The Jordan Algebras of a Lie Algebra

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ABSTRACT: We attach a Jordan algebra L_x to any ad-nilpotent element x of index of nilpotence at most 3 in a Lie algebra L. This Jordan algebra has a behaviour similar to that of the local algebra of a Jordan system at an element. Thus, L_x inherits nice properties from L and keeps relevant information about the element x.

Introduction

Local algebras of a Jordan system (introduced by Meyberg [**25**], used by Zelmanov as a minor part of his brilliant classification of Jordan systems [**31**], and revisited by D'Amour and McCrimmon [**9**]) have played a prominent role in the

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recent structure theory of Jordan systems, mainly due to the fact that niceness properties flow between the system and their local algebras. Thus, D'Amour and McCrimmon extended a substantial part of Zelmanov's results [**31**] to arbitrary quadratic Jordan systems by making use of local algebras [**9**, **10**]; Anquela and Cortés characterized the primitivity of a Jordan system in terms of their local algebras and, as a consequence, gave a full classification of primitive systems [**1**]; the relationship between generalized polynomial identities and the existence of socle in primitive Jordan systems can actually be seen, after the works of D'Amour and McCrimmon [**9**], and Montaner [**26**], as a consequence of the existence of local algebras which are PI; and local algebras of a Jordan system seem to be a crucial notion in order to develop a local Goldie theory for Jordan systems [**13**]. Local algebras (or their related notion of subquotient) have also proved their usefulness in some questions involving Jordan Banach systems. For instance, in the solution given by Loos to the problem on the coincidence of the socle with the largest properly spectrum-finite ideal of a semiprimitive Banach Jordan pair [**21**] (see also [**14**]); in the proof of a structure theorem for Noetherian Banach Jordan pairs [**8**]; and in the solution to the problem on automatic continuity of derivations on semiprimitive Banach Jordan pairs [**15**]. On the other hand, ad-nilpotent elements of index at most 3 (here called Jordan elements) play a fundamental role in the proof of Kostrikin's conjecture that any finite dimensional simple nondegenerate Lie algebra (over a field of characteristic greater than 5) is classical [**6**, **28**]. Jordan elements are also of great importance in the Lie inner ideal structure of associative rings [**5**].

In this paper we show how it is possible to attach a Jordan algebra to any Jordan element x of a Lie algebra L (over a ring of scalars containing $\frac{1}{6}$): define a homotope like product in L by $a \bullet b = \frac{1}{2}[[a, x], b]$ and divide the nonassociative algebra $L^{(x)} = (L, \bullet)$ by the ideal $\ker_L(x) = \{z \in L : [[x, z], x] = 0\}$. Then $L_x = L^{(x)}/\ker_L(x)$ turns out to be a Jordan algebra which we call the Jordan algebra at (the Jordan element) x. These Jordan algebras L_x have a behaviour similar to that of the local algebras of a Jordan system. In fact, if L has a short grading, $L = L_{-n} \oplus \ldots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_n$, then any $x \in L_{\pm n}$ is a Jordan element of L and L_x agrees with the local algebra of the Jordan pair $V = (L_n, L_{-n})$ at x. If L is nondegenerate, so are their Jordan algebras. Moreover, in this case, a Jordan element x is von Neumann regular if and only if L_x is unital. Jordan elements of simple nondegenerate Lie algebras of classical type are determined, what, together with the classification of simple nondegenerate

Lie algebras containing abelian minimal inner ideals, allows us to prove that a Jordan element x of a nondegenerate Lie algebra L (over a field of characteristic 0 or greater than 7) belongs to the socle if and only if L_x has finite capacity. In particular, if L coincides with the socle, any Jordan element is von Neumann regular. Local characterizations of the socle of a nondegenerate Jordan systems were obtained by Loos and Montaner [**20**, **26**].

We would like to think that this local Jordan approach could throw new light on the structure theory of Lie algebras. For the moment, the existence of Jordan algebras in Lie algebras having ad-nilpotent elements seems at least to confirm McCrimmon's assertion: *nine times out of ten, when you open up a Lie algebra you find a Jordan algebra inside which makes it tick.*

1. **Preliminaries**

1.1 Throughout this paper and at least otherwise specified, we will be dealing with Lie algebras L, associative algebras R, Jordan algebras J, and Jordan pairs V, over a ring of scalars Φ containing $\frac{1}{6}$. As usual, $[x, y]$ will denote the Lie bracket of two elements x, y of L, with ad_x the adjoint map determined by x (sometimes we will use capital letters to denote the adjoint operators, i.e., $X := \mathrm{ad}_x$); the product of two elements x, y of R will be written by juxtaposition, xy, or by $x \cdot y$; the Jordan product of two elements x, y of J will be denoted by $x \bullet y$, with U-operator $U_x y = 2x\bullet(x\bullet y) - x^2 \bullet y$; Jordan products of a Jordan pair $V = (V^+, V^-)$ will be written by $Q_x y$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z} y = \{x, y, z\}$. The reader is referred to [**18**, **19**, **24**] for basic results, notation and terminology on Lie algebras, Jordan pairs, and Jordan algebras respectively. Nevertheless, we will stress some notions and basic properties for both Lie algebras and Jordan systems (algebras and pairs).

1.2 Any associative algebra R gives rise to:

(i) a Lie algebra $R^{(-)}$ with Lie bracket $[x, y] := xy - yx$, for all $x, y \in R$,

(ii) a Jordan algebra $R^{(+)}$ with Jordan product $x \bullet y := \frac{1}{2}(xy + yx)$,

(iii) a Jordan pair $V = (R, R)$ with Jordan quadratic operator $Q_x y := xyx$ (in general, any Jordan algebra J yields a Jordan pair (J, J), with $Q_x = U_x$ for all $x \in J$).

1.3 Let $V = (V^+, V^-)$ be a Jordan pair and let L be a Lie algebra.

(i) An element $x \in V^\sigma$, $\sigma = \pm$, is called an *absolute zero divisor* of V if $Q_x = 0$. A Jordan pair V is said to be *nondegenerate* if it has no nonzero absolute zero divisors. Similarly, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$; L is *nondegenerate* if it has no nonzero absolute zero divisors.

(ii) A submodule $B \subset V^\sigma$ is an *inner ideal* of V if $Q_B(V^{-\sigma}) \subset B$. By an inner ideal of a Jordan algebra J we mean an inner ideal of the Jordan pair (J, J). A submodule B of L is an *inner ideal* of L provided $[B, [B, L]] \subset B$.

(iii) Any $x \in V^\sigma$ yields two inner ideals: $[x] := Q_x V^{-\sigma}$ and $(x) := \Phi x + [x]$.

1.4 Let $L = L_{-n} \oplus \ldots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_n$ be a Lie algebra with a $(2n+1)$-grading. Then $V := (L_n, L_{-n})$ is a Jordan pair for the triple product defined by $\{x, y, z\} := [[x, y], z]$ for all $x, z \in L_\sigma$, $y \in L_{-\sigma}$, $\sigma = \pm n$, called the *associated Jordan pair* relative to the grading [**30**, p.351].

