\mathbb{F})^t \to M_{d_\lambda}(\mathbb{F})$. The matrix representing E_n^λ has size $d_\lambda \times td_\lambda$, and the i-th block consists of the representation matrix of the expansion of the i-th identity monomial. We compute a canonical basis for the nullspace of E_n^λ as follows:

- we calculate $\mathrm{RCF}(E_n^\lambda)$, its rank q, and its nullity $td_\lambda - q$;
- we set the free variables (columns without leading 1s) to the standard basis vectors in $\mathbb{F}^{td_\lambda - q}$ and solve for the leading variables;
- we put the resulting basis vectors into a matrix of size $(td_\lambda - q) \times td_\lambda$;
- we compute the RCF of this matrix, which is $\mathrm{RCF}_\lambda(\mathrm{All}(n))$.

Step 3. We compare the results of Steps 1 and 2 to determine whether there exist new multilinear identities satisfied by the tetrad in degree n.

Definition 2.3. The quotient module $\mathrm{New}(n) = \mathrm{All}(n)/\mathrm{Old}(n)$ consists of the new identities for the tetrad in degree n: a complete set of representatives for the equivalence classes of all identities modulo old identities.

To obtain a canonical set of generators for $\mathrm{New}(n)$, for each λ we compare the matrices $\mathrm{RCF}_\lambda(\mathrm{Old}(n))$ and $\mathrm{RCF}_\lambda(\mathrm{All}(n))$. If these two matrices have the same rank, they must be equal; this indicates that every identity in the isotypic component λ follows from identities of lower degree. If the ranks are not equal, then $\mathtt{oldrank}(\lambda)$, the rank of $\mathrm{RCF}_\lambda(\mathrm{Old}(n))$, must be strictly less than $\mathtt{allrank}(\lambda)$, the rank of $\mathrm{RCF}_\lambda(\mathrm{All}(n))$; moreover, the row space of $\mathrm{RCF}_\lambda(\mathrm{Old}(n))$ must be a subspace of the row space of $\mathrm{RCF}_\lambda(\mathrm{All}(n))$. The difference $\mathtt{allrank}(\lambda) - \mathtt{oldrank}(\lambda)$ is the multiplicity of $[\lambda]$ in $\mathrm{New}(n)$. To find S_n-module generators for the isotypic component λ in $\mathrm{New}(n)$, we identify the positions $(i_1, j_1), \dots, (i_{\mathtt{oldrank}}, j_{\mathtt{oldrank}})$ of the leading 1s in $\mathrm{RCF}_\lambda(\mathrm{Old}(n))$, and the analogous positions $(i'_1, j'_1), \dots, (i'_{\mathtt{allrank}}, j'_{\mathtt{allrank}})$ in $\mathrm{RCF}_\lambda(\mathrm{All}(n))$. Comparing these two sets of positions, we obtain

$$\mathcal{J} = \left\{ j_1, \dots, j_{\mathtt{oldrank}} \right\} \subsetneq \left\{ j'_1, \dots, j'_{\mathtt{allrank}} \right\} = \mathcal{J}', \quad \mathcal{J}' \setminus \mathcal{J} = \left\{ j'_{k_1}, \dots, j'_{k_{\mathtt{allrank-oldrank}}} \right\}.$$

Rows $i'_{k_1}, \dots, i'_{k_{\mathtt{allrank-oldrank}}}$ of $\mathrm{RCF}_\lambda(\mathrm{All}(n))$ are the canonical generators of $\mathrm{New}(n)$.

2.3. **Linear algebra over $\mathbb{Q}$, $\mathbb{F}_p$ and $\mathbb{Z}$.** In general, we prefer to do all computations over the field $\mathbb{Q}$ of rational numbers. However, it is well known that, even if the original matrix is very sparse and its entries belong to $\{0, \pm 1\}$, standard algorithms for computing the RCF produce exponential increases in the matrix entries. Even if enough computer memory is available to store the intermediate results, the calculations can take far too much time. It is therefore often convenient to use modular arithmetic, so that each matrix entry uses a fixed small amount of memory. This leads to the problem of rational reconstruction: recovering the correct results over $\mathbb{Q}$ or $\mathbb{Z}$ from the known results over $\mathbb{F}_p$.

In general, rational reconstruction is extremely ill-defined: we want to compute an inverse for a partially-defined infinity-to-one map. Therefore it is only effective when we have a good theoretical understanding of the arithmetical nature of the expected results. In our computations, we may assume that the correct rational coefficients have a common highly composite denominator. The reason is that the vector spaces we study are all modules over the symmetric group S_n, and in the Wedderburn decomposition of the group algebra $\mathbb{Q}S_n$ into a direct sum of full matrix algebras, the matrix units are linear combinations of permutations in which the coefficients have $n!$ as their common denominator. Moreover, $\mathbb{F}S_n$ is semisimple whenever $\mathbb{F}$ has characteristic 0 or $p > n$, so formulas giving the Wedderburn decomposition in rational arithmetic also apply to modular arithmetic with a prime larger than the degree of the multilinear polynomial identities. For a more precise statement of this fact, see [8, Lemma 8].

If we use a large enough prime, say $p > n!$ rather than $p > n$, then we can recognize the common denominator d of the rational coefficients from the distribution of the congruence classes modulo p: the modular coefficients are clustered near the congruence classes representing i/d for $1 \le i \le d-1$. This allows us to recover the rational coefficients; we then multiply by the LCM of the denominators to get integer coefficients, and finally divide by the GCD of the coefficients to get the integer vector with least Euclidean length which is a scalar multiple of the original coefficient vector. Once we have a realistic conjecture for the correct integer coefficients, we can perform a much simpler computation using rational arithmetic to verify the results.

Most of our computations involve finding a basis of integer vectors for the nullspace of a matrix with integer entries. In some cases, the modular methods described in the previous paragraphs give good results, meaning that the basis vectors have small Euclidean lengths. In other cases, we obtain much better results using the Hermite normal form (HNF) of an integer matrix (the analogue over $\mathbb{Z}$ of the RCF) together with the LLL algorithm for lattice basis reduction. If A is an $s \times t$ matrix over $\mathbb{Z}$ then computing the HNF of the transpose produces two matrices over $\mathbb{Z}$: a $t \times s$ matrix H and a $t \times t$ matrix U with $\det(U) = \pm 1$ such that $UA^t = H$. If $\text{rank}(A) = r$ then the bottom $t - r$ rows of U form a lattice basis for the left integer nullspace of A^t, which is the right integer nullspace of A. We then apply the LLL algorithm with increasing values of the parameter to this basis in order to obtain shorter basis vectors. We define a precise measure of the size of a lattice basis consisting of integer vectors $v_1, \dots, v_k$ by the number of decimal digits

in the product of the Euclidean lengths of the vectors:

$$(1) \qquad \sum_{i=1}^{k} \log_{10} \|v_i\|.$$

For a more detailed discussion of the application of HNF and LLL to polynomial identities, including algorithms in pseudocode, see [6, §3] and the monograph [3].

3. Defining identities for Jordan quadruple systems

In this section we determine a complete set of S_n-module generators for the multilinear polynomial identities satisfied by the tetrad in degrees 4 and 7.

Lemma 3.1. *Every multilinear identity in degree 4 satisfied by the tetrad in every associative algebra is a consequence of the symmetry* $\{a, b, c, d\} - \{d, c, b, a\} \equiv 0$.

Proof. Consider two copies of the group algebra $\mathbb{Q}S_4$; the first, denoted Q, is the multilinear subspace of degree 4 in the free quaternary algebra with one operation $\{-, -, -, -\}$; the second, denoted A, is the multilinear subspace of degree 4 in the free associative algebra. Bases of Q and A are the sets $\{\, \{a^\sigma, b^\sigma, c^\sigma, d^\sigma\} \mid \sigma \in S_4 \,\}$ and $\{\, a^\sigma b^\sigma c^\sigma d^\sigma \mid \sigma \in S_4 \,\}$, ordered lexicographically. We initialize the 24×24 expansion matrix E in which the (i, j) entry is the coefficient of the i-th associative monomial in the expansion of the j-th quaternary monomial, and compute its RCF, which has rank 12, and hence nullity 12. We obtain a basis for the nullspace by setting the free variables equal to the standard basis vectors in $\mathbb{Q}^{12}$ and solving for the leading variables. We put these basis vectors into a 12×24 matrix and compute its RCF; every row represents a permutation of the stated symmetry. $\square$

Theorem 3.2. *Every multilinear identity in degree 7 satisfied by the tetrad in every associative algebra is a consequence of the symmetry and these three identities in degree 7:*

$\{\{a,b,c,d\},e,f,g\} + \{\{a,b,f,e\},d,c,g\} + \{\{d,c,f,e\},a,b,g\} - \{g,\{b,a,d,c\},f,e\}$
$- \{g,\{b,a,e,f\},c,d\} - \{g,\{c,d,e,f\},b,a\} \equiv 0,$

$\{\{a,b,c,d\},e,f,g\} - \{\{a,b,g,f\},e,c,d\} + \{\{a,b,d,c\},e,g,f\} - \{\{a,b,f,g\},e,d,c\}$
$+ \{\{a,e,c,d\},b,g,f\} - \{\{a,e,g,f\},b,d,c\} + \{\{a,e,d,c\},b,f,g\} - \{\{a,e,f,g\},b,c,d\}$
$- \{a,\{b,c,d,e\},f,g\} + \{a,\{b,g,f,e\},c,d\} - \{a,\{b,d,c,e\},g,f\} + \{a,\{b,f,g,e\},d,c\} \equiv 0,$

$\{\{a,b,c,d\},e,f,g\} - \{\{a,f,g,c\},b,e,d\} + \{\{c,b,a,d\},e,g,f\} + \{\{f,b,c,e\},g,a,d\}$
$- \{\{f,g,a,e\},c,b,d\} - \{\{f,g,a,d\},b,c,e\} - \{\{f,g,e,d\},a,b,c\} + \{\{g,b,a,e\},f,c,d\}$
$- \{\{g,f,c,e\},a,b,d\} - \{\{g,f,c,d\},b,a,e\} - \{\{g,f,e,d\},c,b,a\} + \{\{e,a,b,d\},c,f,g\}$
$+ \{\{e,c,b,d\},a,g,f\} + \{a,\{b,c,g,f\},e,d\} + \{c,\{b,a,f,g\},e,d\} - \{f,\{b,c,e,g\},a,d\}$
$- \{g,\{b,a,e,f\},c,d\} + \{e,\{a,g,f,c\},b,d\} \equiv 0.$

Proof. By symmetry, we need to consider only two association types in degree 7,

$$\{\{-,-,-,-\},-,-,-\}, \qquad \{-,\{-,-,-,-\},-,-\},$$

each having $7!/2 = 2520$ distinct multilinear monomials, for a total of 5040. A basis of Quad(7) consists of these monomials ordered by association type and then by lex order of the permutation. A basis of the group algebra $\mathbb{Q}S_7$, the multilinear

subspace in the free associative algebra, consists of the permutations in lex order. We record the expansion of the identity monomial in each association type:

$$\{\{a,b,c,d\},e,f,g\} = abcdefg + dcbaefg + gfeabcd + gfedcba,$$

$$\{a,\{b,c,d,e\},f,g\} = abcdefg + aedcbfg + gfbcdea + gfedcba.$$

The other expansions are obtained by permutation of the arguments, since the expansion map $E\colon \mathrm{Quad}(7) \to \mathbb{Q}S_7$ is an S_7-module homomorphism.

