Computing the derivation *Lie* algebra of the quadratic *Jordan* algebra $H_3(O_s, -)$ at any characteristic.

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Abstract. The aim of this work is to create a computational environment in which we can solve an open problem studying some Lie algebras without any restriction of the characteristic of the base field. This problem (see the references) is the study of the simplicity of the Lie algebra $f_4(O_s, -)$ in the cases of characteristic two. We have constructed the quadratic Jordan structure in $H_3(C, -)$ by using the *McCrimmon's* equations. Then, after verifying the axioms for a quadratic Jordan algebra, we use them to determine the generic expression of a matrix in $f_4(O_s, -)$, which is 52-dimensional, where O_s is the alternative algebra of split octonions, and without any restriction on the characteristic of the base field. It is obvious the usefulness of *Mathematica* as we are working with 27×27 matrices and its help has been crucial to solve the problem described. We have used the *Mathematica* 3.0 version for Windows 98.

Algebraic concepts and results

Let $(O_s, -)$ be the *alternative split algebra of octonions* over a field F. This algebra is constituted by the Zorn's matrices

$$\begin{pmatrix} \lambda & (x, y, z) \\ (r, s, t) & \mu \end{pmatrix}$$

where λ and μ are in the field F and (x, y, z), (r, s, t) are in F^3. The involution $-$ is

$$\begin{pmatrix} \lambda & (x, y, z) \\ (r, s, t) & \mu \end{pmatrix} \mapsto \begin{pmatrix} \mu & (-x, -y, -z) \\ (-r, -s, -t) & \lambda \end{pmatrix}$$

and the product is

$$\begin{pmatrix} \lambda & u \\ v & \mu \end{pmatrix} \cdot \begin{pmatrix} \lambda' & u' \\ v' & \mu' \end{pmatrix} \mapsto \begin{pmatrix} \lambda \lambda' + < u, v' > & \lambda u' + \mu' u - v \times v' \\ \lambda' v + \mu v' + u \times u' & \mu \mu' + < v, u' > \end{pmatrix},$$

where $\lambda, \lambda', \mu, \mu' \in F$, (x, y, z), (r, s, t), $u, u', v, v' \in F^3$ and $<>$ and $\times$ are the scalar and vectorial product in F^3. The standard basis of O_s is

$$e_1 = \begin{pmatrix} 1 & (0,0,0) \\ (0,0,0) & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & (0,0,0) \\ (0,0,0) & 1 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & (1,0,0) \\ (0,0,0) & 0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & (0,1,0) \\ (0,0,0) & 0 \end{pmatrix}$$
$$e_5 = \begin{pmatrix} 0 & (0,0,1) \\ (0,0,0) & 0 \end{pmatrix},\ e_6 = \begin{pmatrix} 0 & (0,0,0) \\ (1,0,0) & 0 \end{pmatrix},\ e_7 = \begin{pmatrix} 0 & (0,0,0) \\ (0,1,0) & 0 \end{pmatrix},\ e_8 = \begin{pmatrix} 0 & (0,0,0) \\ (0,0,1) & 0 \end{pmatrix},$$

Then, $H_3(O_s, -)$ is the set of matrices

$$X = \begin{pmatrix} \lambda_1 & a & b \\ \overline{a} & \lambda_2 & c \\ \overline{b} & \overline{c} & \lambda_3 \end{pmatrix}, \text{ with } \lambda_i \in F,\ a, b, c \in O_s$$

For future computations it will be useful to define $\mathrm{tri} : O_s \times O_s \times O_s \longrightarrow O_s$ given by $\mathrm{tri}[a, b, c] := a(b\,c)$. It is usual to express a generic element of $H_3(O_s, -)$ as

$$X = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + X_1(a) + X_2(b) + X_3(c)$$

where the definitions of E_i and X_i are the obvious ones. Then, a basis of $H_3(O_s, -)$ is

$$B = \{E_1, E_2, E_3, \{X_1(e_i)\}_{i=1,\ldots,8}, \{X_2(e_i)\}_{i=1,\ldots,8}, \{X_3(e_i)\}_{i=1,\ldots,8}\}.$$

A *quadratic Jordan algebra* with 1 over an arbitrary commutative ring Φ with 1 is a triple $(J, U, 1)$ where J is a unital left Φ-module, 1 a distinguished element of J, and U is a mapping $a \to U_a$ of J into $\mathrm{Hom}_\Phi(J, J)$ satisfying the following axioms:
i.- U is Φ-quadratic, that is, $U_{\alpha a} = \alpha^2 U_a$, $\alpha \in \Phi$, $a \in J$ and $U_{a,b} := U_{a+b} - U_a - U_b$ is Φ-bilinear in a and b.
ii.- $U_1 = 1$.
iii.- $U_x(T(y, x, z)) = T(x, y, U_x(z))$, $\forall\, x, y, z \in J$, where $T(y, x, z) := U_{y,z}(x)$.
iv.- $U_x U_y U_x = U_{U_x(y)}$, $\forall\, x, y, z \in J$.
v.- Conditions iii. and iv. hold in every scalar extension of J.