(i) If L is nondegenerate, so is V [**30**, Lemma 1.8].

(ii) Because of the grading, a submodule B of $L_{\pm n}$ is an inner ideal of L if and only if it is an inner ideals of V.

1.5 An element e of a Lie algebra L is called *von Neumann regular* if $\mathrm{ad}_e^3 = 0$ and $e \in \mathrm{ad}_e^2(L)$. The notion of von Neumann regularity is compatible with the usual one for associative rings and Jordan algebras (see [**12**, Proposition 2.4]).

1.6 Recall that a pair of elements (e, f) of a Lie algebra L is said to be an *idempotent* if they satisfy: $\mathrm{ad}_e^3 = \mathrm{ad}_f^3 = 0$, $[[e, f], e] = 2e$ and $[[e, f], f] = -2f$. Notice that the two last conditions imply that $(e, [e, f], f)$ is a $\mathfrak{sl}(2)$-*triple*. When $\frac{1}{5} \in \Phi$, any idempotent (e, f) of L gives rise to a 5-grading

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2,$$

where L_i is the i-eigenspace of L relative to $\mathrm{ad}_{[e,f]}$, for $i \in \{-2, -1, 0, 1, 2\}$. As in the associative and Jordan cases, von Neumann regular elements yield idempotents when $\frac{1}{5} \in \Phi$ (see [**29**, V.8.2] or [**11**, Proposition 2.9])

1.7 An *idempotent* of a Jordan pair V is a pair $(x, y) \in V^+ \times V^-$ such that $Q_x y = x$ and $Q_y x = y$. It is a direct consequence of the grading properties that if $L = L_{-n} \oplus \ldots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_n$ is a $(2n+1)$-grading of a Lie algebra L with associated Jordan pair $V = (L_n, L_{-n})$, then every idempotent of V is an idempotent of L.

1.8 (i) Let $V = (V^+, V^-)$ be a Jordan pair, and let $x \in V^{-\sigma}$, $\sigma = \pm$. On the Φ-module V^σ, a product is defined by $a \bullet b := \frac{1}{2}\{a, x, b\}$. With this product, V^σ becomes a Jordan algebra (with U-operator $U_a = Q_a Q_x$) called the *x-homotope* of V and denoted by $V^{(x)}$ [**19**, 1.9]. In particular, any element x of a Jordan algebra J yields the x-homotope $J^{(x)} := V^{(x)}$, V being the Jordan pair defined by J.

(ii) The set $\ker_V(x) = \{a \in V^\sigma : Q_x a = 0\}$ is an ideal of $V^{(x)}$, and the quotient $V_x := V^{(x)}/\ker_V(x)$ is a Jordan algebra called the *local algebra of V at x* [**9**].

(iii) The *local algebra R_x* of an associative algebra R at an element $x \in R$ is defined similarly (see [**16**]): $a \cdot b = axb$ for all $a, b \in R$, and $\ker_R(x) = \{a \in R : xax = 0\}$. Note that for the Jordan pair $V = (R, R)$, V_x agrees with the Jordan algebra $R_x^{(+)}$.

2. The Jordan algebra of a Lie algebra at a Jordan element

2.1 DEFINITION. We say that an element x in a Lie algebra L is a *Jordan element* if x is ad-nilpotent of index at most 3. Note that:

(i) Any nonzero ad-nilpotent element gives rise to a nonzero Jordan element, by a celebrated result due to Kostrikin [**6**, Corollary 1.6].

(ii) Any nonzero finite dimensional Lie algebra over an algebraically closed field of arbitrary characteristic necessarily contains a nonzero ad-nilpotent element [**7**], and therefore a nonzero Jordan element.

As it will be seen later, there is a good reason for calling Jordan elements to those ad-nilpotent elements of index at most 3.

2.2 A natural example of Jordan element is that of a zero-square element of an associative algebra. Let R be an associative algebra and let L be a subalgebra of the Lie algebra $R^{(-)}$. For any $x, y \in L$, we have:

(i) $\mathrm{ad}_x^2(y) = [x, [x, y]] = x^2y - 2xyx + yx^2$,

(ii) $\mathrm{ad}_x^3(y) = x^3y - 3x^2yx + 3xyx^2 - yx^3$.

Thus, if $x^2 = 0$, then $\mathrm{ad}_x^3(L) = 0$, and therefore x is a Jordan element of L. Note also that $\mathrm{ad}_x^2(L) = xLx$ in this case.

In the following lemma and in its proof, we will use capital letters to denote the adjoint operators: $X := \mathrm{ad}_x$, $A := \mathrm{ad}_a$, $B := \mathrm{ad}_b$.

2.3 LEMMA. *Let x be a Jordan element of a Lie algebra L. For any $a, b \in L$, $\alpha \in \Phi$, we have*

(i) $X^2AX = XAX^2$.

(ii) $X^2AX^2 = 0$.

(iii) $X^2A^2XAX^2 = X^2AXA^2X^2$.

(iv) $[X^2(a), X(b)] = [X^2(b), X(a)]$.

(v) $\mathrm{ad}_x^2[[a,x],b] = \mathrm{ad}_x^2[[b,x],a]$.

(vi) $X^2 \,\mathrm{ad}_{[a,X^2(b)]} = \mathrm{ad}_{[X^2(a),b]} X^2$.

(vii) $\mathrm{ad}^2_{X^2(a)} = X^2A^2X^2$.

(viii) αx, $\mathrm{ad}_x^2(a)$ *are Jordan elements.*

Proof. (i), (ii), (vii) are [**6**, Lemma 1.7 (i),(ii), (iii)].

(iii) By (ii),

$$
\begin{aligned}
0 = X^2[A,[A,[A,X]]]X^2 &= X^2(A^3X - 3A^2XA + 3AXA^2 - XA^3)X^2 \\
&= 3X^2AXA^2X^2 - 3X^2A^2XAX^2
\end{aligned},
$$

which proves (iii), because L is 3-torsion free.

(iv) From $\mathrm{ad}_x^3 = 0$, using the Leibniz rule we get

$$0 = X^3[a,b] = 3[X^2(a), X(b)] + 3[X(a), X^2(b)],$$

which implies $[X^2(a), X(b)] = [X^2(b), X(a)]$, because L is 3-torsion free.

(v) Use the Leibniz rule and apply (iv).

(vi) Since $\mathrm{ad}_{[a,X^2(b)]} = [A,[X,[X,B]]$, we get

$$
\begin{aligned}
X^2 \,\mathrm{ad}_{[a,X^2(b)]} &= X^2(A(X^2B + BX^2 - 2XBX) - (X^2B + BX^2 - 2XBX)A) \\
&= X^2ABX^2 - 2X^2AXBX = X^2ABX^2 - 2XAXBX^2 \\
&= ((X^2A + AX^2 - 2XAX)B - B(X^2A + AX^2 - 2XAX))X^2 \\
&= \mathrm{ad}_{[X^2(a),b]} X^2,
\end{aligned}
$$

as required.

