On the identities defining a quadratic Jordan division algebra

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Abstract. Quadratic Jordan algebras are defined by identities that have to hold strictly, i.e that continue to hold in every scalar extension. In this paper we show that strictness is not required for quadratic Jordan division algebras.

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1. Basics

We begin with the classical definition of Jordan algebras.

Definition 1.1. Let k be a field with $\operatorname{char} k \neq 2$.

(a) A commutative, unital k-algebra J is called a (linear) Jordan algebra if $a^2 \cdot (b \cdot a) = (a^2 \cdot b) \cdot a$ holds for all $a, b \in J$.

(b) A non-zero element a of Jordan algebra J is called invertible if there is an element $a^{-1} \in J$ with $a \cdot a^{-1} = 1$ and $a^2 \cdot a^{-1} = a$.

(c) A Jordan algebra is called a Jordan division algebra if every non-zero element is invertible.

The standard example for a Jordan algebra arises in the following way: Let A be an associative k-algebra. Define a new multiplication $\circ$ on A by

$$a \circ b = \frac{1}{2}(ab + ba).$$

Then $A^+ = (J, \circ)$ is a Jordan algebra.

Of course, in this example the constraint that $\operatorname{char} k \neq 2$ is essential. But since Jordan algebras have been a useful tool to describe some algebraic groups that are also defined over fields of characteristic 2, one has to alter the definition of a Jordan algebra to include the case characteristic 2. In 1966 Kevin McCrimmon came up with a new definition for Jordan algebras which works for a field of arbitrary characteristic (see [5]). For convenience, we first introduce some more definitions.

Definition 1.2. Let k be a field of arbitrary characteristic.

(a) If V, W are two vector spaces over k, then a map $Q : V \to W$ is called quadratic if $Q(tv) = t^2 Q(v)$ for all $t \in k, v \in V$ and if there is a k-bilinear map $f : V \times V \to W$ with $Q(v+w) = Q(v) + Q(w) + f(v,w)$ for all $v, w \in V$.

(b) A quadratic algebra over k is a pair (J, Q) where J is a k-vector space and $Q : J \to End_k(J) : a \mapsto Q_a$ is quadratic.

(c) For a quadratic algebra (J, Q) and $a, b \in J$ one defines the maps $Q_{a,b}, V_{a,b} : J \to J$ by $cQ_{a,b} = cQ_{a+b} - cQ_a - cQ_b$ and $cV_{a,b} = bQ_{a,c}$ for $c \in J$. Of course, one has $cV_{a,b} = aV_{c,b}$ for all $a, b, c \in J$. The map $J \times J \to \mathrm{End}_k(J) : (a,b) \mapsto V_{a,b}$ is k-bilinear.

(d) An element $a \in J$ is called invertible if Q_a is invertible. The element $a^{-1} := aQ_a^{-1}$ is called the inverse of a. We denote the set of invertible elements of J by J^*.

(e) If K is an extension field of k, then one defines the quadratic algebra $J_K := (K \otimes_k J, \hat{Q})$ by $\hat{Q}_{\sum_{i=1}^n t_i \otimes a_i} = \sum_{i=1}^n t_i^2 Q_{a_i} + \sum_{i<j} t_i t_j Q_{a_i, a_j}$. We say that an identity in J holds *strictly* if it holds in J_K for all extensions K/k.

Definition 1.3. Let (J, Q) be a quadratic algebra over k and $1 \in J^\# := J \backslash \{0\}$. Then $(J, Q, 1)$ is called a weak quadratic Jordan algebra if the following holds for all $a, b \in J$.

(QJ1) $Q_1 = id_J$.
(QJ2) $Q_a V_{a,b} = V_{b,a} Q_a$.
(QJ3) $Q_{bQ_a} = Q_a Q_b Q_a$.

A weak quadratic Jordan algebra is called a quadratic Jordan algebra if (QJ1)-(QJ3) hold strictly, i.e. if J_K is a weak quadratic Jordan algebra for all extension fields K/k.

Example. (a) Let R be a unital, associative algebra over k. For $a \in R$ define $Q_a : R \to R : b \mapsto bab$. Then $R^+ := (R, Q, 1)$ is a quadratic Jordan algebra. If J is a Jordan subalgebra of R^+, i.e. if J is a subspace of R with $1 \in J$ and $aba \in J$ for all $a, b \in J$, then we can restrict Q to J and hence J is again a quadratic Jordan algebra. A quadratic Jordan algebra is called *special* if it is isomorphic to a Jordan subalgebra of J.

(b) The author doesn't know an example of a weak quadratic Jordan algebra which is not a quadratic Jordan algebra.

The requirement that the identities (QJ1)-(QJ3) have to hold strictly is of course a bit unusual. There is another way to express this condition.

Lemma 1.4. *Let J be a weak quadratic Jordan algebra.*

(a) *J is a quadratic Jordan iff the linearized versions of (QJ2) and (QJ3) hold, i.e. iff*

(QJ2*) $Q_a V_{b,c} + Q_{a,b} V_{a,c} = V_{c,a} Q_{a,b} + V_{c,b} Q_a$.
(QJ3*) $Q_{cQ_a, cQ_{a,b}} = Q_{a,b} Q_c Q_a + Q_a Q_c Q_{a,b}$.
(QJ3**) $Q_{cQ_{a,b}} + Q_{cQ_a, cQ_b} = Q_a Q_c Q_b + Q_b Q_c Q_a + Q_{a,b} Q_c Q_{a,b}$.
hold for all $a, b, c \in J$.

(b) *If (QJ3*) holds, then (QJ3**) holds.*
(c) *If $|k| \geq 3$, then (QJ2*) holds.*
(d) *If $|k| \geq 4$, then (QJ3*) holds.*
(e) *If $|k| \geq 4$, then every weak quadratic Jordan algebra is a quadratic Jordan algebra.*

Proof. (a) See Theorem 1 of Chapter 1.3 in [4], where (QJ2*) equals QJ9, (QJ3*) equals QJ6 and (QJ3**) equals QJ7. Note that Jacobson's identity QJ8 is redundant since it can be deduced from QJ7.

