Algebras of quotients of nonsingular Jordan algebras

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Abstract: We define a Jordan analogue of Johnson's associative algebra of quotients and study it under a suitable condition of nonsingularity of the Jordan algebra, which we call strong nonsingularity. In particular we prove the existence and describe the maximal algebras of quotients of prime strongly nonsingular Jordan algebras.

Keywords: Jordan algebras; algebras of quotients; essential inner ideals; nonsingularity

Introduction

The study of Jordan algebras of quotients originated in the question raised by Jacobson [J1, p. 426] about the possibility of imbedding a Jordan domain in a Jordan division algebra in a way similar to Ore's construction in associative theory. That problem led to the study of suitable algebras of fractions and also to the related problem of adapting Goldie's theory to the Jordan setting, which in turn led to the study of more general algebras of quotients for Jordan algebras.

A complete answer for the problem of finding analogues of Goldie's theorems for linear Jordan algebras was given by Zelmanov in [Z1, Z2] using his fundamental structural results. More recently, Martinez [M] solved the original problem of constructing analogues of Ore's rings of fractions by a different approach. In her work she makes use of the Kantor-Koecher-Tits construction to define algebras of fractions of linear Jordan algebras for a set of denominators satisfying suitable "Ore

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conditions". Zelmanov's and Martinez's results have been extended to quadratic Jordan algebras in [FGM] and [B] respectively, again using similar strategies of those of their original linear versions, namely the structural approach in the case of Goldie's theorems and Faulkner's generalization of the Kantor-Koecher-Tits construction for general algebras of fractions.

Some other work on Jordan algebras of quotients include local orders [FG1, FG2], and Jordan analogues of the Martindale algebra of quotients [GG, AGG]. Both cases follow one of the strategies mentioned above, Zelmanov's structural approach and Martinez's Lie theoretic approach.

An important construction in associative theory is Johnson's algebra of quotients. The aim of this paper is to develop an analogue of that construction for Jordan algebras. Since these algebras are defined for denominators that are essential one sided ideals, the natural choice for its Jordan version is taking essential inner ideals as denominators. Moreover, the associative constructions requires that the algebra be nonsingular, and therefore a Jordan analogue of nonsingularity is needed for a Jordan version of it. There is already a definition of nonsingularity for Jordan algebras given in [FGM], however for this property to relate well with the nonsingularity of associative envelopes, we will need a more stringent version of that concept. Thus we will define what we call strongly nonsingular Jordan algebras. Our main result then asserts that every prime strongly nonsingular Jordan algebra has a maximal algebra of quotients which is analogue to Jonhson's algebra of quotients.

The paper is organized as follows. After a first section of preliminaries, we study essential inner ideals in section 1. In particular, we define the above mentioned notion of nonsingularity and draw some of its consequences. In section 2 we define the kind of algebras of quotients to which the paper is devoted. Among other properties, we prove in 2.9 the transitivity of algebras of quotients for strongly nonsingular algebras. We also define the notion of maximal algebras of quotients and prove their uniqueness for strongly nonsingular algebras. The rest of the paper is devoted to the problem of proving the existence of maximal algebras of quotients for prime strongly nonsingular Jordan algebras. Following a well established strategy in Jordan theory, we consider separately the cases where the algebra satisfies a polynomial identity and the case where it does not, and hence it is in particular a hermitian algebra. In section 3 we deal with PI algebras, where the main tool is the existence of nonzero elements in the weak center, which as we prove, keep on being elements of the weak center of any algebra of quotients. These elements can be used to extend mappings from the centroid of the algebra to the centroid of the algebra of quotients and this allows to prove that the central closure is in this case the maximal algebra of quotients. In section 4 we consider the hermitian case. Here, Lanning's algebra of quotients of a $*$-tight associative $*$-envelope provides a home for the elements of the algebra of quotients of the original Jordan algebra thanks to the good relationship between essential inner ideals of the algebra and essential one sided ideals of the envelope. We finally collect the previous results in the main theorem of the paper asserting that a prime strongly nonsingular Jordan algebra has a maximal algebra of quotients.

0. Preliminaries

0.1. We will work with Jordan algebras over a unital commutative ring of scalars Φ which will be fixed throughout. We refer to [J2, McZ] for notation, terminology, and basic results. In particular, we will make use of the identities proved in [J2], which we will quote with the labellings QJn of that reference. In this section we recall some of those basic results and notations, together with some other that will be used in the paper.

0.2. A Jordan algebra has products U_xy and x^2, quadratic in x and linear in y, whose linearizations are $U_{x,z}y = V_{x,y}z = \{x, y, z\} = U_{x+z}y - U_xy - U_zy$, and $x \circ y = (x + y)^2 - x^2 - y^2$ respectively.

We will denote by $\hat{J}$ the free unital hull $\hat{J} = \Phi 1 \oplus J$ with products $U_{\alpha 1+x}(\beta 1+y) = \alpha^2\beta 1 + \alpha^2 y + \alpha x \circ y + 2\alpha\beta x + \beta x^2 + U_xy$ and $(\alpha 1 + x)^2 = \alpha^2 1 + 2\alpha x + x^2$. (We will also use this notation for the corresponding construction for associative algebras: $\hat{R} = \Phi 1 + R$.)

0.3. Recall that a Φ-submodule K of a Jordan algebra J is an inner ideal if $U_x\hat{J} \subseteq K$ for all $x \in K$, and that an inner ideal $I \subseteq J$ is an ideal if $\{I, J, \hat{J}\}+U_JI \subseteq I$. If I, L are ideals of J, so is their product U_IL, and in particular so is the derived ideal $I^{(1)} = U_II$. An (inner) ideal of J is essential if it has nonzero intersection with any nonzero (inner) ideal of J.

If $X \subseteq J$ is a subset of the Jordan algebra J, the annihilator of X in J is the set $\mathrm{Ann}_J(X)$ of all $z \in J$ which satisfy $U_zx = U_xz = 0$ and $U_xU_z\hat{J} = U_zU_x\hat{J} = V_{x,z}\hat{J} = V_{z,x}\hat{J} = 0$ for all $x \in X$. This is always an inner ideal of J, and it is also an ideal if X is an ideal. If J is nondegenerate and I is an ideal of J, the annihilator of I can be characterized in the following alternative ways (see [Mc2, Mo2]):

$$\mathrm{Ann}_J(I) = \{z \in J \mid U_zI = 0\} = \{z \in J \mid U_Iz = 0\}.$$

0.4. The centroid $\Gamma(J)$ of a Jordan algebra J is the set of all Φ-linear mappings $\gamma : J \to J$ that satisfy: $\gamma(U_xy) = U_x\gamma(y)$, $\gamma^2(U_xz) = U_{\gamma(x)}z$, and $\gamma(\{x, y, z\}) = \{\gamma(x), y, z\}$ for all $x, y \in J$ and all $z \in \hat{J}$. If J is nondegenerate, then $\Gamma(J)$ is a reduced unital commutative ring, and if in addition J is strongly prime, then $\Gamma(J)$ is a domain acting faithfully on J. In that case we can localize to define the central closure $\Gamma(J)^{-1}J$ which is an algebra over the field of fractions $\Gamma(J)^{-1}\Gamma(J)$.

Following [Fu], we define the weak center $C_w(J)$ as the set of all $z \in J$ which have $U_z, V_z \in \Gamma(J)$.

We refer to [Mo2] for the notions of extended centroid and extended central closure. We will denote by $\mathcal{C}(J)$ the extended centroid of J, and by $\mathcal{C}(J)J$ its extended central closure, which is a tight scalar extension of J (see [Mo2]).

0.5. Any associative algebra R gives rise to a Jordan algebra $R^{(+)}$ by taking the products $U_x y = xyx$ and $x^2 = xx$. A Jordan algebra is special if it is isomorphic to a subalgebra of an algebra of the form $R^{(+)}$, and it is called i-special if it satisfies all the identities satisfied by all special algebras. An important class of special algebras are algebras of symmetric elements $H(R,*)$ of associative algebras with involution $(R,*)$, and more generally, ample subspaces $H_0(R,*) \subseteq H(R,*)$ of symmetric elements, subspaces that satisfy: $r + r^*$, rr^* and rhr^* belong to $H_0(R,*)$ for all $r \in R$ and all $h \in H_0(R,*)$.

For a special Jordan algebra J we can always find a associative $*$-envelope, an associative algebra R with involution $*$ such that J is a subalgebra of $H(R,*)$, and R is generated (as an associative algebra) by J. An associative $*$-envelope of J is $*$-tight is any nonzero $*$-ideal I of R hits J: $I \cap J \neq 0$.

A fundamental fact in Jordan theory with important structural consequences for i-special algebras is the existence of hermitian ideals in the free special Jordan algebra $FSJ[X]$, generated by X in the (+)-algebra of the free associative algebra $Ass[X]$ (see [McZ]): for any special Jordan algebra $J \subseteq H(R,*)$ and any a in the associative subalgebra $alg_R(\mathcal{H}(J))$ of R generated by the evaluation $\mathcal{H}(J)$ of $\mathcal{H}(X)$ on J, the trace $a + a^*$ belongs to $\mathcal{H}(X)$. An i-special Jordan algebra J is of hermitian type if $\mathrm{Ann}_J(\sum_{\mathcal{H}} \mathcal{H}(J)) = 0$, where the sum runs on the set of all hermitian ideals.

0.6. We refer to [St, R2] for basic facts about algebras of quotients for associative algebras. We will be interested in algebras of quotients attached to the filters of dense right or left ideals of an associative algebra R, and in particular to the right and left maximal algebra of quotients which we will denote by $Q^r_{max}(R)$ and $Q^l_{max}(R)$ respectively. Recall that a dense left (resp. right) ideal of R is just an essential left (resp. right) ideal of R if R is left (resp. right) nonsingular, that is the left singular ideal $Z_l(R)$ vanishes (resp. the right singular ideal $Z_r(R)$ vanishes). The associative algebras that naturally arise in Jordan theory are associative envelopes and they carry an involution, so it will be important to us to be able to extend involutions to algebras of quotients. This can not be done in general for the one sided maximal algebras of quotients $Q^l_{max}(R)$ and $Q^r_{max}(R)$, so the adequate substitute is the maximal symmetric algebra of quotients $Q_\sigma(R)$ defined by Lanning [L]. Recall that $Q_\sigma(R)$ is the set of elements $q \in Q^r_{max}(R)$ for which there exists a dense left ideal L of R with $Lq \subseteq R$ (or symmetrically, the set of all $q \in Q^l_{max}(R)$ for which there exists a dense right ideal K with $qK \subseteq R$). If R has an involution, this is the biggest subalgebra of the maximal algebra of left (resp. right) quotients to which the involution extends. Another algebra of quotients to which involutions can be extended, and which plays a fundamental role in Zelmanov's structure theory is the Martindale algebra of symmetric quotients $Q_s(R)$ of a semiprime algebra R (see [McZ]). As it is easy to see one has $Q_s(R) \subseteq Q_\sigma(R)$, and $Q_\sigma(Q_s(R)) = Q_\sigma(R)$, so if S is a subalgebra of R and $R \subseteq Q_s(S)$, then $Q_\sigma(R) = Q_\sigma(S)$.

1. Strong nonsingularity

1.1. General localization theory for associative rings is based, among other equivalent sets of data, on the notion of Gabriel filter (see [St]). A Jordan analogue of that notion seems to be difficult to define. However, the weaker notion of linearly topological filter of left (or right) ideals can be easily adapted to the Jordan setting.