Using modular arithmetic ($p = 101$) we initialize a 5040×5040 matrix in which the (i,j) entry is the coefficient of the i-th associative monomial in the expansion of the j-th quaternary monomial. We compute the RCF and find that the rank is 2520, and hence the nullity is 2520. As in the proof of Lemma 3.1, we find a basis of the nullspace; every coefficient belongs to $\{0,1,2,50,51,99,100\}$. For each vector, we multiply by 2 if 50 or 51 is a coefficient, and do nothing otherwise. Since the rows are coefficient vectors of polynomial identities, we can multiply by nonzero scalars to obtain equivalent identities with simpler coefficients. Reducing modulo p using symmetric representatives, the coefficients belong to $\{-4,-2,-1,0,1,2,4\}$. We regard these as integers, and sort the vectors by increasing Euclidean length.

The next step is to extract from the linear basis of 2520 vectors a much smaller set of generators for the nullspace as an S_7-module. Using modular arithmetic, we initialize a zero matrix consisting of upper and lower 5040×5040 blocks. For each vector, regarded as the coefficients of a polynomial identity, we:

- check to see if the identity belongs to the row space of the matrix;
- if not, apply all permutations of the arguments to the identity, and store the results in the rows of the lower block;
- compute the RCF of the matrix; if the rank increases, record the identity as a generator (at this point, the lower block is again zero).

Only four identities increase the rank, and one belongs to the submodule generated by the others, which are independent (none is a consequence of the other two). These are the three identities in the statement of this theorem.   □

Remark 3.3. The identities of Theorem 3.2 can be checked by hand. For example, the expansion of the first identity into the free associative algebra produces

$$abcdefg + gfeabcd + dcbaefg + gfedcba + abfedcg + gcdabfe$$
$$+ efbadcg + gcdefba + dcfeabg + gbadcfe + efcdabg + gbaefcd$$
$$- gbadcfe - efbadcg - gcdabfe - efcdabg - gbaefcd - dcbaefg$$
$$- gfeabcd - dcfeabg - gcdefba - abcdefg - gfedcba - abfedcg = 0.$$

We need computer algebra to prove that these identities are a complete set of S_7-module generators for the tetrad identities in degree 7.

Definition 3.4. A **Jordan quadruple system** (or **JQS**) is a vector space Q over a field $\mathbb{F}$ of characteristic 0 or $p > 7$ with a quadrilinear map $\{-,-,-,-\}\colon Q^4 \to Q$ satisfying the symmetry of Lemma 3.1 and the identities of Theorem 3.2.

Lemma 3.5. *Over a field of characteristic 0 or $p > 7$, column "new" of Table 1 gives the multiplicity of the S_7-module $[\lambda]$ in the kernel of the expansion map for each partition λ.*

Proof. Table 1 was computed using the methods of §2.1. Columns 1 and 2 give the partitions λ and the dimensions d_λ. In degree 7 we consider only the symmetries

		symmetries			expansions				
λ	d_λ	rows	cols	symm	rows	cols	rank	null	new
7	1	3	2	0	2	1	1	1	1
61	6	18	12	4	12	6	3	9	5
52	14	42	28	12	28	14	8	20	8
51^2	15	45	30	16	30	15	6	24	8
43	14	42	28	12	28	14	7	21	9
421	35	105	70	36	70	35	18	52	16
41^3	20	60	40	24	40	20	10	30	6
3^21	21	63	42	20	42	21	9	33	13
32^2	21	63	42	20	42	21	12	30	10
321^2	35	105	70	36	70	35	17	53	17
31^4	15	45	30	16	30	15	9	21	5
2^31	14	42	28	12	28	14	7	21	9
2^21^3	14	42	28	12	28	14	6	22	10
21^5	6	18	12	4	12	6	3	9	5
1^7	1	3	2	0	2	1	0	2	2

TABLE 1. S_7-module multiplicities for the tetrad in degree 7

of the association types, since there are no liftings of identities from lower degrees. There are two association types each with one symmetry:

$$\{\{a_1, a_2, a_3, a_4\}, a_5, a_6, a_7\} - \{\{a_4, a_3, a_2, a_1\}, a_5, a_6, a_7\} \equiv 0,$$
$$\{a_1, \{a_2, a_3, a_4, a_5\}, a_6, a_7\} - \{a_1, \{a_5, a_4, a_3, a_2\}, a_6, a_7\} \equiv 0.$$

Under "symmetries", columns 3 and 4 (rows, cols) give the size of the representation matrix for the symmetries, and column 5 (symm) gives its rank. Under "expansions", columns 6 and 7 give the size of the expansion matrix, column 8 gives its rank, and column 9 gives the nullity of its *transpose*. (For an explanation of using the transpose, see [5, pp. 446-447].) Column "new" is the multiplicity of the module $[\lambda]$ (null minus symm) in the kernel of the expansion map. □

Remark 3.6. The data in Table 1 show that there is no point in searching for possibly simpler nonlinear identities whose linearizations are equivalent to the identities of Theorem 3.2. If $I \equiv 0$ is a nonlinear identity of degree 7, then every monomial of I contains $k \geq 2$ occurrences of some variable, say a. In $L(I)$, the linearization of I, the variable a is replaced by k distinct variables $a_1, \ldots, a_k$ and $L(I)$ is invariant under all permutations of these variables. To obtain the consequences of $L(I)$ in partition 1^7, we evaluate the alternating sum over all permutations of the variables in $L(I)$, but this is clearly 0, contradicting the multiplicity 2 for the module $[1^7]$.

4. SPECIAL IDENTITIES IN DEGREE 10 FOR THE TETRAD

In this section we show that there are identities for the tetrad which do not follow from the defining identities for JQS. (We have not included these identities in Definition 3.4, so that the operad for JQS is quadratic and admits a Koszul dual.)

Definition 4.1. Let A be an associative algebra and let JQS(A) be the underlying vector space of A with the tetrad as the operation. If Q is a JQS, we call Q **special** if

there is an embedding $Q \hookrightarrow \mathrm{JQS}(A)$ for some A; otherwise we call Q **exceptional**. If $I \equiv 0$ is a polynomial identity which is satisfied by the tetrad in every associative algebra but which does not follow from the defining identities for JQS, then we call I a **special identity** for the tetrad.

Open Problem 4.2. Do there exist exceptional JQS?

Theorem 4.3. *For the tetrad, and for each partition λ of 10, columns "symm", "symmlift", "null" and "new" in the corresponding row of Table 2 contain respectively the multiplicity of $[\lambda]$ in the modules* Symm(10), Old(10), All(10) *and* New(10).

Proof. The symmetry in degree 4 implies that every association type for a quadri-linear operation in degree 10 is equivalent to one of the following:

$$
(2) \quad \left\{
\begin{array}{ll}
\{\{\{-,-,-,-\},-,-,-\},-,-,-\}, & \{\{-,\{-,-,-,-\},-,-\},-,-,-\}, \\
\{-,\{\{-,-,-,-\},-,-,-\},-,-\}, & \{-,\{-,\{-,-,-,-\},-,-\},-,-\}, \\
\{\{-,-,-,-\},\{-,-,-,-\},-,-\}, & \{\{-,-,-,-\},-,\{-,-,-,-\},-\}, \\
\{\{-,-,-,-\},-,-,\{-,-,-,-\}\}, & \{-,\{-,-,-,-\},\{-,-,-,-\},-\}.
\end{array}
\right.
$$

These 8 types have respectively 1, 1, 1, 1, 2, 2, 3, 3 symmetries $\iota - \tau \equiv 0$ where ι is the identity monomial and τ is a monomial with a permutation of order 2 of the variables. We list the corresponding monomials τ for each type:

$$
\{\{\{dcba\}efg\}hij\}, \quad \{\{a\{edcb\}fg\}hij\}, \quad \{a\{\{edcb\}fgh\}ij\}, \quad \{a\{b\{fedc\}gh\}ij\},
$$

$$
\{\{dcba\}\{efgh\}ij\}, \quad \{\{abcd\}\{hgfe\}ij\}, \quad \{\{dcba\}e\{fghi\}j\}, \quad \{\{abcd\}e\{ihgf\}j\},
$$

$$
\{\{dcba\}ef\{ghij\}\}, \quad \{\{abcd\}ef\{jihg\}\}, \quad \{\{ghij\}fe\{abcd\}\}, \quad \{a\{edcb\}\{fghi\}j\},
$$

$$
\{a\{bcde\}\{ihgf\}j\}, \quad \{j\{fghi\}\{bcde\}a\}.
$$

For each association type, each symmetry reduces the number of multilinear monomials by a factor of 2, so the total number of monomials is

$$
10! \left(\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} \right) = 9979200.
$$

This number is so large that we must decompose the computation into smaller pieces using the representation theory of S_{10}. (To do the following computation without representation theory would require a 3628800×9979200 matrix.)

The symmetry implies that each identity $I(a,b,c,d,e,f,g)$ from Theorem 3.2 produces 9 liftings in degree 10:

$$
I(\{a,h,i,j\},b,c,d,e,f,g), \quad I(a,\{b,h,i,j\},c,d,e,f,g), \quad \ldots, \quad I(a,b,c,d,e,f,\{g,h,i,j\}),
$$

$$
\{I(a,b,c,d,e,f,g),h,i,j\}, \quad \{h,I(a,b,c,d,e,f,g),i,j\}.
$$

For each partition λ with module $[\lambda]$ of dimension d_λ, we construct a matrix M of size $9d_\lambda \times 8d_\lambda$ ("rows" and "cols" under "symmetries and liftings") consisting of $d_\lambda \times d_\lambda$ blocks. To process an identity, we compute the representation matrices of its terms in each association type [9], store these matrices in the last row of blocks, and compute the RCF of M, so the last row of blocks becomes zero. In this way, we process the symmetries of the association types and compute the rank of M (column "symm"). We retain these results in M and process the liftings; the rank of M increases (column "symmlift") to reach the multiplicity of $[\lambda]$ in the module Old(10). When we are done, the rows of RCF(M) form a canonical set of generators for the isotypic component of $[\lambda]$ in Old(10).

#	λ	d_λ	symmetries and liftings				expansions				new
			rows	cols	symm	symmlift	rows	cols	rank	null	
1	10	1	9	8	0	7	8	1	1	7	·
2	91	9	81	72	30	68	72	9	4	68	·
3	82	35	315	280	140	260	280	35	20	260	·
4	81^2	36	324	288	166	272	288	36	16	272	·
5	73	75	675	600	342	565	600	75	35	565	·
6	721	160	1440	1280	784	1200	1280	160	80	1200	·
7	71^3	84	756	672	442	628	672	84	44	628	·
8	64	90	810	720	416	670	720	90	50	670	·
9	631	315	2835	2520	1594	2365	2520	315	155	2365	·
10	62^2	225	2025	1800	1150	1680	1800	225	120	1680	·
11	621^2	350	3150	2800	1878	2630	2800	350	170	2630	·
12	61^4	126	1134	1008	704	941	1008	126	66	942	1
13	5^2	42	378	336	204	320	336	42	16	320	·
14	541	288	2592	2304	1456	2160	2304	288	144	2160	·
15	532	450	4050	3600	2330	3380	3600	450	220	3380	·
16	531^2	567	5103	4536	3008	4244	4536	567	291	4245	1
17	52^21	525	4725	4200	2808	3939	4200	525	260	3940	1
18	521^3	448	4032	3584	2464	3357	3584	448	224	3360	3
19	51^5	126	1134	1008	708	945	1008	126	60	948	3
20	4^22	252	2268	2016	1282	1880	2016	252	136	1880	·
21	4^21^2	300	2700	2400	1582	2260	2400	300	140	2260	·
22	43^2	210	1890	1680	1092	1580	1680	210	100	1580	·
23	4321	768	6912	6144	4032	5760	6144	768	384	5760	·
24	431^3	525	4725	4200	2802	3933	4200	525	265	3935	2
25	42^3	300	2700	2400	1562	2239	2400	300	160	2240	1
26	42^21^2	567	5103	4536	3022	4260	4536	567	276	4260	·
27	421^4	350	3150	2800	1870	2616	2800	350	180	2620	4
28	41^6	84	756	672	446	631	672	84	40	632	1
29	3^31	210	1890	1680	1080	1569	1680	210	110	1570	1
30	3^22^2	252	2268	2016	1302	1900	2016	252	116	1900	·
31	3^221^2	450	4050	3600	2322	3368	3600	450	230	3370	2
32	3^21^4	225	2025	1800	1164	1693	1800	225	105	1695	2
33	32^31	288	2592	2304	1456	2159	2304	288	144	2160	1
34	32^21^3	315	2835	2520	1588	2358	2520	315	160	2360	2
35	321^5	160	1440	1280	784	1198	1280	160	80	1200	2
36	31^7	36	324	288	162	268	288	36	20	268	·
37	2^5	42	378	336	192	309	336	42	26	310	1
38	2^41^2	90	810	720	428	680	720	90	40	680	·
39	2^31^4	75	675	600	336	559	600	75	40	560	1
40	2^21^6	35	315	280	146	265	280	35	15	265	·
41	21^8	9	81	72	28	67	72	9	5	67	·
42	1^{10}	1	9	8	2	8	8	1	0	8	·