(b) Suppose that (QJ3*) holds. Let $a, b, c \in J$. If we apply (QJ3) with $a+b$ instead of a and c instead of b, we get

$$Q_{cQ_{a+b}} = Q_{a+b} Q_c Q_{a+b},$$

hence

$$Q_{cQ_a + cQ_b + cQ_{a,b}} = (Q_a + Q_b + Q_{a,b}) Q_c (Q_a + Q_b + Q_{a,b})$$

and therefore

$$\begin{aligned}Q_{cQ_a} + Q_{cQ_b} + Q_{cQ_{a,b}} + Q_{cQ_a, cQ_b} + Q_{cQ_a, cQ_{a,b}} + Q_{cQ_b, cQ_{a,b}} =\\ Q_a Q_c Q_a + Q_b Q_c Q_b + Q_a Q_c Q_b + Q_b Q_c Q_a + Q_{a,b} Q_c Q_{a,b} + Q_{a,b} Q_c Q_a +\\ Q_a Q_c Q_{a,b} + Q_b Q_c Q_{a,b} + Q_{a,b} Q_b Q_{a,b}\end{aligned}$$

If we apply (QJ3) for the terms Q_{cQ_a} and Q_{cQb}, we get

$$\begin{aligned}Q_{cQ_{a,b}} + Q_{cQ_a, cQ_b} + Q_{cQ_a, cQ_{a,b}} + Q_{cQ_b, cQ_{a,b}} &=\\ Q_a Q_c Q_b + Q_b Q_c Q_a + Q_{a,b} Q_c Q_{a,b} + Q_{a,b} Q_c Q_a &+\\ Q_a Q_c Q_{a,b} + Q_b Q_c Q_{a,b} + Q_{a,b} Q_c Q_{a,b}&\end{aligned}$$

If we apply (QJ3*) twice, we get

$$Q_{cQ_{a,b}} + Q_{cQ_a, cQ_b} = Q_a Q_c Q_b + Q_b Q_c Q_a + Q_{a,b} Q_c Q_{a,b},$$

which is just identity (QJ3**).

(c) Let $a, b, c \in J$ and $\lambda \in k^*$. If we apply (QJ2) with $a + \lambda c$ instead of a, we get

$$Q_{a+\lambda c} V_{a+\lambda c, b} = V_{b, a+\lambda c} Q_{a+\lambda c},$$

thus

$$\begin{aligned}Q_a V_{a,b} + \lambda(Q_{a,c} V_{a,b} + Q_a V_{c,a}) + \lambda^2 (Q_{a,c} V_{c,b} + Q_c V_{a,c}) + \lambda^3 Q_c V_{c,b} =\\ V_{b,a} Q_a + \lambda(V_{b,a} Q_{a,c} + V_{c,a} Q_a) + \lambda^2 (V_{b,a} Q_c + V_{b,c} Q_{a,c}) + \lambda^3 V_{b,c} Q_c.\end{aligned}$$

If we apply (QJ2) for a and c and divide by λ, we get

$$\begin{aligned}Q_a V_{c,b} + Q_{a,c} V_{a,b} - (V_{b,a} Q_{a,c} + V_{b,c} Q_a) +\\ \lambda(Q_{a,c} V_{c,b} + Q_c V_{a,b} - V_{b,a} Q_c - V_{b,c} Q_{a,c}) = 0.\end{aligned}$$

Since $|k^*| \geq 2$, identity (QJ2*) now follows.

(d) This can be done in a similar way.
(e) This follows from (c) and (d). □

Remark 1.5. (a) If J is a weak quadratic Jordan algebra and $a \in J$ is invertible, then we have $Q_{a^{-1}} = Q_a^{-1}$. Indeed, we have $a = a^{-1}Q_a$ and thus with (QJ3)

$$Q_a = Q_{a^{-1}Q_a} = Q_a Q_{a^{-1}} Q_a,$$

hence $Q_a^{-1} = Q_{a^{-1}}$. Note that if $a, b \in J$ are invertible, then aQ_b and a^{-1} are also invertible.

(b) Let J be a linear Jordan algebra over k. For $a \in J$ define $Q_a : J \to J$ by $bQ_a = -a^2 \cdot b + 2a \cdot (a \cdot b)$. Then $(J, Q, 1)$ is a quadratic Jordan algebra. If $\operatorname{char} k \neq 2$ and J is a quadratic Jordan algebra, then we define a multiplication $\cdot$ on J by $a \cdot b = \frac{1}{2} 1 Q_{a,b}$. Then one can show that J is a linear Jordan algebra. Therefore for $\operatorname{char} k \neq 2$ these two concepts coincide (see for example [5] or [4]). Moreover, an element is invertible in the linear Jordan algebra iff it is invertible in the quadratic Jordan algebra.

We now recall the important concept of an isotope (see also Chapter 1.11 of [4]).

Definition 1.6. Let $J = (J, Q)$ be a quadratic algebra and $a \in J^*$. We define the a-isotope $J^a = (J, Q^a)$ of J by $xQ_y^a = xQ_a^{-1}Q_y$ for all $x, y \in J$.

Lemma 1.7. *Let J be a quadratic algebra and $a, b, c \in J$. Let $V_{b,c}^a$ be the V-map for b and c in J^a, i.e. $xV_{b,c}^a = cQ_{b,x}^a = cQ_a^{-1}Q_{b,x}$ for all $x \in J$.*

(a) $Q_a^a = id_J$.
(b) $Q_{b,c}^a = Q_a^{-1}Q_{b,c}$.
(c) $V_{b,c}^a = V_{b,cQ_a^{-1}}$.

Proof. (a) and (b) are clear. For (c) let $x \in J$. Then we have

$$xV_{b,c}^a = cQ_{b,x}^a = cQ_a^{-1}Q_{b,x} = xV_{b,cQ_a^{-1}}.$$

Thus the claim follows. □

Proposition 1.8. *Let J be a (weak) quadratic Jordan algebra and $a \in J^*$. Then $J^a = (J, Q^a, a)$ is also a (weak) quadratic Jordan algebra.*

Proof. See Chapter 1.11 of [4]. □

In this paper we are mainly interested in (weak) quadratic Jordan division algebras.