(viii) $\mathrm{ad}^3_{\alpha x} = \alpha^3 \,\mathrm{ad}_x^3 = 0$ shows that αx is a Jordan element. Set $w := \mathrm{ad}_x^2(a)$. Using (ii) and (vii) we get, $\mathrm{ad}_w^3 = \mathrm{ad}_{[x,[x,a]]} \mathrm{ad}^2_{\mathrm{ad}_x^2(a)} = (X^2A - 2XAX + AX^2)X^2A^2X^2 = 0$, so $w = \mathrm{ad}_x^2(a)$ is a Jordan element. ■

2.4 THEOREM. *Let L be a Lie algebra and let $x \in L$ be a Jordan element. Then L with the new product defined by $a \bullet b := \frac{1}{2}[[a,x],b]$ is a nonassociative algebra, denoted by $L^{(x)}$, such that:*

(i) $\ker_L(x) := \{a \in L \mid [x,[x,a]] = 0\}$ *is an ideal of* $L^{(x)}$.

(ii) $L_x := L^{(x)}/\ker_L(x)$ *is a Jordan algebra, with U-operator given by*

$$U_{\overline{a}}\overline{b} = \frac{1}{8}\overline{\mathrm{ad}_a^2\,\mathrm{ad}_x^2\,b},$$

for all $a, b \in L$, where $\overline{a}$ denotes the coset of a with respect to $\ker_L(x)$.

Proof. (i) Let $a \in \ker_L(x)$ and $b \in L$. Using the Leibniz rule we get

$$\mathrm{ad}_x^2[[a,x],b] = -[X^3(a),b] - 2[X^2(a),X(b)] - [X(a),X^2(b)] = 0,$$

since $X^2(a) = 0$ and, by (2.3)(iv), $[X(a),X^2(b)] = [X(b),X^2(a)]$. Now we have by (2.3)(v) that $\mathrm{ad}_x^2[[b,x],a] = \mathrm{ad}_x^2[[a,x],b] = 0$, which proves that $\ker_L(x)$ is an ideal of $L^{(x)}$ altogether.

(ii) Consider now the quotient algebra $L_x = L^{(x)}/\ker_L(x)$. It follows from (2.3)(v) that the bullet product in L_x is commutative. Thus we only need to verify the Jordan identity. Let $a, b \in L$ and put $w := [[[a,x],a],x]$. Then

$$\begin{aligned}
8\,(\overline{a}^2 \bullet \overline{b}) \bullet \overline{a} &= \overline{[[[[[[a,x],a],x],b],x],a]} = \overline{[[[w,b],x],a]} = \overline{[w,b]} \bullet \overline{a} = \overline{a} \bullet \overline{[w,b]} \\
&= \overline{[[a,x],[w,b]]} \quad \text{and,} \\
8\,\overline{a}^2 \bullet (\overline{b} \bullet \overline{a}) &= 8\,\overline{a}^2 \bullet (\overline{a} \bullet \overline{b}) = \overline{[[[[a,x],a],x],[[a,x],b]]} = \overline{[w,[[a,x],b]]} \\
&= \overline{[[u,[a,x]],b]} + \overline{[[a,x],[w,b]]} = \overline{[[w,[a,x]],b]} + 8\,(\overline{a}^2 \bullet \overline{b}) \bullet \overline{a}.
\end{aligned}$$

But $[[w,[a,x]],b] \in \ker_L(x)$. Indeed, according to our convention, denote by capital letters the adjoint maps. Then

$$\mathrm{ad}_x^2\,\mathrm{ad}_{[w,[a,x]]} = X^2[W,[A,X]] = X^2WAX - X^2WXA - X^2AXW,$$

since $X^3 = 0$, and where

$$W = 2AXAX - XA^2X - A^2X^2 - 2XAXA + X^2A^2 + XA^2X.$$

Now $\mathrm{ad}_x^2\,\mathrm{ad}_{[w,[a,x]]} = 0$ follows by using (2.3)(i)-(iii). Therefore L_x is a Jordan algebra. Finally, since $[x,L] \subset \ker_L(x)$ and $\overline{a \bullet b} = \overline{b \bullet a}$, we have for all $a,b,c \in L$,

$$
\begin{aligned}
4\,\{\overline{a},\overline{b},\overline{c}\} &:= \overline{a} \bullet (\overline{b} \bullet \overline{c}) + \overline{c} \bullet (\overline{a} \bullet \overline{b}) - \overline{b} \bullet (\overline{a} \bullet \overline{c}) \\
&= \overline{[[a,x],[[b,x],c]]} + \overline{[[c,x],[[a,x],b]]} - \overline{[[b,x],[[a,x],c]]} \\
&= \overline{[[[a,x],[b,x]],c]} + \overline{[[b,x],[[a,x],c]]} + \overline{[[c,x],[[a,x],b]]} - \overline{[[b,x],[[a,x],c]]} \\
&= \overline{[[[a,x],[b,x]],c]} + \overline{[[c,x],[[a,x],b]]} = \overline{[[[a,x],[b,x]],c]} + \overline{[[c,[[a,x],b]],x]} \\
&+ \overline{[c,[x,[[a,x],b]]]} = \overline{[[[a,x],[b,x]],c]} + 0 + \overline{[c,[x,[[a,x],b]]]} \\
&= \overline{[[[a,x],[b,x]],c]} + \overline{[c,[x,[[b,x],a]]]} = \overline{[[[a,x],[b,x]],c]} \\
&+ \overline{[[x,[a,[b,x]]],c]} = \overline{[[a,[x,[b,x]]],c]} = -\overline{[[a,\mathrm{ad}_x^2\,b],c]}.
\end{aligned}
$$

Therefore, $8\,U_{\overline{a}}\overline{b} = 4\,\{\overline{a},\overline{b},\overline{a}\} = \overline{\mathrm{ad}_a^2\,\mathrm{ad}_x^2\,b}$. ∎

2.5 DEFINITION. For any Jordan element x of L, the Jordan algebra L_x we have just introduced will be called the *Jordan algebra of L at x*.

2.6 It deserves to be mentioned that we can define a functor from the category of the pairs (L,a), where L is a Lie algebra over Φ and a is a Jordan element of L, with the morphisms $f:(L,a) \to (M,b)$ being the homomorphisms $f: L \to M$ of Lie algebras such that $f(a) = b$, to the category of the Jordan Φ-algebras, which assigns to each pair (L,a) the Jordan algebra L_a, and to each morphism $f:(L,a) \to (M,b)$ the homomorphism $\overline{f}: L_a \to M_b$ given by $\overline{f}(\overline{x}) = \overline{f(x)}$.

Other elementary facts on Jordan elements and their attached Jordan algebras are listed in the next lemma.

2.7 LEMMA. *Let x be a Jordan element of L.*

(i) $[x] := \mathrm{ad}_x^2(L)$ *and* $(x) := \Phi x + \mathrm{ad}_x^2(L)$ *are both inner ideals of* L.