TABLE 2. S_{10}-module multiplicities for the tetrad in degree 10

For each partition λ, we construct a matrix X of size $8d_\lambda \times d_\lambda$ ("rows" and "cols" under "expansions") consisting of $d_\lambda \times d_\lambda$ blocks; the i-th block contains the representation matrix for the terms of the expansion of the identity monomial in the i-th association type. We compute the RCF of the transpose X^t and find its rank (column "rank") and nullity (column "null); we then extract a basis for the nullspace by setting the free variables to the standard basis vectors and solving for the leading variables. We put the nullspace basis into the rows of a matrix N and compute the RCF. The rows of RCF(N) form a canonical set of generators for the isotypic component of $[\lambda]$ in All(10).

If the multiplicity of $[\lambda]$ in Old(10) coincides with its multiplicity in All(10), then there are no new identities for partition λ. (In this case, we check the consistency of the computations by verifying that the two submodules are equal, which amounts to verifying that the two matrices in RCF are equal.) Otherwise, there are new identities in degree 10 for the tetrad corresponding to partition λ.   □

Open Problem 4.4. Determine a minimal set of multilinear identities in degree 10 for the tetrad which generate New(10) as an S_{10}-module.

In the rest of this section we present some nonlinear special identities in degree 10 for the tetrad. These identities correspond to the partitions $\lambda = s1^t$ ($s + t = 10$) for which column "new" in Table 2 is nonzero, namely $\lambda = 61^4, 51^5, 41^6$. The corresponding Young diagrams have a long first row and a long tail. For these λ, the structure theory of $\mathbb{Q}S_{10}$ allows us to assume that the identity is a symmetric function of the variables in the first row and an alternating function of the variables in the tail, and this reduces the computational problem to a manageable size. We first recall the notion of linearization matrices, called operators in [7, §2].

Definition 4.5. Let $n = n_1 + \cdots + n_k$ ($n \geq n_1 \geq \cdots \geq n_k \geq 1$) be a partition and $\mu = \mu(a_1^{n_1}, \ldots, a_k^{n_k})$ a quaternary monomial of multidegree $(n_1, \ldots, n_k)$. For $i = 1, \ldots, k$ set $N_i = \{n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i\}$ so that $N_1 \cup N_2 \cup \cdots \cup N_k$ is a set partition of $\{1, \ldots, n\}$ with $|N_i| = n_i$. The **basic linearization** of μ is the multilinear monomial $\ell(\mu)$ obtained from μ by first replacing a_i by a_i' for $i = 1, \ldots, k$ and then replacing the n_i occurrences of a_i' by a_j for $j \in N_i$ with j increasing from left to right. The **symmetric** and **alternating linearizations** of μ are

$$L^+(\mu) = \sum_{\sigma_1 \in P_1} \cdots \sum_{\sigma_k \in P_k} \sigma_1 \cdots \sigma_k \ell(\mu),$$

$$L^-(\mu) = \sum_{\sigma_1 \in P_1} \cdots \sum_{\sigma_k \in P_k} \epsilon_i(\sigma_1) \cdots \epsilon_k(\sigma_k) \sigma_1 \cdots \sigma_k \ell(\mu),$$

where P_i is the symmetric group on N_i and $\epsilon_i \colon P_i \to \{\pm 1\}$ is the sign.

Remark 4.6. Over a field of characteristic 0 or $p > n$, the identity I holds if and only if $L^+(I)$ holds; the same is not true for $L^-(I)$.

Definition 4.7. Let λ be another partition of n and let $R_\lambda \colon \mathbb{Q}S_n \to M_d(\mathbb{Q})$ be the corresponding irreducible representation of S_n of dimension d_λ. The **symmetric** and **alternating linearization matrices** for partitions $n_1, \ldots n_k$ and λ are

$$L^+_{n_1, \ldots, n_k; \lambda} = \sum_{\sigma \in P_1 \times \cdots \times P_k} R_\lambda(\sigma), \qquad L^-_{n_1, \ldots, n_k; \lambda} = \sum_{\sigma \in P_1 \times \cdots \times P_k} \epsilon(\sigma) R_\lambda(\sigma).$$

Linearization matrices make it much easier to compute the representation matrix for the linearization of a nonlinear monomial μ: we can use the basic linearization $\ell(\mu)$ of the nonlinear monomial instead of using all the terms in $L^+(\mu)$.

Lemma 4.8. *For all partitions $n_1, \ldots, n_k$ and λ and all monomials μ, we have*

$$R_\lambda(L^+(\mu)) = L^+_{n_1,\ldots,n_k;\lambda} \cdot R_\lambda(\ell(\mu)), \qquad R_\lambda(L^-(\mu)) = L^-_{n_1,\ldots,n_k;\lambda} \cdot R_\lambda(\ell(\mu)).$$

Proof. These equations follow directly from the definitions, the linearity of the maps, and the homomorphism property of a representation. □

Theorem 4.9. *The multihomogeneous identities in Figure 1 are satisfied by the tetrad in every associative algebra but are not consequences of the defining identities for JQS. The sums are over all permutations σ of the non-repeated variables and $\epsilon(\sigma)$ is the sign.*

Proof. The Young tableaux for partitions $\lambda = 61^4, 51^5, 41^6$ are
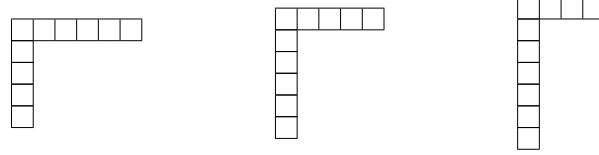
For each λ, the primitive idempotents in the Young basis of the two-sided ideal in the group algebra correspond to standard tableaux; each idempotent is the product of the symmetric sum over row permutations and the alternating sum over column permutations. Hence an identity corresponding to a partition λ with first row of size s and tail of size t is symmetric in the first s variables and alternating in the last t variables. This allows us to reduce the number of monomials by considering only those in which the non-repeated variables appear in increasing order, and identifying these monomials with the corresponding alternating sums. We write in detail the proof for $\lambda = 61^4$; the others are similar.

Let J be the multihomogeneous subspace with variables $a^6 bcde$ in the free quaternary algebra with one operation $\{-,-,-,-\}$ satisfying symmetry in degree 4. For the eight association types (2) we have respectively 134, 134, 134, 134, 86, 86, 49, 52 monomials, and so $\dim J = 809$ (recall that the non-repeated variables appear in increasing order). Let A be the corresponding multihomogeneous subspace in the free associative algebra; we have $\dim A = 5040$. We construct the 809×5040 expansion matrix E in which the (i,j) entry is the (integer) coefficient of the j-th associative monomial in the expansion of the i-th quaternary monomial. Using modular arithmetic, we compute $\mathrm{RCF}(E)$ and obtain rank 110 and nullity 699; the entries of the RCF are 0, ± 1, ± 2, ± 4. We identify the columns which contain leading 1s; these columns $j_1, \ldots, j_{110}$ form a basis of the column space. Using integer arithmetic, we combine these columns of E into a much smaller 809×110 matrix E'. We compute $H = \mathrm{HNF}(E')$ and confirm that the rank and nullity have not changed in characteristic 0; this computation also provides an 809×809 unimodular matrix U such that $UE' = H$. The last 699 rows of U form a lattice basis for the integer nullspace of E'; they have already been reduced using the LLL algorithm, and the entries are 0, ± 1, ± 2. Using formula (1), this basis has size ≈ 607. To get identities as simple as possible, we apply the LLL algorithm again, with standard parameter 3/4; this new basis has size ≈ 537. We sort these 699 vectors by increasing length.

$$\sum_{\sigma \in S_4} \epsilon(\sigma) \Big(\{\{\{a,a,a,b^\sigma\},a,a,c^\sigma\},d^\sigma,a,e^\sigma\} - \{\{\{a,a,a,b^\sigma\},a,a,c^\sigma\},d^\sigma,e^\sigma,a\}$$
$$+ \{\{\{a,a,b^\sigma,a\},c^\sigma,a,a\},d^\sigma,e^\sigma,a\} + \{\{\{a,a,b^\sigma,a\},c^\sigma,a,d^\sigma\},a,e^\sigma,a\}$$
$$+ \{\{\{a,b^\sigma,a,c^\sigma\},a,d^\sigma,a\},a,a,e^\sigma\} - \{\{\{a,b^\sigma,a,c^\sigma\},a,d^\sigma,a\},a,e^\sigma,a\}$$
$$- \{\{a,\{a,a,b^\sigma,a\},a,c^\sigma\},d^\sigma,a,e^\sigma\} - \{\{a,\{a,b^\sigma,a,c^\sigma\},a,d^\sigma\},a,e^\sigma,a\}$$
$$+ \{a,\{\{a,a,b^\sigma,a\},a,c^\sigma,d^\sigma\},a,e^\sigma\} - \{a,\{\{a,a,b^\sigma,c^\sigma\},d^\sigma,a,a\},a,e^\sigma\}$$
$$+ \{a,\{\{a,a,b^\sigma,c^\sigma\},d^\sigma,a,a\},e^\sigma,a\} + \{a,\{a,b^\sigma,a,c^\sigma\},a,d^\sigma,a\},e^\sigma,a\} \Big) \equiv 0$$

$$\sum_{\sigma \in S_5} \epsilon(\sigma) \Big(\{\{\{a,a,b^\sigma,c^\sigma\},a,d^\sigma,a\},a,e^\sigma,f^\sigma\} - \{\{\{a,a,b^\sigma,c^\sigma\},a,d^\sigma,a\},e^\sigma,a,f^\sigma\}$$
$$- \{\{\{a,a,b^\sigma,c^\sigma\},d^\sigma,a,a\},a,e^\sigma,f^\sigma\} + \{\{\{a,a,b^\sigma,c^\sigma\},d^\sigma,a,a\},e^\sigma,a,f^\sigma\}$$
$$- \{\{\{a,b^\sigma,a,c^\sigma\},a,d^\sigma,a\},a,e^\sigma,f^\sigma\} + \{\{\{a,b^\sigma,a,c^\sigma\},a,d^\sigma,a\},e^\sigma,a,f^\sigma\}$$
$$+ \{\{\{a,b^\sigma,a,c^\sigma\},d^\sigma,a,a\},a,e^\sigma,f^\sigma\} - \{\{\{a,b^\sigma,a,c^\sigma\},d^\sigma,a,a\},e^\sigma,a,f^\sigma\}$$
$$- \{\{\{b^\sigma,a,c^\sigma,d^\sigma\},a,e^\sigma,a\},a,f^\sigma,a\} - \{\{\{b^\sigma,a,c^\sigma,d^\sigma\},e^\sigma,a,a\},f^\sigma,a,a\}$$
$$- \{\{\{b^\sigma,c^\sigma,a,d^\sigma\},a,e^\sigma,a\},f^\sigma,a,a\} - \{\{\{b^\sigma,c^\sigma,a,d^\sigma\},e^\sigma,a,a\},a,f^\sigma,a\} \Big) \equiv 0$$