Definition 1.9. A (weak) quadratic Jordan algebra is called a (weak) quadratic Jordan division algebra if every non-zero element in J is invertible.

The theory of quadratic Jordan division algebras is connected with the theory of Moufang sets, which was first noted in [3], see also [2] as an introduction to Moufang sets.

Definition 1.10. A Moufang set consists of a set X with $|X| \geq 3$ and a family $(U_x)_{x\in X}$ of subgroups in $\operatorname{Sym} X$ such that the following holds:

(a) For all $x \in X$ the group U_x fixes x and acts regularly on $X \setminus \{x\}$.
(b) For all $x, y \in X$ and all $g \in U_x$ we have $U_y^g = U_{yg}$.

The groups U_x are called the *root groups* of the Moufang set. The group $G^\dagger := \langle U_x; x \in X\rangle$ is called the *little projective group* of the Moufang set. $G^\dagger$ is a 2-transitive subgroup of $\operatorname{Sym} X$. The Moufang set is called *proper* if $G^\dagger$ is not sharply 2-transitive and *improper* else.

Every Moufang set can be constructed as follows: Let $(U, +)$ be a (not necessarily abelian) group, let ∞ be a new symbol and set $X := U \cup \{\infty\}$. For $a \in U$ let $\alpha_a \in \operatorname{Sym} X$ be defined by $b\alpha_a = b + a$ for $b \in U$ and $\infty\alpha_a = \infty$. Then $U_\infty := \{\alpha_a; a \in U\}$ is a subgroup of $\operatorname{Sym} X$ isomorphic to U. Let τ be a permutation of X which interchanges 0 and ∞. We set $U_0 := U_\infty^\tau$, $U_a := U_0^{\alpha_a}$ for $a \in U$ and $\mathbb{M}(U, \tau) := (X, (U_x)_{x\in X})$. By [3] $\mathbb{M}(U, \tau)$ is a Moufang set iff for all $a \in U^\#$ the map $h_a := \tau\alpha_a\alpha^\tau_{-a\tau^{-1}}\alpha_{-(-a\tau^{-1})\tau}$ induces an isomorphism on U. The map h_a is called the *Hua map* correspondig to a.

Example. (a) Let X be a set with at least 3 elements, $G \leq \operatorname{Sym} X$ be a sharply 2-transitive group. Then $(X, (G_x)_{x\in X})$ is an improper Moufang set with little projective group $G^\dagger = G$.
(b) Let k be a field, $X := \mathbb{P}^1(k)$ and for $x \in X$ let U_x be the subgroup of $\mathrm{PSL}_2(k) \leq \operatorname{Sym} X$ induced by the group of unipotent matrices that fix x. Then $(X, (U_x)_{x\in X})$ is a Moufang set with little projective group $G^\dagger = \mathrm{PSL}_2(k)$. It is proper iff $|k| \geq 4$.

The second construction can be generalized to weak quadratic Jordan division algebras. In [3] the authors showed the following.

Theorem 1.11. *Let J be a weak quadratic Jordan division algebra, $X = J \cup \{\infty\}$ and $\tau \in \operatorname{Sym} X$ with $\infty\tau = 0, 0\tau = \infty$ and $a\tau = -a^{-1}$ for $a \in J^\#$. Then for all $a \in J^\#, x \in J$ we have $xh_a = xQ_a$ and thus $\mathbb{M}(J) := \mathbb{M}(J, \tau)$ is a Moufang set.*

De Medts and Weiss didn't use the concept of a weak quadratic Jordan algebra and formulated their theorem for quadratic Jordan division algebras, but their proof doesn't make use of the strictness of (QJ1)-(QJ3), so it also holds for weak Jordan division algebras.
One of the big open problems concerning Moufang sets is the following conjecture:

Conjecture 1.12. If $(X, (U_x)_{x\in X})$ is a proper Moufang set with U_x abelian for all $x \in X$, then there is a field k and a quadratic Jordan division algebra J over k such that $(X, (U_x)_{x\in X})$ is isomorphic to $\mathbb{M}(J)$.

If 1.12 is true, then one has a classification of proper Moufang sets with abelian root groups since quadratic Jordan division algebras have been classified by McCrimmon and Zel'manov (see [7]). The proof follows from the classification of strongly prime quadratic Jordan algebras over an algebraically closed field, therefore it is essential that scalar extensions are allowed. There has been progress in proving 1.12 (see [1]), but in general this conjecture is still open. The requirement that (QJ2) and (QJ3) have to hold strictly is of course an obstacle. If there would be be a weak quadratic Jordan division algebra which is not a quadratic Jordan algebra, then conjecture 1.12 would be false. Such an algebra could a priori exist over $\mathbb{F}_2$ or $\mathbb{F}_3$. In this paper we will prove that no such algebra exists.

MAIN THEOREM *Every weak quadratic Jordan division algebra is a quadratic Jordan algebra.*

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2. Some useful identities

In the following let $(J, Q, 1)$ be a weak quadratic Jordan algebra. We collect some identities we will need later. These identities are already known for quadratic Jordan algebras and can be found for instance in [4].

Lemma 2.1. $yQ_{aQ_x,x} = aQ_{yQ_x,x}$ *for all* $a, x, y \in J$.

Proof. (QJ2) implies $xQ_{a,y}Q_x = yV_{a,x}Q_x = yQ_xV_{x,a} = aQ_{x,yQ_x}$. Since the first expression is symmetric in a and y, so is the second. Hence we get $aQ_{x,yQ_x} = yQ_{aQ_x,x}$. □

Lemma 2.2. *For all* $x \in J$ *we have* $Q_{x,1} = V_{x,1} = V_{1,x}$.