Let J be a Jordan algebra, K be an inner ideal of J and $a \in J$, we define

$$(K : a) = \{x \in K \mid x \circ a \in K, U_a x \in K\}.$$

The set $(K : a)$ is again an inner ideal. Indeed, $(K : a)$ is clearly a Φ-submodule of J, and if $x \in (K : a)$, $z \in J$, we have $(U_x z) \circ a = \{x, z, x \circ a\} - U_x(z \circ a) \in K$ and $U_a U_x z = U_{a\circ x} z - U_x U_a z - a \circ U_x(z \circ a) + \{x, z, U_a x\} \in K$.

A family $\mathcal{F}$ of inner ideals of a Jordan algebra J will be called a *linearly topological filter* of inner ideals if it satisfies:

LTFI. Any inner ideal of J which contains an element from $\mathcal{F}$ belongs to $\mathcal{F}$.

LTFII. If $K, L \in \mathcal{F}$, then $K \cap L \in \mathcal{F}$.

LTFIII. If $K \in \mathcal{F}$ and $a \in J$, then $(K : a) \in \mathcal{F}$.

Since we will be interested in essential inner ideals, we first prove that in a nondegenerate Jordan algebra they form a linearly topological filter.

1.2. Lemma. *Let J be a nondegenerate Jordan algebra. If K is an essential inner ideal of J and $a \in J$, then $(K : a)$ is again essential.*

Proof. Let N be a nonzero inner ideal of J, and let us prove that $N \cap (K : a) \neq 0$. By essentiality of K, we can assume that $N \subseteq K$. Now, let us see that we can assume that $U_aN \subseteq K$. Indeed, if $U_aN \nsubseteq K$, then $U_aN \neq 0$ and we can choose $x \in N$ with $0 \neq U_ax \in U_aN \cap K$. Replacing N by U_xU_aJ, which is obviously nonzero and is contained in N, we obtain $U_aN \subseteq K$. Note that under this assumption we have $N \cap (K : a) = \{x \in N \mid x \circ a \in K\}$.

Next consider the inner ideal $U_{1-a}N$. If this is nonzero, there is some $k = U_{1-a}x \in U_{1-a}N \cap K$ with $0 \neq x \in N$, and obviously this is also true if $U_{1-a}N = 0$. Thus, in both cases we can take a nonzero $x \in N$ with $U_{1-a}x \in K$. Then $x \circ a = x + U_ax - U_{1-a}x \in K$, hence $x \in (K : a) \cap N$. ■

1.3. The definition of a ring of quotients based on the filter of essential left (or right) ideals in associative theory involves the nonsingularity of the ring. In Jordan theory, an analogue of the notion of nonsingularity was introduced in [FGM]: a Jordan algebra J is nonsingular if for any essential inner ideal K of J the annihilator $\mathrm{Ann}_J(K)$ vanishes. As it will become apparent later, we will need a more stringent notion of nonsingularity based in a weaker annihilation for essential inner ideals, so we will consider the following property of an essential inner ideal K of J:

$$(*) \qquad \text{for all } a \in J, \quad U_aK = 0 \Rightarrow a = 0$$

An algebra J will be called *strongly nonsingular* if every essential inner ideal K satisfies $(*)$. Although the word ‘strongly’ departs here from its common usage in Jordan theory, where it usually means nondegenerate, it is not very far from that meaning since it is easy to see that an essential inner ideal K of a Jordan algebra J satisfies $(*)$ if and only if K is nondegenerate as a Jordan algebra. Indeed, the ‘only if’ is obvious, and if $U_aK = 0$ for some nonzero $a \in J$, then $L = \Phi a + U_a\hat{J}$ is a nonzero inner ideal of J, hence there is a nonzero $k \in L \cap K$, and it is easy to see that k is an absolute zero divisor of K, contradicting the nondegeneracy of K. Thus, a Jordan algebra J is strongly nonsingular if and only if every essential inner ideal of J is a nondegenerate algebra.

As it is well known, the product of two essential left ideals in a left nonsingular associative ring is again essential. We next prove an analogous fact for essential inner ideals of strongly nonsingular Jordan algebras.

1.4. Lemma. *For a Jordan algebra J and a Φ-submodule $A \subseteq J$ of J, the set*

$$\mathcal{K}_J(A) = \{a \in A \mid U_aJ + \{a, J, A\} \subseteq A\}$$

is an inner ideal of J.

Proof. The set $\mathcal{K}_J(A)$ is easily seen to be a Φ-submodule of J. Now, if $a \in \mathcal{K}(A)$ and $x \in J$, for all $b \in A$ and $y \in J$ we have $U_{U_a x} J \subseteq U_a J \subseteq A$, and

$$\begin{aligned}\{U_a x, y, b\} &= \{a, x, \{a, y, b\}\} - U_a\{x, b, y\} \in \\ &\in \{a, J, \{a, J, A\}\} + U_a J \subseteq \\ &\subseteq \{a, J, A\} + A \subseteq A\end{aligned}$$

Therefore $U_a x \in \mathcal{K}_J(A)$, and $\mathcal{K}_J(A)$ is an inner ideal of J. ∎

1.5. Proposition. *Let J be a strongly nonsingular Jordan algebra and let K be an inner ideal of J. If K is essential, then $\mathcal{K}_J(U_K K)$ is also essential.*

Proof. We first show that $U_{U_x y} z \in \mathcal{K}_J(U_K K)$ for all $x, y, z \in K$. Indeed, $U_{U_{U_x y} z} J \subseteq U_x U_y J \subseteq U_K K$, and for all $t \in K$, $a \in J$,

$$\{U_{U_x y} z, a, t\} = \{U_x y, z, \{U_x y, a, t\}\} - U_x U_y U_x\{z, t, a\} \in \{K, K, K\} + U_K K = U_K K.$$

Now let L be a nonzero inner ideal of J. Since K is essential, there is a nonzero $x \in K \cap L$. Now, $U_x K \neq 0$ by the strong nonsingularity of J, hence there is $y \in K$ with $0 \neq U_x y$. Next, $U_{U_x y} K \neq 0$ again by the essentiality of K and the strong nonsingularity of J, so there is $z \in K$ with $a = U_{U_x y} z \neq 0$. Clearly $a \in L$ and $a \in \mathcal{K}_J(U_K K)$ by what we proved before. Therefore $L \cap \mathcal{K}_J(K) \neq 0$ proving the essentiality of $\mathcal{K}_J(U_K K)$. ∎

We apply next the $\mathcal{K}$-construction to show that strong nonsingularity is inherited by essential ideals.

1.6. Lemma. *Let J be a Jordan algebra and let I be an essential ideal of J, then J is strongly nonsingular if and only if I is a strongly nonsingular algebra.*

Proof. Suppose first that I is strongly nonsingular and let K be an essential inner ideal of J. If $a \in J$ has $U_a K = 0$, then $U_{U_a y} K = 0$ for any $y \in I$. Now it is easy to see that $K \cap I$ is an essential inner ideal of I, hence $U_{U_a y}(K \cap I) = 0$ implies that $U_a y = 0$, hence $U_a I = 0$ and $a = 0$ by the essentiality of I.

Assume now that J is strongly nonsingular and let K be an essential ideal of I. We claim that $\mathcal{K}_J(K)$ is an essential ideal of J. Indeed, if L is a nonzero inner ideal of J, then $U_L I \subseteq L \cap I$ is nonzero by the essentiality of I. Thus $I \cap L$ is a nonzero inner ideal of I hence $K \cap L = K \cap (I \cap L)$ is nonzero and we can take a nonzero $x \in K \cap L$. From the essentiality of I it easily follows that we can find elements $y, z \in I$ with $U_{U_x y} z \neq 0$.

Now we have $U_{U_{U_x y} z} J \subseteq U_x U_y J \subseteq U_K I \subseteq K$, and for all $a \in J$ and $k \in K$, applying QJ15 we also have $\{U_{U_x y} z, a, k\} = \{U_x y, \{z, U_x y, a\}, k\} - \{U_x U_y U_x a, z, k\} \in$

$\{K, I, K\} \subseteq K$. Therefore $U_{U_x y} z \in \mathcal{K}_J(K)$ and, since $U_{U_x y} z \in L$, we get $\mathcal{K}_J(K) \cap L \neq 0$. This proves the essentiality of $\mathcal{K}_J(K)$.

Finally, if $U_a K = 0$ for some $a \in I$, then $U_a \mathcal{K}_J(K) = 0$, hence $a = 0$ by the essentiality of $\mathcal{K}_J(K)$ and the strong nonsingularity of J. ■

2. Algebras of quotients

2.1. Let $\tilde{J}$ be a Jordan algebra, let J be a subalgebra of $\tilde{J}$ and let $\tilde{a} \in \tilde{J}$. Recall from [Mo2] that an element $x \in J$ is a J-denominator of $\tilde{a}$ if the following multiplications take $\tilde{a}$ back into J:

(Di)	$U_x \tilde{a}$	(Dii)	$U_{\tilde{a}} x$	(Diii)	$U_{\tilde{a}} U_x \hat{J}$
(Diii')	$U_x U_{\tilde{a}} \hat{J}$	(Div)	$V_{x,\tilde{a}} \hat{J}$	(Div')	$V_{\tilde{a},x} \hat{J}$

We will denote the set of J-denominators of $\tilde{a}$ by $\mathcal{D}_J(\tilde{a})$. It has been proved in [Mo2, 4.2] that $\mathcal{D}_J(\tilde{a})$ is an inner ideal of J. We remark (see [FGM, p.410]) that any $x \in J$ satisfying (Di), (Dii), (Diii) and (Div) belongs to $\mathcal{D}_J(\tilde{a})$.

2.2. Let J be a subalgebra of a Jordan algebra Q. We will say that Q is an *algebra of quotients* of J if the following conditions hold:

(i) $\mathcal{D}_J(q)$ is an essential inner ideal of J for all $q \in Q$.

(ii) $U_q \mathcal{D}_J(q) \neq 0$ for any nonzero $q \in Q$.

Clearly, any nondegenerate algebra J is its own algebra of quotients since its inner ideal of denominators $\mathcal{D}_J(x) = J$ is essential for all $x \in J$, and the nondegeneracy of J implies $U_x \mathcal{D}_J(x) = U_x J \neq 0$. Reciprocally, any Jordan algebra having an algebra of quotients is nondegenerate by property (ii).

2.3. Examples.

1. We have already mentioned that a nondegenerate Jordan algebra J is an algebra of quotients of J itself. More generally, if I is an essential ideal of J which is nondegenerate as a Jordan algebra, then J is an algebra of quotients of I. Indeed, any $x \in J$ has $\mathcal{D}_J(x) = I$ essential, and $U_x \mathcal{D}_J(x) = U_x I \neq 0$.
2. If J is a strongly nonsingular Jordan algebra and K is an essential inner ideal of J, then J is an algebra of quotients of K. Indeed, it is easy to see that $\mathcal{D}_J(x) = (K : x)$ for all $x \in J$, which is an essential inner ideal of J by 1.2. Now, if N is a nonzero inner ideal of K, for any $0 \neq y \in N$, $U_y K \subseteq N$ is a nonzero inner ideal of J by strong nonsingularity of J, hence $0 \neq U_y K \cap (K : x) \subseteq N \cap (K : x)$, and $(K : x)$ is an essential inner ideal of K. The condition $U_x(K : x) \neq 0$

follows from the essentiality of $(K : x)$ as an inner ideal of J, and the strong nonsingularity of J.

