DIRICHLET FORMS AND MARKOV SEMIGROUPS ON NON-ASSOCIATIVE VECTOR BUNDLES

CHO-HO CHU AND ZHONGMIN QIAN

ABSTRACT. We introduce non-associative vector bundles and study Dirichlet forms and the associated Markov semigroups on these bundles.

1. INTRODUCTION

A non-commutative theory of Dirichlet forms and Markov semigroups has been developed in [1, 8, 9, 10]. Two forms of non-commutative theory are usually considered: either the domains of the Dirichlet forms are furnished by some non-commutative C*-algebras, typically, the non-commutative $L^p(\mathcal{A})$ spaces of a semifinite von Neumann algebra $\mathcal{A}$, or, one considers the semigroups acting on sections of vector bundles over Riemannian manifolds, with non-commutative fibres. In [9, 10], the latter case has been studied for C*-bundles over compact manifolds whose fibres are finite-dimensional real C*-algebras. To be precise, the Dirichlet forms in both cases are defined in terms of the Hermitian part of the relevant spaces, namely, either the Hermitian part

$$L^2_h(\mathcal{A}) = \{x \in L^2(\mathcal{A}) : x^* = x\}$$

of the non-commutative space $L^2(\mathcal{A})$, as in [1, p. 177], or the section $L^2(\mathfrak{A}_h)$ with bundle $\mathfrak{A}_h$ whose fibres are the Hermitian part

$$A_h = \{x \in A : x^* = x\}$$

of a finite-dimensional real C*-algebra A, equipped with the L_2-norm of a trace, as in [9, Theorem 2]. It was also noted in [9] that a natural alternative approach would be to consider bundles whose fibres have the structure of a compact Jordan algebra.

In this paper, we consider more general vector bundles modelled on the non-associative L^p-spaces, usually infinite dimensional, of a semifinite Jordan von Neumann algebra. This includes the bundles $\mathfrak{A}_h$ considered in [9] as well as the alternative approach proposed in [9] and mentioned above. We describe a framework for a non-associative theory of Dirichlet forms on these bundles and extend to this setting some contractivity results concerning the associated Markov semigroups (cf. [9, 10, 17]).

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We begin by describing the non-associative L^p-spaces, constructed from a Jordan algebra. We recall that a real, but not necessarily associative, algebra $\mathcal{A}$ is called a *Jordan algebra* if its algebraic product satisfies

$$xy = yx \qquad \text{and} \qquad x^2(yx) = (x^2y)x \qquad (x, y \in \mathcal{A}).$$

By a *Jordan von Neumann algebra* $\mathcal{A}$, we mean a real Banach space $\mathcal{A}$ which is also a Jordan algebra, with a (necessarily unique) *separable* predual $\mathcal{A}_*$, such that

$$\begin{aligned} \|xy\| &\leq \|x\|\|y\| \\ \|x^2\| &= \|x\|^2 \\ \|x^2\| &\leq \|x^2 + y^2\| \end{aligned}$$

for $x, y \in \mathcal{A}$. Without the separability condition on the predual, these algebras are known as *JBW-algebras* in literature [19]. The *weak topology* on $\mathcal{A}$ is the topology $\sigma(\mathcal{A}, \mathcal{A}_*)$. We note that $\mathcal{A}$ contains an identity $\mathbf{1}$ and the order in $\mathcal{A}$ is induced by the closed cone

$$\mathcal{A}^+ = \{x^2 : x \in \mathcal{A}\}$$

and we have $\mathcal{A} = \mathcal{A}^+ - \mathcal{A}^+$. Given $x \in \mathcal{A}$, one can define its modulus $|x| = (x^2)^{1/2} \in \mathcal{A}^+$. Each $x \in \mathcal{A}$ has a polar decomposition

$$x = s|x|$$

where s is a symmetry in $\mathcal{A}$ which means that $s^2 = \mathbf{1}$.

Example 1.1. Let $\mathcal{A}$ be a (complex) von Neumann algebra with a separable predual, for instance, the algebra $B(H)$ of bounded linear operators on a complex separable Hilbert space H. Then the Hermitian part

$$\mathcal{A}_h = \{T \in \mathcal{A} : T^* = T\}$$

is a Jordan von Neumann algebra, with the Jordan product defined by

$$T \circ S = \frac{1}{2}(TS + ST)$$

where the product on the right is the original product in $\mathcal{A}$. The positive cone $\mathcal{A}^+ = \{T^*T : T \in \mathcal{A}\}$ coincides with $\mathcal{A}_h^+$.

Example 1.2. Let A be a real C*-algebra. Then its complexification $\widetilde{A} = A + iA$ can be given a norm so that it becomes a (complex) C*-algebra, and A embeds isometrically as a real C*-subalgebra of $\widetilde{A}$ [15, 15.4]. We note that A is generally not identical with the Hermitian part of $\widetilde{A}$. If A has a separable predual, then its Hermitian part

$$A_h = \{x \in A : x^* = x\}$$

is a Jordan von Neumann algebra, with the Jordan product defined by

$$x \circ y = \frac{1}{2}(xy + yx)$$

where the associative product on the right is the original product in A.