Proof. By (QJ2) we have $V_{x,1} = V_{x,1}Q_1 = Q_1V_{1,x} = V_{1,x}$. We have $xQ_{1,y} = yV_{1,x} = yV_{x,1} = 1Q_{x,y}$ for all $y \in J$. Since the last expression is symmetric in x and y, so is the first. Thus we have $yV_{1,x} = xQ_{1,y} = yQ_{1,x}$. □

Lemma 2.3. *If* $a \in J^*$ *and* $x \in J$, *then we have* $V_{x,a^{-1}} = V_{a,xQ_a^{-1}} = Q_a^{-1}Q_{x,a}$.

Proof. We apply 2.2 for the isotope J^a and have

$$V^a_{x,a} = V^a_{a,x} = Q^a_{a,x}$$

and therefore

$V_{x,a^{-1}} = V_{a,xQ_a^{-1}} = Q_a^{-1}Q_{a,x}$. □

Lemma 2.4. $Q_{1,x}Q_x = Q_xQ_{1,x}$ *for all* $x \in J$.

Proof. We have $Q_xQ_{1,x} = Q_xV_{x,1} = V_{1,x}Q_x = Q_{1,x}Q_x$ by (QJ2) and 2.2. □

Lemma 2.3 for $x = 1$ and a^{-1} in place of a and Lemma 2.4 together imply

Lemma 2.5. *For all* $a \in J^*$ *we have* $Q_{1,a^{-1}} = V_{a,1}Q_a^{-1}$.

Lemma 2.6. *If* $x \in J^*$, *then* $Q_x^{-1}V_{a,x} = V_{x,a}Q_x^{-1} = Q_{a,x^{-1}}$ *for all* $a \in J$.

Proof. We have $aQ_{yQ_x,x} = yQ_{aQ_x,x} = yQ_xQ_{a,x^{-1}}Q_x$ for all $a \in J$ by (QJ3) and 2.1. Replacing y by yQ_x^{-1}, we get $yV_{x,a} = aQ_{y,x} = yQ_{a,x^{-1}}Q_x$. Thus the second equation follows. The first now follows from (QJ2). □

Lemma 2.7. *If $x, y \in J^*$, then we have*

$$Q_x^{-1}Q_{x+y}Q_y^{-1} = Q_{x^{-1}+y^{-1}}.$$

Proof. Using the previous lemma for x^{-1} in place of a and y in place of x, and then with x^{-1} instead of x and y instead of a, we have

$$Q_x^{-1}Q_{x,y}Q_y^{-1} = V_{y,x^{-1}}Q_y^{-1} = Q_{y^{-1},x^{-1}}.$$

Since $Q_x^{-1}Q_xQ_y^{-1} = Q_y^{-1}$ and $Q_x^{-1}Q_yQ_y^{-1} = Q_x^{-1}$, we get

$$Q_x^{-1}Q_{x+y}Q_y^{-1} = Q_x^{-1}(Q_x + Q_y + Q_{x,y})Q_y^{-1} =$$

$Q_y^{-1} + Q_x^{-1} + Q_{x^{-1},y^{-1}} = Q_{x^{-1}+y^{-1}}$. □

We will also make use of the following "Hua-identity" for weak quadratic Jordan division algebras. It was proved by De Medts and Weiss in [3] in order to show that a quadratic Jordan division algebra defines a Moufang set. As mentioned before, the proof doesn't make use of the strictness of (QJ1)-(QJ3), so it still holds for weak quadratic Jordan division algebras.

Theorem 2.8. *Let J be a weak quadratic Jordan division algebra and $a, b \in J^*$ with $a \neq b^{-1}$. Then we have*

$$aQ_b = b - (b^{-1} - (b - a^{-1})^{-1})^{-1}.$$

Proof. See 4.1 of [3]. □

3. Derivations and anti-derivations of weak quadratic Jordan algebras

Definition 3.1. Let J be a weak quadratic Jordan algebra and $\epsilon \in \{+, -\}$. A linear map $\delta : J \to J$ is called an ϵ-derivation if $\delta(aQ_b) = \epsilon\delta(a)Q_b + aQ_{b,\delta(b)}$ holds for all $a, b \in J$.

We will call the +-derivations just *derivations* and the −-derivations *anti-derivations.*

Remark 3.2. Our definition of a derivation differs slightly from the usual definition of a derivation of a quadratic Jordan algebra which additionally invokes that $\delta(1) = 0$ (see [6]). To the author's knowledge the concept of an anti-derivation of a quadratic Jordan algebra hasn't been considered before.

Example. Let A be an associative algebra and $J \subseteq A$ a special quadratic Jordan algebra. If $\delta : A \to A$ is an (anti-)derivation of A with $\delta(J) \leq J$, then δ induces an (anti-)derivation of J. Indeed, for $a, b \in J$ we have $\delta(aQ_b) = \delta(bab) = \delta(b)ab + \epsilon b\delta(ab) = \delta(b)ab + \epsilon b\delta(a)b + \epsilon^2 ba\delta(b) = \epsilon\delta(a)Q_b + aQ_{b,\delta(b)}$ with $\epsilon = +$ if δ is a derivation and $\epsilon = -$ for δ an anti-derivation.

Lemma 3.3. *Let δ be an ϵ-derivation for $\epsilon = \pm$ and $a, b, c \in J$. Then we have:*

(a) *$\delta(aQ_{b,c}) = \epsilon\delta(a)Q_{b,c} + aQ_{\delta(b),c} + aQ_{b,\delta(c)}$ and $\delta(aV_{b,c}) = \delta(a)V_{b,c} + aV_{\delta(b),c} + \epsilon aV_{b,\delta(c)}$ for all $a, b, c \in J$.*
(b) *If $a \in J^*$, then $\delta(a^{-1}) = -\epsilon\delta(a)Q_a^{-1}$.*
(c) *The identity is an anti-derivation.*
(d) *If $\operatorname{char} k \neq 2$ and δ is a derivation, then $\delta(1) = 0$.*
(e) *If $\operatorname{char} k = 2$, then $Q_{1,\delta(1)} = 0$.*
(f) *If $\operatorname{char} k \neq 2$ and δ is an anti-derivation, then $\delta(a) = \frac{1}{2}aQ_{1,\delta(1)}$.*