3. Suppose that J is strongly prime and let $\Gamma^{-1}J$ be the central closure of J. Then for any element $q = \gamma^{-1}a \in \Gamma^{-1}J$ we have $\gamma^2 J \subseteq \mathcal{D}_J(q)$ [FGM, p. 409]. Moreover, since J is strongly prime, J has no Γ-torsion by 0.4, hence for any $0 \neq \gamma \in \Gamma$, the set γJ is a nonzero ideal of J which is essential as an inner ideal. By the nondegeneracy of J, we have $U_q\mathcal{D}_J(q) \supseteq U_{\gamma^{-1}a}\gamma^2 J \supseteq U_a J \neq 0$ (see [FGM 4.2]). Hence $\Gamma^{-1}J$ is an algebra of quotients of J.

4. The extended central closure $\mathcal{C}(J)J$ of a nondegenerate Jordan algebra J is an algebra of quotients of J. Indeed, since for any $x \in \mathcal{C}(J)J$ there is an essential ideal of J contained in $\mathcal{D}_J(x)$ by [Mo2, 4.3(ii)], we get $U_x\mathcal{D}_J(x) \neq 0$ by [FGM, 4.3].

5. Let J be a Jordan algebra. Recall that an element $s \in J$ is said to be injective if the mapping U_s is injective over J. Following [FGM] we denote by $\mathrm{Inj}(J)$ the set of injective elements of J. A set $S \subseteq \mathrm{Inj}(J)$ is a monad if $U_s t, s^2 \in S$ for any $s, t \in S$ (see [Z1, Z2, FGM]). A monad S is said to be an Ore monad if $U_s S \cap U_t S \neq \emptyset$ for any $s, t \in S$. An algebra Q containing J as a subalgebra is an algebra of S-quotients (and J is an S-order of Q) if all elements of S are invertible in Q and for all $q \in Q$, $\mathcal{D}_J(q) \cap S \neq \emptyset$. It has been proved in [M,B] that a necessary condition for such an algebra Q to exist is that S satisfies the Ore condition in J: for any $x \in J$ and any $s \in S$ there exists $t \in U_s S$ such that $t \circ x \in K_s = \Phi s + U_s\hat{J}$. Note that for such an element t, we have $U_x t^2 = (x \circ t)^2 + U_t x^2 - \{x \circ t, x, t\} \in K_s$, hence $t^2 \in S \cap (K_s : x)$. Moreover, if $r \in S \cap (K_s : x)$, then any $t \in U_s S \cap U_r S$ has $t \in U_s S$ and $t \circ x \in K_s$. Thus the Ore condition can be rephrased: for any $x \in J$ and any inner ideal K of J, $K \cap S \neq \emptyset$ implies $(K : x) \cap S \neq \emptyset$.

   If J is a nondegenerate Jordan algebra and $S \subseteq \mathrm{Inv}(J)$ is an Ore monad, any algebra of S-quotients Q of J is an algebra of quotients in the sense of 2.2, that is, any element from S becomes invertible in Q: $S \subseteq \mathrm{Inv}(Q)$, and for any $q \in Q$, $\mathcal{D}_J(q) \cap S \neq \emptyset$. Indeed, if K is an inner ideal of J with $K \cap S \neq \emptyset$, then, for any nonzero $x \in J$, $U_x K = 0$ implies $U_x U_s J = 0$, hence $U_{U_s x} J = 0$, and $U_s x = 0$ by the nondegeneracy of J. Hence $x = 0$, a contradiction. In particular $0 \neq U_x(K : x) \subseteq U_x J \cap K$, which proves that K is essential. Then $\mathcal{D}_J(q)$ is essential for any $q \in Q$, and if $U_q\mathcal{D}_J(q) = 0$, then for any $s \in \mathcal{D}_J(q) \cap S$, $U_{U_s q} J \subseteq U_s U_q \mathcal{D}_J(q) = 0$, hence $U_s q = 0$ because $U_s q \in J$. Thus $q = 0$ since s is invertible in Q.

2.4. Lemma. *Let Q be an algebra of quotients of the Jordan algebra J. Then:*

(i) *Q is nondegenerate,*

(ii) *For any $q \in Q$, $U_qJ \cap J \neq 0$,*

(iii) *Any nonzero inner ideal of Q hits J nontrivially,*

If J is strongly nonsingular, then:

(iv) *If K is an essential inner ideal of K, then $U_qK \neq 0$ for any nonzero $q \in Q$,*

(v) *If L is an inner ideal of Q, then L is essential if and only if $L \cap J$ is an essential inner ideal of J,*

(vi) *Q is strongly nonsingular.*

Proof. (i) and (ii) are obvious since $0 \neq U_q\mathcal{D}_J(q) \subseteq U_qJ \cap J \subseteq U_qQ$ for any nonzero $q \in Q$, and (iii) easily follows from this.

Now if K is an essential inner ideal of J and $U_qK = 0$ for some $q \in Q$, then $U_{U_qx}K = U_qU_xU_qK = 0$ for any $x \in \mathcal{D}_J(q)$, hence $U_qx = 0$ since $U_qx \in J$ and J is strongly nonsingular. Thus $U_q\mathcal{D}_J(q) = 0$, hence $q = 0$, which proves (iv).

Next assume that L is an essential inner ideal of Q and let K be a nonzero inner ideal of J and take a nonzero $a \in K$. Then U_aQ is a nonzero inner ideal of Q hence there is $q \in Q$ such that $0 \neq U_aq \in U_aQ \cap L$ by the essentiality of L. Now $U_{U_aq}(\mathcal{D}_J(q) : a) \subseteq U_aU_q\mathcal{D}_J(q) \cap L \subseteq U_aJ \cap L \subseteq K \cap L$, and $U_{U_aq}(\mathcal{D}_J(q) : a) \neq 0$ by (iv) since $(\mathcal{D}_J(q) : a)$ is essential by 1.2. On the other hand, if L is an inner ideal of Q and $L \cap J$ is essential. For any nonzero inner ideal N of Q, and any nonzero $q \in N$ we have $0 \neq U_qQ \cap J$ by (iii), hence $N \cap J$ is a nonzero inner ideal of J and thus $0 \neq (N \cap J) \cap (L \cap J) \subseteq N \cap L$, hence L is essential.

Finally, (vi) is straightforward from (iii) and (iv). ∎

2.5. Lemma. *Let J be a Jordan algebra and let $M \subseteq J$ be a Φ-submodule. If $U_zM \neq 0$ for all $0 \neq z \in J$, then $U_Mz \neq 0$ for all $0 \neq z \in J$.*

Proof. Consider the polynomial algebra $J[t] = J \otimes_\Phi \Phi[t]$ and the submodule $M[t] \subseteq J[t]$ of polynomials whose coefficients belong to M. Then, for any $0 \neq p \in J[t]$ we have $U_pM[t] \neq 0$. Indeed, if $U_pM[t] = 0$ for some nonzero $p \in J[t]$, we can choose such a p of minimal degree. If $zt^n \neq 0$ is the leading term of p, for any $m \in M$, the term of degree $2n$ in U_pm is $U_zm = 0$, hence $U_zM = 0$ and thus $z = 0$, a contradiction.

Now assume that $U_Mz = 0$ for some $z \in J$. If $m = m_0 + m_1t + \ldots + m_nt^n \in M[t]$, we have $U_mz = \sum_{i=1}^n U_{m_i}zt^{2i} + \sum_{1\le i<j\le n}\{m_i, z, m_j\} \in (U_Mz)[t] = 0$, hence $U_{M[t]}z = 0$.

Now take $x, y \in M$ and set $m = x + yt \in M[t]$ and $a = U_mU_zm$. Then we have

$U_a = U_m U_z U_m U_z U_m = U_{U_m z} U_z U_m = 0$ and, since the condition satisfied by $M[t]$ implies that $J[t]$ is nondegenerate, we get $a = U_m U_z m = 0$. Then the coefficient in degree 1 of a is $U_x U_z y + \{x, U_z x, y\} = 0$, but since $\{x, U_z x, y\} = \{U_x z, x, y\} = 0$, we obtain $U_x U_z y = 0$ for all $x, y \in M$. Then $U_{U_z x} M = U_z U_x U_z M = 0$ for all $x \in M$, hence $U_z x = 0$ by the hypothesis on M. Thus $U_z M = 0$ hence, again by the hypothesis on M, we get $z = 0$. ■

2.6. Lemma. *Let J be a strongly nonsingular Jordan algebra and let Q be an algebra of quotients of J. If K is an essential inner ideal of J, then $U_K q \neq 0$ for all $0 \neq q \in Q$.*

Proof. This is straightforward from 2.5 and 2.4(iv). ■

Our next result will be useful to check essentiality of inner ideals of denominators without going through all conditions Di-iv.

2.7. Lemma. *Let $\tilde{J}$ be a Jordan algebra, let J be a subalgebra of $\tilde{J}$ and $\tilde{a} \in \tilde{J}$. If J is strongly nonsingular and there is an essential inner ideal K of J such that $x \circ \tilde{a}$ and $U_x \tilde{a}$ are in J for all $x \in K$, then $\mathcal{D}_J(q)$ is an essential inner ideal.*

Proof. Take $x, y \in K$, and set $z = U_x y$. Note that $z \in K$, hence $z \circ \tilde{a}$ and $U_z \tilde{a}$ belong to J. Next, for all $c \in J$, we have $\{z, \tilde{a}, c\} = \{U_x y, \tilde{a}, c\} = \{x, y \circ (x \circ \tilde{a}), c\} - \{x, \{y, \tilde{a}, x\}, c\} - \{U_x \tilde{a}, y, c\} \in J$. Also, $U_z U_{\tilde{a}} z = U_x U_y U_x U_{\tilde{a}} U_x y = U_x U_y U_{U_x \tilde{a}} y \in J$.

Now, take $b, c \in J$ and set $d = U_{U_z b} c$. Since $d \in U_x K$, we have $d \circ \tilde{a} \in J$, $U_d \tilde{a} \in J$ and $\{d, \tilde{a}, J\} \subseteq J$. On the other hand, the identity QJ6 in the z-homotope yields $U_{\tilde{a}} d = U_{\tilde{a}} U_z U_b U_z c = U_{\{\tilde{a}, z, b\}} U_z c - U_b U_{U_z \tilde{a}} c - \{\{\tilde{a}, U_z b, c\}, U_z \tilde{a}, b\} + \{c, z, U_b U_z U_{\tilde{a}} z\} \in J$. Moreover, since $U_d J \subseteq U_{U_z b} J$ we also have $U_{\tilde{a}} U_d J \subseteq J$. Thus d satisfies (Di), (Dii), (Diii) and (Div) of 2.1, hence $d \in \mathcal{D}_J(\tilde{a})$.