We refer to [19] for other examples of Jordan von Neumann algebras which are not the Hermitian part of a real or complex C*-algebra.

We recall that a Jordan von Neumann algebra $\mathcal{A}$ is *semifinite* if it admits a faithful semifinite normal trace. A *trace* on $\mathcal{A}$ is an additive function $\tau : \mathcal{A}^+ \longrightarrow [0,\infty]$ satisfying

(i) $\tau(\alpha x) = \alpha\tau(x) \qquad (\alpha \geq 0)$

(ii) $\tau(sxs) = \tau(x) \qquad$ (s is a symmetry).

A trace τ is *faithful* if $\tau(x) = 0$ implies $x = 0$. It is called *semifinite* if for any $x \in \mathcal{A}^+\backslash\{0\}$, there exists $y \in \mathcal{A}^+\backslash\{0\}$ such that $y \leq x$ and $\tau(y) < \infty$. If τ preserves monotone convergence, then it is called *normal*.

A prototypic example of a semifinite Jordan von Neumann algebra is the Hermitian part $B(H)_h$ of the algebra $B(H)$ of bounded operators on a separable Hilbert space H, with the canonical trace; but important examples include Hermitian parts of all finite von Neumann algebras with separable predual, in particular, the group von Neumann algebras of infinite-conjugacy-class groups which are type II_1 factors (cf. [27, p.367]).

In the sequel, $\mathcal{A}$ will denote a semifinite Jordan von Neumann algebra with a faithful semifinite normal trace τ. There is a weakly dense ideal of $\mathcal{A}$ associated with τ, namely,

$$\mathcal{N}_\tau = \mathcal{N}_\tau^+ - \mathcal{N}_\tau^+$$

where

$$\mathcal{N}_\tau^+ = \{a \in \mathcal{A}^+ : \tau(a) < \infty\}$$

and the trace τ can be extended to a linear functional on $\mathcal{N}_\tau$, still denoted by τ. For $1 \leq p < \infty$, we define the L^p-norm

$$|||x|||_p = \tau(|x|^p)^{1/p} \qquad (x \in \mathcal{N}_\tau)$$

where $|x|^p \in \mathcal{N}_\tau^+$ is defined by function calculus. The completion of the normed space $(\mathcal{N}_\tau, |||\cdot|||_p)$ is denoted by $L^p(\mathcal{A},\tau)$, called the *non-associative L^p-space of $\mathcal{A}$ with respect to τ*. The space $L^1(\mathcal{A},\tau)$ is linearly isometric to $\mathcal{A}_*$ and $L^2(\mathcal{A},\tau)$ is a Hilbert space with inner product denoted by $\langle\cdot,\cdot\rangle_\tau$. We define $L^\infty(\mathcal{A},\tau) = \mathcal{A}$ and refer to [20] for further details of these L^p spaces.

One can construct a *non-commutative* L^p-space $L^p(\mathcal{M},\tau_0)$ of a (complex) von Neumann algebra $\mathcal{M}$ with a faithful semifinite normal trace τ_0. If $\mathcal{M}$ has a separable predual, then the Hermitian part $\mathcal{A} = \mathcal{M}_h$ of $\mathcal{M}$ is a Jordan von Neumann algebra with trace τ which is the restriction of τ_0 to $\mathcal{A}^+$, and $L^p(\mathcal{A},\tau)$ identifies with the Hermitian part $L^p_h(\mathcal{M},\tau_0)$ of $L^p(\mathcal{M},\tau_0)$ [2].

Example 1.3. If $\mathcal{A} = \mathcal{B}(H)_h$ is the Hermitian part of the algebra of bounded operators on a separable Hilbert space H, with the canonical trace τ, then $L^2(\mathcal{A},\tau) = \mathcal{N}_\tau$ is the space of self-adjoint Hilbert-Schmidt operators on H and is separable.

Example 1.4. If A is a finite-dimensional real C*-algebra, then $L^2(A_h,\tau) = (A_h, |||\cdot|||_2)$ for any trace τ on A_h. This is the space considered in [9].

2. Non-associative vector bundles and Dirichlet forms

In this section, we introduce non-associative vector bundles on Riemannian manifolds and the setting for a non-associative theory of Dirichlet forms. These bundles are vector bundles whose fibres have Jordan algebraic structures, more precisely, the fibres of these bundles are real Hilbert spaces isometric to a non-associative Hilbert space of a semifinite Jordan von Neumann algebra.