$$\sum_{\sigma \in S_5} \epsilon(\sigma) \Big(\{\{\{a,a,a,b^\sigma\},c^\sigma,d^\sigma,a\},e^\sigma,a,f^\sigma\} - \{\{\{a,a,a,b^\sigma\},c^\sigma,d^\sigma,a\},e^\sigma,f^\sigma,a\}$$
$$+ \{\{\{a,b^\sigma,a,c^\sigma\},d^\sigma,e^\sigma,a\},a,a,f^\sigma\} - \{\{\{a,b^\sigma,a,c^\sigma\},d^\sigma,e^\sigma,a\},a,f^\sigma,a\}$$
$$- \{\{a,\{a,a,b^\sigma,c^\sigma\},a,d^\sigma\},e^\sigma,f^\sigma,a\} - \{\{a,\{a,a,b^\sigma,c^\sigma\},d^\sigma,a\},e^\sigma,a,f^\sigma\}$$
$$+ \{\{a,\{a,a,b^\sigma,c^\sigma\},d^\sigma,a\},e^\sigma,f^\sigma,a\} + \{\{a,\{a,b^\sigma,a,c^\sigma\},a,d^\sigma\},e^\sigma,f^\sigma,a\}$$
$$- \{\{a,\{b^\sigma,a,c^\sigma,d^\sigma\},e^\sigma,a\},a,a,f^\sigma\} - \{\{a,\{b^\sigma,c^\sigma,a,d^\sigma\},e^\sigma,a\},a,f^\sigma,a\}$$
$$+ \{a,\{\{a,a,b^\sigma,c^\sigma\},a,d^\sigma,e^\sigma\},f^\sigma,a\} - \{a,\{\{a,b^\sigma,a,c^\sigma\},a,d^\sigma,e^\sigma\},f^\sigma,a\} \Big) \equiv 0$$

$$\sum_{\sigma \in S_5} \epsilon(\sigma) \Big(\{\{\{a,a,b^\sigma,a\},c^\sigma,a,d^\sigma\},e^\sigma,f^\sigma,a\} - \{\{\{a,b^\sigma,a,c^\sigma\},a,a,d^\sigma\},e^\sigma,f^\sigma,a\}$$
$$- \{\{\{a,b^\sigma,a,c^\sigma\},a,d^\sigma,a\},e^\sigma,f^\sigma,a\} + \{\{\{a,b^\sigma,a,c^\sigma\},d^\sigma,a,e^\sigma\},a,a,f^\sigma\}$$
$$+ \{\{\{a,b^\sigma,a,c^\sigma\},d^\sigma,e^\sigma,a\},a,a,f^\sigma\} + \{\{a,\{a,a,b^\sigma,c^\sigma\},d^\sigma,a\},e^\sigma,a,f^\sigma\}$$
$$+ \{\{a,\{a,b^\sigma,a,c^\sigma\},a,d^\sigma\},e^\sigma,a,f^\sigma\} + \{\{a,\{a,b^\sigma,a,c^\sigma\},d^\sigma,a\},e^\sigma,f^\sigma,a\}$$
$$- \{\{a,\{b^\sigma,a,c^\sigma,d^\sigma\},a,e^\sigma\},a,a,f^\sigma\} - \{\{a,\{b^\sigma,c^\sigma,a,d^\sigma\},a,e^\sigma\},a,f^\sigma,a\}$$
$$+ \{\{b^\sigma,\{a,a,c^\sigma,a\},d^\sigma,a\},e^\sigma,a,f^\sigma\} - \{a,\{\{a,a,b^\sigma,c^\sigma\},d^\sigma,a,e^\sigma\},a,f^\sigma\}$$
$$- \{a,\{\{a,b^\sigma,a,c^\sigma\},a,d^\sigma,e^\sigma\},a,f^\sigma\} - \{a,\{\{a,b^\sigma,a,c^\sigma\},a,d^\sigma,e^\sigma\},f^\sigma,a\}$$
$$- \{a,\{\{a,b^\sigma,a,c^\sigma\},d^\sigma,a,e^\sigma\},f^\sigma,a\} \Big) \equiv 0$$

$$\sum_{\sigma \in S_6} \epsilon(\sigma) \Big(\{\{\{b^\sigma,a,c^\sigma,d^\sigma\},a,a,e^\sigma\},a,f^\sigma,g^\sigma\} - \{\{\{b^\sigma,a,c^\sigma,d^\sigma\},a,a,e^\sigma\},f^\sigma,a,g^\sigma\}$$
$$+ \{\{\{b^\sigma,a,c^\sigma,d^\sigma\},e^\sigma,a,f^\sigma\},a,a,g^\sigma\} - \{\{\{b^\sigma,c^\sigma,a,d^\sigma\},a,a,e^\sigma\},a,f^\sigma,g^\sigma\}$$
$$+ \{\{\{b^\sigma,c^\sigma,a,d^\sigma\},a,a,e^\sigma\},f^\sigma,a,g^\sigma\} + \{\{\{b^\sigma,c^\sigma,a,d^\sigma\},a,e^\sigma,f^\sigma\},a,a,g^\sigma\} \Big) \equiv 0$$

FIGURE 1. Some special identities for the tetrad in degree 10

We proceed as in the proof of Theorem 4.3 using modular arithmetic ($p = 101$) but considering only partition $\lambda = 61^4$. The matrix has size 1134×1008. We first compute the rank of the symmetries (704) and the symmetries with the liftings (941). The usual next step would be to process the linearizations of the 699 nonlinear identities obtained in the previous paragraph; this however produces multilinear identities with very large numbers of terms. In order to reduce the size of the computation, we use the equations of Lemma 4.8. We require a combination of symmetric linearization over the first 6 variables (in the first row of the Young diagram) with alternating linearization over the last 4 variables (in the tail). This involves a slight modification of the linearization matrices in Definition 4.7. Our combined linearization matrix is the sum of $6!4! = 17280$ terms, but it factors as the product of two partial linearization matrices with respectively 6! terms (permutations $\sigma \in S_6$ acting on $\{1, \dots, 6\}$) and 4! terms (permutations $\tau \in S_4$ acting on $\{7, \dots, 10\}$), for a total of only $6! + 4! = 744$ terms:

$$\sum_{\sigma \in S_6, \tau \in S_4} \epsilon(\tau) R_\lambda(\sigma\tau) = \sum_{\sigma \in S_6, \tau \in S_4} \epsilon(\tau) R_\lambda(\sigma) R_\lambda(\tau) = \sum_{\sigma \in S_6} R_\lambda(\sigma) \cdot \sum_{\tau \in S_4} \epsilon(\tau) R_\lambda(\tau).$$

This 126×126 matrix is surprisingly simple: it has rank 1 and 21 nonzero entries. Since the rank of all identities for $\lambda = 61^4$ is 942, we expect exactly one new identity. After processing the 699 nonlinear identities, we find that only one increases the rank. This is the identity in the statement of the theorem. □

5. Four families of finite dimensional Jordan quadruple systems; examples of universal associative envelopes

In this section we consider some small examples of special JQS, and determine the structure of their universal associative envelopes. We first define four infinite families A_n, B_n, C_{pqr}, D_{pqq} of finite dimensional JQS which are subspaces of matrix algebras which are closed under the tetrad.

- A_n consists of all $n \times n$ matrices; the dimension is n^2.
- B_n consists of all $n \times n$ symmetric matrices; the dimension is $\frac{1}{2}n(n+1)$.

For the remaining systems, we recall the classification [10] of simple associative n-ary algebras in the case $n = 4$. We choose integers $p, q, r \geq 1$ and consider block matrices of size $p + q + r$, where M_{pq} is a matrix of size $p \times q$:

$$(3) \qquad \begin{bmatrix} 0 & 0 & M_{pr} \\ M_{qp} & 0 & 0 \\ 0 & M_{rq} & 0 \end{bmatrix}$$

This space is closed under the associative quadruple product of matrices.

- C_{pqr} ($p \geq q, r$) consists of all matrices (3); the dimension is $pq + qr + rp$.
- D_{pq} consists of all matrices (3) with $q = r$ where $M_{pq} = M_{qp}^t$ and M_{qq} is symmetric; the dimension is $pq + \frac{1}{2}q(q+1)$.

We consider the smallest non-trivial system J in each family: A_2, B_2, C_{111}, D_{11} with dimensions 4, 3, 3, 2 respectively. To construct the universal associative envelope $U(J)$, we use noncommutative Gröbner bases in free associative algebras [4]: $U(J)$ is the quotient $F\langle B \rangle / I(G)$ of the free associative algebra $F\langle B \rangle$ on a basis B of J by the ideal $I(G)$ generated by the following set of relations:

$$G = \Big\{ abcd + dcba - \{a,b,c,d\} \mid a, b, c, d \in B \Big\}.$$

We find a Gröbner basis of $I(G)$ and identify the monomials in $F\langle B\rangle$ which do not have the leading monomial of any Gröbner basis element as a subword. The cosets of these monomials form a basis for $U(J)$. If $U(J)$ is finite dimensional, then we can determine its structure using the Wedderburn decomposition of associative algebras [2]. We consider the four smallest systems by increasing dimension.

Proposition 5.1. *We have $U(D_{11}) \cong \mathbb{Q} \oplus M_3(\mathbb{Q})$, and hence up to isomorphism, D_{11} has only two finite dimensional irreducible representations, the 1-dimensional trivial representation and the 3-dimensional natural representation.*

Proof. In the natural representation by 3×3 matrices, D_{11} has this basis:

$$a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The only nonzero quadruple products are $\{a, b, a, a\} = a$ and $\{b, a, a, b\} = b$. Hence $U(D_{11}) = F\langle a, b\rangle / I(G)$ where $I(G)$ is generated by the following self-reduced set $\{g_1, \ldots, g_{10}\}$ in lex order of leading monomials with $a \prec b$:

$$a^4, \quad aba^2 + a^2ba - a, \quad ab^2a, \quad ba^3 + a^3b, \quad ba^2b - b,$$

$$baba + abab, \quad b^2a^2 + a^2b^2, \quad b^2ab + bab^2, \quad b^3a + ab^3, \quad b^4.$$

We make the Gröbner basis algorithm deterministic by considering the pairs (g_i, g_j) in lex order of the subscripts (i, j); whenever their composition is reducible, we reduce the highest term in deglex order using the generator whose leading monomial is lowest in deglex order. For example, (g_2, g_1) produces this composition:

$$g_2a^2 - abg_1 = (aba^2 + a^2ba - a)a^2 - aba^4 = aba^4 + a^2ba^3 - a^3 - aba^4 = a^2ba^3 - a^3.$$

We reduce this composition in three steps using g_2, g_2, g_1 as follows:

$$a^2ba^3 - a^3 - ag_2a = a^2ba^3 - a^3 - a(aba^2 + a^2ba - a)a = -a^3ba^2,$$

$$-a^3ba^2 + a^2g_2 = -a^3ba^2 + a^2(aba^2 + a^2ba - a) = a^4ba - a^3,$$

$$a^4ba - a^3 - g_1ba = a^4ba - a^3 - a^4ba = -a^3.$$

We cannot reduce further, so we record a^3 as a (monic) composition. This process gives the following set $\{h_1, \ldots, h_{24}\}$ of distinct nonzero reduced compositions:

$$a^3, \quad ab^2, \quad bab, \quad b^2a, \quad b^3, \quad a^3b, \quad a^2b^2, \quad abab, \quad ab^3, \quad bab^2, \quad a^3ba,$$

$$a^3b^2, \quad a^2bab, \quad a^2b^3 + b^2, \quad abab^2, \quad bab^3, \quad a^3bab, \quad a^2bab^2 - a^3b^3 - ab^2,$$

$$a^2bab^2 - bab, \quad a^2bab^2, \quad abab^3, \quad a^3bab^2 - 2a^2b^2, \quad a^2bab^3, \quad a^2bab^3 - ab^3.$$

We sort $\{g_1, \ldots, g_{10}, h_1, \ldots, h_{24}\}$ by deglex order of leading monomials, and compute the normal form of each element with respect to the preceding elements. Many normal forms are 0, showing that $I(G)$ is generated by these five elements:

$$b^2, \qquad a^3, \qquad bab, \qquad aba^2 + a^2ba - a, \qquad ba^2b - b.$$

All compositions of these elements reduce to 0, so we have a Gröbner basis for $I(G)$. Only $d = 10$ monomials in $F\langle a, b\rangle$ are not divisible any of the leading monomials:

$$1, \quad a, \quad b, \quad a^2, \quad ab, \quad ba, \quad a^2b, \quad aba, \quad ba^2, \quad a^2ba.$$

The cosets of these monomials modulo $I(G)$ form a basis for $U(D_{11})$. We calculate the structure constants for $U(D_{11})$ by computing the normal forms of the products of these monomials with respect to the Gröbner basis; see Table 3.