Now let N be a nonzero inner ideal of J. Since K is essential we can choose $0 \neq x \in K \cap N$, and since J is strongly nonsingular, we can choose $y \in K$ with $z = U_x y \neq 0$. Finally, from the nondegeneracy of J, it follows that there exist $b, c \in J$ with $0 \neq U_z b$ and $d = U_{U_z b} c \neq 0$. Since $d \in \mathcal{D}_J(\tilde{a}) \cap U_x J \subseteq \mathcal{D}_J(\tilde{a}) \cap K$, this proves the essentiality of K. ■

2.8. Lemma. *Let Q be an algebra of quotients of a strongly nonsingular Jordan algebra J and assume that Q is a subalgebra of a Jordan algebra $\tilde{Q}$. If $\tilde{q} \in \tilde{Q}$ has an essential inner ideal of denominators $\mathcal{D}_J(\tilde{q})$, then $\mathcal{D}_Q(\tilde{q})$ is essential. Moreover, if $U_{\tilde{q}} \mathcal{D}_J(\tilde{q}) \neq 0$, then $U_{\tilde{q}} \mathcal{D}_Q(\tilde{q}) \neq 0$.*

Proof. For any $x, y \in \mathcal{D}_J(\tilde{q})$ and any $p \in Q$ we have by QJ15,

$$\{\tilde{q}, U_x y, p\} = \{\{\tilde{q}, x, y\}, x, p\} - \{y, U_x \tilde{q}, p\} \in \{J, J, Q\} \subseteq Q.$$

Moreover, by QJ6,

$$U_{\tilde{q}}U_{U_xy}p = U_{\{\tilde{q},x,y\}}U_xp - U_yU_{U_x\tilde{q}}p - \{\{\tilde{q}, U_xy, p\}, U_x\tilde{q}, y\} + \{p, x, U_yU_xU_{\tilde{q}}x\} \in$$

$$U_JU_JQ + \{\{J, J, Q\}, J, J\} + \{Q, J, J\} \subseteq Q.$$

Therefore, $U_xy \in \mathcal{D}_Q(\tilde{q})$ for any $x, y \in \mathcal{D}_J(\tilde{q})$, hence $K = \mathcal{K}_J(U_{\mathcal{D}_J(\tilde{q})}\mathcal{D}_J(\tilde{q})) \subseteq \mathcal{D}_Q(\tilde{q})$. Since K is essential in J by 1.5, the essentiality of $\mathcal{D}_Q(\tilde{q})$ follows from 2.4(v).

Now, if $U_{\tilde{q}}\mathcal{D}_J(\tilde{q}) = 0$, then, with the previous notation, $U_{\tilde{q}}K = 0$. Hence for any $p \in \mathcal{D}_Q(\tilde{q})$ we have $U_{U_{\tilde{q}}p}K = 0$, and since $U_{\tilde{q}}p \in Q$ and K is an essential ideal of J, we get $U_{\tilde{q}}p = 0$ for all $p \in \mathcal{D}_Q(\tilde{q})$ by 2.4(iv), that is $U_{\tilde{q}}\mathcal{D}_Q(\tilde{q}) = 0$. ■

2.9. Proposition. *Let $J_1 \subseteq J_2 \subseteq J_3$ be Jordan algebras, each a subalgebra of the next one, and assume that J_1 is strongly nonsingular. Then J_3 is an algebra of quotients of J_1 if and only if J_3 is an algebra of quotients of J_2 and J_2 is an algebra of quotients of J_1.*

Proof. If J_3 is an algebra of quotients of J_1, it is obvious that J_2 is also an algebra of quotients of J_1, and it follows from 2.8 that J_3 is an algebra of quotients of J_2.

Now assume that J_2 is an algebra of quotients of J_1, and J_3 is an algebra of quotients of J_2. Take $q \in J_3$ and consider the set

$$N = \{x \in J_1 \mid x \circ q \in J_1, U_xq \in J_1, \{x, q, J_1\} \subseteq J_1\}.$$

The set N is then an inner ideal of J_1. Indeed, if $x, y \in N$, it is clear that $(x + y) \circ q = x \circ q + y \circ q \in J_1$, $\{x + y, q, J_1\} \subseteq \{x, q, J_1\} + \{y, q, J_1\} \subseteq J_1$ and $U_{x+y}q = \{x, q, y\} - U_xq - U_yq \in \{x, q, J_1\} + J_1 \subseteq J_1$, hence $x + y \in J_1$, and N is a Φ-submodule of J_1. On the other hand, if $x \in N$ and $z \in J_1$, then $U_xz \circ q = x \circ (z \circ (x \circ q)) - x \circ \{x, q, z\} - U_xq \circ z \in J_1$, $U_{U_xz}q = U_xU_zU_xq \in J_1$, and for all $w \in J_1$, $\{U_xz, q, w\} = \{x, z \circ (x \circ q), w\} - \{x, \{z, q, x\}, w\} - \{U_xq, z, w\} \in J_1$, hence $U_xz \in N$, and N is an inner ideal.

Now $\mathcal{D}_{J_2}(q) \cap J_1$ is an essential inner ideal of J_1 by 2.4(v). Thus for any nonzero inner ideal K of J_1 there is a nonzero $x \in \mathcal{D}_{J_2}(q) \cap K$. Now, since J_1 is strongly nonsingular, and $\mathcal{D}_{J_1}(U_xq)$ is essential (note that $U_xq \in J_2$), there is $y \in \mathcal{D}_{J_1}(U_xq)$ with $U_xy \neq 0$. Then $\{q, x, y\}$ and $\{q, x, y\} \circ x$ belong to J_2, hence $\mathcal{D}_{J_1}(\{q, x, y\})$ and $\mathcal{D}_{J_1}(\{q, x, y\} \circ x)$ are both essential, hence so is $\mathcal{D}_{J_1}(\{q, x, y\}) \cap \mathcal{D}_{J_1}(\{q, x, y\} \circ x)$, and by the strong nonsingularity of J_1 there is $z \in \mathcal{D}_{J_1}(\{q, x, y\}) \cap \mathcal{D}_{J_1}(\{q, x, y\} \circ x)$ with $a = U_{U_xy}z \neq 0$. Now, for any $b \in \hat{J}_1$ we have:

$$
\begin{aligned}
\{q,a,b\} &= \{q,U_{U_xy}z,b\} = \{\{q,U_xy,z\},U_xy,b\} - \{z,U_{U_xy}q,b\} = && \text{by QJ15}\\
&= \{\{\{q,x,y\},x,z\},U_xy,b\}-\\
&- \{\{y,U_xq,z\},U_xy,b\} - \{z,U_{U_xy}q,b\} = && \text{by QJ15}\\
&= \{(\{q,x,y\}\circ x)\circ z,U_xy,b\} - \{\{x,\{q,x,y\},z\},U_xy,b\}-\\
&- \{\{y,U_xq,z\},U_xy,b\} - \{z,U_xU_yU_xq,b\} \in && \text{by QJ14}\\
&\in \{(\{q,x,y\}\circ x)\circ \mathcal{D}_{J_1}(\{q,x,y\}\circ x),J_1,J_1\}+\\
&+ \{\{J_1,\{q,x,y\},\mathcal{D}_{J_1}(\{q,x,y\})\},J_1,\hat{J}_1\}+\\
&+ \{\{\mathcal{D}_{J_1}(U_xq),U_xq,J_1\},J_1,\hat{J}_1\}+\\
&+ \{J_1,U_{J_1}U_{\mathcal{D}_{J_1}}(U_xq)\}U_xq,\hat{J}_1\} \subseteq J_1
\end{aligned}
$$

and

$$U_aq = U_{U_xy}U_zU_xU_yU_xq \in U^3_{J_1}U_{\mathcal{D}_{J_1}(U_xq)}U_xq \in J_1.$$

Therefore $a \in N$ and obviously $a \in K$, hence $N \cap K \neq 0$, which proves that N is essential, and by 2.7, that $\mathcal{D}_{J_1}(q)$ is essential.

Now, if $U_q\mathcal{D}_{J_1}(q) = 0$, then for any $p \in \mathcal{D}_{J_2}(q)$ we have $U_{U_pq}\mathcal{D}_{J_1}(q) = 0$, hence $U_pq = 0$ by 2.4(iv) and the essentiality of $\mathcal{D}_{J_1}(q)$. Thus $U_{\mathcal{D}_{J_2}(q)}q = 0$, hence $q = 0$ by 2.6. This proves that J_3 is an algebra of quotients of J_1. ■

2.10. We will say that an algebra of quotients Q of a Jordan algebra J is a *maximal algebra of quotients* if for any other algebra of quotients $Q' \supseteq J$ there exists a homomorphism $\alpha : Q' \to Q$ whose restriction to J is the identity mapping: $\alpha(x) = x$ for all $x \in J$.

2.11. Remark. If Q and Q' are algebras of quotients of a Jordan algebra J and $\alpha : Q' \to Q$ is a homomorphism which restricts to the identity on J, then α is injective. Indeed, if $q \in Q$ has $\alpha(q) = 0$, then $U_q\mathcal{D}_J(q) = \alpha(U_q\mathcal{D}_J(q))$ (since $U_q\mathcal{D}_J(q) \subseteq J) = U_{\alpha(q)}\alpha(\mathcal{D}_J(q)) = 0$, hence $q = 0$.

2.12. Lemma. *Let Q and Q' be algebras of quotients of a strongly nonsingular Jordan algebra J. If $\alpha, \beta : Q' \to Q$ are homomorphisms whose restriction to J is the identity mapping, then $\alpha = \beta$.*

Proof. Take $q \in Q$ and $k \in \mathcal{D}_J(q)$ and set $p = \alpha(q) - \beta(q)$. We have $U_kp = U_k\alpha(q) - U_k\beta(q) = U_{\alpha(k)}\alpha(q) - U_{\beta(k)}\beta(q) = \alpha(U_kq) - \beta(U_kq) = U_kq - U_kq = 0$. Thus $U_{\mathcal{D}_J(q)}p = 0$, hence $p = 0$ by 2.6. ■

2.13. Lemma. *If Q and Q' are maximal algebras of quotients of a strongly nonsingular Jordan algebra J, then there exists a unique isomorphism $\alpha : Q \to Q'$ that extends the identity mapping $J \to J$.*

Proof. This is straightforward from 2.12. ■

In view of this result, if a strongly nonsingular Jordan algebra J has a maximal algebra of quotients, such an algebra is unique up to an isomorphism extending the identity on J. We will then denote this algebra by $Q_{max}(J)$ and will refer to it as *the* maximal algebra of quotients of J.

3. Algebras of quotients of PI algebras

We show in this section that the maximal algebra of quotients of a strongly prime PI-algebra is just its central closure. Two main ingredient for the proof of that fact is the good behavior of the weak center with respect to algebras of quotients and the fact that an inner ideal of such an algebra is essential if and only if it contains an essential ideal. We begin by proving the latter assertion and next study the weak center.

3.1. Lemma. *Let J be a strongly prime PI Jordan algebra. Then*

(a) *Any essential inner ideal of J hits nontrivially the weak center of J.*

(b) *An inner ideal of J is essential if and only if it contains an essential ideal of J.*

(c) *J is strongly nonsingular.*

Proof. (a) Since J is a strongly prime PI algebra, its central closure $\tilde{J} = \Gamma(J)^{-1}J$ is simple of finite capacity [ACM, 1.1]. In particular, the only essential inner ideal of $\tilde{J}$ is $\tilde{J}$ itself. Now, the span $\tilde{K} = \Gamma^{-1}K \subseteq \tilde{J}$ of K over $\Gamma(J)^{-1}\Gamma(J)$ is easily seen to be an essential inner ideal of $\tilde{J}$, hence $\tilde{K} = \tilde{J}$. Since $\tilde{J}$ is unital, we have $1 \in \tilde{K}$, hence there are $\gamma \in \Gamma(J)$ and $k \in K$ such that $1 = \gamma^{-1}k$, and $\gamma 1 = k \in K$ clearly has $k \in C_w(J)$.

(b) Since J is strongly prime, every nonzero ideal is essential, so it suffices to show that if K is an essential inner ideal of J, then it contains a nonzero ideal of J. Now, by (a), there is a nonzero $z \in K \cap C_w(J)$, and it is straightforward that $U_zJ \subseteq K$ is a nonzero ideal of J.