In order to construct a quadratic Jordan structure in $H_3(O_s, -)$ we use the *McCrimmon's* equations

$$\begin{aligned}
&U_{a[i\,i]}\, b[i\,i] = a\,b\,a\,[i\,i] \\
&U_{a[i\,j]}\, b[i\,i] = \overline{a}\,b\,a\,[j\,j] \\
&U_{a[i\,j]}\, b[i\,j] = a\,\overline{b}\,a\,[i\,j] \\
&T(a\,[i\,i], b\,[i\,j], c\,[j\,j]) = a\,b\,c\,[i\,j] \\
&T(a\,[i\,i], b\,[i\,j], c\,[j\,i]) = (a\,b\,c + \overline{a\,b\,c})\,[i\,i] \\
&T(a\,[i\,i], b\,[i\,j], c\,[j\,k]) = a\,b\,c\,[i\,k] \\
&T(a\,[i\,i], b\,[i\,i], c\,[i\,j]) = a\,b\,c\,[i\,j] \\
&T(a\,[i\,j], b\,[j\,j], c[j\,k]) = a\,b\,c\,[i\,k] \\
&T(a\,[i\,j], b\,[j\,i], c\,[i\,k]) = a(b\,c)\,[i\,k] \\
&T(a\,[i\,j], b\,[j\,k], c\,[k\,i]) = (a(b\,c) + \overline{a(b\,c)})\,[i\,i]
\end{aligned}$$

where $a\,[i\,j]$ represents an element of $H_3(O_s, -)$ filled with zeros except for an octonion a at the (i, j) position and it is understood that all the U formulas not covered by these and $a\,[j\,i] = \overline{a}\,[i\,j]$ are 0.

A *derivation* in this context is a linear map $D : J \longrightarrow J$ satisfying

i.- $D(1) = 0$,
ii.- $D(U_x(y)) = T(D(x), y, x) + U_x(D(y))$, for all x, y.

It is easy to prove that the set of all derivations is a Lie algebra with the product $[D_1, D_2] := D_1 D_2 - D_2 D_1$ for any two derivations D_1 and D_2. Then, we write $f_4(O_s, -)$ to be the *Lie algebra of the derivation of the quadratic Jordan algebra* $H_3(O_s, -)$. It is important to note that this construction does not depend on the characteristic of the base field.

Implementing the quadratic Jordan structure

■ Commands and definitions

We are going to implement the quadratic Jordan structure of $H_3(O_s, -)$. We have to follow *McCrimmon's* and define the U operator and the triple product T almost "element by element". We begin with the U operator depending of the concrete place of the particular elements involved. Afterwards, we will use certain relations to define the operator over a generic element. We start without the definitions of the structure of O_s and $H_3(O_s, -)$ in order to clarify the process. The element $U_{a[i,j]}\, b[k, l]$ in *McCrimmon's* equations will be denoted in *Mathematica* syntax by $U_{\text{a\_,i\_,j\_}}$[b_, k_, l_]. We make then

```
Ua_,i_,j_@b_, k_, l_D := Which@
  i == k == l, Ej@tri@σ@aD, b, aDD,
  i == k && j == l && i ≠ j, Xi,j@tri@a, σ@bD, aDD,
  i == l && j == k && i ≠ j, Xi,j@tri@a, b, aDD,
  j == k == l, Ei@tri@a, b, σ@aDDD,
  True, E1@0DD
```

in terms of $\boldsymbol{E}_i$, $\boldsymbol{X}_{i,j}$, which are as X_i but refering to the place (i, j), and the product of three octonions with **tri** and the conjugation $\boldsymbol{\sigma}$. We have not defined yet the basis of $H_3(O_s, -)$. The T operator is a little bit more delicate. We have to begin creating an instruction, **GivenQ**, to recognize the particular *McCrimmon's* equation we have to use:

```
GivenQ@a_, b_, c_, d_, e_, f_D := Which@a == b == c && d != a && d == e == f,
  8True, 1<, a == b == c && d != a && e == d && f == a,
  8True, 2<, a == b == c && d != a && d == e && f != e && f != a,
  8True, 3<, a == b == c == d == e && f != a,
  8True, 4<, a != b && b == c == d == e && f != a && f != b,
  8True, 5<, a != b && b == c && a == d == e && f != a && f != b, 8True, 6<,
  a != b && a == f && b == c && d != a && d != b && d == e, 8True, 7<, True,
  8False, 0<D
```