	1	a	b	a^2	ab	ba	a^2b	aba	ba^2	a^2ba
1	1	a	b	a^2	ab	ba	a^2b	aba	ba^2	a^2ba
a	a	a^2	ab	0	a^2b	aba	0	a^2ba	$-a^2ba+a$	0
b	b	ba	0	ba^2	0	0	b	0	0	ba
a^2	a^2	0	a^2b	0	0	a^2ba	0	0	a^2	0
ab	ab	aba	0	$-a^2ba+a$	0	0	ab	0	0	aba
ba	ba	ba^2	0	0	b	0	0	ba	0	0
a^2b	a^2b	a^2ba	0	a^2	0	0	a^2b	0	0	a^2ba
aba	aba	$-a^2ba+a$	0	0	ab	0	0	aba	0	0
ba^2	ba^2	0	b	0	0	ba	0	0	ba^2	0
a^2ba	a^2ba	a^2	0	0	a^2b	0	0	a^2ba	0	0

TABLE 3. Structure constants for $U(D_{11})$

The radical of $U(D_{11})$ is the nullspace of the Dickson matrix [4, Corollary 12]: the $d \times d$ matrix Δ defined in terms of the structure constants c_{ij}^k for $U(D_{11})$:

$$\Delta = \begin{bmatrix} 10 & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 3 & 3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 3 & 3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\ \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \left(\Delta_{ij} = \sum_{k=1}^{d} \sum_{\ell=1}^{d} c_{ji}^k c_{k\ell}^\ell \right)$$

This matrix has full rank, so $U(D_{11})$ is semisimple. The center $Z(U(D_{11}))$ is the nullspace of the $d^2 \times d$ matrix from [4, Corollary 15]; the RCF of this matrix is

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

Hence a basis for $Z(U(D_{11}))$ consists of (the cosets of) 1 and $c = a^2b + aba + ba^2$. We calculate $c^2 = c$, so $x(x-1)$ is the minimal polynomial of c. Hence a basis for $Z(U(D_{11}))$ consisting of orthogonal primitive idempotents is $\{c, 1-c\}$.

We compute the two-sided ideals in $U(D_{11})$ generated by c and $1-c$. We find that $\dim\langle 1-c \rangle = 1$ so that $\langle 1-c \rangle \cong \mathbb{F}$ with basis $\mathbb{1} = 1 - a^2b - aba - ba^2$. Moreover, $\dim\langle c \rangle = 9$, so we expect that $\langle c \rangle \cong M_3(\mathbb{F})$. To obtain an isomorphism, we compute a basis for $\langle c \rangle$, and find that it consists of the leading monomials of the Gröbner basis (excluding 1). The coset of b generates a 3-dimensional left ideal in $\langle c \rangle$ with basis

$e_1 = b$, $e_2 = ab$, $e_3 = a^2 b$. We identify these with the standard basis vectors in $\mathbb{F}^3$ and determine the elements of $\langle c \rangle$ corresponding to the matrix units E_{ij} ($1 \le i, j \le 3$). For each i, j we determine $c_1, \dots, c_9 \in \mathbb{Q}$ so that the coset of the generic element,

$$E = c_1 a + c_2 b + c_3 a^2 + c_4 ab + c_5 ba + c_6 a^2 b + c_7 aba + c_8 ba^2 + c_9 a^2 ba,$$

acts like the matrix unit E_{ij}; that is, $E_{ij} e_k = \delta_{jk} e_i$. From Table 3 we obtain:

$$Eb = c_8 b + c_1 ab + c_3 a^2 b, \quad Eab = c_5 b + c_7 ab + (c_1 + c_9) a^2 b, \quad Ea^2 b = c_2 b + c_4 ab + c_6 a^2 b.$$

From this we obtain linear systems which determine $c_1, \dots, c_9$ for each E_{ij}, and solving these systems gives this isomorphism $\langle c \rangle \cong M_3(\mathbb{F})$:

$$\left[E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33} \right] \longleftrightarrow \left[ba^2, ba, b, a - a^2 ba, aba, ab, a^2, a^2 ba, a^2 b \right].$$

The set $\{\, \mathbb{1},\, E_{ij} \mid 1 \le i, j \le 3 \,\}$ is a basis for $U(D_{11})$ which reflects the decomposition into simple two-sided ideals, and proves the proposition. $\square$

Open Problem 5.2. Determine the structure of $U(D_{pq})$ for all p, q. In particular, do we always have the isomorphism $U(D_{pq}) \cong \mathbb{Q} \oplus M_n(\mathbb{Q})$ where $n = pq + \frac{1}{2} q(q+1)$?

Proposition 5.3. *We have $U(C_{111}) \cong \mathbb{Q} \oplus 2\, M_3(\mathbb{Q})$, and hence up to isomorphism, C_{111} has only three finite dimensional irreducible representations: 1-dimensional trivial, 3-dimensional natural, and another 3-dimensional.*

Proof. Our basis for C_{111} consists of matrices defining the natural representation:

$$a = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The nonzero quadrilinear products are: $\{a, c, b, a\} = a$, $\{b, c, a, b\} = b$, $\{c, b, a, c\} = c$. The original generating set for $I(G)$ contains 45 elements and is already self-reduced. We obtain 290 distinct nontrivial compositions; the resulting set of 335 generators collapses to a self-reduced set of only 13 elements, which is a Gröbner basis:

$$a^2, \quad b^2, \quad c^2, \quad aba, \quad aca, \quad bab, \quad bcb, \quad cac, \quad cbc,$$
$$acba + abca - a, \quad bcab + bacb - b, \quad cbac + cabc - c, \quad cabca - ca.$$

Only 19 monomials in $F\langle a, b, c \rangle$ do not have a leading monomial of a Gröbner basis element as a subword:

$$1, \ a, \ b, \ c, \ ab, \ ac, \ ba, \ bc, \ ca, \ cb, \ abc, \ acb, \ bac, \ bca, \ cab, \ cba, \ abca, \ bacb, \ cabc.$$

Hence $U(C_{111})$ has dimension 19; as in the proof of Proposition 5.1 we find that $U(C_{111})$ is semisimple, and its center has dimension 3. The cosets of these elements form a basis of the center consisting of orthogonal primitive idempotents:

$$1 - abc - acb - bac - bca - cab - cba, \qquad abc + bca + cab, \qquad acb + bac + cba.$$

These elements generate simple two-sided ideals of dimensions 1, 9, 9 respectively. We omit the isomorphisms of each simple ideal with a full matrix algebra. $\square$

Open Problem 5.4. Determine the structure of the universal associative envelope of C_{pqr} for all p, q, r. Do we always have the isomorphism $U(C_{pqr}) \cong \mathbb{Q} \oplus 2\, M_n(\mathbb{Q})$ where $n = pq + qr + rp$?

Proposition 5.5. *We have $U(B_2) \otimes_{\mathbb{Q}} \mathbb{K} \cong \mathbb{K} \oplus 3\, M_2(\mathbb{K})$ where $\mathbb{K} = \mathbb{Q}(\beta)$ with $\beta = 1 + \sqrt{-3}$. Hence over this quadratic extension field, B_2 has only four finite dimensional irreducible representations up to isomorphism: 1-dimensional trivial, 2-dimensional natural, and another two 2-dimensional.*

Proof. Our basis for B_2 consists of matrices defining the natural representation:

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

There are 19 nonzero quadrilinear products of these generators (which we omit). The original generating set for $I(G)$ contains 45 elements and is already self-reduced. We obtain 533 distinct nontrivial compositions; the resulting set of 578 generators collapses to a self-reduced set of only 8 elements, which is a Gröbner basis:

$$ab, \quad ba, \quad ca - bc, \quad cb - ac, \quad c^2 - b^2 - a^2, \quad a^4 - a, \quad b^4 - b, \quad b^3 c + a^3 c - c.$$

Only 13 monomials in $F\langle a, b, c \rangle$ do not have a leading monomial of a Gröbner basis element as a subword:

$$1, \quad a, \quad b, \quad c, \quad a^2, \quad ac, \quad b^2, \quad bc, \quad a^3, \quad a^2 c, \quad b^3, \quad b^2 c, \quad a^3 c.$$

Hence $U(B_2)$ has dimension 13; as before, we find that $U(B_2)$ is semisimple. The center has dimension 4 and the following basis: 1, $a + b$, $a^2 + b^2$, $a^3 + b^3$. These elements are not orthogonal primitive idempotents; unlike the previous two cases, to split the center we must extend the base field.

- First splitting: set $e = 1$ and $x = a + b$. The minimal polynomial of x is $t^4 - t = t(t^3 - 1)$, so we set $y = x$ and $z = x^3 - e$. Then y generates a 3-dimensional central ideal with identity element $a^3 + b^3$, and $\zeta_1 = -z$ is a primitive idempotent.
- Second splitting: set $e = a^3 + b^3$ and $x = a + b$. The minimal polynomial of x is $t^3 - 1 = (t - 1)(t^2 + t + 1)$, so we set $y = x - e$ and $z = x^2 + x + e$. Then y generates a 2-dimensional central ideal with identity element $-\frac{1}{3}(a + b) - \frac{1}{3}(a^2 + b^2) + \frac{2}{3}(a^3 + b^3)$, and $\zeta_2 = \frac{1}{3}z$ is a primitive idempotent.
- Third splitting: set $e = -\frac{1}{3}(a+b) - \frac{1}{3}(a^2+b^2) + \frac{2}{3}(a^3+b^3)$ and $x = (a+b) - (a^3+b^3)$. The minimal polynomial of x is $t^2 + 3t + 3 = (t - \alpha)(t - \bar{\alpha})$ where $\alpha = \frac{1}{2}(-3 + \sqrt{-3})$. We set $y = x - \alpha e$ and $z = x - \bar{\alpha} e$; each generates a 1-dimensional central ideal. We get the idempotents $\zeta_3 = \frac{-1}{\sqrt{-3}} y$ and $\zeta_4 = \frac{1}{\sqrt{-3}} z$.