Proof. (a) The first equation follows by linearizing the defining property of an ϵ-derivation. The second equation can be obtained by the first.
(b) We have

$$\delta(a^{-1}) = \delta(aQ_{a^{-1}}) = \epsilon\delta(a)Q_{a^{-1}} + aQ_{a^{-1},\delta(a^{-1})}.$$

Now $aQ_{a^{-1},\delta(a^{-1})} = aV_{a,\delta(a^{-1})}Q_a^{-1} = \delta(a^{-1})Q_{a,a}Q_a^{-1} = 2\delta(a^{-1})$ by 2.6. Thus we get $-\delta(a^{-1}) = \epsilon\delta(a)Q_{a^{-1}} = \epsilon\delta(a)Q_a^{-1}$.
(c) We have $id(aQ_b) = aQ_b = -id(a)Q_b + aQ_{b,id(b)}$, which shows that the identity is an anti-derivation.
(d) We have $\delta(1) = \delta(1^{-1}) = -\delta(1)Q_1^{-1} = -\delta(1)$, thus the claim follows.
(e) For all $a \in J$ we have

$$\delta(a) = \delta(aQ_1) = \delta(a)Q_1 + aQ_{1,\delta(1)} = \delta(a) + aQ_{1,\delta(1)},$$

hence the claim follows.
(f) We have

$$\delta(a) = \delta(aQ_1) = -\delta(a)Q_1 + aQ_{1,\delta(1)} = -\delta(a) + aQ_{1,\delta(1)}$$

and thus $\delta(a) = \frac{1}{2}aQ_{1,\delta(1)}$. □

We set $\mathfrak{D}_\epsilon(J) := \{\delta \in \operatorname{End}_k(J); \delta \text{ is an } \epsilon\text{-derivation of } J\}$ and $\mathfrak{D}(J) = \mathfrak{D}_+(J) + \mathfrak{D}_-(J)$. If $\operatorname{char} k = 2$, then we have $\mathfrak{D}_+(J) = \mathfrak{D}_-(J) = \mathfrak{D}(J)$, while for $\operatorname{char} k \neq 2$ we have $\mathfrak{D}(J) = \mathfrak{D}_+(J) \oplus \mathfrak{D}_-(J)$. We call the elements of $\mathfrak{D}(J)$ *generalized derivations.*

Lemma 3.4. *For $\epsilon_1, \epsilon_2 \in \{+, -\}$ we have $[\mathfrak{D}_{\epsilon_1}(J), \mathfrak{D}_{\epsilon_2}(J)] \subseteq \mathfrak{D}_{\epsilon_1\epsilon_2}(J)$. In particular $\mathfrak{D}_+(J)$ and $\mathfrak{D}(J)$ are Lie subalgebras of $\operatorname{End}_k(J)$, and if $\operatorname{char} k \neq 2$, then $\mathfrak{D}(J)$ is Z_2-graded.*

Proof. For $i = 1, 2$ let $\delta_i \in \mathfrak{D}_{\epsilon_i}(J)$. For $a, b \in J$ we have

$$\delta_1(\delta_2(aQ_b)) = \delta_1(\epsilon_2\delta_2(a)Q_b + aQ_{b,\delta_2(b)}) = \epsilon_1\epsilon_2\delta_1(\delta_2(a))Q_b + \epsilon_2\delta_2(a)Q_{b,\delta_1(b)} + \epsilon_1\delta_1(a)Q_{b,\delta_2(b)} + aQ_{\delta_1(b),\delta_2(b)} + aQ_{\delta_1(\delta_2(b)),b}$$

and analogously

$$\delta_2(\delta_1(aQ_b)) = \epsilon_1\epsilon_2\delta_2(\delta_1(a))Q_b + \epsilon_1\delta_1(a)Q_{b,\delta_2(b)} + \epsilon_2\delta_2(a)Q_{b,\delta_1(b)} + aQ_{\delta_1(b),\delta_2(b)} + aQ_{\delta_2(\delta_1(b)),b}$$

Thus we get

$$\begin{aligned}[\delta_1, \delta_2](aQ_b) = &\quad \epsilon_1\epsilon_2(\delta_1(\delta_2(a)) - \delta_2(\delta_1(a)))Q_b + aQ_{\delta_1(\delta_2(b)),b} - aQ_{\delta_1(\delta_2(b)),b} \\ = &\quad \epsilon_1\epsilon_2[\delta_1, \delta_2](a)Q_b + aQ_{[\delta_1,\delta_2](b),b}.\end{aligned}$$

Hence the claim follows. □

Lemma 3.5. *If J is a quadratic Jordan algebra, then for all $a \in J$ the map $Q_{1,a}$ is an anti-derivation of J.*

Proof. $Q_{1,a}$ is an anti-derivation iff for all $b \in J$ we have

$$Q_b Q_{1,a} = -Q_{1,a} Q_b + Q_{b,bQ_{1,a}}.$$

But this is just identity (QJ3*) with 1 in place of a and a in place of b which holds by definition in a quadratic Jordan algebra. □

Remark 3.6. As an application of 3.5 and 3.4 one gets that for a linear Jordan algebra J and $a, c \in J$ the map $b \mapsto [a, b, c]$ is a derivation of J, where $[a, b, c] = (ab)c - a(bc)$ is the *associator* of a, b and c. This fact was already noted in [4].