Let M be a Riemannian manifold with the Riemannian measure dx and let $L^2(\mathcal{A},\tau)$ be a non-associative Hilbert space as before. We denote by $L^2(M, L^2(\mathcal{A},\tau))$ the real Hilbert space of (equivalence classes of) $L^2(\mathcal{A},\tau)$-valued Bochner integrable functions f on M satisfying

$$\|f\|_2 = \left(\int_M |||f(x)|||_2^2 dx\right)^{\frac{1}{2}} < \infty$$

(cf. [13, p.97]), with inner product

$$\langle f, g\rangle = \int_M \langle f(x), g(x)\rangle_\tau dx.$$

Let $C_c^\infty(M, L^2(\mathcal{A},\tau))$ be the space of smooth $L^2(\mathcal{A},\tau)$-valued functions on M with compact support. Standard arguments show that $C_c^\infty(M, L^2(\mathcal{A},\tau))$ is $\|\cdot\|_2$-dense in $L^2(M, L^2(\mathcal{A},\tau))$.

A vector bundle $\pi : E \longrightarrow M$ is called a *non-associative bundle* if its fibres E_x are all real Hilbert spaces linearly isometric to the non-assoicative Hilbert space $L^2(\mathcal{A},\tau)$ of a Jordan von Neumann algebra $\mathcal{A}$ with a faithful semifinite normal trace τ. In this case, E is a Hilbert manifold modeled on the real Hilbert space $L^2(\mathcal{A},\tau)\times\mathbb{R}^n$ where $n = \dim M$. We denote the inner product in E_x by $\langle\cdot,\cdot\rangle_x$. Given the linear isometry

$$\gamma_x : E_x \longrightarrow L^2(\mathcal{A},\tau)$$

we have $\langle\xi,\zeta\rangle_x = \langle\gamma_x(\xi),\gamma_x(\zeta)\rangle_\tau$. The set $C_c^\infty(E)$ of smooth sections on M with compact support is a vector space with inner product and norm:

$$\langle\varphi,\psi\rangle = \int_M \langle\varphi(x),\psi(x)\rangle_x dx$$

$$\|\varphi\|_2 = \langle\varphi,\varphi\rangle^{1/2}.$$

The completion $\mathcal{L}^2(E)$ of $C_c^\infty(E)$ with respect to the above norm identifies with the real Hilbert space $L^2(M, L^2(\mathcal{A},\tau))$. More generally, for $1 \le p < \infty$, we denote by $\mathcal{L}^p(E)$ the completion of $C_c^\infty(E)$ with respect to the following norm:

$$\|\varphi\|_p = \left(\int_M \langle\varphi(x),\varphi(x)\rangle_x^{p/2} dx\right)^{1/p}.$$

Let $\mathcal{L}^\infty(E)$ be the space of (essentially) bounded sections on M.

The L^p-space $L^p(\mathcal{A},\tau)$ can be partially ordered by the cone $L^p(\mathcal{A},\tau)^+$ which is defined to be the $|||\cdot|||_p$-closure of $\mathcal{N}_\tau^+$. For $p\in(1,\infty)$, the norm $|||\cdot|||_p$ is Fréchet differentiable except at 0. Given a map $f : \mathbb{R} \longrightarrow L^p(\mathcal{A}, \tau)^+$, differentiable at $t_0 \in \mathbb{R}$ with $f(t_0) \neq 0$, we have, by [20, Lemma 14],

$$\frac{d}{dt}\tau\left(f(t)^p\right)|_{t=t_0} = p\tau\left(f(t_0)^{p-1}\frac{d}{dt}f(t)\,|_{t=t_0}\right).$$

For $z, w \in L^2(\mathcal{A}, \tau)^+$, we have $\langle z, w\rangle_\tau \geq 0$ (cf. [20, Lemma 1]). Every $z \in L^2(\mathcal{A}, \tau)$ has a decomposition $z = z^+ - z^-$ with $z^+, z^- \geq 0$ and $z^+z^- = 0$. The modulus of z is defined to be $|z| = z^+ + z^-$.

Each fibre E_x of the non-associative vector bundle $\pi : E \longrightarrow M$ carries the above order and Jordan algebraic structures of $L^2(\mathcal{A}, \tau)$ via the isometry $\gamma_x : E_x \longrightarrow L^2(\mathcal{A}, \tau)$. A section φ of E is said to be *positive* if $\varphi(x) \geq 0$ for almost all $x \in M$. We denote this by $\varphi \geq 0$.

Let $\Gamma(E)$ be the space of smooth sections of E. Given $\varphi \in \Gamma(E)$, we define $\varphi^{\pm}(x) = \varphi(x)^{\pm}$ and $|\varphi|(x) = |\varphi(x)|$ for $x \in M$. Then $\varphi = \varphi^+ - \varphi^-$ and $|\varphi| = \varphi^+ + \varphi^-$. We have $\langle \varphi^+, \varphi^-\rangle = 0$, in fact,

$$\int_M \langle \varphi(x)^+, \varphi(x)^-\rangle_x d\mu(x) = 0$$

for any measure μ on M. The above order structures can be extended to the completion $\mathcal{L}^2(E) \backsimeq L^2(M, L^2(\mathcal{A}, \tau))$. A linear map $P : \mathcal{L}^2(E) \longrightarrow \mathcal{L}^2(E)$ is called *positive*, in symbol, $P \geq 0$, if $\varphi \geq 0$ implies $P\varphi \geq 0$.