Using the fact that $ab = ba = 0$ in $U(B_2)$ since these monomials are in the Gröbner basis, we obtain the central basis of orthogonal primitive idempotents:

$$\zeta_1 = 1 - a^3 - b^3, \qquad \zeta_2 = \tfrac{1}{3}(a + b + a^2 + b^2 + a^3 + b^3),$$

$$\zeta_3 = -\tfrac{1}{6}\Big(\beta(a + b) + \bar{\beta}(a^2 + b^2) - 2(a^3 + b^3)\Big),$$

$$\zeta_4 = -\tfrac{1}{6}\Big(\bar{\beta}(a + b) + \beta(a^2 + b^2) - 2(a^3 + b^3)\Big),$$

where $\beta = 1 + \sqrt{-3}$. These four elements generate simple two-sided ideals of dimensions 1, 4, 4, 4 respectively. We omit the calculation of the isomorphisms of each simple ideal with a full matrix algebra. $\qquad \square$

Open Problem 5.6. Determine the structure of the universal associative envelope of B_n for all n. In particular, after a quadratic extension $\mathbb{K}$ of $\mathbb{Q}$, do we always have the isomorphism $U(B_n) \otimes_{\mathbb{Q}} \mathbb{K} \cong \mathbb{K} \oplus 3\, M_n(\mathbb{K})$?

Proposition 5.7. *We have $U(A_2) \otimes_{\mathbb{Q}} \mathbb{K} \cong \mathbb{K} \oplus 6\,M_2(\mathbb{K})$ where $\mathbb{K} = \mathbb{Q}(\beta)$, $\beta = 1 + \sqrt{-3}$. Hence over this quadratic extension, A_2 has exactly seven finite dimensional irreducible representations up to isomorphism: 1-dimensional trivial, 2-dimensional natural, and another five 2-dimensional.*

Proof. Our basis for A_2 consists of these matrices in the natural representation:

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad d = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are 31 nonzero quadrilinear products of these generators (which we omit). The original generating set of $I(G)$ contains 136 elements and is already self-reduced. We obtain 2769 distinct nontrivial compositions; the resulting 2905 generators collapses to a self-reduced set of 25 elements, which is a Gröbner basis:

$$ad, \quad b^2, \quad bd - ab, \quad c^2, \quad cd - ac, \quad da, \quad db - ba, \quad dc - ca, \quad d^2 - cb - bc + a^2,$$

$$aba, \quad aca, \quad acb + abc - a^3, \quad bab, \quad bca - abc, \quad bcb - ba^2 - a^2b, \quad cac,$$

$$cba + abc - a^3, \quad cbc - ca^2 - a^2c, \quad a^4 - a, \quad ba^3 + a^3b - b, \quad ba^2b, \quad ca^3 + a^3c - c,$$

$$ca^2b + ba^2c - d, \quad ca^2c, \quad cabc + a^3c - c.$$

Only 25 monomials in $F\langle a, b, c, d \rangle$ do not have a leading monomial of a Gröbner basis element as a subword:

$$1, \quad a, \quad b, \quad c, \quad d, \quad a^2, \quad ab, \quad ac, \quad ba, \quad bc, \quad ca, \quad cb, \quad a^3, \quad a^2b,$$

$$a^2c, \quad abc, \quad ba^2, \quad bac, \quad ca^2, \quad cab, \quad a^3b, \quad a^3c, \quad a^2bc, \quad ba^2c, \quad a^3bc.$$

Hence $U(A_2)$ has dimension 25; as before, we find that $U(A_2)$ is semisimple. The center has dimension 7 and the following basis:

$$1, \quad a+d, \quad bc + cb, \quad a^3 - abc + bac, \quad abc + cab, \quad a - a^2bc + ba^2c, \quad -\tfrac{1}{2}(a^2 + bc) + a^3bc.$$

These elements are not orthogonal primitive idempotents; as in the previous case, to split the center we must extend the base field. We obtain the following central basis consisting of orthogonal primitive idempotents:

$$1 - a^3 - bac - cab,$$

$$\tfrac{1}{3}\big(d - a^2 + cb + abc + cab + a^2bc - ba^2c + 2a^3bc\big),$$

$$\tfrac{1}{3}\big(a + a^2 + bc + a^3 - abc + bac - a^2bc + ba^2c - 2a^3bc\big),$$

$$-\tfrac{1}{6}\big(\beta a + \bar{\beta}a^2 + \bar{\beta}bc - 2a^3 + 2abc - 2bac - \beta a^2bc + \beta ba^2c - 2\bar{\beta}a^3bc\big),$$

$$-\tfrac{1}{6}\big(\beta d - \bar{\beta}a^2 + \bar{\beta}cb - 2abc - 2cab + \beta a^2bc - \beta ba^2c + 2\bar{\beta}a^3bc\big),$$

$$-\tfrac{1}{6}\big(\bar{\beta}a + \beta a^2 + \beta bc - 2a^3 + 2abc - 2bac - \bar{\beta}a^2bc + \bar{\beta}ba^2c - 2\beta a^3bc\big),$$

$$-\tfrac{1}{6}\big(\bar{\beta}d - \beta a^2 + \beta cb - 2abc - 2cab + \bar{\beta}a^2bc - \bar{\beta}ba^2c + 2\beta a^3bc\big),$$

where $\beta = 1 + \sqrt{-3}$. These seven elements generate simple two-sided ideals of dimensions $1, 4, 4, 4, 4, 4, 4$ respectively. We omit the calculation of the isomorphisms of each simple ideal with a full matrix algebra. $\qquad \square$

Open Problem 5.8. Determine the structure of the universal associative envelope for A_n for all n. In particular, after making a quadratic extension $\mathbb{K}$ of $\mathbb{Q}$, do we always have the isomorphism $U(A_n) \otimes_{\mathbb{Q}} \mathbb{K} \cong \mathbb{K} \oplus 6\,M_n(\mathbb{K})$?

6. Defining identities for anti-Jordan quadruple systems

In this and the next two sections we present analogous results for the anti-tetrad $[a, b, c, d] = abcd - dcba$. Since the methods are very similar, we omit most details.

Lemma 6.1. *Every multilinear identity in degree 4 satisfied by the anti-tetrad is a consequence of the skew-symmetry* $[a, b, c, d] + [d, c, b, a] \equiv 0$.

Theorem 6.2. *Every multilinear identity in degree 7 satisfied by the anti-tetrad is a consequence of skew-symmetry in degree 4 and these two identities in degree 7:*

$$[[a, b, c, d], e, f, g] - [[a, b, f, e], d, c, g] + [[d, c, f, e], a, b, g] + [g, [b, a, d, c], f, e]$$
$$- [g, [b, a, e, f], c, d] + [g, [c, d, e, f], b, a] \equiv 0,$$

$$[[a, b, c, d], e, f, g] - [[a, f, g, c], b, e, d] + [[c, a, b, d], g, e, f] - [[c, b, a, f], g, e, d]$$
$$- [[c, g, d, b], a, e, f] - [[c, g, e, f], a, b, d] + [[c, g, e, d], b, a, f] - [[b, a, c, f], e, d, g]$$
$$- [[b, c, f, g], a, e, d] + [[b, e, d, g], a, c, f] - [[g, f, e, d], c, b, a] + [[g, d, e, f], c, a, b]$$
$$- [[f, a, b, d], e, g, c] - [[f, c, e, d], g, a, b] - [a, [b, c, g, f], e, d] - [c, [a, b, d, g], e, f]$$
$$+ [c, [b, a, f, g], e, d] + [b, [a, c, g, d], e, f] - [b, [a, g, f, c], e, d] + [b, [a, g, d, e], c, f]$$
$$+ [d, [c, b, a, e], f, g] \equiv 0.$$

Proof. For the anti-tetrad the rank is 2519 and the nullity is 2521. □

Definition 6.3. An **anti-Jordan quadruple system** (**AJQS**) is a vector space Q over a field $\mathbb{F}$ of characteristic 0 or $p > 7$ with a quadrilinear map $[-, -, -, -]: Q^4 \to Q$ satisfying the skew-symmetry of Lemma 6.1 and the identities of Theorem 6.2.

		symmetries			expansions				
λ	d_λ	rows	cols	sym	rows	cols	rank	null	new
7	1	3	2	2	2	1	0	2	0
61	6	18	12	8	12	6	3	9	1
52	14	42	28	16	28	14	8	20	4
51^2	15	45	30	14	30	15	6	24	10
43	14	42	28	16	28	14	7	21	5
421	35	105	70	34	70	35	18	52	18
41^3	20	60	40	16	40	20	10	30	14
3^21	21	63	42	22	42	21	9	33	11
32^2	21	63	42	22	42	21	12	30	8
321^2	35	105	70	34	70	35	17	53	19
31^4	15	45	30	14	30	15	9	21	7
2^31	14	42	28	16	28	14	7	21	5
2^21^3	14	42	28	16	28	14	6	22	6
21^5	6	18	12	8	12	6	3	9	1
1^7	1	3	2	2	2	1	0	2	0

TABLE 4. S_7-module multiplicities for the anti-tetrad in degree 7

Lemma 6.4. *Over a field of characteristic 0 or $p > 7$, for each partition λ column "new" of Table 4 gives the multiplicity of $[\lambda]$ in the kernel of the expansion map for the anti-tetrad.*

7. Special identities in degree 10 for the anti-tetrad

The notions of special and exceptional for AJQS, and special identities for the anti-tetrad, are the obvious analogues of Definition 4.1.

Open Problem 7.1. Do there exist exceptional AJQS?

Theorem 7.2. *For the anti-tetrad, and for each partition λ of 10, columns "symm", "symmlift", "null" and "new" in the corresponding row of Table 5 contain respectively the multiplicity of $[\lambda]$ in the modules* Symm(10), Old(10), All(10) *and* New(10).

Open Problem 7.3. Determine a minimal set of multilinear identities in degree 10 for the anti-tetrad which generate New(10) as an S_{10}-module.

In the rest of this section we present some nonlinear special identities in degree 10 for the anti-tetrad. These identities correspond to the partitions $\lambda = st$ ($s + t = 10$) for which column "new" in Table 5 is nonzero, namely $\lambda = 82, 73, 64, 5^2$. The corresponding Young diagrams have only two rows of lengths s and t. For these λ, the structure theory of $\mathbb{Q}S_{10}$ allows us to assume that the identity is a symmetric function of two disjoint sets of variables, so we may reduce the size of the computations by working with nonlinear monomials which have only two variables a and b occurring respectively s and t times.

Theorem 7.4. *The multihomogeneous identities in Figure 2 are satisfied by the anti-tetrad in every associative algebra but are not consequences of the defining identities for AJQS.*

Remark 7.5. For the anti-tetrad, we obtained better results using the LLL algorithm with a higher value of the reduction parameter. We illustrate with $\lambda = 64$; the underlying variables are a^6b^4 which produce 210 associative monomials. The eight association types produce respectively 82, 82, 82, 82, 32, 32, 14, 14 nonlinear quaternary monomials for a total of 420. The 210×420 expansion matrix has rank 99 and nullity 321. Computing the HNF with an initial application of LLL produces a lattice basis of the nullspace with size ≈ 421 using equation (1); the coefficients belong to $\{0, \pm1, \pm2, -3\}$. Applying LLL with parameter 3/4 reduces the basis size to ≈ 389 with coefficients $\{0, \pm1, \pm2\}$. Applying LLL again with parameter 99/100 reduces the basis size to ≈ 337 with coefficients $\{0, \pm1, -2\}$. We obtain three special nonlinear identities for this partition.

8. Four families of finite dimensional anti-Jordan quadruple systems; examples of universal associative envelopes

We define four families of finite dimensional AJQS, denoted A_n^-, B_n^-, C_{pqr}^-, D_{pq}^-; these are subspaces of matrix algebras which are closed under the anti-tetrad.

- A_n^- consists of all $n \times n$ matrices; the dimension is n^2.
- B_n^- consists of all $n \times n$ skew-symmetric matrices; the dimension is $\frac{1}{2}n(n-1)$.