(ii) *For any inner ideal B of L, $B_x := (B/\ker_L(x) \cap B, \bullet)$ is a subalgebra of L_x.*

(iii) *If I is an ideal of L and $x \in I$ is von Neumann regular, then both Jordan algebras I_x and L_x agree.*

(iv) *If $L = B \oplus C$ is a direct sum of ideals and $x = b + c$ with respect to this decomposition, then $L_x \cong B_b \times C_c$.*

Proof. (i). It follows from [**6**, Lemma 1.10] and its proof, while (ii) and (iv) are straightforward. Now to prove (iii) it suffices to show that any coset $\overline{a}$ in L_x is equal to a coset $\overline{b}$ in I_x. Write $x = \mathrm{ad}_x^2(y)$ for some $y \in L$. Then, by (2.3(vii)),

$\mathrm{ad}_x^2 = \mathrm{ad}^2_{\mathrm{ad}_x^2(y)} = \mathrm{ad}_x^2 \mathrm{ad}_y^2 \mathrm{ad}_x^2$ and hence, for any $a \in L$, $\mathrm{ad}_x^2(a) = \mathrm{ad}_x^2(b)$, where $b = \mathrm{ad}_y^2 \mathrm{ad}_x^2(a) \in I$. ■

2.8 PROPOSITION. *Let x be a Jordan element of a Lie algebra L. For any $y \in L$ we have that $w := \frac{1}{4}[[x, y], x]$ is a Jordan element and $(L_x)_{\overline{y}} \cong L_w$.*

Proof. That w is a Jordan element follows from (2.3)(viii). For any $a \in L$, denote by $\overline{a}$ and $\widetilde{a}$ the cosets of a in L_x and L_w respectively. First we note:

(1) $\quad \overline{a} = 0 \Leftrightarrow \mathrm{ad}_x^2(a) = 0 \Rightarrow \mathrm{ad}_w^2(a) = \frac{1}{16} \mathrm{ad}_x^2 \mathrm{ad}_y^2 \mathrm{ad}_x^2(a) = 0,$

which proves that $\overline{a} \mapsto \widetilde{a}$ defines a map, say φ, of L_x onto L_w. We claim that φ is actually a homomorphism of the $\overline{y}$-homotope $L_x^{(\overline{y})}$ onto L_w, with $\ker(\varphi) = \ker(U_{\overline{y}})$, and therefore φ induces an isomorphism of $(L_x)_{\overline{y}}$ onto L_w. Since φ is clearly linear, to show that it is a homomorphism of Jordan algebras it suffices to verify that $\varphi(\overline{a}^2) = \widetilde{a}^2$, for any $a \in L$. By (2.4)(ii),

(2) $\quad \varphi(\overline{a}^2) = \varphi(U_{\overline{a}}\overline{y}) = \frac{1}{8}\widetilde{\mathrm{ad}_a^2 \mathrm{ad}_x^2}(y) = \frac{1}{2}[\widetilde{[a, w]}, a] = \widetilde{a} \bullet \widetilde{a}.$

Finally,

(3) $\quad \widetilde{a} = 0 \Leftrightarrow \mathrm{ad}_w^2(a) = \frac{1}{16} \mathrm{ad}_x^2 \mathrm{ad}_y^2 \mathrm{ad}_x^2(a) = 0 \Leftrightarrow U_{\overline{y}}\overline{a} = 0 \Leftrightarrow \overline{a} \in \ker(U_{\overline{y}}).$ ■

2.9 PROPOSITION. *Let $L = L_{-n} \oplus \ldots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_n$ be a Lie algebra with a $(2n+1)$-grading, and let $V = (L_n, L_{-n})$ be its associated Jordan pair. For any $x \in L_{-n}$, x is a Jordan element and L_x is isomorphic to the local algebra V_x of V at x.*

Proof. Because of the grading, x is a Jordan element of L. Denote by π_n the projection of L onto L_n. For any $a \in L$, $[x, [x, a]] = [x, [x, \pi_n(a)]] = -\{x, \pi_n(a), x\}$, so $a \in \ker_L(x)$ if and only if $\pi_n(a) \in \ker_V(x)$. Hence $\pi_n : L \to L_n$ induces an isomorphism of L_x onto V_x. ■

2.10 When $\frac{1}{5} \in \Phi$, given an idempotent (e, f) of a Lie algebra L, denote by $V(e, f)$ its associated Jordan pair $(L_2, L_{-2}) = (\mathrm{ad}_e^2(L), \mathrm{ad}_f^2(L))$, and by $\mathrm{J}(e, f)$ the unital Jordan algebra defined on the Φ-module $\mathrm{ad}_e^2(L)$ by the product $x \bullet y := [[x, f], y]$, for all $x, y \in \mathrm{ad}_e^2(L)$, (see [**6**, Lemmas 2.1 and 2.2] or [**11**, Proposition 2.9]).

2.11 PROPOSITION. *Let (e, f) be an idempotent of a Lie algebra L with $\frac{1}{5} \in \Phi$. Then the Jordan algebra L_e of L at e is isomorphic to the Jordan algebra $\mathrm{J}(e, f)$.*

Proof. Since (e, f) remains idempotent in the Jordan pair $V(e, f)$, it follows from [**9**, (1.7)] that $\mathrm{J}(e,f)$ is isomorphic to the local algebra of $V(e,f)$ at x. Now (2.9) applies. ■

In the following corollary we get that any unital Jordan algebra can be regarded as the Jordan algebra of a Lie algebra at an element. For a Jordan pair V, we denote by $\mathrm{TKK}(V)$ the Tits-Kantor-Koecher algebra of V.

2.12 COROLLARY. *Let J is a unital Jordan algebra, $V = (J, J)$ and $L = \mathrm{TKK}(V)$. If $\frac{1}{5} \in \Phi$, the Jordan algebra of L at 1 is isomorphic to J.*

Proof. The unit element $1 \in J$ is part of an idempotent $(1, \tilde{1}) \in L = \mathrm{TKK}(V)$, so by (2.11) $L_1 \cong J(1, \tilde{1}) = J$. ■

2.13 (i) Recall [**20**] that the *socle* of a nondegenerate Jordan pair V is defined as $\mathrm{Soc}(V) = (\mathrm{Soc}(V^+), \mathrm{Soc}(V^-))$, where $\mathrm{Soc}(V^\sigma)$ is the sum of all the minimal inner ideals of V contained in V^σ; the socle of a nondegenerate Jordan algebra J, denoted by $\mathrm{Soc}(J)$, is defined as the sum of all its minimal inner ideals. For a nondegenerate Lie algebra L, the socle, $\mathrm{Soc}(L)$, is defined as the sum of all minimal inner ideals of L [**11**].

(ii) The socle of a nondegenerate Jordan pair, Jordan algebra or Lie algebra is an ideal which is a direct sum of simple ideals ([**20**, Theorem 2], [**27**, Theorem 17], [**11**, Theorem 3.6]).

(iii) It follows from [**16**, Corollary 2.2] that for a semiprime associative algebra R, the *associative socle* of R agrees with the *Jordan socle* of $R^{(+)}$. (Note that $R^{(+)}$ is nondegenerate if and only if R is semiprime.)