It brings out an answer like {True, i} if the equation is the i-th one. Then we can define T: (1) depending on the place of the octonions; (2) being symmetric; and (3) in terms of the U operator in such cases that we have a relation between them:

```
T@a_, i_, j_, b_, k_, l_, c_, m_, n_D := Which@
  i == m && j == n, U_{a+c,i,j}@b, k, lD - U_{a,i,j}@b, k, lD - U_{c,i,j}@b, k, lD,
  GivenQ@i, j, k, l, m, nD@@1DD,
  F_{GivenQ@i,j,k,l,m,nD@@2DD}@a, i, j, b, k, l, c, m, nD,
  GivenQ@j, i, k, l, m, nD@@1DD,
  F_{GivenQ@j,i,k,l,m,nD@@2DD}@σ@aD, j, i, b, k, l, c, m, nD,
  GivenQ@i, j, l, k, m, nD@@1DD,
  F_{GivenQ@i,j,l,k,m,nD@@2DD}@a, i, j, σ@bD, l, k, c, m, nD,
  GivenQ@i, j, k, l, n, mD@@1DD,
  F_{GivenQ@i,j,k,l,n,mD@@2DD}@a, i, j, b, k, l, σ@cD, n, mD,
  GivenQ@j, i, l, k, m, nD@@1DD,
  F_{GivenQ@j,i,l,k,m,nD@@2DD}@σ@aD, j, i, σ@bD, l, k, c, m, nD,
  GivenQ@j, i, k, l, n, mD@@1DD,
  F_{GivenQ@j,i,k,l,n,mD@@2DD}@σ@aD, j, i, b, k, l, σ@cD, n, mD,
  GivenQ@i, j, l, k, n, mD@@1DD,
  F_{GivenQ@i,j,l,k,n,mD@@2DD}@a, i, j, σ@bD, l, k, σ@cD, n, mD,
  GivenQ@j, i, l, k, n, mD@@1DD, F_{GivenQ@j,i,l,k,n,mD@@2DD}@σ@aD,
   j, i, σ@bD, l, k, σ@cD, n, mD, GivenQ@m, n, k, l, i, jD@@1DD,
  F_{GivenQ@m,n,k,l,i,jD@@2DD}@c, m, n, b, k, l, a, i, jD,
  GivenQ@n, m, k, l, i, jD@@1DD,
  F_{GivenQ@n,m,k,l,i,jD@@2DD}@σ@cD, n, m, b, k, l, a, i, jD,
  GivenQ@m, n, l, k, i, jD@@1DD,
  F_{GivenQ@m,n,l,k,i,jD@@2DD}@c, m, n, σ@bD, l, k, a, i, jD,
  GivenQ@m, n, k, l, j, iD@@1DD,
  F_{GivenQ@m,n,k,l,j,iD@@2DD}@c, m, n, b, k, l, σ@aD, j, iD,
  GivenQ@n, m, l, k, i, jD@@1DD,
  F_{GivenQ@n,m,l,k,i,jD@@2DD}@σ@cD, n, m, σ@bD, l, k, a, i, jD,
  GivenQ@n, m, k, l, j, iD@@1DD,
  F_{GivenQ@n,m,k,l,j,iD@@2DD}@σ@cD, n, m, b, k, l, σ@aD, j, iD,
  GivenQ@m, n, l, k, j, iD@@1DD,
  F_{GivenQ@m,n,l,k,j,iD@@2DD}@c, m, n, σ@bD, l, k, σ@aD, j, iD,
  GivenQ@n, m, l, k, j, iD@@1DD,
  F_{GivenQ@n,m,l,k,j,iD@@2DD}@σ@cD, n, m, σ@bD, l, k, σ@aD, j, iD,
  True, E_1@0DD
```

The terms F_{GivenQ} represent each one of the *McCrimmon's* equations that we present right now:

```
F_1@a_, i_, i_, b_, i_, j_, c_, j_, j_D := X_{i,j}@tri@a, b, cDD;
F_2@a_, i_, i_, b_, i_, j_, c_, j_, i_D := E_i@tri@a, b, cD + σ@tri@a, b, cDDD;
F_3@a_, i_, i_, b_, i_, j_, c_, j_, k_D := X_{i,k}@tri@a, b, cDD;
F_4@a_, i_, i_, b_, i_, i_, c_, i_, j_D := X_{i,j}@tri@a, b, cDD;
F_5@a_, i_, j_, b_, j_, j_, c_, j_, k_D := X_{i,k}@tri@a, b, cDD;
F_6@a_, i_, j_, b_, j_, i_, c_, i_, k_D := X_{i,k}@tri@a, b, cDD;
F_7@a_, i_, j_, b_, j_, k_, c_, k_, i_D := E_i@tri@a, b, cD + σ@tri@a, b, cDDD;
```