(c) This immediately follows from (b). ■

3.2. Lemma. *Let J be a nondegenerate Jordan algebra.*

(a) *If J is unital and Q is an algebra of quotients of J, then Q is also unital with the same unit as J.*

(b) *The socle of J is contained in every essential inner ideal of J.*

(c) *If J has finite capacity, then $J = Q_{max}(J)$.*

Proof. (a) If $1 \in J$ is the unit of J, then 1 is an idempotent of J, hence of Q. Thus it gives rise to a Peirce decomposition $Q = Q_0(1) + Q_1(1) + Q_2(1)$. Now, $Q_0(1)$ is an inner ideal of Q, and $Q_0(1) \cap J = 0$, hence it follow from 2.4(iii) that $Q_0(1) = 0$. Then $Q_1(1)$ is an inner ideal of Q, and again $Q_1(1) \cap J = 0$ implies $Q_1(1) = 0$. Therefore $Q = Q_2(1)$, and 1 is the unit of Q.

(b) The socle of J is the sum of all minimal inner ideals of J and if K is an essential inner ideal of J, it is easy to see that it contains every minimal inner ideal of J, hence it contains the socle.

(c) If J has finite capacity, then it is unital and coincides with its socle. Thus, if K is a essential inner ideal of K, then $K = J$ and thus $1 \in K$. Now, for any algebra of quotients Q of J and any $q \in Q$, the inner ideal $\mathcal{D}_J(q)$ is essential, hence $1 \in \mathcal{D}_J(q)$ and $q = U_1 q \in J$. Therefore $J = Q$. ∎

3.3. Lemma. *The following identity holds in any Jordan algebra J:*

$$(**) \qquad U_{U_a b} U_c d + U_{U_a c} U_b d + \{a, \{b, a, c\}, \{d, b, U_a c\}\} =$$

$$= U_a U_{\{b,a,c\}} d + \{U_a c, b, \{d, b, U_a c\}\} + \{a, U_b U_a U_c a, d\}.$$

Proof. First note that by JQ15 we have

$$\{U_a b, c, \{d, b, U_a c\}\} - \{a, \{b, a, c\}, \{d, b, U_a c\}\} = \{U_a c, b, \{d, b, U_a c\}\},$$

hence identity $(**)$ can be rewritten as

$$U_{U_a b} U_c d + U_{U_a c} U_b d + \{U_a b, c, \{d, b, U_a c\}\} =$$

$$= U_a U_{\{b,a,c\}} d + \{a, U_b U_a U_c a, d\}.$$

Now consider the polynomial algebra $J[t]$. Evaluating the identity $U_x U_{U_y x} z = U_{U_x y} U_y z$ in $x = a$, $y = b + ct$ and $z = d$ and comparing coefficients in t^2 we get

$$U_{U_a b} U_c d + U_{U_a c} U_b d + \{U_a b, \{b, d, c\}, U_a c\} =$$

$$= U_a U_{\{b,a,c\}} d + U_a \{U_b a, d, U_c a\}.$$

Thus we have to prove the identity

$$\{U_a b, \{b, d, c\}, U_a c\} + \{a, U_b U_a U_c a, d\} =$$

$$\{U_ab, c, \{d, b, U_ac\}\} + U_a\{U_ba, d, U_ca\}.$$

Now, by the partial linearization of QJ8', we have

$$\{U_ab, \{b, d, c\}, U_ac\} - \{U_ab, c, \{d, b, U_ac\}\} =$$

$$\{U_ab, b, \{d, c, U_ac\}\} - \{U_ab, \{c, U_ac, b\}, d\}.$$

On the other hand, by QJ11 we get

$$U_a\{U_ba, d, U_ca\} = \{U_ab, b, \{d, U_ac, c\}\} - \{U_aU_ba, U_ac, d\}.$$

So gathering all the above information, it only remains to prove the identity

$$\{U_ab, \{U_ca, a, b\}, d\} = \{a, U_bU_aU_ca, d\} + \{U_aU_ba, U_ca, d\}.$$

Set $x = U_ca$. Using identity QJ15 several times we obtain

$$\begin{aligned}\{U_ab, \{x, a, b\}, d\} &= \{a, \{b, a, \{x, a, b\}\}, d\} - \{U_a\{x, a, b\}, b, d\} = \\ &= 2\{a, U_bU_ax, d\} + \{a, \{U_ba, a, x\}, d\} - \{\{U_ax, b, a\}, b, d\} = \\ &= 2\{a, U_bU_ax, d\} + \{U_aU_ba, x, d\} + \{U_ax, U_ba, d\} - \\ &- \{\{U_ax, b, a, \}, b, d\} = 2\{a, U_bU_ax, d\} + \{U_aU_ba, x, d\} - \\ &- \{a, U_bU_ax, d\} = \{a, U_bU_ax, d\} + \{a, U_aU_bx, d\},\end{aligned}$$

and this finishes the proof of identity $(**)$. ■

3.4. Proposition. *If Q is an algebra of quotients of a strongly nonsingular Jordan algebra J, then $C_w(J) = C_w(Q) \cap J$.*

Proof. The containment $C_w(Q) \cap J \subseteq C_w(J)$ is obvious.

Take $z \in C_w(J)$, and let $q \in \hat{Q}$, $k \in \mathcal{D}_J(q)$. Then, for all $x \in J$ we have by QJ15

$$\begin{aligned}U_k\{x, z, q\} &= \{\{k, x, z\}, q, k\} - \{z, x, U_kq\} = \\ &= \{\{k, z, x\}, q, k\} - \{x, z, U_kq\} = \\ &= U_k\{z, x, q\},\end{aligned}$$

hence $U_{\mathcal{D}_J(q)}(\{x, z, q\} - \{z, x, q\}) = 0$, which implies by 2.6

(1) $$\{x, z, q\} = \{z, x, q\} \quad \text{for all } x \in J,\ q \in \hat{Q}.$$

Now, for all $x \in \hat{J} \subseteq \hat{Q}$, $q \in \hat{Q}$ and $k \in K = \mathcal{D}_J(q) \cap \mathcal{D}_J(\{z, x, q\})$, we have

$$\begin{aligned} U_k U_z U_x q &= U_{\{k,z,x\}} q - U_x U_z U_k q - \\ &- \{x, z, U_k\{z, x, q\}\} + \{U_x U_z k, q, k\} = \quad \text{by JP21 of [Lo]} \\ &= U_{\{k,x,z\}} q - U_z U_x U_k q - \\ &- \{z, x, U_k\{x, z, q\}\} + \{U_z U_x k, q, k\} = \\ &= U_k U_x U_z q. \end{aligned}$$

Hence $U_K(U_z U_x q - U_x U_z q) = 0$. thus, by 2.6 we get

$$U_z U_x q = U_x U_z q \quad \text{for all } x \in \hat{J},\ q \in \hat{Q}. \tag{2}$$

Next, using (2) we get $U_z\{x, y, q\} = U_z(x \circ (y \circ q) - \{x, q, y\}) = x \circ (y \circ U_z q) - \{x, U_z q, y\} = \{x, y, U_z q\}$, hence

$$U_z\{x, y, q\} = \{x, y, U_z q\} \quad \text{for all } x, y \in J,\ q \in \hat{Q}. \tag{3}$$

Now, write the identity $(**)$ of 3.3 as $U_{U_a b} U_c d = f(a, b, c, d)$ and take $p, q \in \hat{Q}$ and $s, t \in \mathcal{D}_J(p)$, and set also $k = U_s t$. Then, using (2) and (3) we have

$$\begin{aligned} U_k U_z U_p q &= U_z U_k U_p q = U_z U_{U_s t} U_p q = \\ &= U_z f(s, t, p, q) = f(s, t, p, U_z q) = \\ &= U_{U_s t} U_p U_z q = U_k U_p U_z q. \end{aligned}$$

Therefore $U_L(U_z U_p q - U_p U_z q) = 0$, where $L = \mathcal{K}_J(U_{\mathcal{D}_J(p)} \mathcal{D}_J(p))$. Since L is essential by 1.5, we get from 2.6

$$U_z U_p q = U_p U_z q \quad \text{for all } p, q \in \hat{Q}. \tag{4}$$

This implies that $U_z \in \Gamma(Q)$, since if $p \in Q$ and $q \in \hat{Q}$, we have $U_z^2 U_p q = U_z U_p U_z q = U_{U_z p} q$.

Now, since $z^2 \in C_w(J)$, we have $U_z V_z = U_{z,z^2} = U_{z+z^2} - U_z - U_{z^2} \in \Gamma(Q)$. Thus, for any $p, q \in \hat{Q}$ we get $U_z(V_z U_p q - U_p V_z q) = U_z V_z U_p q - U_p U_z V_z q = 0$ and $U_z^2(V_z^2 U_p q - U_{V_z p} q) = (U_z V_z)^2 U_p q - U_{U_z V_z p} q = 0$, hence $U_z(V_z^2 U_p q - U_{V_z p} q) = 0$ (since $\gamma^2(w) = 0$ implies $\gamma(w) = 0$ by the nondegeneracy of Q). Then, setting $w = V_z U_p q - U_p V_z q$ or $V_z^2 U_p q - U_{V_z p} q$ it is easy to see that w belongs to the annihilator of the ideal $I = \Phi z + U_z \hat{Q}$, generated by z in Q, and since $w \in I$, the semiprimeness of Q implies that $w = 0$, hence $V_zU_pq = U_pV_zq$ and $V_z^2U_pq = U_{V_zp}q$ for all $p, q \in \hat{Q}$, and therefore $V_z \in \Gamma(Q)$ hence $z \in \mathcal{C}_w(Q)$. ■

3.5. Theorem. *Let J be a strongly prime PI Jordan algebra. Then the central closure $\Gamma(J)^{-1}J$ is the maximal algebra of quotients of J.*

Proof. We first note that J is strongly nonsingular by 3.1 and therefore the algebra of quotients, if it exists, is unique up to isomorphism by 2.13, and also that $\Gamma(J)^{-1}J$ is an algebra of quotients of J by 2.3.3.

Let now Q be an algebra of quotients of J. Since J is strongly prime and Q is tight over J by 2.4(iii), Q is also strongly prime. Set $\tilde{Q} = \Gamma(Q)^{-1}Q$, the central closure of Q, and note that since $\tilde{Q}$ is an algebra of quotients of Q, it is also an algebra of quotients of J by 2.9. We claim that there exists a monomorphism $\phi : \Gamma(J) \to \Gamma(\tilde{Q})$ which satisfies $\phi(\gamma)(x) = \gamma(x)$ for all $x \in J$ and $\gamma \in \Gamma(J)$. To define ϕ, take $\gamma \in \Gamma(J)$. Then, if $q \in \tilde{Q}$, $\mathcal{D}_J(q)$ is essential, hence there is a nonzero $z \in \mathcal{D}_J(q) \cap C_w(J)$ by 3.1(a), and $z \in C_w(\tilde{Q})$ by 3.4. We set

$$\phi(\gamma)(q) := U_z^{-1}\gamma(U_zq)$$

which makes sense since $U_zq \in J$ and $U_z \in \Gamma(\tilde{Q})$ is invertible in $\Gamma(Q)^{-1}\Gamma(Q) \subseteq \Gamma(\tilde{Q})$.