Let Q be a closable non-negative quadratic form with domain $C_c^\infty(E) \subset \mathcal{L}^2(E)$. Then there is a positive self-adjoint operator L in $\mathcal{L}^2(E)$ such that

$$Q(\varphi, \psi) = \langle L\varphi, \psi\rangle \qquad (\varphi, \psi \in C_c^\infty(E)).$$

We write $Q(\varphi)$ for $Q(\varphi, \varphi)$. The proof of the following result is similar to [9, Theorem 1].

Theorem 2.1. *Let $Q(\cdot) = \langle L^{1/2}(\cdot), L^{1/2}(\cdot)\rangle$ be a quadratic form where $L : \mathcal{D}(L) \longrightarrow \mathcal{L}^2(E)$ is a self-adjoint, positive operator which generates a semigroup $(P_t)_{t\geq 0}$ on $\mathcal{L}^2(E)$. The following conditions are equivalent.*

(i) $P_t \geq 0$ *for* $t > 0$.
(ii) *Given $\varphi \in \mathcal{D}(L^{1/2})$, we have $|\varphi| \in \mathcal{D}(L^{1/2})$ and $Q(|\varphi|) \leq Q(\varphi)$.*
(iii) *Given $\varphi \in \mathcal{D}(L^{1/2})$, we have $|\varphi| \in \mathcal{D}(L^{1/2})$ and $Q(\varphi^+, \varphi^-) \leq 0$.*
(iv) *For $\varphi \in \mathcal{L}^2(E)$ and $\varphi \geq 0$, we have $(\alpha + L)^{-1}(\varphi) \geq 0$ for all $\alpha > 0$.*

Proof. (i) $\Rightarrow$ (ii). Let $\varphi \in \mathcal{D}(L^{1/2})$. Then by positivity of P_t, we have

$$\begin{aligned}
\langle P_t\varphi, \varphi\rangle &= \langle P_t\varphi^+ - P_t\varphi^-, \varphi^+ - \varphi^-\rangle \\
&= \langle P_t\varphi^+, \varphi^+\rangle + \langle P_t\varphi^-, \varphi^-\rangle - \langle P_t\varphi^+, \varphi^-\rangle - \langle P_t\varphi^-, \varphi^+\rangle \\
&\leq \langle P_t|\varphi|, |\varphi|\rangle.
\end{aligned}$$

Hence

$$\frac{1}{t}\langle (I - P_t)|\varphi|, |\varphi|\rangle \leq \frac{1}{t}\langle (I - P_t)\varphi, \varphi\rangle$$

and $\limsup_{t\to 0}\frac{1}{t}\langle (I-P_t)|\varphi|, |\varphi|\rangle \le \langle L^{1/2}\varphi, L^{1/2}\varphi\rangle$. It follows that $|\varphi| \in \mathcal{D}(L^{1/2})$ and $Q(|\varphi|) \le Q(\varphi)$.

(ii) $\Leftrightarrow$ (iii). This follows from

$$4Q(\varphi^+, \varphi^-) = Q(|\varphi|) - Q(\varphi)$$

where $\varphi, |\varphi| \in \mathcal{D}(L^{1/2})$ implies that $\varphi^{\pm} \in \mathcal{D}(L^{1/2})$.

(iii) $\Rightarrow$ (iv). Fix $\alpha > 0$. Denote $K = \mathcal{D}(L^{1/2})$ which is a Hilbert space with respect to the inner product

$$\langle \psi, \varphi\rangle_1 = \langle L^{1/2}\psi, L^{1/2}\varphi\rangle + \alpha\langle \psi, \varphi\rangle.$$

Let $J : K \longrightarrow \mathcal{L}^2(E)$ be the natural embedding. Then, for $\psi \in K$, $\varphi \in \mathcal{L}^2(E)$, we have

$$\begin{aligned}\langle \psi, (\alpha+L)^{-1}\varphi\rangle_1 &= \langle L^{1/2}\psi, L^{1/2}(\alpha+L)^{-1}\varphi\rangle \\ &\quad + \alpha\,\langle \psi, (\alpha+L)^{-1}\varphi\rangle \\ &= \langle (\alpha+L)\psi, (\alpha+L)^{-1}\varphi)\rangle \\ &= \langle \psi, \varphi\rangle = \langle J\psi, \varphi\rangle.\end{aligned}$$

Therefore $J^*\varphi = (\alpha+L)^{-1}\varphi$. Let $\psi = J^*\varphi$. We have

$$\begin{aligned}\langle |\psi|, |\psi|\rangle_1 &= Q(|\psi|) + \alpha\langle |\psi|, |\psi|\rangle \\ &\le Q(\psi) + \alpha\langle \psi, \psi\rangle = \langle \psi, \psi\rangle_1.\end{aligned}$$

Let $\varphi \ge 0$. Then

$$\begin{aligned}\langle |\psi|, \psi\rangle_1 &= \langle |\psi|, J^*\varphi\rangle_1 \\ &= \langle |\psi|, \varphi\rangle \\ &\ge \langle \psi, \varphi\rangle = \langle \psi, J^*\varphi\rangle_1 = \langle \psi, \psi\rangle_1.\end{aligned}$$

Hence $(\alpha+L)^{-1}\varphi = J^*\varphi = \psi = |\psi| \ge 0$.