It is time to define the operatores that have been inactive untill now:

```
vectorial@8x_, y_, z_<, 8u_, v_, w_<D =
 8y*w - z*v, z*u - x*w, x*v - y*u<;
scalar@8x_, y_, z_<, 8u_, v_, w_<D =
  x*u + y*v + z*w;
prodA  α_   x_  ,  γ_   z_  E :=
       y_   β_     t_   δ_
J  α*γ + scalar@x, tD            α*z + δ*x - vectorial@y, tD  N;
   γ*y + β*t + vectorial@x, zD   β*δ + scalar@y, zD
σ@x_D := J  x@@2, 2DD   -x@@1, 2DD  N;
           -x@@2, 1DD    x@@1, 1DD
e1 = J   1         80, 0, 0<  N;
       80, 0, 0<   0
e2 = J   0         80, 0, 0<  N; e3 = J   0         81, 0, 0<  N;
       80, 0, 0<   1                    80, 0, 0<   0
e4 = J   0         80, 1, 0<  N;
       80, 0, 0<   0
e5 = J   0         80, 0, 1<  N; e6 = J   0         80, 0, 0<  N;
       80, 0, 0<   0                    81, 0, 0<   0
e7 = J   0         80, 0, 0<  N; e8 = J   0         80, 0, 0<  N;
       80, 1, 0<   0                    80, 0, 1<   0
Id = J   1         80, 0, 0<  N; zero = 0*Id;
       80, 0, 0<   1
E1@x_D :=  x Id  zero  zero  ;
           zero  zero  zero
           zero  zero  zero
E2@x_D :=  zero  zero  zero  ; E3@x_D :=  zero  zero  zero  ;
           zero  x Id  zero               zero  zero  zero
           zero  zero  zero               zero  zero  x Id
X1,2@ξ_D :=  zero    ξ     zero  ;
             σ@ξD   zero   zero
             zero   zero   zero
X1,3@ξ_D :=  zero   zero   ξ     ; X2,1@x_D := X1,2@σ@xDD;
             zero   zero   zero
             σ@ξD   zero   zero
X2,3@ξ_D :=  zero   zero   zero  ;
             zero   zero   ξ
             zero   σ@ξD   zero
X3,1@ξ_D :=  zero   zero   σ@ξD  ; X3,2@x_D := X2,3@σ@xDD;
             zero   zero   zero
             ξ      zero   zero
ZERO = E1@0D;
δi_,j_ := If@i == j, 1, 0D;
octo@x_D := Sum@xi * ei, 8i, 8<D;
tri@a_, b_, c_D := prod@a, prod@b, cDD;
```

The next definition constructs the *T* operator for a generic element by using the fact that it is trilinear

```
T@x_, y_, z_D := Sum@T@x@@i, jDD, i, j, y@@k, lDD, k, l, z@@m, nDD, m, nD,
  8i, 1, 3<, 8j, i, 3<, 8k, 1, 3<, 8l, k, 3<, 8m, 1, 3<, 8n, m, 3<D
```

and also we define the *U* operator, first with a generic matrix as its subindex

```
U α_  a_  c_ @x_D := T@E1@αD + E2@βD + E3@γD, x, X1,2@aD + X2,3@bD + X1,3@cDD +
  _   β_  b_
  _   _   γ_
  T@E1@αD + E2@βD, x, E3@γDD + T@E1@αD, x, E2@βDD + Uα,1,1@xD + Uβ,2,2@xD +
  Uγ,3,3@xD + T@X1,2@aD + X2,3@bD, x, X1,3@cDD + T@X1,2@aD, x, X2,3@bDD +
  Ua,1,2@xD + Ub,2,3@xD + Uc,1,3@xD
```

and then with a generic matrix as its argument

```
              α_  a_  c_
U_{x_,i_,j_}A  _   β_  b_  E := U_{x,i,j}@α, 1, 1D + U_{x,i,j}@β, 2, 2D +
              k _   _   ϒ_{
  U_{x,i,j}@ϒ, 3, 3D + U_{x,i,j}@a, 1, 2D + U_{x,i,j}@b, 2, 3D + U_{x,i,j}@c, 1, 3D
```

■ Verifying identities

Once we have defined the necessary commands and instructions to construct the quadratic Jordan structure in $H_3(O_s, -)$, it is time to verify the corresponding identities. We remark at this point that the implementation of the U operator made here, ensures the codition *i*) in the definition of quadratic Jordan algebra. We construct a basis by making

```
b_1 = E_1@1D; b_2 = E_2@1D; b_3 = E_3@1D;
Do@b_i = Which@3 < i ≤ 11, X_{1,2}@e_{i-3}D,
   11 < i ≤ 19, X_{2,3}@e_{i-11}D, True, X_{1,3}@e_{i-19}DD, 8i, 4, 27<D;
ONE = b_1 + b_2 + b_3;
```

and three generic elements:

```
gener = Sum@λ_i * b_i, 8i, 1, 27<D;

gener2 = Sum@μ_i * b_i, 8i, 1, 27<D;

gener3 = Sum@δ_i * b_i, 8i, 1, 27<D;
```

we can verify that T is, as it must be, symmetric with respect to the first and third variables:

```
Expand@T@gener, gener3, gener2D - T@gener2, gener3, generDD == ZERO
```

True

The first identity is

$$U_{Id} = Id \tag{1}$$

that is, $U_1(x) = x$ for all x in $H_3(O_s, -)$

```
U_ONE@generD == gener
```

True

The second one is

$$U_x(T(y, x, z)) = T(x, y, U_x(z)), \ \forall\, x, y, z \in H_3(O_s, -) \tag{2}$$

whose verification can be shown, by using that it is linear in z, in

```
Do@Print@
  Expand@
    U_gener@T@gener2, gener, b_iDD - T@gener, gener2, U_gener@b_iDDD == ZEROD,
 8i, 27<D
```

True

. . .

True

We only have then to confirm that

$$U_x(U_y(U_x(z))) = U_{U_x(y)}(z), \ \forall\, x, y, z \in H_3(O_s, -) \tag{3}$$

For that, we make

```
Do@
 Print@
  Expand@Ugener@Ugener2@Ugener@biDDD - UUgener@gener2D@biDD == ZEROD, 8i, 27<D
```

True

. . .