Let us first show that the above expression is independent of the choice of z, or more generally, that any $z' \in C_w(J)$ with $U_{z'}q \in J$ will give the same result. Indeed, for such a z' we also have $U_{z'}^{-1} \in \Gamma(\tilde{Q})$, and

$$\begin{aligned} U_{z'}^{-1}\gamma(U_{z'}q) =& U_{z'}^{-1}U_z^{-1}U_z\gamma(U_{z'}q) = U_{z'}^{-1}U_z^{-1}\gamma(U_zU_{z'}q) = \\ =& U_{z'}^{-1}U_z^{-1}\gamma(U_{z'}U_zq) = U_{z'}^{-1}U_z^{-1}U_{z'}\gamma(U_zq) = \\ =& U_z^{-1}\gamma(U_zq). \end{aligned}$$

Now take $p, q \in \tilde{Q}$. Then $K = \mathcal{D}_J(p) \cap \mathcal{D}_J(q) \cap \mathcal{D}_J(p+q)$ is essential in J hence there exists a nonzero $z \in K \cap C_w(J)$, an we have:

$$\begin{aligned} \phi(\gamma)(p+q) =& U_z^{-1}\gamma(U_z(p+q)) = U_z^{-1}\gamma(U_zp + U_zq) = \\ =& U_z^{-1}\gamma(U_zp) + U_z^{-1}\gamma(U_zq) = \phi(\gamma)(p) + \phi(\gamma)(q), \end{aligned}$$

hence ϕ is linear.

Next note that $z^3 \in C_w(J)$ has $U_{z^3}U_pq = U_{z^2}U_pU_zq = U_{U_zp}U_zq \in J$, and we have

$$\begin{aligned} \phi(\gamma)(U_pq) =& U_{z^3}^{-1}\gamma(U_{z^3}U_pq) = U_{z^3}^{-1}\gamma(U_{U_zp}U_zq) = \\ =& U_{z^3}^{-1}U_{U_zp}\gamma(U_zq) = U_{z^3}^{-1}U_z^2U_p\gamma(U_zq) = \\ =& U_z^{-1}U_p\gamma(U_zq) = U_pU_z^{-1}\gamma(U_zq) = \\ =& U_p\phi(\gamma)(q). \end{aligned}$$

Analogous computations using z^3 as a denominator show that $\phi(\gamma)^2(U_p q) = U_{\phi(\gamma)(p)}q$, and using z^2, that $\phi(\gamma)(p \circ q) = p \circ \phi(\gamma)(q)$ and $\phi(\gamma)^2(q^2) = (\phi(\gamma)(q))^2$. Therefore ϕ maps $\Gamma(J)$ into $\Gamma(\tilde{Q})$. It only remains to show that ϕ is a ring homomorphism. To prove that take $\gamma, \delta \in \Gamma(J)$. It is quite straightforward that $\phi(\gamma+\delta) = \phi(\gamma)+\phi(\delta)$. On the other hand, if $q \in \tilde{Q}$ and $0 \neq z \in \mathcal{D}_J(q)$, we have $U_z\phi(\delta)(q) = U_zU_z^{-1}\delta(U_zq) = \delta(U_zq) \in J$, and

$$\phi(\gamma)\phi(\delta)(q) = U_z^{-1}\gamma(U_z\phi(\delta)(q)) =$$

$$= U_z^{-1}\gamma(\delta(U_zq)) = U_z^{-1}(\gamma\delta)(U_zq) = \phi(\gamma\delta)(q),$$

hence $\phi(\gamma)\phi(\delta) = \phi(\gamma\delta)$, and ϕ is a homomorphism.

The mapping ϕ gives $\tilde{Q}$ a structure of $\Gamma(J)$-algebra, and it is clear that $\tilde{\tilde{Q}} = \Gamma(\tilde{Q})^{-1}\tilde{Q}$ is then a $\Gamma(J)^{-1}\Gamma(J)$-algebra. Thus there exists a monomorphism of $\Gamma(J)^{-1}J$ into $\tilde{\tilde{Q}}$ which extends the inclusion $J \subseteq \tilde{Q} \subseteq \tilde{\tilde{Q}}$, and we can view $\Gamma(J)^{-1}J$ as a subalgebra of $\tilde{\tilde{Q}}$. Now, $\tilde{Q}$ is an algebra of quotients of J and $\tilde{\tilde{Q}}$ is an algebra of quotients of $\tilde{Q}$, hence $\tilde{\tilde{Q}}$ is an algebra of quotients of J by 2.9. Therefore $\tilde{\tilde{Q}}$ is an algebra of quotients of $\Gamma(J)^{-1}J$, again by 2.9. On the other hand, since J is PI and strongly prime, $\Gamma(J)^{-1}J$ is simple of finite capacity by [ACM], hence it is its own maximal algebra of quotients by 3.2(c). Then $\Gamma(J)^{-1}J = \tilde{\tilde{Q}}$ and $Q \subseteq \Gamma(J)^{-1}J$, which proves that $Q_{max}(J) = \Gamma(J)^{-1}J$. ∎

4. Algebras of hermitian type and general case

Since algebras of hermitian type are special we can make use of associative envelopes to transfer our problems to the associative setting. In the case of algebras of quotients this requires first to have a good relationship between essential inner ideals of the Jordan algebra and essential one sided ideals of its associative envelopes.

4.1. Let J be a Jordan algebra, recall that an element $a \in J$ gives rise to a local algebra J_a defined as the quotient of the a-homotope by the ideal $\mathrm{Ker}a = \{x \in J \mid U_ax = U_aU_xa = 0\}$. following [Mo1] we denote by $\mathrm{PI}(J)$ the set of all $a \in J$ such that J_a is a PI-algebra. It is proved in [Mo1] that if J is nondegenerate $\mathrm{PI}(J)$ is an ideal of J and if J is strongly prime and $\mathrm{PI}(J) \neq 0$, then the extended central closure $\mathcal{C}(J)J$ as nonzero socle and $\mathrm{Soc}(\mathcal{C}(J)J) \cap J = \mathrm{PI}(J)$. Similar notions can be defined for associative algebras where we again use the notation $\mathrm{PI}(R)$.

4.2. Lemma. *Let J be a strongly prime special Jordan algebra and let R be a $*$-tight associative $*$-envelope of J.*

(i) *If L is an essential left ideal of R, then $L \cap J$ is an essential inner ideal of J.*

(ii) *If K is an essential inner ideal of J, then $\hat{R}K$ is an essential left ideal of R.*

Proof. (i) Consider first the case where $\mathrm{PI}(J) \neq 0$. Since J is nondegenerate it suffices to show that $U_a J \cap (L \cap J) \neq 0$ for all $0 \neq a \in J$. Since $\mathrm{PI}(J) = \mathrm{PI}(R) \cap J$ by [Mo1, 6.5], we have $\mathrm{PI}(R) \neq 0$, hence R satisfies a GPI. Then the socle $\mathrm{Soc}(\tilde{R})$ of the central closure $\tilde{R} = \mathcal{C}_*(R)R$ is nonzero and $\mathrm{PI}(R) = R \cap \mathrm{Soc}(\tilde{R})$ by [R1, Ej. 7.6.2, p.287] and the $*$-primeness of R. Now the essentiality of L in R implies the essentiality of the left ideal $\tilde{L} = \mathcal{C}_*(R)L$ of $\tilde{R}$, and this implies $\mathrm{Soc}(\tilde{R}) \subseteq \tilde{L}$ (arguing as in 3.2(b)). Now $U_a\mathrm{PI}(J) \neq 0$ since J is strongly prime and $\mathrm{PI}(J)$ is a nonzero ideal. Thus we can take a nonzero $b \in U_a\mathrm{PI}(J)$. Then $b \in \mathrm{PI}(J) \subseteq \mathrm{PI}(R) \subseteq \mathrm{Soc}(\tilde{R}) \subseteq \tilde{L}$, so b can be written as $b = \sum_{i=1}^{n} \lambda_i x_i$ for some $\lambda_1, \ldots, \lambda_n \in \mathcal{C}_*(R)$ and $x_1, \ldots, x_n \in L$. Now there exists a nonzero $*$-ideal I of R with $\lambda_i I \subseteq R$ for all i. Then $I \cap J$ is a nonzero ideal of J since R is $*$-tight over J, hence $N = U_{I\cap J}(I \cap J)$ is a nonzero ideal of J by primeness. Now, if $x, y \in I \cap J$, we have $U_b U_x y = \sum_{i,j} x_i(\lambda_i x)z(x\lambda_j)x_j \in \sum_j Rx_j \subseteq L$, hence $U_b N \subseteq L$. On the other hand $U_b N \subseteq U_a J$, and this is nonzero since $N \neq 0$ and J is strongly prime, so we have $0 \neq U_b N \subseteq L \cap U_a J$ which proves the essentiality of $L \cap J$.

The assertion for case where $\mathrm{PI}(J) = 0$ has been proved in [FGM, p.467].

(ii) Again, we consider first the case where $\mathrm{PI}(J) \neq 0$. If J is PI, then K contains an essential ideal $I \subseteq K$ by 3.1(b), hence $\hat{R}K$ contains the nonzero $*$-ideal $\hat{R}I$ generated by I in R. Thus, if $N \subseteq R$ is a nonzero left ideal we have $0 \neq \hat{R}IN$ (by $*$-primeness of R)$\subseteq \hat{R}K \cap N$, which proves that $\hat{R}K$ is essential. Now assume that J is not PI and put $\tilde{J} = \mathcal{C}(J)J$, the extended central closure of J, and $\tilde{K} = \mathcal{C}(J)K$, the $\mathcal{C}(J)$-span of K in $\tilde{J}$. Take a $*$-tight associative $*$-envelope $\tilde{R}$ of $\tilde{J}$. Since J is not PI, by [Mc3, 2.2] there exists a unique $*$-homomorphism $R \to \tilde{R}$ which extends the inclusion $J \subseteq R$, moreover, that homomorphism is injective since its kernel is a $*$-ideal of R which intersects J trivially, hence it is zero by $*$-tightness of R over J. Thus, we can assume that $R \subseteq \tilde{R}$. Now, the inner ideal $\tilde{K}$ of $\tilde{J}$ is easily seen to be essential, and since $\tilde{J}$ has nonzero socle 3.2(b) gives $\mathrm{Soc}(\tilde{J}) \subseteq \tilde{K}$. Now, if N is a nonzero left ideal of R, arguing as in the PI case, we get $0 \neq \tilde{R}\mathrm{Soc}(J) \cap \tilde{R}N \subseteq \tilde{R}K \cap \tilde{R}N$. We claim that for any $\tilde{x} \in \tilde{R}$ there is a nonzero ideal I of J such that $I\tilde{x} \subseteq R$. Indeed, since $\tilde{R}$ is generated by $\tilde{J}$, the element $\tilde{x}$ is a sum of products of elements from $\tilde{J}$. So first finding an ideal in those conditions for each summand, and then taking the intersection of them all, produces an ideal of the required kind for $\tilde{x}$. Thus we can assume that $\tilde{x} = \tilde{y}_1 \cdots \tilde{y}_n$ for some $\tilde{y}_i \in \tilde{J}$. We carry on an induction on n. If $\tilde{x} = \tilde{y}_1 \in \tilde{J}$, by [Mo2,4.3(ii)] there exists an ideal $I \subseteq \mathcal{D}_J(\tilde{x})$, hence $(U_I J)\tilde{x} \subseteq I\{J, I, \tilde{x}\} + (U_I\tilde{x})J \subseteq JJ \subseteq R$, and the ideal $U_I J$ works. If the result holds up to $n-1$ factors, set $\tilde{z} = \tilde{y}_1 \cdots \tilde{y}_{n-1}$, so that $\tilde{x} = \tilde{z}\tilde{y}_n$

and take an ideal I_0 of J with $I_0\tilde{z} \subseteq R$, and an ideal I_1 of J with $I_1\tilde{y}_n \subseteq R$. Then $U_{I_0}I_1\tilde{x} \subseteq I_0I_1I_0\tilde{z}\tilde{y}_n \subseteq I_0I_1R\tilde{y}_n \subseteq I_0\hat{R}I_1\tilde{y}_n$ (since $I_1R \subseteq \hat{R}I_1$ by [FGM, 1.12 (i)]) $\subseteq I_0RR \subseteq R$.