Theorem 3.7. *Let J be a weak quadratic Jordan algebra. Suppose that for all $a, y \in J^{\#}$ there is a generalized derivation δ with $\delta(a) = y$. Then J is a quadratic Jordan algebra.*

Proof. Let $a \in J$. We set

$$\begin{aligned} L_J(a) \quad := \quad & \{y \in J : V_{b,y} Q_a + V_{b,a} Q_{a,y} = Q_a V_{y,b} + Q_{a,y} V_{a,b} \text{ and} \\ & Q_{bQ_a, bQ_{a,y}} = Q_a Q_b Q_{a,y} + Q_{a,y} Q_b Q_a \text{ for all } b \in J\}. \end{aligned}$$

Then $L_J(a)$ is a subspace of J. Now let $b, x \in J$, $\epsilon = \pm$ and $\delta \in \mathfrak{D}_\epsilon(J)$. Then we have

$$\begin{aligned} \delta(xQ_a Q_b Q_a) = \epsilon\delta(xQ_a Q_b) Q_a + xQ_a Q_b Q_{a,\delta(a)} = \\ \delta(xQ_a) Q_b Q_a + \epsilon x Q_a Q_{b,\delta(b)} Q_a + xQ_a Q_b Q_{a,\delta(a)} = \\ \epsilon\delta(x) Q_a Q_b Q_a + xQ_{a,\delta(a)} Q_b Q_a + \epsilon x Q_a Q_{b,\delta(b)} Q_a + xQ_a Q_b Q_{a,\delta(a)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \delta(xQ_a Q_b Q_a) = \delta(xQ_{bQ_a}) = \epsilon\delta(x) Q_{bQ_a} + xQ_{bQ_a,\delta(bQ_a)} = \\ \epsilon\delta(x) Q_a Q_b Q_a + xQ_{bQ_a,\epsilon\delta(b)Q_a} + xQ_{bQ_a,bQ_{a,\delta(a)}} = \\ \epsilon\delta(x) Q_a Q_b Q_a + \epsilon x Q_a Q_{b,\delta(b)} Q_a + xQ_{bQ_a,bQ_{a,\delta(a)}}. \end{aligned}$$

Hence we get

$$Q_{a,\delta(a)} Q_b Q_a + Q_a Q_b Q_{a,\delta(a)} = Q_{bQ_a, bQ_{a,\delta(a)}}.$$

Moreover, we have

$$\begin{aligned} \delta(xV_{b,a} Q_a) = \epsilon\delta(xV_{b,a}) Q_a + xV_{b,a} Q_{a,\delta(a)} = \\ \epsilon\delta(x) V_{b,a} Q_a + xV_{b,\delta(a)} Q_a + \epsilon x V_{\delta(b),a} Q_a + xV_{b,a} Q_{a,\delta(a)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \delta(xV_{b,a} Q_a) = \delta(xQ_a V_{a,b}) = \delta(xQ_a) V_{a,b} + xQ_a V_{\delta(a),b} + \epsilon x Q_a V_{a,\delta(b)} = \\ \epsilon\delta(x) Q_a V_{a,b} + xQ_{a,\delta(a)} V_{a,b} + xQ_a V_{\delta(a),b} + \epsilon x Q_a V_{a,\delta(b)}. \end{aligned}$$

Thus we get

$$V_{b,\delta(a)} Q_a + V_{b,a} Q_{a,\delta(a)} = Q_a V_{\delta(a),b} + Q_{a,\delta(a)} V_{a,b}.$$

This shows that $\delta(a) \in L_J(a)$ for all $\delta \in \mathfrak{D}_\epsilon(J)$, $\epsilon = \pm$. Thus the claim follows. $\square$

4. The proof of the main theorem

Theorem 4.1. *Let J be a weak Jordan division algebra, $\epsilon \in \{+,-\}$. Then $\delta \in \mathrm{End}_k(J)$ is an ϵ-derivation of J if and only if $\delta(a^{-1}) = -\epsilon\delta(a)Q_a^{-1}$ for all $a \in J^*$.*

Proof. For $\delta \in \mathfrak{D}_\epsilon(J)$ it follows from 3.3 (b) that $\delta(a^{-1}) = -\epsilon\delta(a)Q_a^{-1}$.
Now we assume that $\delta : J \to J$ be an epimorphism with $\delta(a^{-1}) = -\epsilon\delta(a)Q_a^{-1}$.
Let $a, b \in J^*$ with $a \neq b^{-1}$. Then we have by the Hua-identity 2.8

$$aQ_b = b - (b^{-1} - (b - a^{-1})^{-1})^{-1}.$$

Thus we get

$$\begin{aligned}
\delta(aQ_b) &= \delta(b) - \delta(b^{-1} - (b - a^{-1})^{-1})^{-1} \\
&= \delta(b) + \epsilon(\delta(b^{-1}) - \delta((b - a^{-1})^{-1}))Q^{-1}_{b^{-1}-(b-a^{-1})^{-1}} \\
&= \delta(b) + \epsilon(-\epsilon\delta(b)Q_b^{-1} + \epsilon\delta(b - a^{-1})Q^{-1}_{b-a^{-1}})Q^{-1}_{b^{-1}-(b-a^{-1})^{-1}} \\
&= \delta(b) - \delta(b)Q_b^{-1}Q^{-1}_{b^{-1}-(b-a^{-1})^{-1}} + \delta(b)Q^{-1}_{b-a^{-1}}Q^{-1}_{b^{-1}-(b-a^{-1})^{-1}} \\
&\quad +\epsilon\delta(a)Q_a^{-1}Q^{-1}_{b-a^{-1}}Q^{-1}_{b^{-1}-(b-a^{-1})^{-1}}.
\end{aligned}$$

Now for $x = b$ and $y = -(b - a^{-1})$ we have by 2.7

$$Q_{b^{-1}-(b-a^{-1})^{-1}} = Q_{b^{-1}}Q_{b-(b-a^{-1})}Q_{-(b-a^{-1})^{-1}} = Q_b^{-1}Q_a^{-1}Q^{-1}_{b-a^{-1}}$$

and with $x = -(b - a^{-1})$ and $y = b$ we get

$$Q_{b^{-1}-(b-a^{-1})^{-1}} = Q_{-(b-a^{-1})^{-1}+b^{-1}} = Q^{-1}_{b-a^{-1}}Q_a^{-1}Q_b^{-1}.$$