For the remaining systems, we choose integers $p, q, r \geq 1$ and consider block matrices of size $p + q + r$ as in equation (3).

- C_{pqr}^- ($p \geq q, r$) consists of all matrices (3); the dimension is $pq + qr + rp$.
- D_{pq}^- consists of all matrices (3) with $q = r$ where $M_{pq} = -M_{qp}^t$ and M_{qq} is skew-symmetric; the dimension is $pq + \frac{1}{2}q(q-1)$.

The smallest non-trivial systems in each family are A_2^-, B_3^-, C_{111}^-, D_{21}^- with dimensions 4, 3, 3, 2 respectively. We consider these systems by increasing dimension.

			symmetries and liftings				expansions				
#	λ	d_λ	rows	cols	sym	lift	rows	cols	rank	null	new
1	10	1	9	8	8	8	8	1	0	8	·
2	91	9	81	72	64	68	72	9	4	68	·
3	82	35	315	280	222	264	280	35	15	265	1
4	81^2	36	324	288	218	269	288	36	19	269	·
5	73	75	675	600	442	555	600	75	40	560	5
6	721	160	1440	1280	896	1197	1280	160	80	1200	3
7	71^3	84	756	672	446	632	672	84	40	632	·
8	64	90	810	720	524	677	720	90	40	680	3
9	631	315	2835	2520	1722	2352	2520	315	160	2360	8
10	62^2	225	2025	1800	1200	1693	1800	225	105	1695	2
11	621^2	350	3150	2800	1798	2619	2800	350	180	2620	1
12	61^4	126	1134	1008	612	948	1008	126	60	948	·
13	5^2	42	378	336	236	309	336	42	26	310	1
14	541	288	2592	2304	1568	2155	2304	288	144	2160	5
15	532	450	4050	3600	2390	3365	3600	450	230	3370	5
16	531^2	567	5103	4536	2960	4258	4536	567	276	4260	2
17	52^21	525	4725	4200	2698	3935	4200	525	265	3935	·
18	521^3	448	4032	3584	2240	3360	3584	448	224	3360	·
19	51^5	126	1134	1008	604	942	1008	126	66	942	·
20	4^22	252	2268	2016	1358	1899	2016	252	116	1900	1
21	4^21^2	300	2700	2400	1562	2239	2400	300	160	2240	1
22	43^2	210	1890	1680	1124	1568	1680	210	110	1570	2
23	4321	768	6912	6144	4032	5760	6144	768	384	5760	·
24	431^3	525	4725	4200	2702	3940	4200	525	260	3940	·
25	42^3	300	2700	2400	1582	2260	2400	300	140	2260	·
26	42^21^2	567	5103	4536	2944	4245	4536	567	291	4245	·
27	421^4	350	3150	2800	1810	2630	2800	350	170	2630	·
28	41^6	84	756	672	442	627	672	84	44	628	1
29	3^31	210	1890	1680	1132	1580	1680	210	100	1580	·
30	3^22^2	252	2268	2016	1338	1880	2016	252	136	1880	·
31	3^221^2	450	4050	3600	2402	3380	3600	450	220	3380	·
32	3^21^4	225	2025	1800	1184	1680	1800	225	120	1680	·
33	32^31	288	2592	2304	1568	2160	2304	288	144	2160	·
34	32^21^3	315	2835	2520	1726	2365	2520	315	155	2365	·
35	321^5	160	1440	1280	896	1200	1280	160	80	1200	·
36	31^7	36	324	288	222	271	288	36	16	272	1
37	2^5	42	378	336	244	320	336	42	16	320	·
38	2^41^2	90	810	720	516	670	720	90	50	670	·
39	2^31^4	75	675	600	446	565	600	75	35	565	·
40	2^21^6	35	315	280	218	260	280	35	20	260	·
41	21^8	9	81	72	64	68	72	9	4	68	·
42	10	1	9	8	8	8	8	1	0	8	·

TABLE 5. S_{10}-module multiplicities for the anti-tetrad in degree 10

$[[[a, a, b, a], a, b, a], a, a, a] - [a, [[a, a, a, b], a, b, a], a, a] - [a, [a, [a, a, b, a], a, a], b, a]$
$+ [a, [a, [a, a, b, a], b, a], a, a] \equiv 0,$

$[[[a, a, b, a], a, b, a], b, a, a] + [[a, [a, b, a, b], a, a], b, a, a] - [a, [[a, a, b, a], b, a, a], b, a]$
$- [a, [[a, b, a, b], a, a, b], a, a] - [a, [[a, b, a, b], a, b, a], a, a] - [[a, a, b, a], [a, b, a, b], a, a] \equiv 0,$

$[[[a, a, b, a], b, b, a], a, a, a] - [a, [[a, a, a, b], a, b, b], a, a] + [a, [[a, a, b, a], a, b, b], a, a]$
$- [a, [[a, a, b, a], b, b, a], a, a] + [a, [[a, a, b, b], a, a, b], a, a] - [a, [[a, a, b, b], a, b, a], a, a] \equiv 0,$

$[[[a, a, b, a], a, b, a], a, a, b] - [a, [[a, a, a, b], a, b, a], a, b] + [a, [[a, a, b, a], a, a, b], a, b]$
$- [b, [[a, a, b, a], a, b, a], a, a] - [a, [a, [a, a, b, a], a, b], a, b] + [a, [a, [a, a, b, a], b, a], a, b] \equiv 0,$

$[[a, [a, a, b, a], a, a], b, b, a] - [[a, [a, a, b, a], a, b], a, a, b] + [[a, [a, a, b, a], a, b], b, a, a]$
$+ [[a, [a, a, b, b], a, a], b, a, a] - [a, [[a, a, b, a], a, b, b], a, a] - [a, [[a, a, b, b], a, b, a], a, a]$
$+ [[a, a, a, b], [a, a, b, a], a, b] - [[a, a, a, b], b, [a, a, b, a], a] \equiv 0,$

$[[[a, a, a, b], a, a, a], b, b, a] - [[[a, a, a, b], a, b, b], a, a, a] + [[a, [a, a, a, b], a, b], b, a, a]$
$- [[a, [a, a, b, a], a, a], b, b, a] + [[a, [a, a, b, b], a, a], a, a, b] - [[a, [a, a, b, b], b, a], a, a, a]$
$+ [a, [[a, a, b, a], b, b, a], a, a] - [a, [[a, a, b, b], a, a, b], a, a] + [b, [a, [a, a, a, b], a, b], a, a]$
$- [[a, a, a, b], [a, a, b, b], a, a] \equiv 0,$

$[[b, [a, a, a, b], a, b], a, a, b] + [[b, [a, a, b, a], a, b], a, a, b] - [b, [[a, a, a, b], b, a, a], a, b]$
$- [b, [[a, a, b, a], a, a, b], a, b] \equiv 0,$

$[[[a, a, b, a], b, b, a], b, a, a] - [[a, [a, b, a, b], a, a], b, b, a] - [[a, [a, b, a, b], b, a], a, b, a]$
$- [a, [[a, a, b, a], b, a, b], b, a] + [a, [[a, b, a, b], a, a, b], b, a] \equiv 0,$

$[[[a, a, b, a], b, b, a], b, a, a] - [a, [[a, a, b, a], b, b, a], b, a] - [a, [[a, b, a, b], a, b, b], a, a]$
$+ [a, [[b, a, b, b], a, a, b], a, a] - [a, [[b, a, b, b], a, b, a], a, a] \equiv 0,$

$[[[a, a, b, a], b, b, a], b, a, b] - [a, [[a, b, a, b], a, b, b], a, b] + [a, [[b, a, b, b], a, a, b], a, b]$
$- [a, [[b, a, b, b], a, b, a], a, b] - [b, [[a, a, b, a], b, b, a], b, a] \equiv 0.$

FIGURE 2. Some special identities for the anti-tetrad in degree 10

Proposition 8.1. *The universal associative envelope $U(D_{21}^-)$ is $\mathbb{Z}$-graded by degree and infinite dimensional; in degrees $n \geq 6$ it is linearly isomorphic to the (commutative) polynomial algebra $\mathbb{F}[a, b]$.*

Proof. This is the basis for D_{21}^- which defines the natural representation:

$$a = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Every quadrilinear product in these generators is 0, so the original 6 ideal generators have no terms of degree 1; it follows that $U(D_{21}^-)$ is graded by degree:

$$aba^2 - a^2ba, \quad ba^3 - a^3b, \quad baba - abab, \quad b^2a^2 - a^2b^2, \quad b^2ab - bab^2, \quad b^3a - ab^3.$$

These generators produce 12 distinct non-trivial compositions:

$$a^3ba - a^4b, \quad a^2b^2a - a^3b^2, \quad ba^2ba - a^2bab, \quad ba^2b^2 - a^2b^3,$$

$$bab^2a - abab^2, \quad bab^3 - ab^4, \quad a^2bab^2 - a^3b^3, \quad ba^2bab - abab^2a,$$

$$a^4bab - a^5b^2, \quad a^3bab^2 - a^4b^3, \quad abab^4 - a^2b^5, \quad ba^2bab^2 - a^3b^4.$$

After self-reduction, the 18 generators become a Gröbner basis of 13 elements:

$$aba^2 - a^2ba, \quad ba^3 - a^3b, \quad baba - abab, \quad b^2a^2 - a^2b^2, \quad b^2ab - bab^2, \quad b^3a - ab^3, \quad a^3ba - a^4b,$$

$$a^2b^2a - a^3b^2, \quad ba^2ba - a^2bab, \quad ba^2b^2 - a^2b^3, \quad bab^2a - abab^2, \quad bab^3 - ab^4, \quad a^2bab^2 - a^3b^3.$$

There is an infinite set of monomials in $F\langle a, b\rangle$ which are not divisible by the leading monomial of any element of the Gröbner basis, and $U(D_{21}^-)$ is infinite dimensional. Here are the basis monomials for $U(D_{21}^-)$ in degrees ≤ 5:

$$1, \quad a, \quad b, \quad a^2, \quad ab, \quad ba, \quad b^2, \quad a^3, \quad a^2b, \quad aba, \quad ab^2, \quad ba^2, \quad bab, \quad b^2a, \quad b^3,$$

$$a^4, \quad a^3b, \quad a^2ba, \quad a^2b^2, \quad abab, \quad ab^2a, \quad ab^3, \quad ba^2b, \quad bab^2, \quad b^4,$$

$$a^5, \quad a^4b, \quad a^3b^2, \quad a^2bab, \quad a^2b^3, \quad abab^2, \quad ab^4, \quad b^5.$$

In degree $n \geq 6$ a basis consists of the monomials $a^{n-i}b^i$ for $i = 0, \ldots, n$. To see this, note that every element of the Gröbner basis has the form $m_1 - m_2$, and replacing m_1 by m_2 moves a to the left and b to the right; now use induction on n. Thus in degrees $n \geq 6$, $U(D_{21}^-)$ is isomorphic to the (commutative) polynomial algebra $\mathbb{Q}[a, b]$, but there is some noncommutativity in degrees $n \leq 5$. $\qquad\square$

Remark 8.2. We did similar computations for D_{31}^-. The original set of 36 homogeneous generators produced 278 distinct non-trivial compositions; after self-reduction the 314 generators became a Gröbner basis of 94 elements. It follows that $U(D_{31}^-)$ is $\mathbb{Z}$-graded and infinite dimensional, and that for degree $n \geq 6$, the dimension of the homogeneous subspace is $\binom{n+2}{2}$.