(iv) $\Rightarrow$ (i). This follows from

$$P_t = \lim_{n\to\infty}\left(I + \frac{t}{n}L\right)^{-n}.$$

$\square$

A quadratic form Q in $\mathcal{L}^2(E)$ satisfying the conditions in Theorem 2.1 and generating a contractive semigroup (P_t) on $\mathcal{L}^p(E)$ for $p \in [1,\infty]$ is called a *Dirichlet form*, where P_t is called a *contraction* on $\mathcal{L}^p(E)$ if it maps $\mathcal{L}^2(E)\cap\mathcal{L}^p(E)$ into $\mathcal{L}^2(E)\cap\mathcal{L}^p(E)$, and is contractive in the L^p-norm.

From now on, we fix a non-associative vector bundle $\pi : E \longrightarrow M$ with fibres isometric to the real Hilbert space $L^2(\mathcal{A},\tau)$ of a Jordan von Neumann algebra $\mathcal{A}$ with a faithful semifinite normal trace τ. By [21, Theorem 1.8.19], the vector bundle $\pi : E \longrightarrow M$ has a Riemannian metric, that is, the inner products $\langle \cdot, \cdot \rangle_x$ on E_x can be chosen to depend smoothly on $x \in M$. Let TE be the total tangent space of E. By [21, Theorem 1.8.23], the above vector bundle possesses a metric connection $K : TE \longrightarrow E$, compatible with the Riemannian structure such that, for each $\varphi \in \Gamma(E)$,

$$D_X\varphi(x) := K \circ d\varphi_x(X) \in E$$

is the associated covariant derivation of φ in the direction $X \in T_xM$, where $d\varphi_x : T_xM \longrightarrow T_{\varphi(x)}E$ is the differential of φ at $x \in M$. For any vector field X on M, $D_X\varphi$ is a smooth section of E (cf.[21, p.49]) and

$$X\langle \varphi, \psi \rangle = \langle D_X\varphi, \psi \rangle + \langle \varphi, D_X\psi \rangle.$$

We note that $K \circ d\varphi_x \in L(T_xM, E_x)$, the space of linear maps between T_xM and E_x, and the tensor product $E_x \otimes T_x^*M$ is dense in $L(T_xM, E_x)$ in the compact open topology (cf. [13, p.240]). If the fibre E_x is finite-dimensional, then $L(T_xM, E_x) = E_x \otimes T_x^*M$ and we have the connection $D : \Gamma(E) \longrightarrow \Gamma(E) \otimes \Gamma(T^*M)$ given by

$$D\varphi = K \circ d\varphi.$$

For $\varphi, \psi \in C_c^\infty(E)$, we define

$$\langle D\varphi(x), D\psi(x) \rangle_\tau = \sum_{i=1}^n \langle D_{X_i}\varphi(x), D_{X_i}\psi(x) \rangle_x$$

where $\{X_1, \ldots, X_n\}$ is an orthonormal moving frame on M.

Given $\pi : E \longrightarrow M$ endowed with a Riemannian structure and a compatible affine connection D, the qudratic form

$$\mathcal{E}(\varphi, \psi) = \int_M \langle D\varphi, D\psi \rangle_\tau d\mu \qquad (\varphi, \psi \in C_c^\infty(E))$$

satisfies the conditions in Theorem 2.1 since $\mathcal{E}(\varphi^+, \varphi^-) = 0$.

3. Hypercontractivity

The theory of hypercontractive semigroups was introduced in a fundamental paper of Nelson [24] who discovered that the Ornstein-Uhlenbeck semigroup $P_t : L^p(\mathbb{R}^d, \mu) \longrightarrow L^q(\mathbb{R}^d, \mu)$ is bounded if p, q and t are properly related, where μ is the Gaussian measure. After important improvements in [14, 26], the precise minimum time t for contractivity from L^p to L^q was established in [25].

In his seminal paper [17], Gross proved the equivalence of hypercontractivity and a logarithmic Sobolev inequality for diffusion semigroups which may be stated as follows. Let $(P_t)_{t \geq 0}$ be the diffusion semigroup associated to a local Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mathcal{X}, \mu)$ for some σ-finite measure $(X, \mathcal{X}, \mu)$. Let

$$\text{Ent}(f) = \int_X (f \ln f)\, d\mu - \left(\int_X f d\mu \right) \left(\ln \int_X f d\mu \right) \tag{1}$$

denote the entropy of f. Let $a > 0$ and $b \geq 0$. Define

$$p(t) = 1 + (p-1)e^{4t/a}; \quad m(t) = b\left(p^{-1} - p(t)^{-1}\right).$$

Then the following logarithmic Sobolev inequality

$$\mathrm{Ent}(f^2) \leq a\mathcal{E}(f,f) + b||f||_2^2 \qquad (f \in \mathcal{F}) \tag{2}$$

holds if, and only if,

$$||P_t f||_{p(t)} \leq e^{m(t)}||f||_p \tag{3}$$

for all $f \in L^p(\mu)$, $p \in (1,\infty)$ and $t > 0$. We refer to [3, 6, 11, 12, 17, 18] for the evolution of this form of Gross's theorem. We also refer to [7] for a bibliographic review of hyercontractivity.

