Hopf duals, algebraic groups, and Jordan pairs

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Abstract

The *Hom* functor is used to construct various algebras and coalgebras, including the continuous dual of a Hopf algebra with a residually finitely generated projective linear topology. The dual is used to construct a k-group scheme. The results are applied to the study of algebraic groups associated with Jordan pairs.

1 Introduction

In [2], we developed a strong connection between certain Hopf algebras and Jordan pairs. The Hopf algebras involved are cocommutative, so by taking a suitable dual one obtains commutative Hopf algebras which can serve as coordinate algebras for algebraic groups. In developing the results, we need to use different ways of forming the dual of a Hopf algebra for different purposes. We present here a unified approach based on a topological formulation. This development gives quite general results about Hopf algebras and algebraic groups with the applications to Jordan pairs delayed until the final section.

The obstacle to endowing the dual $\mathcal{H}^*$ of a Hopf algebra $\mathcal{H}$ with a Hopf algebra structure is that, in general, $(\mathcal{H}\otimes\mathcal{H})^* \ncong \mathcal{H}^*\otimes\mathcal{H}^*$. The multiplication map $\mathcal{H}\otimes\mathcal{H}\to\mathcal{H}$ induces a map $\mathcal{H}^*\to(\mathcal{H}\otimes\mathcal{H})^*$, but we need a coproduct $\mathcal{H}^*\to\mathcal{H}^*\otimes\mathcal{H}^*$. The usual remedy for this problem is to use only part of $\mathcal{H}^*$; e.g., all $f\in\mathcal{H}^*$ which vanish on a subspace of finite codimension. There are several other ways of choosing which part of the dual to use, but they can be unified by specifying a topology $\mathcal{T}$ on $\mathcal{H}$ and taking the continuous dual $\mathcal{H}^*_{\mathcal{T}}$; i.e., all continuous linear functions on $\mathcal{H}$.

A more general formulation of the obstacle described above is

$$Hom(\mathcal{V}_1\otimes\mathcal{V}_2,\mathcal{W}_1\otimes\mathcal{W}_2)\ncong Hom(\mathcal{V}_1,\mathcal{W}_1)\otimes Hom(\mathcal{V}_2,\mathcal{W}_2).$$

In §2, we observe that we have an isomorphism if each $\mathcal{V}_i$ is a finitely generated projective module. More generally, in Lemma 1, we show that if we equip $\mathcal{V}_i$ with a linear topology which is residually finitely generated projective, and consider only continuous homomorphisms, then we still have an isomorphism.

In Theorem 3, we show, among other things, that if $\mathcal{H}_{\mathcal{T}}$ is a residually finitely generated projective Hopf algebra and $\mathcal{H}'$ is a Hopf algebra, then $Hom(\mathcal{H}_{\mathcal{T}}, \mathcal{H}')$ is a Hopf algebra. In particular, we can take $\mathcal{H}' = k$ to see that $\mathcal{H}_{\mathcal{T}}^*$ is a Hopf algebra. The remainder of §2 is devoted to special examples giving the standard methods of taking a dual of Hopf algebra and to various consequences of the construction including a double dual.

In §3, we construct group schemes with $k[G] = \mathcal{H}_{\mathcal{T}}^*$. We also show how the construction of the distribution algebra (hyperalgebra) of a group scheme fits into our framework. In §4, we relate vector groups to binomial divided power maps. We also look at modules for group schemes. In particular, we give the equivalence of k_m-modules and $\mathbb{Z}$-gradings and examine the dual of a G-module. In §5, we make the applications to Jordan pairs. On one hand, in Theorem 19, we construct an algebraic group G from a finite dimensional Jordan pair $\mathcal{V}$ over a field k. We also construct a subgroup H acting as automorphisms of $\mathcal{V}$. In Theorem 20, we use our methods to prove a result of Loos [5] constructing a Jordan pair from a group with an elementary action.

2 A Hopf algebra construction

We first set some notation and recall a few results. Let Mod_k be the category of a modules over a commutative associative ring k. Recall the functor

$$Hom : (Mod_k)^{op} \times Mod_k \to Mod_k$$

has $Hom(\alpha, \beta)(\phi) = \beta \circ \phi \circ \alpha$ for $\alpha \in Hom(\mathcal{V}', \mathcal{V})$, $\beta \in Hom(\mathcal{W}, \mathcal{W}')$, and $\phi \in Hom(\mathcal{V}, \mathcal{W})$. We view $\mathcal{V}$ in Mod_k as either a right or left module and note that the multiplication maps

$$\begin{aligned} \rho &: \mathcal{V} \otimes k \to \mathcal{V}, \\ \lambda &: k \otimes \mathcal{V} \to \mathcal{V} \end{aligned}$$

are isomorphisms. If $\mathcal{V} = k$, then $\rho = \lambda = \mu_k$, the multiplication map for k. The evaluation map $f \to f(1)$ is also an isomorphism

$$\xi_{\mathcal{V}} : Hom(k, \mathcal{V}) \to \mathcal{V}.$$

Let $\mathcal{V}^* = Hom(\mathcal{V}, k)$ denote the linear dual of $\mathcal{V}$. If $f \in \mathcal{V}^*$ and $w \in \mathcal{W}$, let $wf \in Hom(\mathcal{V}, \mathcal{W})$ with $(wf)(v) = wf(v)$ for $v \in \mathcal{V}$. We also let

$$(w_1 \otimes w_2)(f_1 \otimes f_2) = w_1 f_1 \otimes w_2 f_2.$$

Recall that $\mathcal{V}$ is a finitely generated projective module if and only if there are *dual generating sets*; i.e., generators $\{v_i : 1 \le i \le n\}$ of $\mathcal{V}$ and $\{f_i : 1 \le i \le n\}$ of $\mathcal{V}^*$ with