Thus we get using 2.6

$$\begin{aligned}
\delta(aQ_b) &= \delta(b) - \delta(b)Q_aQ_{b-a^{-1}} + \delta(b)Q_aQ_b + \epsilon\delta(a)Q_b \\
&= \epsilon\delta(a)Q_b + \delta(b)Q_a(Q_{-a^{-1}} - Q_{b-a^{-1}} + Q_b) \\
&= \epsilon\delta(a)Q_b - \delta(b)Q_aQ_{b,-a^{-1}} \\
&= \epsilon\delta(a)Q_b + \delta(b)Q_aQ_{a^{-1},b} \\
&= \epsilon\delta(a)Q_b + \delta(b)V_{b,a} \\
&= \epsilon\delta(a)Q_b + aQ_{b,\delta(b)},
\end{aligned}$$

as desired.
We still have to prove $\delta(aQ_b) = \epsilon\delta(a)Q_b + aQ_{b,\delta(b)}$ for $b \in \{0, a^{-1}\}$. The statement is clear for $b = 0$, while we have by 2.6

$$\epsilon\delta(a)Q_{a^{-1}} + aQ_{a^{-1},\delta(a^{-1})} = \epsilon\delta(a)Q_a^{-1} + aV_{a,\delta(a^{-1})}Q_a^{-1} =$$
$$-\delta(a^{-1}) + \delta(a^{-1})Q_{a,a}Q_a^{-1} = \delta(a^{-1}) = \delta(aQ_{a^{-1}}).$$ $\square$

Lemma 4.2. *Let J be a weak quadratic Jordan division algebra. Then for all $a \in J$ the map $\delta_a = Q_{1,a}$ is an anti-derivation of J.*

Proof. We have by 2.2, 2.5 and 2.6

$$\delta_a(x^{-1}) = x^{-1}Q_{1,a} = x^{-1}V_{1,a} = aQ_{1,x^{-1}} = aV_{x,1}Q_x^{-1} = aV_{1,x}Q_x^{-1} = xQ_{1,a}Q_x^{-1} = \delta_a(x)Q_x^{-1}$$

for all $x \in J^*$. The result now follows from 4.1. □

Corollary 4.3. *For all $a \in J^*$ and all $b \in J$ the map $Q^a_{a,b} = V^a_{a,b} = V^a_{b,a}$ is an anti-derivation of J^a.*

For characteristic not 2 we can show that the converse of 3.5 holds. Thus 4.2 implies that a weak quadratic Jordan division algebra in characteristic not 2 is a quadratic Jordan algebra.

Theorem 4.4. *Let J be a weak quadratic Jordan algebra over a field k with $\operatorname{char} k \neq 2$. Suppose that for all $a \in J$ the map $Q_{1,a}$ is an anti-derivation of J. For $a, b \in J$ define $a \cdot b = \frac{1}{2}aQ_{1,b}$. Then $(J, +, \cdot)$ is a linear Jordan division algebra. Thus J is a quadratic Jordan algebra.*

Proof. We have $a \cdot b = \frac{1}{2}aQ_{1,b} = \frac{1}{2}aV_{1,b} = \frac{1}{2}bQ_{1,a} = b \cdot a$ by 2.2, so $\cdot$ is commutative. Moreover, we have $1 \cdot a = a \cdot 1 = \frac{1}{2}aQ_{1,1} = \frac{1}{2} \cdot 2aQ_1 = a$, so 1 is the neutral element. It remains to show that $a^2 \cdot (b \cdot a) = (a^2 \cdot b) \cdot a$ holds for all $a, b \in J$. Note that $a^2 = \frac{1}{2}aQ_{1,a} = \frac{1}{2}1Q_{a,a} = 1Q_a$. Since $Q_{1,a\cdot b}$ is an anti-derivation, we have

$$a^2 \cdot (a \cdot b) = \frac{1}{2}1Q_aQ_{1,a\cdot b} = -\frac{1}{2}1Q_{1,a\cdot b}Q_a + \frac{1}{2}1Q_{a,aQ_{1,a\cdot b}} = -(a \cdot b)Q_a + \frac{1}{2}aQ_{1,2a\cdot(a\cdot b)} = -(a \cdot b)Q_a + 2a \cdot (a \cdot (a \cdot b)).$$

Moreover, we have

$$(a^2 \cdot b) \cdot a = \frac{1}{4}1Q_aQ_{1,b}Q_{1,a} = -\frac{1}{4}1Q_{1,b}Q_aQ_{1,a} + \frac{1}{4}1Q_{a,aQ_{1,b}}Q_{1,a} = -\frac{1}{2}bQ_{1,a}Q_a + \frac{1}{4}aQ_{1,2a\cdot b}Q_{1,a} = -(a \cdot b)Q_a + (a \cdot (a \cdot b))Q_{1,a} = -(a \cdot b)Q_a + 2(a \cdot (a \cdot b)) \cdot a = -(a \cdot b)Q_a + 2a \cdot (a \cdot (a \cdot b)).$$

Thus $(J, \cdot)$ is a linear Jordan algebra. □

The following proof works for a field in arbitrary characteristic.

Theorem 4.5. *A weak quadratic Jordan division algebra is a Jordan division algebra.*

Proof. We have to show (QJ2*) and (QJ3*), the first only for $k = \mathbb{F}_2$. Since these equalities automatically hold if one of the elements involved is zero, we only have to show them for non-zero elements. So let $a, b, c \in J^*$. Since $Q^c_{b,c}$ is an anti-derivation of J^c by 4.3, we have

$$Q^c_aQ^c_{b,c} = -Q^c_{b,c}Q^c_a + Q^c_{aQ^c_{b,c},a},$$

hence

$$Q_c^{-1}Q_aQ_c^{-1}Q_{b,c} = -Q_c^{-1}Q_{b,c}Q_c^{-1}Q_a + Q_c^{-1}Q_{aQ_c^{-1}Q_{b,c},a}.$$

Multiplying Q_c on the left yields

$$Q_a Q_c^{-1} Q_{b,c} + Q_{b,c} Q_c^{-1} Q_a = Q_{a Q_c^{-1} Q_{b,c}, a}. \tag{4.1}$$