Take then a nonzero $\tilde{x} \in \tilde{R}N \cap \tilde{R}K$. Then $\tilde{x} = \sum_{i=1}^{n} \tilde{r}_i x_i = \sum_{i=1}^{m} \tilde{s}_i k_i$ for some $\tilde{r}_1, \ldots, \tilde{r}_n, \tilde{s}_1, \ldots, \tilde{s}_m \in \tilde{R}$, $x_1, \ldots, x_n \in N$ and $k_1, \ldots, k_m \in K$. By what we have just proved there exist nonzero ideals I_i and I'_j of J with $I_i\tilde{r}_i \subseteq R$ and $I'_j\tilde{s}_j \subseteq R$ for all i and j. Then $I = I_1 \cap \cdots \cap I_n \cap I'_1 \cap \cdots I'_m$ is a nonzero ideal of J by primeness, and we have $I\tilde{x} \subseteq \sum_{i=1}^{n} I\tilde{r}_i x_i \subseteq \sum_{i=1}^{n} Rx_i \subseteq N$, and similarly $I\tilde{x} \subseteq RK$. Now, if $I\tilde{x} = 0$, then $(U_I\tilde{J})\tilde{x} = 0$ but $U_I\tilde{J} = U_{\mathcal{C}(J)I}\tilde{J}$ is a nonzero ideal of $\tilde{J}$, hence $0 \neq \tilde{x}$ annihilates the ideal $\tilde{R}(U_I\tilde{J})$ generated by it. But this is impossible since $\tilde{R}$ is $*$-tight over $\tilde{J}$. Thus we conclude that $0 \neq I\tilde{x} \subseteq \hat{R}K \cap N$, and this proves that $\hat{R}K$ is essential.

Finally, the case where $\mathrm{PI}(J) = 0$ has been proved in [FGM, 10.10]. ∎

4.3. Lemma. *Let J be a strongly prime special Jordan algebra and let R be a $*$-tight associative $*$-envelope of J. Then J is strongly nonsingular if and only if R is (left and right) nonsingular.*

Proof. Suppose first that J is strongly nonsingular. Then it is in particular, nonsingular, and by [FGM, 6.14] we have $Z_l(R)\cap J = 0$, hence $(Z_l(R)\cap Z_r(R))\cap J = 0$ since $Z_l(R)^* = Z_r(R)$, were $Z_l(R)$ (resp. $Z_r(R)$) denotes the left (resp. right) singular ideal of R. Now, the ideal $Z_l(R) \cap Z_r(R)$ is $*$-invariant, hence $Z_l(R) \cap Z_r(R) = 0$ by $*$-tightness of R over J. Now, if $a \in (Z_l(R) + Z_r(R)) \cap J$, since $Z_l(R) \cap Z_r(R) = 0$, a can be uniquely written as $a = b + b^*$ with $b \in Z_l(R)$. Then there exists an essential left ideal L of R with $Lb = 0$ (hence also $b^*L^* = 0$). Now $L \cap J$ and $L^* \cap J$ are essential inner ideals of J by 4.2(i), hence $L \cap L^* \cap J$ is also an essential inner ideal of J. Now we have $U_{L\cap L^*\cap J}a \subseteq LaL^* = L(b+b^*)L^* = 0$, hence $a = 0$ by 2.6. Therefore $(Z_l(R) + Z_r(R)) \cap J = 0$, hence $Z_l(R) + Z_r(R) = 0$ by $*$-tightness of R.

Now assume that R is (left and right) nonsingular and let K be an essential inner ideal of J. If $\mathrm{PI}(J) \neq 0$, then $\mathcal{C}(J)J$ has nonzero socle $\mathrm{Soc}(\mathcal{C}(J)J)$ by [Mo2, 5.1]. The $\mathcal{C}(J)$-span $\mathcal{C}(J)K$ of $K \subseteq J \subseteq \mathcal{C}(J)J$ is then an essential inner ideal of $\mathcal{C}(J)J$, hence $\mathrm{Soc}(\mathcal{C}(J)J) \subseteq \mathcal{C}(J)K$. Thus, if $U_aK = 0$ for some $a \in J$, then $U_a\mathrm{Soc}(\mathcal{C}(J)J) = 0$, hence $a \in \mathrm{Ann}_{\mathcal{C}(J)J}(\mathrm{Soc}(\mathcal{C}(J)J)) = 0$, and therefore J is strongly nonsingular.

On the other hand, if $\mathrm{PI}(J) = 0$, and $U_aK = 0$ for the essential inner ideal $K \subseteq J$ and some $a \in J$, then J is special, and in a $*$-tight associative $*$-envelope R of J we have $akJka = U_aU_kJ \subseteq U_aK = 0$ for all $k \in K$. Then the ideal $I = id_R(ka)$ generated by ka in R has $I \cap I^* = 0$ by [FGM, 5.2(ii)]. If R is prime, this implies $I = 0$, hence $ka = 0$ and therefore $Ka = 0$ which yields $K \subseteq \mathrm{Ann}_J(a)$. Thus $a \in \Theta(J)$, the singular ideal of J (see [FGM]), and $\Theta(J) = 0$ by [FGM, 6.14], hence $a = 0$.

Therefore we can assume that R is not prime. We now denote by rann_R (resp. lrann_R) the right (resp. left) annihilator in R. By [FGM, 5.2,5.3], we get $Ka \subseteq P$ for a $*$-splitting ideal P of R. Then $KaK \subseteq P \cap P^* = 0$, hence $K \subseteq \mathrm{rann}_R(Ka)$ and $\mathrm{rann}_R(Ka)$ is essential by 4.2(ii), hence $Ka \subseteq Z_r(R) = 0$, and we get $K \subseteq \mathrm{lann}_R(a)$. So again by 4.2(ii), $\mathrm{lann}_R(a)$ is essential, and $a \in Z_l(R) = 0$. ∎

4.4. Lemma. *Let J be a strongly prime Jordan algebra and let Q be an algebra of quotients of J. Assume that Q is special and let A be a $*$-tight associative $*$-envelope of Q. Denote by $T = alg_A(J)$ the associative subalgebra of A generated by J. Then:*

(i) *For any $a \in A$ there exists an essential inner ideal K of J such that $Ka \subseteq T$,*

(ii) *T is a $*$-tight associative $*$-envelope of J.*

Proof. (i) Since every $a \in A$ is a sum of products of elements of Q, if the result holds for products of elements of Q, it will hold for arbitrary elements $a \in A$ by taking the intersection of the inner ideals corresponding to each summand in which a decomposes. Thus we can assume that $a = q_1 \cdots q_n$ with $q_i \in Q$. We prove the result by induction on n.

If $n = 1$, so that $a = q_1 \in Q$, for any $x, y \in \mathcal{D}_J(a)$ we have $(U_x y)q = x\{y, x, q\}x - y(U_x q) \in JJ \subseteq T$, so $K = \mathcal{K}_J(U_{\mathcal{D}_J(a)}\mathcal{D}_J(a))$ has the desired property since it is essential by 1.5.

If the result holds for a product of at most $n-1$ elements of Q, write $b = q_2 \cdots q_n$, so that $a = q_1 b$. Then there is an essential inner ideal N of J such that $Nb \subseteq T$. Set $L = \{x \in T \mid xq_1 \in \hat{T}N\}$, and $K = L \cap J$. Then it is clear that L is a left ideal of T, hence K is an inner ideal of J, and $Ka \subseteq La = Lq_1 b \subseteq TNb \subseteq T$, so it suffices to prove that K is essential. Now, since N is an essential inner ideal of J, $\hat{T}N$ is an essential left ideal of T by 4.2(ii). Then $L = (\hat{T}N : q_1)$ is essential [R2, 3.3.3], hence K is essential by 4.2(i).

(ii) Let I be a nonzero $*$-ideal of T. Then clearly $\tilde{I} = \hat{A}I\hat{A}$ is a nonzero $*$-ideal of A, and by tightness, there is a nonzero $q \in \tilde{I} \cap Q$. Now we can write $q = \sum_i a_i y_i b_i \in \hat{A}I\hat{A}$ for some $a_i, b_i \in \hat{A}$ and $y_i \in I$. By (i), for each i there are an essential inner ideals K_i and N_i of J with $K_i a_i + N_i b_i^* \subseteq T$, hence $L_i = K_i \cap N_i$, which is again essential, has $L_i a_i + b_i L_i \subseteq T$. Then, the essential inner ideal $K = \bigcap_i L_i$ satisfies $Ka_i + b_i K \subseteq T$ for all i. Now put $N = K \cap \mathcal{D}_J(q)$, which is again an essential inner ideal. Then we have $U_N q \subseteq U_{\mathcal{D}_J(q)} q \subseteq J$, and $U_N q \subseteq \sum_i K a_i y_i b_i K \subseteq \sum_i TIT \subseteq I$. By 2.6 $U_N q \neq 0$, hence $0 \neq I \cap J$ and T is $*$-tight over J. ∎

4.5. Remark. Let J be a strongly prime special Jordan algebra, and let I be a nonzero ideal of J. If R is a $*$-tight associative $*$-envelope of J and $S = alg_R(I)$ is the subalgebra of R generated by I, we can assume by [McZ] that $R \subseteq Q_s(S)$ where Q_s denotes the algebra of symmetric quotients, and therefore $Q_s(S) = Q_s(R)$, hence $Q_\sigma(R) = Q_\sigma(S)$ by 0.6.

4.6. Proposition. *Let J be a prime strongly nonsingular Jordan algebra of hermitian type and let R be a $*$-tight associative $*$-envelope of J. Then, the set*

$$\mathcal{Q} = \{q \in H(Q_\sigma(R), *) \mid \mathcal{D}_J(q) \text{ is essential in } J\}$$

is an ample subspace of symmetric elements of the maximal algebra of symmetric quotients $Q_\sigma(R)$ of R.

Proof. Since J is of hermitian type, it is special and there is a hermitian ideal $\mathcal{H}(X)$ in the free special Jordan algebra $FSJ[X]$ on a denumerable set of generators with $\mathcal{H}(J) \neq 0$. Denote by S the subalgebra $alg_R(\mathcal{H}(J))$ generated by $I = \mathcal{H}(J)$, which is a $*$-tight associative $*$-envelope of I [McZ]. We consider the set $\mathcal{Q}(I) = \{q \in H(Q_\sigma(S), *) \mid \mathcal{D}_I(q)$ is essential in $I\}$. We claim that $\mathcal{Q}(I)$ is an ample subspace of symmetric elements of $Q_\sigma(I)$ (which is the particular case $I = J$ of the proposition).

If $p, q \in \mathcal{Q}(I)$, for any $x \in \mathcal{D}_I(p) \cap \mathcal{D}_I(q)$ we have $x \circ (p+q) = x \circ p + x \circ q \in I$ and $U_x(p+q) = U_x p + U_x q \in I$, so it follows from 2.7 and the essentiality of $\mathcal{D}_I(p) \cap \mathcal{D}_I(q)$ that $\mathcal{D}_I(p+q)$ is essential, hence $p + q \in \mathcal{Q}(I)$.