2.14 Note that if x is a Jordan element of L, then the map $\overline{a} \mapsto \mathrm{ad}_x^2(a)$, defines a bijection from L_x onto the inner ideal $[x]$ of L. In fact, this map induces an one-to-one order-preserving correspondence between the set of inner ideals $[\overline{a}]$ of the Jordan algebra L_x and the set of inner ideals $[y]$ of L, $y \in [x]$.

2.15 PROPOSITION. *Let L be a nondegenerate Lie algebra and let x be a Jordan element of L. Then*

(i) *L_x is a nondegenerate Jordan algebra.*

(ii) *If $\frac{1}{5} \in \Phi$, L_x is unital if and only if x is von Neumann regular.*

(iii) *$\overline{a} \in \mathrm{Soc}(L_x)$ if and only if $\mathrm{ad}_x^2(a) \in \mathrm{Soc}(L)$.*

Proof. (i) If $U_{\overline{a}}\overline{b} = \overline{0}$ for every $\overline{b} \in L_x$, then $0 = \mathrm{ad}_x^2 \mathrm{ad}_a^2 \mathrm{ad}_x^2 b = \mathrm{ad}^2_{\mathrm{ad}_x^2 a} b$ for every $b \in L$ (2.3)(vii), which implies, since L is nondegenerate, that $\mathrm{ad}_x^2 a = 0$, i.e., $\overline{a} = \overline{0}$.

(ii) Suppose that there exists $y \in L$ such that $\{x, [x, y], y\}$ is a $\mathfrak{sl}_2$-triple with $\mathrm{ad}_y^3 = 0$. Then $\overline{y}$ is the unit of L_x because

$$[x, [x, [[y, x], a]]] = [x, [[x, [y, x]], a]] = [[x, [y, x]], [x, a]] = 2[x, [x, a]]$$

therefore $\overline{y} \bullet \overline{a} = \frac{1}{2}\overline{[[y, x], a]} = \overline{a}$.

Suppose, conversely, that L_x is a unital Jordan algebra and let $\bar{t}$ be its unit. Put $z := [x, [x, -t]] - 2x$. Then for every $a \in L$,

$$\begin{aligned} 0 &= [x, [x, [[t, x], a] - 2a]] = -[x, [[x, [x, t]], a] + 2[x, a]] \\ &= [[x, [x, -t]], [x, a]] - 2[x, [x, a]] = [z, [x, a]] \end{aligned}$$

but $[z, x] = 0$ and $0 = \mathrm{ad}_{[z,[x,a]]} = -\mathrm{ad}_z \mathrm{ad}_a \mathrm{ad}_x - \mathrm{ad}_x \mathrm{ad}_a \mathrm{ad}_z$ which implies that

$$\begin{aligned} [z, [[x, [x, -t]], a]] &= [[x, [x, -t]], [z, a]] = [x, [x, [-t, [z, a]]]] \\ &= -[x, [z, [-t, [x, a]]]] = 0. \end{aligned}$$

Now, $[z, [z, a]] = 0$ for every $a \in L$ and therefore, since L is nondegenerate, $z = 0$, i.e., $[x, [x, -t]] = 2x$. Finally, by [**29**, Lemma V.8.2] (or [**11**, 2.9]), there exists $y \in L$ such that $\{x, [x, y], y\}$ is a $\mathfrak{sl}_2$-triple with $\mathrm{ad}_y^3 = 0$.

(iii) Since the socle of a nondegenerate Jordan system or Lie algebra is a sum of minimal inner ideals, it follows from (2.14) that $\overline{a}$ belongs to $\mathrm{Soc}(L_x)$ if and only if $\mathrm{ad}_x^2(a)$ lies in $\mathrm{Soc}(L)$. ■

3. Jordan algebras of simple special Lie algebras

3.1 Let R be a (not necessarily unital) associative algebra with associated Lie algebra $R^{(-)}$. We have the Lie algebras $R' := [R, R]$, $\overline{R} := R^{(-)}/Z(R)$ and $\overline{R}' := R'/R' \cap Z(R)$, where $Z(R)$ stands for the center of R. We write π to denote the canonical projection of $R^{(-)}$ onto $\overline{R}$. Note that $\overline{R}'$ can be regarded as an ideal of $\overline{R}$.

3.2 LEMMA. *Let $y \in R$ be a nonzero element such that $y^2 = 0$. Then*

(i) *y is a Jordan element of $R^{(-)}$ and we have a canonical isomorphism of $R_y^{(-)}$ onto $R_y^{(+)}$, the local algebra of the Jordan algebra $R^{(+)}$ at y.*

(ii) *If R is semiprime, then $\overline{R}_{\pi(y)} \cong R_y^{(-)} \cong R_y^{(+)}$.*

(iii) $\overline{R}_{\pi(y)}$ *is simple whenever* R *is simple.*

Proof. (i) $y^2 = 0$ implies $[y, [y, a]] = -2yay$ for all $a \in R$. Hence $\mathrm{ad}_y^3 = 0$ and therefore y is a Jordan element of $R^{(-)}$. Moreover, for any $a, b \in R$, we have $y[[a, y], b]y = y(ayb + bya)y$ which implies that the map $a + \ker_{R^{(-)}}(y) \to a + \ker_{R^{(+)}}(y)$ defines an isomorphism of $R_y^{(-)}$ onto $R_y^{(+)}$.

(ii) As pointed out in (2.6), $\pi : R^{(-)} \to \overline{R}$ induces the epimorphism $\overline{\pi} : R_y^{(-)} \to \overline{R}_{\pi(y)}$. We claim that $\overline{\pi}$ is actually an isomorphism. Indeed, let $a \in R$ be such that $[\pi(y), [\pi(y), \pi(a)]] = \pi(0)$. Then $[y, [y, a]] \in Z(R)$. Hence $[y, [y, a]] = -2yay = 0$ because R cannot contain nonzero central nilpotent elements, by semiprimeness. Now we have by (i) that $\overline{R}_{\pi(y)} \cong R_y^{(-)} \cong R_y^{(+)}$.

(iii) If R is simple, then $R^{(+)}$ is a simple Jordan algebra [**17**, Theorem 1.1]. Hence, by local theory of Jordan systems [**2**, Corollary 3.5], $R_y^{(+)}$ is also simple. ∎

3.3 THEOREM. *Let* R *be a simple associative algebra and* $x \in R'$. *If* $\pi(x)$ *is a nonzero Jordan element of* $\overline{R}'$, *then* $\overline{R}'_{\pi(x)}$ *is a simple Jordan algebra. In fact,* $\overline{R}'_{\pi(x)} \cong R_y^{(+)}$, *for some* $y \in R$ *such that* $y^2 = 0$.