True

This last one has been the most complex one under a computational view. Not in vain, in a PC Pentium - 233 MHz - 64 Mb Ram it has taken almost 4 hours. We not only have implemented the quadratic Jordan structure of $H_3(O_s, -)$, but also *without any restriction on the characteristic*. This will be important to the next section, where we will present the generic expression of an element of $f_4(O_s, -)$ at any characteristic.

Generic element of $f_4(O_s, -)$ at any characteristic

Once we have implemented the quadratic Jordan structure of $H_3(O_s, -)$, we have used it to determine the generic expression of an element in $f_4(O_s, -)$. As we have obtained a matrix of high order (27×27) with 52 paramateres, the work has been too long to present it all here. Instead of this, we are going to show two representative examples of how to proceed.

■ Example 1: the image of E_1

Let Δ be any derivation of $H_3(O_s, -)$. We define the image of E_1 (represented by $E_1[1]$) to be a generic element of $H_3(O_s, -)$, thus

```
Δ@E1@1DD = gener;
```

where we have used the notation Δ to represent the derivation. The numbering of the input/output depends on the particular moment of the session. As we have that $U_{E_1}(1) = E_1$, we have that $D(E_1[1]) = D(U_{E_1[1]}[1]) = \{D(E_1[1]), 1, E_1[1]\} + U_{E_1[1]}[D(1)] = \{D(E_1[1]), 1, E_1[1]\}$ and then it must be null

```
Δ@E1@1DD - T@Δ@E1@1DD, ONE, E1@1DD êê Expand
```

$$\begin{pmatrix} \begin{pmatrix} -\lambda_1 & \{0,0,0\} \\ \{0,0,0\} & -\lambda_1 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} \lambda_2 & \{0,0,0\} \\ \{0,0,0\} & \lambda_2 \end{pmatrix} & \begin{pmatrix} \lambda_{12} & \{\lambda_{14},\lambda_{15},\lambda_{16}\} \\ \{\lambda_{17},\lambda_{18},\lambda_{19}\} & \lambda_{13} \end{pmatrix} \\ \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} \lambda_{13} & \{-\lambda_{14},-\lambda_{15},-\lambda_{16}\} \\ \{-\lambda_{17},-\lambda_{18},-\lambda_{19}\} & \lambda_{12} \end{pmatrix} & \begin{pmatrix} \lambda_3 & \{0,0,0\} \\ \{0,0,0\} & \lambda_3 \end{pmatrix} \end{pmatrix}$$

That is why we obtain the following conditions for the parameters:

```
Δ@E1@1DD = Δ@E1@1DD ê. 8λ1 → 0, λ2 → 0, λ3 → 0, λ12 → 0, λ13 → 0,
    λ14 → 0, λ15 → 0, λ16 → 0, λ17 → 0, λ18 → 0, λ19 → 0<
```

$$\begin{pmatrix} \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} \lambda_4 & \{\lambda_6,\lambda_7,\lambda_8\} \\ \{\lambda_9,\lambda_{10},\lambda_{11}\} & \lambda_5 \end{pmatrix} & \begin{pmatrix} \lambda_{20} & \{\lambda_{22},\lambda_{23},\lambda_{24}\} \\ \{\lambda_{25},\lambda_{26},\lambda_{27}\} & \lambda_{21} \end{pmatrix} \\ \begin{pmatrix} \lambda_5 & \{-\lambda_6,-\lambda_7,-\lambda_8\} \\ \{-\lambda_9,-\lambda_{10},-\lambda_{11}\} & \lambda_4 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} \\ \begin{pmatrix} \lambda_{21} & \{-\lambda_{22},-\lambda_{23},-\lambda_{24}\} \\ \{-\lambda_{25},-\lambda_{26},-\lambda_{27}\} & \lambda_{20} \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} \end{pmatrix}$$

and the image of E_1 can be written in terms of two octonions (16 parameters). We then rename these parameters in order to fix them.

■ Example 2: the image of $X_1(e_1)$

We first define the image of $X_1[e_1]$ by a derivation Δ of $H_3(O_s, -)$ as a generic element

```
Δ@X1@e1DD = gener;
```

We use then the U operator with $X_1[e_1]$ and $E_1[1]$, whose image we have already determined, to verify that

```
UX1@e1D@E1@1DD == CERO
```

True

and then, if we construct, from the definition of derivation,

```
defin@x_, y_D := T@Δ@xD, y, xD + Ux@Δ@yDD
```

it must be null

```
defin@X1@e1D, E1@1DD ±± Expand
```

$$\begin{pmatrix} \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} \alpha_2+\lambda_1 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & \alpha_2+\lambda_1 \end{pmatrix} & \begin{pmatrix} \lambda_5 & \{0,0,0\} \\ \{0,0,0\} & \lambda_5 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{\lambda_{25},\lambda_{26},\lambda_{27}\} & \lambda_{21} \end{pmatrix} \\ \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} \lambda_{21} & \{0,0,0\} \\ \{-\lambda_{25},-\lambda_{26},-\lambda_{27}\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} \end{pmatrix}$$