Let $\pi : E \to M$ be a non-associative vector bundle, endowed with a Riemannian structure and a compatible affine connection D. Let

$$\mathcal{E}(\varphi,\psi) = \int_M \langle D\varphi, D\psi\rangle_\tau d\mu \qquad (\varphi,\psi \in C_c^\infty(E) \subset \mathcal{L}^2(E))$$

be a Dirichlet form. Let $(P_t)_{t\geq 0}$ be the diffusion semigroup of the vector bundle E with generator L defined by $\mathcal{E}$. That is, $P_t = e^{-tL}$ and the self-adjoint operator L is determined via integration by parts

$$\int_M \langle D\varphi, D\psi\rangle_\tau d\mu = \int_M \langle L\varphi, \psi\rangle_\tau d\mu = \int_M \langle L\varphi(x), \psi(x)\rangle_x d\mu(x) \ .$$

As before, let $||\varphi||_p$ denote the L^p-norm of $|\varphi|_\tau$, where we define

$$|\varphi|_\tau(x) = \langle \varphi(x), \varphi(x)\rangle_x^{1/2} \qquad (x \in M)$$

which is abbreviated to

$$|\varphi|_\tau^2 = \langle \varphi, \varphi\rangle_\tau$$

if no confusion is likely.

In the following result for non-associative vector bundles, the special case for line bundles is implicit in the fundamental work of Gross [17]. The proof uses an argument of Bakry [4].

Proposition 3.1. *Let $a > 0$, $b \geq 0$. The following two conditions are equivalent.*

(i) $(P_t)_{t\geq 0}$ *possesses hypercontractivity, that is,*

$$||P_t\varphi||_{p(t)} \leq e^{m(t)}||\varphi||_p \qquad (\varphi \in C_c^\infty(E)) \quad \textit{with} \tag{4}$$

$$p(t) = 1 + (p-1)e^{\frac{4}{a}t}\ , \quad m(t) = b\left(p^{-1} - p(t)^{-1}\right) \quad (t > 0,\ p > 1). \tag{5}$$

(ii) *For all $p > 1$, we have*

$$\mathit{Ent}(|\varphi|_\tau^p) \leq -\frac{ap^2}{8(p-1)}\int_M |\varphi|_\tau^{p-2}\left.\frac{d}{dt}\right|_{t=0}|P_t\varphi|_\tau^2 + b||\varphi||_p^p\,. \tag{6}$$

Proof. Consider the function $F(t) = e^{-m(t)}||P_t\varphi||_{p(t)}$ where $m(0) = 0$ and $p(0) = p$. We have $F(0) = ||\varphi||_p$. A straightforward computation shows that

$$\begin{aligned} \frac{d}{dt}\log F(t) &= -m'(t) + \frac{p'(t)}{p(t)^2}\frac{1}{||P_t\varphi||_{p(t)}^{p(t)}}\ \mathrm{Ent}\left(|P_t\varphi|_\tau^{p(t)}\right) \\ &\quad + \frac{1}{2||P_t\varphi||_{p(t)}^{p(t)}}\int_M |P_t\varphi|_\tau^{p(t)-2}\frac{d}{dt}|P_t\varphi|_\tau^2\ . \end{aligned} \tag{7}$$

Multiplying both sides by $||P_t\varphi||_{p(t)}^{p(t)}$, we obtain

$$\begin{aligned} & ||P_t\varphi||_{p(t)}^{p(t)}\left(\frac{d}{dt}\log F(t)\right) \\ = & \frac{p'(t)}{p^2(t)}\left[\mathrm{Ent}\left(|P_t\varphi|_\tau^{p(t)}\right) + \frac{p(t)^2}{2p'(t)}\int_M |P_t\varphi|_\tau^{p(t)-2}\frac{d|P_t\varphi|_\tau^2}{dt} - \frac{m'(t)p(t)^2}{p'(t)}||P_t\varphi||_{p(t)}^{p(t)}\right] \end{aligned} \tag{8}$$

By definition, $p(t)$ and $m(t)$ are chosen to solve the following differential equations:

$$\frac{p(t)^2}{p'(t)} = \frac{ap^2}{4(p-1)}\ , \quad p(0) = p$$

and

$$\frac{m'(t)p(t)^2}{p'(t)} = b\ , \qquad m(0) = 0\ .$$

Assume (i). Since $F(0) = ||\varphi||_p$, the hypercontractivity of (P_t) implies $F'(0) \le 0$ which gives, via (8),

$$\mathrm{Ent}\,(|\varphi|_\tau^p) + \frac{p^2}{2p'(0)}\int_M |\varphi|_\tau^{p-2}\ \frac{d}{dt}\bigg|_{t=0}|P_t\varphi|_\tau^2 - \frac{m'(0)p^2}{p'(0)}||\varphi||_p^p \le 0\ .$$

Together with (5), this shows (6) holds.