Proof. By [**11**, Lemma 5.2], $\overline{R}'$ is nondegenerate and therefore $\overline{R}'_{\pi(x)} \neq 0$. Now $\mathrm{ad}^3_{\pi(x)} = 0$ implies $\mathrm{ad}_x^4 = 0$, and hence, by [**5**, Theorem 3.2], $(x - z)^2 = 0$ for some $z \in Z(R)$. Put $y := x - z$ which is nonzero since $\pi(x) \neq 0$. Then $\pi(x) = \pi(y)$ and $\overline{R}'_{\pi(x)} = \overline{R}'_{\pi(y)}$. Now (3.2)(iii) applies since $\overline{R}'_{\pi(y)}$ can be regarded as a *nonzero* ideal of $\overline{R}_{\pi(y)}$. ∎

3.4 Assume now that R has an involution $*$. Denote by $K = \mathrm{Skew}(R, *)$ the Lie algebra of the skew-symmetric elements of R. We also consider the Lie algebras $K' = [K, K]$, $\overline{K} = K/K \cap Z(R)$ and $\overline{K}' = K'/K' \cap Z(R)$. Notice that $\overline{K}' \cong [\overline{K}, \overline{K}]$. If R is simple and either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16, then $\overline{K}$ is nondegenerate and $\overline{K}'$ is a simple nondegenerate Lie algebra [**11**, Lemma 5.13].

3.5 LEMMA. *Let* R *be a simple associative algebra with involution* $*$, *and let* $a \in K$ *be a nonzero skew-symmetric element such that* $a^2 = 0$. *Then*

(i) *the map* $-*$ *induces an involution* $\star$ *on* R_a, *the associative local algebra of* R *at* a, *given by* $\overline{x}^\star = -\overline{x^*}$, $x \in R$,

(ii) $\pi(a)$ *is a nonzero Jordan element of* $\overline{K}$ *and* $\overline{K}_{\pi(a)} \cong \mathrm{Sym}(R_a, \star)$,

(iii) *the Jordan algebra* $\overline{K}_{\pi(a)}$ *is simple.*

Proof. (i) It is straightforward. (ii) By (3.2(i)), a is a Jordan element of $R^{(-)}$ and hence also of K (since a is skew). Then $\pi(a)$ is a Jordan element of $\overline{K}$, which is nonzero by simplicity of R. Moreover, the canonical isomorphism of $R_a^{(-)}$ onto $R_a^{(+)}$ given in (3.2(ii)) restricts to an isomorphism of $\overline{K}_{\pi(a)}$ onto $\mathrm{Sym}(R_a, \star)$. (iii) Finally, R_a inherits simplicity of R and hence $\mathrm{Sym}(R_a, \star)$ is also simple by [**17**, Theorem 2.1]. ■

3.6 THEOREM. *Let R be a simple associative algebra with involution $*$ of the second kind (over the centroid of R). Suppose that either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16. Let $b \in K'$ be such that $\pi(b)$ is a nonzero Jordan element of $\overline{K}'$. Then*

(i) *there is an $\alpha \in K \cap Z(R)$ such that $(b - \alpha)^2 = 0$,*

(ii) *$\overline{K}_{\pi(b)} \cong \overline{K}_{\pi(a)} \cong \mathrm{Sym}(R_a, \star)$, for $a = b - \alpha$, and*

(iii) *the Jordan algebra $\overline{K}'_{\pi(b)}$ is simple.*

Proof. (i) By (2.7)(i), $(\pi(b)) = \Phi\pi(b) + \mathrm{ad}^2_{\pi(b)}(\overline{K}')$ is an abelian inner ideal of $\overline{K}'$, so $(\pi(b)) = \pi(B)$, where $B = \pi^{-1}((\pi(b)))$ is a proper inner ideal of K'. Since $b \in B$, it follows from [**6**, Theorem 4.26] that there is an $\alpha \in Z(R)$ such that $(b - \alpha)^2 = 0$. Write $\alpha = \alpha_s + \alpha_k$, where $\alpha_s \in \mathrm{Sym}(Z(R), *)$ and $\alpha_k \in \mathrm{Skew}(Z(R), *)$. Then $(b - (\alpha_s + \alpha_k))^2 = 0$ and $((b - (\alpha_s + \alpha_k))^2)^* = (-b - (\alpha_s - \alpha_k))^2 = 0$. Hence

$$0 = (b - (\alpha_s + \alpha_k))^2 - (b + (\alpha_s - \alpha_k))^2 = -4\alpha_s b + 4\alpha_s\alpha_k = -4\alpha_s(b - \alpha_k),$$

which implies $\alpha_s = 0$ or $b = \alpha_k$. But the latter does not hold since $\pi(b) \neq 0$. Thus, $\alpha_s = 0$ and therefore $\alpha = \alpha_k \in K \cap Z(R)$.

(ii) and (iii). Put $a = b - \alpha$ as in (i). Then $a \in K$ with $a^2 = 0$. Hence, by (3.5)(ii), $\overline{K}_{\pi(b)} = \overline{K}_{\pi(a)} \cong \mathrm{Sym}(R_a, \star)$, which is a simple Jordan algebra by (3.5)(iii). Since $\overline{K}'_{\pi(b)}$ can be regarded as a *nonzero* ideal of $\overline{K}_{\pi(b)}$, $\overline{K}'_{\pi(b)} \cong \overline{K}_{\pi(b)}$ by simplicity. ■

Before dealing with the case where the involution is of the first kind, we recall some notation on pairs of dual vector spaces.

3.7 Let $\mathcal{P} = (X, Y, g)$ be a dual pair of vector spaces over a division algebra Δ. A linear operator $a : X \to X$ is *continuous* (relative to $\mathcal{P}$) if there exists $a^\# : Y \to Y$, necessarily unique, such that $g(ax, y) = g(x, a^\# y)$ for all $x \in X$, $y \in Y$. Denote by $\mathcal{L}_Y(X)$ the Φ-algebra of all continuous operators on X, and by $\mathcal{F}_Y(X)$ the ideal of $\mathcal{L}_Y(X)$ consisting of all finite rank continuous operators.

3.8 Let $\mathcal{P} = (X, Y, g)$ be as above. For $x \in X$, $y \in Y$, write y^*x to denote the linear operator defined by $y^*x(x') = g(x', y)x$ for all $x' \in X$. Note that y^*x is continuous with adjoint yx^* given by $yx^*(y') = yg(x, y')$ for all $y' \in Y$. Every $a \in \mathcal{F}_Y(X)$ can be expressed as a sum $a = \sum y_i^* x_i$ of these rank one operators.

3.9 (i) Let R be a simple associative algebra coinciding with its socle. By [**4**, Theorem 4.3.9], we can regard R as the algebra $\mathcal{F}_Y(X)$, relative to a dual pair $\mathcal{P} = (X, Y, g)$ over a division algebra Δ.

(ii) If R has an involution $*$, then the dual pair $\mathcal{P}$ comes from a self-dual vector space (X, h), where h is a nondegenerate Hermitian or skew-Hermitian form h over a division algebra with involution $(\Delta, -)$. Then R can be regarded as $\mathcal{F}_X(X)$, with $*$ being the adjoint involution with respect to h [**4**, Theorem 4.6.8]. Actually, we may assume, without loss of generality, that either h is symmetric (in this case Δ is a field with the identity as involution), or h is skew-Hermitian. In the first case we say that $*$ is *orthogonal*, and *skew-Hermitian* in the second case.