Then we make

```
Δ@X1@e1DD =
  Δ@X1@e1DD ±±. 8λ1 -> -α2, λ5 -> 0, λ21 -> 0, λ25 -> 0, λ26 -> 0, λ27 -> 0<
```

$$\begin{pmatrix} \begin{pmatrix} -\alpha_2 & \{0,0,0\} \\ \{0,0,0\} & -\alpha_2 \end{pmatrix} & \begin{pmatrix} \lambda_4 & \{\lambda_6,\lambda_7,\lambda_8\} \\ \{\lambda_9,\lambda_{10},\lambda_{11}\} & 0 \end{pmatrix} & \begin{pmatrix} \lambda_{20} & \{\lambda_{22},\lambda_{23},\lambda_{24}\} \\ \{0,0,0\} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \{-\lambda_6,-\lambda_7,-\lambda_8\} \\ \{-\lambda_9,-\lambda_{10},-\lambda_{11}\} & \lambda_4 \end{pmatrix} & \begin{pmatrix} \lambda_2 & \{0,0,0\} \\ \{0,0,0\} & \lambda_2 \end{pmatrix} & \begin{pmatrix} \lambda_{12} & \{\lambda_{14},\lambda_{15},\lambda_{16}\} \\ \{\lambda_{17},\lambda_{18},\lambda_{19}\} & \lambda_{13} \end{pmatrix} \\ \begin{pmatrix} 0 & \{-\lambda_{22},-\lambda_{23},-\lambda_{24}\} \\ \{0,0,0\} & \lambda_{20} \end{pmatrix} & \begin{pmatrix} \lambda_{13} & \{-\lambda_{14},-\lambda_{15},-\lambda_{16}\} \\ \{-\lambda_{17},-\lambda_{18},-\lambda_{19}\} & \lambda_{12} \end{pmatrix} & \begin{pmatrix} \lambda_3 & \{0,0,0\} \\ \{0,0,0\} & \lambda_3 \end{pmatrix} \end{pmatrix}$$

where the parameter α_2 comes from a previous fixed image. We have used these kind of relations between elements in $H_3(O_s, -)$ with the U and T operator to minimizied the number of free parameters. We have also used the *Leibnitz* rule

```
Leib@x_, y_, z_D := T@Δ@xD, y, zD + T@x, Δ@yD, zD + T@x, y, Δ@zDD
```

It can be used as follows. For instance, we have

```
T@E1@1D, E1@1D, X1@e1DD == X1@e1D
```

True

and then, by applying the derivation, it must be zero

```
Δ@X1@e1DD - Leib@E1@1D, E1@1D, X1@e1DD ±± Expand
```

$$\begin{pmatrix} \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{\lambda_{17}-\beta_6,\lambda_{18}-\beta_7,\lambda_{19}-\beta_8\} & \lambda_{13}-\beta_2 \end{pmatrix} \\ \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} & \begin{pmatrix} \lambda_{13}-\beta_2 & \{0,0,0\} \\ \{\beta_6-\lambda_{17},\beta_7-\lambda_{18},\beta_8-\lambda_{19}\} & 0 \end{pmatrix} & \begin{pmatrix} \lambda_3 & \{0,0,0\} \\ \{0,0,0\} & \lambda_3 \end{pmatrix} \end{pmatrix}$$

Then we can write

Δ@X1@e1DD = Δ@X1@e1DD ±±. 8λ3 → 0, λ13 → β2, λ17 → β6, λ18 → β7, λ19 → β8<

$$\begin{pmatrix} \begin{pmatrix} -\alpha_2 & \{0,0,0\} \\ \{0,0,0\} & -\alpha_2 \end{pmatrix} & \begin{pmatrix} \lambda_4 & \{\lambda_6,\lambda_7,\lambda_8\} \\ \{\lambda_9,\lambda_{10},\lambda_{11}\} & 0 \end{pmatrix} & \begin{pmatrix} \lambda_{20} & \{\lambda_{22},\lambda_{23},\lambda_{24}\} \\ \{0,0,0\} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \{-\lambda_6,-\lambda_7,-\lambda_8\} \\ \{-\lambda_9,-\lambda_{10},-\lambda_{11}\} & \lambda_4 \end{pmatrix} & \begin{pmatrix} \alpha_2 & \{0,0,0\} \\ \{0,0,0\} & \alpha_2 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{\beta_6,\beta_7,\beta_8\} & \beta_2 \end{pmatrix} \\ \begin{pmatrix} 0 & \{-\lambda_{22},-\lambda_{23},-\lambda_{24}\} \\ \{0,0,0\} & \lambda_{20} \end{pmatrix} & \begin{pmatrix} \beta_2 & \{0,0,0\} \\ \{-\beta_6,-\beta_7,-\beta_8\} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \{0,0,0\} \\ \{0,0,0\} & 0 \end{pmatrix} \end{pmatrix}$$

where the α_i and β_i parameters come from another previous fixed images.