Replacing a by aQ_c and applying (QJ3) yields

$$Q_c Q_a Q_{b,c} + Q_{b,c} Q_a Q_c = Q_{a Q_{b,c}, a Q_c}.$$

This shows (QJ3*).
We now show (QJ2*). If we replace c by c^{-1} in (4.1), we get

$$Q_a Q_c Q_{b,c^{-1}} + Q_{b,c^{-1}} Q_c Q_a = Q_{a Q_c Q_{b,c^{-1}}, a}.$$

By Lemma 2.6 we have $Q_c Q_{b,c^{-1}} = V_{b,c}$ and $Q_{b,c^{-1}} Q_c = V_{c,b}$, therefore we get

$$Q_a V_{b,c} + V_{c,b} Q_a = Q_{a V_{b,c}, a} = Q_{a, c Q_{b,a}}$$

and thus

$$Q_a V_{b,c} = -V_{c,b} Q_a + Q_{a, c Q_{b,a}}. \tag{4.2}$$

Moreover, since $Q^a_{a,c} = V^a_{a,c} = V^a_{c,a}$ is an anti-derivation of J^a, we have

$$Q^a_{a,b} V^a_{a,c} = -V^a_{a,c} Q^a_{a,b} + Q^a_{a V^a_{a,c}, b} + Q^a_{a, b V^a_{a,c}} =$$
$$-V^a_{c,a} Q^a_{a,b} + Q^a_{c Q^a_{a,a}, b} + Q^a_{a, b V^a_{a,c}} = -V^a_{c,a} Q^a_{a,b} + Q^a_{2c,b} + Q^a_{a, b V^a_{a,c}}.$$

Hence we have

$$Q_a^{-1} Q_{a,b} V_{a, c Q_a^{-1}} = -V_{c,a^{-1}} Q_a^{-1} Q_{a,b} + Q_a^{-1} Q_{2c,b} + Q_a^{-1} Q_{a, b V_{a, c Q_a^{-1}}}.$$

Using (QJ2) and multiplying Q_a on the left yields

$$Q_{a,b} V_{a, c Q_a^{-1}} = -V_{a^{-1}, c} Q_{a,b} + 2 Q_{c,b} + Q_{a, c Q_a^{-1} Q_{a,b}}.$$

Replacing c by cQ_a yields

$$Q_{a,b} V_{a,c} = -V_{a^{-1}, c Q_a} Q_{a,b} + 2 Q_{c Q_a, b} + Q_{a, c Q_{a,b}}.$$

By 2.3 we have $V_{a^{-1}, c Q_a} = V_{c,a}$. Hence we get

$$Q_{a,b} V_{a,c} = -V_{c,a} Q_{a,b} + 2 Q_{c Q_a, b} + Q_{a, c Q_{a,b}}. \tag{4.3}$$

Adding (4.2) and (4.3) yields

$$Q_{a,b} V_{a,c} + Q_a V_{b,c} = -V_{c,b} Q_a - V_{c,a} Q_{a,b} + 2 Q_{c Q_a, b} + 2 Q_{a, c Q_{a,b}}.$$

This gives (QJ2*) for char $k = 2$. □

5. An application for Moufang sets

Let $\mathbb{M}(U,\tau)$ be a proper Moufang set with abelian root groups. We may suppose that $\tau = \mu_e$ for an element $e \in U^{\#}$, where μ_e is the unique element in the double coset $U_0\alpha_e U_0$ which interchanges 0 and ∞.
By [8] $\mathbb{M}(U,\tau)$ is special, so by [9], Thm. 5.2(a) U is either torsion free and uniquely divisible or an elementary-abelian p-group for a prime p. We write $\operatorname{char} U = 0$ in the first case and $\operatorname{char} U = p$ in the second. We can view U as a k-vector space for $k = \mathbb{Q}$ if $\operatorname{char} U = 0$ and $k = \mathbb{F}_p$ if $\operatorname{char} U = p$.
In order to give U the structure of a quadratic Jordan algebra, we need a quadratic map between U and $\operatorname{End}_k(U)$. There is a natural candidate for this map. For $a \in U^{\#}$ let h_a be the Hua map corresponding to a and set $h_0 = 0$. Let $\mathcal{H} : U \to End_k(U) : a \mapsto h_a$. Then $(U, \mathcal{H}, e)$ satisfies (QJ1) and (QJ3) and one has $h_{a\tau} = h_a^{-1}$ for all $a \in U^{\#}$ and $h_{a\cdot s} = h_a \cdot s^2$ for $a \in U$ and all $s \in k$. Moreover, if $(U, \mathcal{H}, e)$ is a quadratic Jordan division algebra, then $\mathbb{M}(U) \cong \mathbb{M}(U,\tau)$. It remains to show that (QJ2) holds and that the map $(a,b) \mapsto h_{a,b} := h_{a+b} - h_a - h_b$ is biadditive. In [1] the authors showed the following:

Theorem 5.1. *Let $M(U,\tau)$ be a proper Moufang set with U abelian. Assume that* $\operatorname{char} U \neq 2, 3$. *If (QJ2) holds, then $(U, \mathcal{H}, e)$ is a quadratic Jordan division algebra.*

The authors had to exclude the case $\operatorname{char} U \in \{2,3\}$ because (QJ1)-(QJ3) are required to hold strictly, which was only guaranteed if $|k| \geq 4$. But our main theorem shows that this is always the case for weak Jordan division algebras. Thus we get

Corollary 5.2. 5.1 *also holds for* $\operatorname{char} U \in \{2,3\}$.

Remark 5.3. (a) It is sufficient to prove a weaker version of axiom (QJ2) which has to hold in all isotopes of $(U, \mathcal{H}, e)$, i.e. for all choices of $e \in U^{\#}$, compare 5.6 of [1].

(b) If $\operatorname{char} U \neq 2, 3$, then in order prove that $(U, \mathcal{H}, e)$ is a quadratic Jordan division algebra, it is also sufficient to prove that the map $(a,b) \mapsto h_{a,b}$ is biadditive (5.12 of [1]). In this case however the strictness is not the only obstacle for $\operatorname{char} U \in \{2,3\}$ and therefore it is not yet clear if the statement is also true in this case.

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