3.10 THEOREM. *Let R be a simple associative algebra with involution $*$ of the first kind (over the centroid of R). Suppose that either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16. Let $b \in K'$ be such that $\pi(b)$ is a nonzero Jordan element of $\overline{K}'$. Then $b^3 = 0$ and one of the following holds:*

(i) $b^2 = 0$. *Then* $\overline{K}'_{\pi(b)} \cong \mathrm{Sym}(R_b, \star)$.

(ii) $b^2 \neq 0$. *Then R has nonzero socle with orthogonal involution, $\overline{K}' = K$ is a finitary orthogonal Lie algebra, and $\overline{K}'_{\pi(b)}$ is isomorphic to the Jordan algebra defined by a nondegenerate quadratic form with base point on a vector space of dimension greater than* 2.

(iii) *In any case, $\overline{K}'_{\pi(b)}$ is simple.*

Proof. By (2.7)(i), $(\pi(b)) = \Phi\pi(b) + \mathrm{ad}^2_{\pi(b)}(\overline{K}')$ is an abelian inner ideal of $\overline{K}'$, so $(\pi(b)) = \pi(B)$, where $B = \pi^{-1}((\pi(b)))$ is a proper inner ideal of K'. Then $[B, B] = 0$ by [**5**, Theorem 4.21]. Since $b \in B$, it follows from [**5**, Theorem 4.21] that $b^3 = 0$. Moreover, $[B, B] = 0$ implies $\mathrm{ad}^3_b(K') = [b, \mathrm{ad}^2_b(K')] = 0$ and hence $\mathrm{ad}^4_b(K) = 0$. Then $6b^2Kb^2 = \mathrm{ad}^4_b(K) = 0$. Since $\frac{1}{6} \in \Phi$, R satisfies the generalized polynomial identity with involution $b^2(X - X^*)b^2 = 0$. (See [**4**] for definition of generalized polynomial identity with involution ($*$GPI)).

(i) Suppose that $b^2 = 0$. By (3.5)(ii), $\overline{K}_{\pi(b)} \cong \mathrm{Sym}(R_b, \star)$, which is a simple Jordan algebra by (3.5)(iii). Since $\overline{K}'_{\pi(b)}$ can be regarded as a *nonzero* ideal of

$\overline{K}_{\pi(b)}$, $\overline{K}'_{\pi(b)} \cong \overline{K}_{\pi(b)}$ by simplicity.

(ii) If $b^2 \neq 0$, the $*$GPI $b^2(X - X^*)b^2 = 0$ is nontrivial and hence R satisfies a nonzero generalized polynomial identity [**4**, Corollary 6.25]. Then, by [**4**, Corollary 6.1.7], R has nonzero socle and R is isomorphic to $\mathcal{F}_X(X)$, relative to a nondegenerate symmetric or skew-Hermitian form h form over a division algebra with involution $(\Delta, -)$. We claim that the involution is necessarily orthogonal, that is, the skew-Hermitian case cannot occur. Otherwise, for any nonzero $x \in X$, the rank one operator x^*x is skew. Hence $0 = b^2(x^*x)b^2 = (b^2(x))^*b^2(x)$, which implies $b^2(x) = 0$ for all $x \in X$, a contradiction. Therefore, K is isomorphic to the finitary orthogonal Lie algebra [**3**] $\mathfrak{fo}(X, h) = \mathrm{Skew}(\mathcal{F}_X(X), *)$. Finally, it follows from [**12**, (3.7) and (3.8)] (where actually only characteristic different from 2 is required) that $b = x^*z - z^*x$, where x is a nonzero isotropic vector of an hyperbolic plane H of X and $z \in H^\perp$ is not isotropic. By [**11**, (8)], taking an isotropic vector y in H such that $h(x, y) = 1$, and putting $c = -2h(z, z)^{-1}(y^*z - z^*y)$, we obtain an idempotent (b, c) of $\overline{K}' = K' = K = \mathfrak{fo}(X, h)$, with associated Jordan pair $V(b, c)$ isomorphic to the Jordan pair defined by the restriction of the quadratic form $q(v) := h(v, v)$, $v \in H^\perp$, to $H^\perp$. By (2.11), $\overline{K}'_{\pi(b)} = K_b$ is isomorphic to $\mathtt{J}(b, c)$, the latter being isomorphic to the Jordan algebra defined by a scalar multiple of the quadratic form q [**24**, 7.3.1]. ■

4. Jordan algebras of a Lie algebra which coincides with its socle

The main result of this section proves that for an element of a nondegenerate Lie algebra coinciding with its socle, being Jordan is equivalent to being von Neumann regular. In fact, a Jordan element of a nondegenerate Lie algebra belongs to the socle if and only if the Jordan algebra attached to this element has finite capacity.

4.1 Recall that a nondegenerate Jordan algebra J is said to have *finite capacity* if it has a unit 1 which can be written as a finite sum of orthogonal division idempotents: $1 = e_1 + \cdots + e_n$ with $e_i^2 = e_i$, $e_i \bullet e_j = 0$ for $i \neq j$, $U_{e_i}J$ a division Jordan algebra [**24**, p.96]. It is known that a nondegenerate Jordan algebra J has finite capacity if and only if satisfies both descending and ascending conditions on all principal inner ideals $[x]$, $x \in J$, [**23**, 1.15].

4.2 Theorem. *Let L be a nondegenerate Lie algebra over a field of characteristic 0 or greater than 7. For a nonzero Jordan element x of L, the following conditions are equivalent:*

(1) *$x \in \mathrm{Soc}(L)$.*

(2) *x is von Neumann regular and L_x has finite capacity.*

(3) *x is von Neumann regular and L satisfies the descending chain condition for all inner ideals $[y]$, $y \in (x)$.*

Proof. (1) ⇒ (2). Since Jordan algebras having finite capacity are unital, it suffices to show by (2.15)(ii) that L_x has finite capacity. Let us first assume that L is a simple nondegenerate Lie algebra containing an abelian minimal inner ideal (the general case will be considered later). By [**11**, Theorem 6.1], we have the following possibilities for L: (i) L is finite dimensional over its centroid; (ii) there exists a simple associative algebra R which coincides with its socle and which is not a division algebra such that $L = [R,R]/Z(R) \cap [R,R]$; (iii) L is isomorphic to $[K,K]/Z(R) \cap [K,K]$, where $K = \mathrm{Skew}(R,*)$, for R being a simple associative algebra with isotropic involution ($a^*a = 0$ for some nonzero element $a \in R$) which coincides with its socle, and where either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16. Let F denote the centroid of L, which is a field since L is simple. If (i), then L_x is a nondegenerate (2.15)(i) finite dimensional Jordan algebra, so L_x has finite capacity, as required. Suppose now that $L = [R,R]/Z(R) \cap [R,R]$ as in (ii). By (3.3), $L_x \cong R_y^{(+)}$ for some nonzero element $y \in R$ such that $y^2 = 0$. Hence, by the local characterization of the socle [**26**, (0.7)], L_x has finite capacity. Consider finally the case when $L = [K,K]/Z(R) \cap [K,K]$ as in (iii). We can still distinguish between two cases. If the involution $*$ of R is of the second kind (over F), we have by (3.6) that L_x is a simple Jordan algebra (in fact, $L_x \cong \mathrm{Sym}(R_a,\star)$, with a being a nonzero skew-symmetric element of R). Then R_a is Artinian by [**16**, Theorem 2.3], and hence $L_x \cong \mathrm{Sym}(R_a,\star)$ has finite capacity by [**22**, Proposition 4]. Suppose now that the involution $*$ of R is of the first kind. By (3.10), either $L_x \cong \mathrm{Sym}(R_b,\star)$, where b is a nonzero skew-symmetric element of R, or L_x is a Jordan algebra isomorphic to the Jordan algebra defined by a nondegenerate quadratic form with base point on a vector space of dimension greater than 2. In both cases, L_x is simple and has finite capacity.