■ The matrix

Proceeding this way, we have used the multiplication rules in $H_3(O_s, -)$ in terms of the U and T operators, combining them with the derivation, to characterize the Lie algebra $f_4(O_s, -)$. As a result, any element of the algebra is as

matrix

$$\begin{pmatrix}
0&0&0&\alpha_1&\alpha_2&\alpha_3&\alpha_4&\alpha_5&\alpha_6&\alpha_7&\alpha_8&0&0&0&0&0&0&0&0&\beta_1&\beta_2&\beta_3&\beta_4&\beta_5&\beta_6&\beta_7&\beta_8\\
0&0&0&-\alpha_1&-\alpha_2&-\alpha_3&-\alpha_4&-\alpha_5&-\alpha_6&-\alpha_7&-\alpha_8&\gamma_1&\gamma_2&\gamma_3&\gamma_4&\gamma_5&\gamma_6&\gamma_7&\gamma_8&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-\gamma_1&-\gamma_2&-\gamma_3&-\gamma_4&-\gamma_5&-\gamma_6&-\gamma_7&-\gamma_8&-\beta_1&-\beta_2&-\beta_3&-\beta_4&-\beta_5&-\beta_6&-\beta_7&-\beta_8\\
-\alpha_2&\alpha_2&0&\delta_1&0&\delta_2&\delta_3&\delta_4&\delta_5&\delta_6&\delta_7&0&\beta_2&0&0&0&\beta_6&\beta_7&\beta_8&\gamma_1&0&\gamma_3&\gamma_4&\gamma_5&0&0&0\\
-\alpha_1&\alpha_1&0&0&-\delta_1&\rho_1&\rho_2&\rho_3&\rho_4&\rho_5&\rho_6&\beta_1&0&\beta_3&\beta_4&\beta_5&0&0&0&0&\gamma_2&0&0&0&\gamma_6&\gamma_7&\gamma_8\\
\alpha_6&-\alpha_6&0&\rho_4&\delta_5&\eta_1&\eta_2&\eta_3&0&\eta_4&\eta_5&-\beta_6&0&-\beta_2&0&0&0&\beta_5&-\beta_4&\gamma_6&0&\gamma_2&0&0&0&-\gamma_5&\gamma_4\\
\alpha_7&-\alpha_7&0&\rho_5&\delta_6&\epsilon_1&\epsilon_2&\epsilon_3&-\eta_4&0&\epsilon_4&-\beta_7&0&0&-\beta_2&0&-\beta_5&0&\beta_3&\gamma_7&0&0&\gamma_2&0&\gamma_5&0&-\gamma_3\\
\alpha_8&-\alpha_8&0&\rho_6&\delta_7&\phi_1&\phi_2&2\xi+\delta_1-\epsilon_2-\eta_1&-\eta_5&-\epsilon_4&0&-\beta_8&0&0&0&-\beta_2&\beta_4&-\beta_3&0&\gamma_8&0&0&0&\gamma_2&-\gamma_4&\gamma_3&0\\
\alpha_3&-\alpha_3&0&\rho_1&\delta_2&0&\psi_1&\psi_2&-\eta_1&-\epsilon_1&-\phi_1&0&-\beta_3&0&-\beta_8&\beta_7&-\beta_1&0&0&0&\gamma_3&0&\gamma_8&-\gamma_7&\gamma_1&0&0\\
\alpha_4&-\alpha_4&0&\rho_2&\delta_3&-\psi_1&0&\chi_1&-\eta_2&-\epsilon_2&-\phi_2&0&-\beta_4&\beta_8&0&-\beta_6&0&-\beta_1&0&0&\gamma_4&-\gamma_8&0&\gamma_6&0&\gamma_1&0\\
\alpha_5&-\alpha_5&0&\rho_3&\delta_4&-\psi_2&-\chi_1&0&-\eta_3&-\epsilon_3&-2\xi-\delta_1+\epsilon_2+\eta_1&0&-\beta_5&-\beta_7&\beta_6&0&0&0&-\beta_1&0&\gamma_5&\gamma_7&-\gamma_6&0&0&0&\gamma_1\\
0&-\gamma_2&\gamma_2&0&-\beta_2&-\beta_3&-\beta_4&-\beta_5&0&0&0&\xi&0&\epsilon_4&-\eta_5&\eta_4&-\rho_4&-\rho_5&-\rho_6&-\alpha_1&0&0&0&0&-\alpha_6&-\alpha_7&-\alpha_8\\
0&-\gamma_1&\gamma_1&-\beta_1&0&0&0&0&-\beta_6&-\beta_7&-\beta_8&0&-\xi&-\delta_2&-\delta_3&-\delta_4&-\chi_1&\psi_2&-\psi_1&0&-\alpha_2&-\alpha_3&-\alpha_4&-\alpha_5&0&0&0\\
0&\gamma_6&-\gamma_6&0&\beta_6&\beta_1&0&0&0&\beta_5&-\beta_4&-\chi_1&-\rho_4&-\xi-\delta_1+\eta_1&\eta_2&\eta_3&0&-\rho_3&\rho_2&0&-\alpha_6&-\alpha_1&0&0&0&-\alpha_5&\alpha_4\\
0&\gamma_7&-\gamma_7&0&\beta_7&0&\beta_1&0&-\beta_5&0&\beta_3&\psi_2&-\rho_5&\epsilon_1&-\xi-\delta_1+\epsilon_2&\epsilon_3&\rho_3&0&-\rho_1&0&-\alpha_7&0&-\alpha_1&0&\alpha_5&0&-\alpha_3\\