Conversely, assume (ii). Applying (6) to $P_t\varphi$ and using (8), we see that (6) implies $\frac{d}{dt}\log F(t) \le 0$, so $F'(t) \le 0$. Therefore $F(t) \le F(0)$ which in turn yields the hypercontractivity of $(P_t)_{t\ge 0}$. □

Theorem 3.2. *Let $(P_t)_{t\ge 0}$ be the diffusion semigroup on a non-associative vector bundle $E \longrightarrow M$ with the generator L associated with the Dirichlet form*

$$\mathcal{E}(\varphi,\psi) = \int_M \langle D\varphi, D\psi\rangle_\tau d\mu \qquad\qquad (\varphi,\psi \in C_c^\infty(E)).$$

Then the hypercontractivity of $(P_t)_{t\ge 0}$ is equivalent to the following log-Sobolev inequality

$$\mathit{Ent}\left(|\varphi|_\tau^2\right) \le a\int_M |D\varphi|_\tau^2 + b||\varphi||_2^2\ . \tag{9}$$

Proof. As

$$\left.\frac{d}{dt}\right|_{t=0} |P_t\varphi|_\tau^2(x) = \left.\frac{d}{dt}\right|_{t=0} \langle P_t\varphi(x), P_t\varphi(x)\rangle_x = 2\langle L\varphi(x), \varphi(x)\rangle_x \,,$$

we have

$$-\int_M |\varphi|_\tau^{p-2} \left.\frac{d}{dt}\right|_{t=0} |P_t\varphi|_\tau^2 d\mu = 2\int_M \langle D\varphi, D(|\varphi|_\tau^{p-2}\varphi)\rangle_\tau d\mu \,. \tag{10}$$

For any $\beta > 0$, we have by the product rule,

$$D(|\varphi|_\tau^\beta \varphi) = \left(d|\varphi|_\tau^\beta\right)\varphi + |\varphi|_\tau^\beta D\varphi$$

so that

$$\begin{aligned} |D(|\varphi|_\tau^\beta\varphi)|_\tau^2 &= \langle \left(d|\varphi|_\tau^\beta\right)\varphi + |\varphi|_\tau^\beta D\varphi, \left(d|\varphi|_\tau^\beta\right)\varphi + |\varphi|_\tau^\beta D\varphi\rangle_\tau \\ &= |d|\varphi|_\tau^\beta|^2|\varphi|_\tau^2 + |\varphi|_\tau^{2\beta}|D\varphi|_\tau^2 + \langle D\varphi, \left(d|\varphi|_\tau^{2\beta}\right)\varphi\rangle_\tau \,. \end{aligned}$$

While

$$\langle D\varphi, D(|\varphi|_\tau^{p-2}\varphi)\rangle_\tau = \langle D\varphi, \left(d|\varphi|_\tau^{p-2}\right)\varphi\rangle_\tau + |\varphi|_\tau^{p-2}|D\varphi|_\tau^2 \,,$$

and therefore, with $\beta = (p-2)/2$, we have

$$\begin{aligned} \langle D\varphi, D(|\varphi|_\tau^{p-2}\varphi)\rangle_\tau &= |D(|\varphi|_\tau^\beta\varphi)|_\tau^2 - |d|\varphi|_\tau^\beta|^2|\varphi|_\tau^2 \\ &= |D(|\varphi|_\tau^{\frac{p}{2}-1}\varphi)|_\tau^2 - \frac{(p-2)^2}{p^2}|d|\varphi|_\tau^{\frac{p}{2}}|^2 \,. \end{aligned}$$

Hence, by Proposition 3.1, the hypercontractivity of $(P_t)_{t\geq 0}$ is equivalent to the following entropy inequality:

$$\mathrm{Ent}\left(|\varphi|_\tau^2\right) \leq \frac{ap^2}{4(p-1)}\int_M \left(|D\varphi|_\tau^2 - \frac{(p-2)^2}{p^2}|d|\varphi|_\tau|^2\right) + b||\varphi||_2^2$$

for all $p > 1$ and $\varphi \in C_c^\infty(E)$. Our claim will follow if we can show for any given φ, the right-hand side is minimized when $p = 2$. To this end we consider

$$\begin{aligned} U(p) &= \frac{p^2}{p-1}\int_M \left(|D\varphi|_\tau^2 - \frac{(p-2)^2}{p^2}|d|\varphi|_\tau|^2\right) \\ &= \frac{p^2}{p-1}\int_M |D\varphi|_\tau^2 - \frac{(p-2)^2}{p-1}\int_M |d|\varphi|_\tau|^2 \,, \end{aligned}$$

where it is clear that

$$U'(p) = \frac{p(p-2)}{(p-1)^2}\left(\int_M |D\varphi|_\tau^2 - \int_M |d|\varphi|_\tau|^2\right) \,.$$