Next, if $q \in Q_\sigma(S)$, then there exists a dense (hence essential) left ideal L of S that satisfies $Lq + Lq^* \subseteq S$. Then $K = L \cap I$ is also essential by 4.2(i). Now we have $kq \in Lq \subseteq S$ and $kq^* \in Lq^a st \subseteq S$ for any $k \in K$, hence $k \circ (q+q^*) = (kq) + (k^*q)^* + (qk) + (qk^*)^* = (kq) + (kq)^* + (qk) + (qk)^*$ is a sum of two traces of elements of S, hence it belongs to the ample subspace I. Also $U_k(q + q^*) = U_k q + (U_k q)^*$ is a trace of an element of S, so again it belongs to I. Since S is $*$-tight over S, it follows from 4.2(1) that K is essential in I, hence $\mathcal{D}_I(q+q^*)$ is essential by 2.7, and $q+q^* \in \mathcal{Q}(I)$.

Now take $q \in Q_\sigma(S)$ and $h \in \mathcal{Q}(I)$. The inner ideal $\mathcal{D}_I(h)$ is essential in I, hence $N = \mathcal{K}_I(U_{\mathcal{D}_I(h)}\mathcal{D}_I(h)$ is also essential by 1.5 and the strong nonsingularity of I due to 1.6. Then $\hat{S}N$ (resp. $N\hat{S}$) is an essential left (resp. right) ideal of S by 4.2(2). Since S is nonsingular by 4.3 and the strong nonsingularity of I, the ideal $\hat{S}N$ is dense, and there exists a dense (hence essential) left ideal L of S with $Lh \subseteq \hat{S}N$. On the other hand, there exists a dense (hence essential) left ideal L' of S with $L'qhq^* \subseteq \hat{S}N$, so by taking the intersection of L and L' we can assume that $Lqhq^* \subseteq \hat{S}N$.

Now set $K = L \cap I$, which is an essential inner ideal of I by 4.2(1). For any $k \in K$ we have $kqhq^* \in Lqhq^* \subseteq S$ and $qhq^*k = (kqhq^*)^* \in S^* = S$, hence $k \circ (qhq^*) = (kqhq^* + (kqhq^*)^*) + (qhq^*k + (qhq^*k)^*) \in I$, since this is a sum of traces of elements of S and I is an ample subspace of symmetric elements of S. On the other hand, since $kq \in \hat{S}N$, there exist elements $s_i \in \hat{S}$ and $x_i, y_i \in \mathcal{D}_I(h)$ with $kq = \sum_i s_i U_{x_i} y_i$. Thus, using the notation $\{s\} = s + s^*$ for $s \in S$, we have

$$U_k(qhq^*) = kqh(kq)^* = \sum_i s_i(U_{x_i}y_i)q(U_{x_i}y_i)s_i^* + \sum_{i<j}\{s_i(U_{x_i}y_i)q(U_{x_j}y_j)s_j^*\} =$$
$$= \sum_i s_i(U_{U_{x_i}y_i}q)s_i^* + \sum_{i<j}\{s_i x_i\{y_i, x_i, q\}(U_{x_j}y_j)s_j^*\} -$$
$$- \{s_i(U_{x_i}q)y_i(U_{x_j}y_j)s_j^*\} \in I,$$

since the elements $U_{U_{x_i}y_i}q \in S$, $s_i x_i\{y_i, x_i, q\}(U_{x_j}y_j)s_j^*$ and $s_i(U_{x_i}q)y_i(U_{x_j}y_j)s_j^*$ belong to S, and I is an ample subspace of symmetric elements of S.

Since K is essential, 2.7 implies that $\mathcal{D}_I(qhq^*)$ is essential, hence $qhq^* \in \mathcal{Q}(I)$. Note also that $Q_\sigma(S)$ is unital, and if 1 is its unit element $1 = 1^*$ obviously has $\mathcal{D}_I(1) = I$, hence $1 \in \mathcal{Q}(I)$. Thus we have proved that all elements $q+q^*$, $qq^* = q1q^*$ and qhq^* belong to $\mathcal{Q}(I)$, for all $q \in Q_\sigma(S)$ and all $h \in \mathcal{Q}(I)$, and therefore $\mathcal{Q}(I)$ is an ample subspace of symmetric elements of $\mathcal{Q}_\sigma(S)$, and in particular it is a Jordan subalgebra of $H(Q_\sigma(S), *)$.

It is clear now that $\mathcal{Q}(S)$ will be an algebra of quotients of I if $U_q\mathcal{D}_I(q) \neq 0$ for all nonzero $q \in \mathcal{Q}(I)$. So suppose that $q \in \mathcal{Q}(I)$ has $U_q\mathcal{D}_I(q) = 0$. Then for any $x \in \mathcal{D}_I(q)$ we have $U_{U_xq}\mathcal{D}_I(q) = 0$ hence $U_xq = 0$ since $\mathcal{D}_I(q)$ is essential, $U_xq \in I$, and I is strongly nonsingular. Thus $U_{\mathcal{D}_I(q)}q = 0$, and for any $x, y \in \mathcal{D}_I(q)$ we have $x \circ q \in I$, and $U_{x\circ q}y = U_qU_xy + U_xU_qy + \{q, x, \{y, q, x\}\} - (U_qx^2)\circ y = 0$. Again, since $\mathcal{D}_I(q)$ is essential, $x \circ q \in I$, and I is strongly nonsingular, we get $x \circ q = 0$, hence $q \circ \mathcal{D}_I(q) = 0$. Now, for any $x, y \in \mathcal{D}_I(q)$, we have $\{y, x, q\} = y \circ (x \circ q) - \{y, q, x\} = 0$, and $(U_xy)q = x\{y, x, q\} - (U_xq)y = 0$. Therefore, setting $K = \mathcal{K}_I(U_{\mathcal{D}_I(q)}\mathcal{D}_I(q))$ we have $\hat{S}Kq = 0$. But since K is essential by 1.5, $\hat{S}K$ is an essential left ideal of S by 4.2(2), hence a dense left ideal by the nonsingularity of S, and therefore [L, 2.1] gives $q = 0$.

To complete the proof it only remains to show that $\mathcal{Q}(I) = \mathcal{Q}$. First note that we have $Q_\sigma(R) = Q_\sigma(S)$ by 4.5, hence $J \subseteq H(Q_\sigma(S), *) = H(Q_\sigma(R), *)$ and $\mathcal{Q}(I) \subseteq H(Q_\sigma(R), *)$. Since clearly $\mathcal{D}_I(x) = I$ is essential for any $x \in J$, we have $J \subseteq \mathcal{Q}(I)$, hence $\mathcal{Q}(I)$ is an algebra of quotients of J by 2.9. In particular, $\mathcal{D}_J(q)$ is essential in J for any $q \in \mathcal{Q}(I)$, hence $\mathcal{Q}(I) \subseteq \mathcal{Q}$. Reciprocally, if $q \in \mathcal{Q}$, then $K = \mathcal{D}_J(q) \cap I$ is an essential inner ideal of I, hence $N = \mathcal{K}_I(U_KK)$ is also an essential inner ideal of I. Now, if $x, y \in K$ and $z \in \hat{I}$ we have $U_{U_xy}q = U_xU_yU_xq \in U_xU_yJ \subseteq I$, $\{U_xy, q, z\} = \{x, \{y, x, q\}, z\} - \{U_xq, y, z\} \in \{I, J, \hat{I}\} + \{J, I, \hat{I}\} \subseteq I$, so in particular, $N \circ q \subseteq I$ and $U_Nq \subseteq I$. Then $\mathcal{D}_I(q)$ is essential by 2.7, and $q \in \mathcal{Q}(I)$. ■

4.7. Theorem. *Let J be a prime strongly nonsingular Jordan algebra of hermitian type, and let $\mathcal{Q}$ be as in 4.6. Then the algebra $\mathcal{Q} = Q_{max}(J)$ is the maximal* *algebra of quotients of J.*

Proof. Let Q be an algebra of quotients of J. Since $J \subseteq Q$ is of hermitian type, so is Q, and in particular it is special. Let A and R be $*$-tight associative $*$-envelopes of Q and J respectively. Set $T = alg_A(J)$, the associative subalgebra of A generated by J. By 4.4(ii), T is a $*$-tight associative $*$-envelope of J, hence by [Mc3, 2.3] (see remark below) the identity $J \to J$ uniquely extends to a $*$-isomorphism $R \to T$ since J is of hermitian type. Thus we can assume that $R = T$.

Now by 4.4(i), for any $a \in A$ there exists a essential inner ideals K_1 and K_2 of J with $K_1 a + K_2 a^* \subseteq T$, so $Ka + aK \subseteq T$ for the essential inner ideal $K = K_1 \cap K_2$ of J. then the left ideal $L = \hat{T}K$ of T, which is essential by 4.2, satisfies $La + aL^* \subseteq T$. Since T is left (and right) nonsingular by 4.3, L is a dense left ideal. Moreover, if N is a dense left ideal of T and $Na = 0$, then $AN \cap Q \supseteq N \cap J$, and $N \cap J$ is essential by 4.2(i), hence AN is an essential left ideal of A by 4.2(ii), and therefore it is dense since A is nonsingular by 4.3 and 2.4(vi), hence $a = 0$.

Thus, by [L, 2.1], $A \subseteq Q_\sigma(T)$ and $Q \subseteq H(Q_\sigma(T), *)$. Since all elements of Q have essential inner ideal of J-denominators, we get $Q \subseteq \mathcal{Q}$, and this proves the maximality of $\mathcal{Q}$. ■

4.8. Remark. To apply the Prime Zelmanov Extension Theorem as it is stated in [Mc3, 2.3], we would need that $\mathcal{Z}(J) \neq 0$ for the particular hermitian ideal generated by the polynomial Z_{48} mentioned in [Mc3, 0.4], however it is easy to see that the only condition that is needed in its proof is that the hermitian ideal satisfies, in addition to being hermitian, the eating property $\{y_1 \cdots y_r \mathcal{Z}(X)^{(m)} y_r + 2 \cdots y_n\} \subseteq FSJ[X \cup T]$ for $m \geq n - 4$, and the ampleness property: If $J \subseteq H(A, *)$, then $\mathcal{Z}(J) = H_0(A, *)$ is an ample subspace of symmetric elements in the subalgebra $A_0 \subseteq A$ it generates. These two conditions hold for any hermitian ideal $\mathcal{H}(X)$ by [McZ, 2.3, 1.3].

4.9. Theorem. *Let J be a strongly prime Jordan algebra. If J is strongly nonsingular, then J has a maximal algebra of quotients $Q_{max}(J)$. More precisely,*

(a) *If J is PI, then $Q_{max}(J) = \Gamma(J)^{-1}J$ is the central closure of J.*

(b) *If J is not PI (hence it is special), and R is a $*$-tight associative $*$-envelope of J, then $Q_{max}(J) = \{q \in H(Q_\sigma(R), *) \mid \mathcal{D}_J(q)$ is essential in $J\}$, which is an ample subspace of symmetric elements of the maximal symmetric algebra of quotients $Q_\sigma(R)$ of R.*

Proof. The assertion (a) about PI algebras is proved in 3.5, and if J is not PI, then it is of hermitian type and 4.7 gives (b). ■

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