0&\gamma_8&-\gamma_8&0&\beta_8&0&0&\beta_1&\beta_4&-\beta_3&0&-\psi_1&-\rho_6&\phi_1&\phi_2&\xi-\epsilon_2-\eta_1&-\rho_2&\rho_1&0&0&-\alpha_8&0&0&-\alpha_1&-\alpha_4&\alpha_3&0\\
0&\gamma_3&-\gamma_3&\beta_3&0&0&-\beta_8&\beta_7&\beta_2&0&0&-\delta_2&\epsilon_4&0&\delta_7&-\delta_6&\xi+\delta_1-\eta_1&-\epsilon_1&-\phi_1&-\alpha_3&0&0&\alpha_8&-\alpha_7&-\alpha_2&0&0\\
0&\gamma_4&-\gamma_4&\beta_4&0&\beta_8&0&-\beta_6&0&\beta_2&0&-\delta_3&-\eta_5&-\delta_7&0&\delta_5&-\eta_2&\xi+\delta_1-\epsilon_2&-\phi_2&-\alpha_4&0&-\alpha_8&0&\alpha_6&0&-\alpha_2&0\\
0&\gamma_5&-\gamma_5&\beta_5&0&-\beta_7&\beta_6&0&0&0&\beta_2&-\delta_4&\eta_4&\delta_6&-\delta_5&0&-\eta_3&-\epsilon_3&-\xi+\epsilon_2+\eta_1&-\alpha_5&0&\alpha_7&-\alpha_6&0&0&0&-\alpha_2\\
-\beta_2&0&\beta_2&-\gamma_2&0&\gamma_3&\gamma_4&\gamma_5&0&0&0&\alpha_2&0&0&0&0&-\alpha_6&-\alpha_7&-\alpha_8&\xi+\delta_1&0&\epsilon_4&-\eta_5&\eta_4&\delta_5&\delta_6&\delta_7\\
-\beta_1&0&\beta_1&0&-\gamma_1&0&0&0&\gamma_6&\gamma_7&\gamma_8&0&\alpha_1&-\alpha_3&-\alpha_4&-\alpha_5&0&0&0&0&-\xi-\delta_1&\rho_1&\rho_2&\rho_3&-\chi_1&\psi_2&-\psi_1\\
\beta_6&0&-\beta_6&\gamma_6&0&-\gamma_1&0&0&0&-\gamma_5&\gamma_4&0&-\alpha_6&\alpha_2&0&0&0&-\alpha_5&\alpha_4&-\chi_1&\delta_5&\eta_1-\xi&\eta_2&\eta_3&0&\delta_4&-\delta_3\\
\beta_7&0&-\beta_7&\gamma_7&0&0&-\gamma_1&0&\gamma_5&0&-\gamma_3&0&-\alpha_7&0&\alpha_2&0&\alpha_5&0&-\alpha_3&\psi_2&\delta_6&\epsilon_1&\epsilon_2-\xi&\epsilon_3&-\delta_4&0&\delta_2\\
\beta_8&0&-\beta_8&\gamma_8&0&0&0&-\gamma_1&-\gamma_4&\gamma_3&0&0&-\alpha_8&0&0&\alpha_2&-\alpha_4&\alpha_3&0&-\psi_1&\delta_7&\phi_1&\phi_2&\xi+\delta_1-\epsilon_2-\eta_1&\delta_3&-\delta_2&0\\
\beta_3&0&-\beta_3&0&\gamma_3&0&\gamma_8&-\gamma_7&-\gamma_2&0&0&-\alpha_3&0&0&\alpha_8&-\alpha_7&\alpha_1&0&0&\rho_1&\epsilon_4&0&-\rho_6&\rho_5&\xi-\eta_1&-\epsilon_1&-\phi_1\\
\beta_4&0&-\beta_4&0&\gamma_4&-\gamma_8&0&\gamma_6&0&-\gamma_2&0&-\alpha_4&0&-\alpha_8&0&\alpha_6&0&\alpha_1&0&\rho_2&-\eta_5&\rho_6&0&-\rho_4&-\eta_2&\xi-\epsilon_2&-\phi_2\\
\beta_5&0&-\beta_5&0&\gamma_5&\gamma_7&-\gamma_6&0&0&0&-\gamma_2&-\alpha_5&0&\alpha_7&-\alpha_6&0&0&0&\alpha_1&\rho_3&\eta_4&-\rho_5&\rho_4&0&-\eta_3&-\epsilon_3&-\xi-\delta_1+\epsilon_2+\eta_1
\end{pmatrix}$$

Futher utility

Following the generic expression in **matrix**, it is easy to determine a basis of this Lie algebra. We have made it and proved that the Lie algebra is not simple if the characteristic of the base field is 2. Working with characteristic 2, we have shown that there exists a unique proper nonzero ideal which is 26-dimensional (see the references). We have also present a new proof of the simplicity of this algebra out of the characteristic two case.

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