Therefore $U(p)$ takes its minimum value at $p = 2$, or at $\int_M |D\varphi|_\tau^2 = \int_M |d|\varphi|_\tau|^2$, where in the latter case, $U(p)$ is constant. In both cases, the minimum value of $U(p)$ is $4\int_M |D\varphi|_\tau^2$ which proves our claim. $\square$

In the scalar case, the reduction in (6) from any value p to $p = 2$ (logarithmic Sobolev inequality) is achieved by the simple fact that $\int_M |D\varphi|^2 = \int_M |d|\varphi|_\tau|^2$. The latter is no longer true for sections of vector bundles. Our only contribution is the observation that, nevertheless, such a reduction can still be obtained via a max-min argument instead.

Corollary 3.3. *Let μ be a σ-finite measure on a Riemannian manifold M. If a logarithmic Sobolev inequality holds for functions:*

$$\mathit{Ent}\left(f^2\right) \leq a \int_M |\nabla f|^2 + b||f||_2^2 \qquad \textit{for all } f \in C_c^\infty(M) \ , \tag{11}$$

then the semigroup $(P_t)_{t\geq 0}$ on a non-associative vector bundle $E \longrightarrow M$ as in Theorem 3.2 possesses hypercontractivity.

Proof. Since D is compatible with the Riemannian structure on E, we have

$$d|\varphi|_\tau^2 = 2\langle D\varphi, \varphi\rangle_\tau$$

so that $|d|\varphi|_\tau^2| \leq 2|D\varphi|_\tau|\varphi|_\tau$ which implies that $|d|\varphi|_\tau| \leq |D\varphi|_\tau$. However $|d|\varphi|_\tau| = |\nabla|\varphi|_\tau|$, therefore by applying (11) to $|\varphi|_\tau$, we obtain

$$\begin{aligned} \mathrm{Ent}\left(|\varphi|_\tau^2\right) &\leq a \int_M |d|\varphi|_\tau|^2 + b||\varphi||_2^2 \\ &\leq a \int_M |D\varphi|_\tau^2 + b||\varphi||_2^2 \ . \end{aligned}$$

The conclusion now follows from the above theorem immediately. ☐

4. Harmonic functions

To conclude, we discuss harmonic functions with respect to a Dirichlet Laplacian in the scalar case on Lie groups. We show, not surprisingly, the absence of a nontrivial L^p harmonic function for $1 \leq p < \infty$.

Let G be a Lie group with a right invariant Haar measure λ, and let $L^p(G)$ be the Lebesgue spaces with respect to the Haar measure λ. Given a Dirichlet form $\mathcal{E}$ on $L^2(G)$, we consider the associated positive self-adjoint operator L in $L^2(G)$, the *Dirichlet Laplacian* of $\mathcal{E}$, satisfying

$$\mathcal{E}(\varphi, \psi) = \langle L\varphi, \psi\rangle \qquad (\varphi, \psi \in \mathcal{D}(L)).$$

We assume that L commutes with right translations of G:

$$Lr_a = r_a L \qquad (a \in G)$$

where $r_a : x \mapsto xa \in G$ is a right translation by a. In this case, the Markov semigroup

$$P_t : L^2(G) \longrightarrow L^2(G) \qquad (t \geq 0)$$

generated by L, commutes with right translations of G and is a convolution semigroup:

$$P_t(f) = f * \sigma_t \qquad (f \in L^2(G))$$

where $(\sigma_t)_{t\geq 0}$ is a family of probability measures on G and the support of each σ_t generates the group G. A complex function $f \in \mathcal{D}(L)$ is called *L-harmonic* if $Lf = 0$.

Theorem 4.1. *Let $1 \leq p < \infty$ and let $f \in L^p(G)$. If f is L-harmonic, then f is constant.*

Proof. Let $(\sigma_t)_{t\geq 0}$ be the induced convolution semigroup of probability measures on G. Then we have $f * \sigma_t = f$ and since the support of σ_t generates G, by [5, Theorem 3.12], f is constant. □

We note that, given a complete Riemannian manifold M and the Laplace operator Δ of its Riemannian metric, it is a well-known result of Yau [28] that all L^p Δ-harmonic functions on M are constant, for $1 < p < \infty$, and if in addition, M has non-negative Ricci curvature, then all L^1 harmonic functions on M are also constant [29, 22] (see also [16]). Yau's result applies to Lie groups for $1 < p < \infty$, however, it has been shown by Milnor [23] that for almost all left-invariant Riemannian metrics on a Lie group, the Ricci curvature changes sign and in this case, the above L^1 result does not apply directly although Theorem 4.1 shows that it is still true for all Lie groups.

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School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, UK

Mathematical Institute, University of Oxford, Oxford OX1 3LB, UK