Let us now deal with the general case $x \in \mathrm{Soc}(L)$. Since $\mathrm{Soc}(L)$ is a direct sum of simple ideals, we can write $x = x_1 + \cdots + x_n$, where each x_i is nonzero and belongs to an ideal M_i, which is simple as a Lie algebra and coincides with its socle; the M_i being mutually orthogonal. Hence, for each $1 \le i \le n$, x_i is a Jordan element of M_i. This implies, by (2.7)(i), that M_i contains a nonzero abelian inner ideal, and hence an abelian minimal inner ideal, by the classification of minimal inner ideals [**6**, Theorem 1.12]. Then, by the above, for each $1 \leq i \leq n$, $(M_i)_{x_i}$ is a nondegenerate Jordan algebra of finite capacity. Set $M = M_1 \oplus \cdots \oplus M_n$. By (2.7)(iv), $M_x \cong (M_1)_{x_1} \times \cdots \times (M_n)_{x_n}$, so M has finite capacity and x is von Neumann regular in M. Hence, by (2.7)(iii), $L_x \cong M_x$ has finite capacity.

(2) $\Rightarrow$ (3). By (2.14), L satisfies the descending chain condition for all inner ideals $[y]$, $y \in (x)$, since L_x has dcc on principal inner ideals.

(3) $\Rightarrow$ (1). It was proved in [**11**, Corollary 4.3]. ∎

REFERENCES

1. J.A. Anquela, T. Cortés. Primitive Jordan pairs and triple systems. J. Algebra **184**, (1995), 632-678.

2. J.A. Anquela, T. Cortés. Local and subquotient inheritance of simplicity in Jordan systems. J. Algebra **240**, (2001), 680-704.

3. A.A. Baranov. Finitary simple Lie algebras. J. Algebra **219**, (1999), 299-329.

4. K.I. Beidar, W.S. Martindale, A.V. Mikhalev. Rings with generalized identities; Pure and Applied Mathematics, **196**, Marcel Dekker, 1996.

5. G. Benkart. The Lie inner ideal structure of associative rings. J. Algebra **43**, (1976), 561-584.

6. G. Benkart. On inner ideals and ad-nilpotent elements of Lie algebras. Trans. Amer. Math. Soc. **232**, (1977), 61-81.

7. G. Benkart, I.M. Isaacs. On the existence of ad-nilpotent elements. Proc. Amer. Math. Soc. **63**, (1977), 39-40.

8. N. Boudi, A. Fernández López, E. García Rus, H. Marhnine, C. Zarhouti. Noetherian Jordan Banach pairs. Math. Proc. Camb. Phil. Soc. **130**, (2001), 25-36.

9. A. D'Amour, K. McCrimmon. The local algebras of Jordan systems. J. Algebra **177**, (1995), 199-239.

10. A. D'Amour, K. McCrimmon. The structure of quadratic Jordan systems of Clifford type. J. Algebra **234**, (2000), 31-89.

11. C. Draper, A. Fernández López, E. García, M. Gómez Lozano. The socle of a nondegenerate Lie algebra (preprint).

12. A. FERNÁNDEZ LÓPEZ, E. GARCÍA, M. GÓMEZ LOZANO. Inner ideals of finitary simple Lie algebras. J. Lie Theory **16**(1), (2006), 97-114.

13. A. FERNÁNDEZ LÓPEZ, E. GARCÍA RUS, O. JAA. Towards a Goldie theory for Jordan pairs. Manuscripta Math. **95**, (1998), 79-90.

14. A. FERNÁNDEZ LÓPEZ, O. JAA. Local algebras and the largest spectrum finite ideal. Extracta Mathematicae **13**, (1998), 61-67.

15. A. FERNÁNDEZ LÓPEZ, H. MARHNINE, C. ZARHOUTI. Derivations on Banach-Jordan pairs. Quart. J. Math. **52**, (2001), 269-283.

16. A. FERNÁNDEZ LÓPEZ, M. TOCÓN. The Local Algebras of an Associative Algebra with Their Applications: a survey in: Applicable Mathematics in the Golden Age (Editor J.C. Misra); Narosa Publishing House (2003), 254-275.

17 I.N. HERSTEIN. Topics in Ring Theory; Chicago Lectures in Mathematics; The University of Chicago Press, 1969.

18. N. JACOBSON. Lie Algebras; Interscience Publishers; New York, 1962.

19. O. LOOS. Jordan Pairs. Lecture Notes in Mathematics; **460**, Springer-Verlag; New York, 1975.

20. O. LOOS. On the socle of a Jordan pair. Collect. Math. bf 40(2), (1989), 109-125.

21 O. LOOS. Properly algebraic and spectrum-finite ideals in Jordan systems. Math. Proc. Camb. Phil. Soc **114**, (1993), 149-161.

22. K. MCCRIMMON. Inner ideals in quadratic Jordan algebras. Trans. Amer. Math. Soc. **159**, (1971), 445-468.

23. K. MCCRIMMON. Zelmanov's prime theorem for quadratic Jordan algebras. J. Algebra **76**, (1982), 297-326.

24. K. MCCRIMMON. A Taste of Jordan Algebras. Springer (Universititext), 2004.

25. K. MEYBERG. Lectures on Algebras and Triple Systems. Univ. of Virginia Lecture Notes, Charlottesville, 1972.

26. F. MONTANER. Local PI theory of Jordan systems. J. Algebra **216**, (1999), 302-327.

27. J.M. OSBORN, M. RACINE. Jordan rings with nonzero socle. Trans. Amer. Math. Soc. **251**, (1979), 375-387.

28. A.A. PREMET. Lie algebras without strong degeneration. Math USSR Sbornik **57** (1987), 151-164.

29. G.B. SELIGMAN. Modular Lie Algebras. Ergebnisse Math. Grenzgebiete, **40**, Springer-Verlag, Berlin and New York, 1967.

30. E.I. ZELMANOV. Lie algebras with a finite grading. Math. USSR Sbornik **52**(2), (1985), 537-552.

31. E.I. ZELMANOV. Prime Jordan triple systems I; II; III. Siber. Math. J. **18** (1983), 23-67; **25** (1984), 42-49; **26** (1985), 71-82.