Conjecture 8.3. *For all $p \geq 2$, the universal associative envelope $U(D_{p1}^-)$ is $\mathbb{Z}$-graded and infinite dimensional, and for degree $n \geq 6$, the dimension of the homogeneous component is $\binom{n+p-1}{p-1}$, equal to that of the (commutative) polynomial algebra in p variables.*

Open Problem 8.4. Study the representation theory of the anti-Jordan triple systems D_{p1}^- and their universal associative envelopes $U(D_{p1}^-)$. Comparing our results with [14] suggests that $U(D_{p1}^-)$ is likely to be a (generalized) down-up algebra.

Remark 8.5. We did similar computations for D_{12}^-. The original set of 36 non-homogeneous generators produced 341 distinct non-trivial compositions; after self-reduction the 377 generators became 19 elements. This set produced one more non-trivial composition, resulting in a Gröbner basis of 20 elements:

$$c^2, \quad a^3, \quad a^2b, \quad aba, \quad ab^2, \quad aca, \quad ba^2, \quad bab, \quad b^2a, \quad b^3,$$

$$bca + acb, \quad bcb, \quad cac, \quad cbc, \quad acba + a^2cb - a, \quad bacb + acb^2 - b,$$

$$cbac - cabc - c, \quad a^2cb^2 - ab, \quad acb^2c - bc, \quad ca^2cb - ca.$$

From this it follows that $U(D_{12}^-)$ has dimension 26.

Open Problem 8.6. Study the representation theory of the anti-Jordan triple systems D_{pq}^- for $q \geq 2$. In particular, do we always have $U(D_{pq}^-) \cong \mathbb{Q} \oplus M_{p+2q}(\mathbb{Q})$?

Proposition 8.7. *We have $U(C_{111}^-) \cong \mathbb{Q} \oplus 2\,M_3(\mathbb{Q})$, and hence up to isomorphism, C_{111}^- has only three finite dimensional irreducible representations: 1-dimensional trivial, 3-dimensional natural, and another 3-dimensional.*

Proof. Our standard basis for C_{111}^- is the same as that for C_{111}; see the proof of Proposition 5.3. Only three quadrilinear products are nonzero:

$$[a,c,b,a] = a, \qquad [b,c,a,b] = -b, \qquad [c,b,a,c] = c.$$

The original set G of generators for the ideal $I(G)$ contains 36 elements and is already self-reduced. We obtain 333 distinct nontrivial compositions of these generators; the resulting set of 369 generators collapses to a self-reduced set of only 13 elements, which is a Gröbner basis:

$$a^2, \quad b^2, \quad c^2, \quad aba, \quad aca, \quad bab, \quad bcb, \quad cac, \quad cbc,$$
$$acba - abca - a, \quad bcab - bacb + b, \quad cbac - cabc - c, \quad cabca + ca.$$

The leading monomials of these elements are the same as those of the Gröbner basis in the proof of Proposition 5.3. Hence $U(C_{111}^-)$ has dimension 19, and a basis consists of the same monomials as for $U(C_{111})$ in the proof of Proposition 5.3. We find that $U(C_{111}^-)$ is semisimple, and its center has dimension 3. The cosets of these elements form a central basis of orthogonal primitive idempotents:

$$1 + abc - acb - bac + bca + cab - cba, \qquad -abc - bca - cab, \qquad acb + bac + cba.$$

These elements generate simple two-sided ideals of dimensions 1, 9, 9 respectively. We omit the isomorphisms of each simple ideal with a full matrix algebra.   □

Open Problem 8.8. Study the representation theory of the anti-Jordan triple systems C_{pqr}^-. In particular, do we always have $U(C_{pqr}^-) \cong \mathbb{Q} \oplus 2\,M_{p+q+r}(\mathbb{Q})$?

Proposition 8.9. *We have $U(B_3^-) \otimes_{\mathbb{Q}} \mathbb{K} \cong \mathbb{K} \oplus 3\,M_3(\mathbb{K})$ where $\mathbb{K} = \mathbb{Q}(\beta)$, $\beta = 1 + \sqrt{-3}$. Hence up to isomorphism, B_3^- has four finite dimensional irreducible representations over $\mathbb{K}$: 1-dimensional trivial, 3-dimensional natural, and two other 3-dimensional.*

Proof. Our standard basis for B_3^- consists of these three 3×3 matrices defining the natural representation:

$$a = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The original set G of generators for the ideal $I(G)$ contains 36 elements and is already self-reduced. We obtain 385 distinct nontrivial compositions of these generators; the resulting set of 421 generators collapses to a self-reduced set of 30 elements, which is a Gröbner basis:

$$aba, \quad aca, \quad ac^2 + ab^2 - a^3, \quad bab, \quad b^2a + ab^2 - a^3, \quad b^3 - ba^2 - a^2b, \quad bca + acb,$$
$$bcb, \quad bc^2 - a^2b, \quad ca^2 - b^2c, \quad cab + bac, \quad cac, \quad cba + abc, \quad cb^2 - a^2c, \quad cbc,$$
$$c^2a - ab^2, \quad c^2b - ba^2, \quad c^3 - b^2c - a^2c, \quad a^2cb - a^2bc - a, \quad ab^2c, \quad ba^3 - a^3b - c,$$
$$ba^2c, \quad bacb + a^3c - b, \quad a^5 + cb - bc, \quad a^4b + ac, \quad a^4c - ab, \quad a^3b^2 + cb,$$
$$a^3bc + \tfrac{1}{2}c^2 - \tfrac{1}{2}b^2 + \tfrac{1}{2}a^2, \quad ba^2b^2 - ca, \quad ba^2bc + ba.$$

Only 28 monomials in $F\langle a, b, c \rangle$ do not have a leading monomial of a Gröbner basis element as a subword:

$$1, \quad a, \quad b, \quad c, \quad a^2, \quad ab, \quad ac, \quad ba, \quad b^2, \quad bc, \quad ca, \quad cb, \quad c^2, \quad a^3, \quad a^2b, \quad a^2c,$$
$$ab^2, \quad abc, \quad acb, \quad ba^2, \quad bac, \quad b^2c, \quad a^4, \quad a^3b, \quad a^3c, \quad a^2b^2, \quad a^2bc, \quad ba^2b.$$

Hence $U(B_3^-)$ has dimension 28; as before, we find that $U(B_3^-)$ is semisimple. The center has dimension 4; finding a basis of orthogonal primitive idempotents requires a quadratic extension of $\mathbb{Q}$. The required basis is

$$1 + abc - acb - bac,$$
$$-\tfrac{1}{6}\big(a^2 + b^2 + c^2 + 2abc - 2acb - 2bac - 2a^4 - 2ba^2b\big),$$
$$\tfrac{1}{12}\big(\beta(a^2 + b^2 + c^2) - 4abc + 4acb + 4bac - 2\bar{\beta}(a^4 + ba^2b)\big),$$
$$\tfrac{1}{12}\big(\bar{\beta}(a^2 + b^2 + c^2) - 4abc + 4acb + 4bac - 2\beta(a^4 + ba^2b)\big),$$

where $\beta = 1 + \sqrt{-3}$. These four elements generate simple two-sided ideals of dimensions 1, 4, 4, 4 respectively. We omit the calculation of the isomorphisms of each simple ideal with a full matrix algebra. □

Open Problem 8.10. Study the representation theory of the anti-Jordan triple systems B_n^-. In particular, over a quadratic extension $\mathbb{K}$ of $\mathbb{Q}$ do we always have $U(B_n^-) \otimes_{\mathbb{Q}} \mathbb{K} \cong \mathbb{K} \oplus 3 M_n(\mathbb{K})$?

Proposition 8.11. *We have* $U(A_2^-) \otimes_{\mathbb{Q}} \mathbb{K} \cong \mathbb{K} \oplus 6 M_2(\mathbb{K})$ *where* $\mathbb{K} = F(\beta)$, $\beta = 1 + \sqrt{-3}$. *Hence up to isomorphism, A_2^- has seven finite dimensional irreducible representations over* $\mathbb{K}$: *1-dimensional trivial, 2-dimensional natural, and five other 2-dimensional.*

Proof. Our standard basis for A_2^- consists of the four 2×2 matrix units as in the proof of Proposition 5.7. The original set G of generators for the ideal $I(G)$ contains 120 elements and is already self-reduced. We obtain 2821 distinct nontrivial compositions of these generators; the resulting set of 2941 generators collapses after three iterations to a self-reduced set of only 28 elements, which is a Gröbner basis:

$$ad, \quad b^2, \quad bd - ab, \quad c^2, \quad cd - ac, \quad da, \quad db - ba, \quad dc - ca,$$
$$d^2 - cb - bc + a^2, \quad aba, \quad aca, \quad acb + abc - a^3, \quad bab, \quad bca - abc,$$
$$bcb - ba^2 - a^2b, \quad cac, \quad cba + abc - a^3, \quad cbc - ca^2 - a^2c, \quad a^2bc - \tfrac{1}{2}a^4 - \tfrac{1}{2}a,$$
$$ba^3 - a^3b + b, \quad ba^2b, \quad ca^3 - a^3c - c, \quad ca^2b - ba^2c - d, \quad ca^2c,$$
$$cabc - a^3c - c, \quad a^4b - ab, \quad a^4c + ac, \quad a^6 - 2abc + a^3.$$

Only 25 monomials in $F\langle a, b, c, d \rangle$ do not have a leading monomial of a Gröbner basis element as a subword:

$$1, \quad a, \quad b, \quad c, \quad d, \quad a^2, \quad ab, \quad ac, \quad ba, \quad bc, \quad ca, \quad cb, \quad a^3, \quad a^2b,$$
$$a^2c, \quad abc, \quad ba^2, \quad bac, \quad ca^2, \quad cab, \quad a^4, \quad a^3b, \quad a^3c, \quad ba^2c, \quad a^5.$$

Hence $U(A_2^-)$ has dimension 25; as before, we find that $U(A_2^-)$ is semisimple. The center has dimension 7; splitting it requires a quadratic extension of $\mathbb{Q}$. We obtain this central basis of orthogonal primitive idempotents:

$$1 + a^3 - 2abc + bac - cab,$$

$$\tfrac{1}{6}\big(a + 2d + 2cb + 2abc + 2cab + a^4 + 2ba^2c + 2a^5\big),$$

$$-\tfrac{1}{6}\big(a - 2bc + 2a^3 - 2abc + 2bac - a^4 - 2ba^2c + 2a^5\big),$$

$$\tfrac{1}{12}\big(\beta a - 2\bar{\beta}bc - 4a^3 + 4abc - 4bac - \beta a^4 - 2\beta ba^2c + 2\bar{\beta}a^5\big),$$

$$\tfrac{1}{12}\big(\bar{\beta}a - 2\beta bc - 4a^3 + 4abc - 4bac - \bar{\beta}a^4 - 2\bar{\beta}ba^2c + 2\beta a^5\big),$$

$$-\tfrac{1}{12}\big(\beta a + 2\beta d + 2\bar{\beta}cb - 4abc - 4cab + \beta a^4 + 2\beta ba^2c + 2\bar{\beta}a^5\big),$$

$$-\tfrac{1}{12}\big(\bar{\beta}a + 2\bar{\beta}d + 2\beta cb - 4abc - 4cab + \bar{\beta}a^4 + 2\bar{\beta}ba^2c + 2\beta a^5\big),$$

where $\beta = 1 + \sqrt{-3}$. These seven elements generate simple two-sided ideals of dimensions 1, 4, 4, 4, 4, 4, 4 respectively. We omit the calculation of the isomorphisms of each simple ideal with a full matrix algebra. $\qquad\square$

Open Problem 8.12. Study the representation theory of the anti-Jordan triple systems A_n^-. In particular, over a quadratic extension $\mathbb{K}$ of $\mathbb{Q}$ do we always have $U(A_n^-) \otimes_{\mathbb{Q}} \mathbb{K} \cong \mathbb{K} \oplus 6\,M_n(\mathbb{K})$?

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