$$\sum_{i=1}^{n} v_i f_i = Id_{\mathcal{V}}.$$

If $\phi_i \in Hom(\mathcal{V}_i, \mathcal{W}_i)$, we shall need to distinguish between

$$\phi_1 \otimes \phi_2 \in Hom(\mathcal{V}_1, \mathcal{W}_1) \otimes Hom(\mathcal{V}_2, \mathcal{W}_2)$$

and

$$\phi_1 \otimes' \phi_2 \in Hom(\mathcal{V}_1 \otimes \mathcal{V}_2, \mathcal{W}_1 \otimes \mathcal{W}_2)$$

given by

$$(\phi_1 \otimes' \phi_2)(v_1 \otimes v_2) = \phi_1(v_1) \otimes \phi_2(v_2).$$

Clearly,

$$\chi : \phi_1 \otimes \phi_2 \to \phi_1 \otimes' \phi_2$$

defines a homomorphism. Moreover, if $\alpha_i \in Hom(\mathcal{V}'_i, \mathcal{V}_i)$ and $\beta_i \in Hom(\mathcal{W}_i, \mathcal{W}'_i)$, then

$$\begin{aligned} Hom(\alpha_1 \otimes' \alpha_2, \beta_1 \otimes' \beta_2)(\phi_1 \otimes' \phi_2) &= (\beta_1 \otimes' \beta_2) \circ (\phi_1 \otimes' \phi_2) \circ (\alpha_1 \otimes' \alpha_2) \\ &= Hom(\alpha_1, \beta_1)(\phi_1) \otimes' Hom(\alpha_2, \beta_2)(\phi_2), \end{aligned}$$

i.e.

$$Hom(\alpha_1 \otimes' \alpha_2, \beta_1 \otimes' \beta_2) \circ \chi = \chi \circ Hom(\alpha_1, \beta_1) \otimes' Hom(\alpha_2, \beta_2). \qquad (1)$$

This shows that χ gives a morphism (natural transformation) of functors

$$\chi : Hom(-,-) \otimes Hom(-,-) \to Hom(- \otimes -, - \otimes -).$$

If $\mathcal{V}_1$ and $\mathcal{V}_2$ are finitely generated projective modules, then χ is an isomorphism. Indeed, using dual generating sets v_i, f_i for $\mathcal{V}_1$ and u_j, g_j for $\mathcal{V}_2$, we have

$$\chi^{-1}(\phi) = \sum_{i,j} \phi(v_i \otimes u_j)(f_i \otimes g_j).$$

In general, χ is not an isomorphism, but we can get a more general isomorphism result by expanding the category.

Let $\mathcal{V}_\mathcal{T}$ be a linear topological k-module; i.e., $\mathcal{V}$ is a topological k-module whose topology $\mathcal{T}$ has a *linear base* $\mathcal{B}$ of neighborhoods of 0 consisting of submodules of $\mathcal{V}$. We let $Hom(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{S})$ consist of the continuous linear maps from $\mathcal{V}$ to $\mathcal{W}$, giving a category $LTopMod_k$. We can view Hom as a functor

$$Hom : (LTopMod_k)^{op} \times LTopMod_k \to Mod_k.$$

We view $\mathcal{V}$ as a topological module with the discrete topology (with linear base $\{0\}$), so Mod_k is a full subcategory of $LTopMod_k$. If $\mathcal{B}$ is a linear base for $\mathcal{V}_\mathcal{T}$, we note that

$$Hom(\mathcal{V}_\mathcal{T}, \mathcal{W}) = \{\phi \in Hom(\mathcal{V}, \mathcal{W}) : \phi(I) = 0 \text{ for some } I \in \mathcal{B}\}.$$

If I is a submodule of $\mathcal{V}$, let $\pi_I : \mathcal{V} \to \mathcal{V}/I$ be the canonical homomorphism. If $I \subset \mathcal{V}$ and $J \subset \mathcal{W}$ are submodules, set

$$I * J = \ker(\pi_I \otimes' \pi_J) \subset \mathcal{V} \otimes \mathcal{W}.$$

We note that $I * J$ is the sum K of the images of $I \otimes \mathcal{W}$ and $\mathcal{V} \otimes J$ in $\mathcal{V} \otimes \mathcal{W}$, since we can map $\mathcal{V}/I \otimes \mathcal{W}/J \to (\mathcal{V} \otimes \mathcal{W})/K$ with $(v+I) \otimes (w+J) \to v \otimes w + K$.

Given $\mathcal{V}_{\mathcal{T}}$ with linear base $\mathcal{B}$ and $\mathcal{W}_{\mathcal{S}}$ with linear base $\mathcal{C}$, it is easy to see that

$$\mathcal{B} * \mathcal{C} = \{I * J : I \in \mathcal{B}, J \in \mathcal{C}\}$$

is a linear base for a topology $\mathcal{U}$ on $\mathcal{V} \otimes \mathcal{W}$. Moreover, $\mathcal{U}$ does not depend on the choice of linear bases for $\mathcal{T}$ and $\mathcal{S}$, so we may write $\mathcal{V}_{\mathcal{T}} \otimes \mathcal{W}_{\mathcal{S}}$ for $(\mathcal{V} \otimes \mathcal{W})_{\mathcal{U}}$. If $\phi_i \in Hom(\mathcal{V}_{i\mathcal{T}_i}, \mathcal{W}_{i\mathcal{S}_i})$, then $\phi_1 \otimes' \phi_2$ is continuous; i.e.,

$$\phi_1 \otimes' \phi_2 \in Hom(\mathcal{V}_{1\mathcal{T}_1} \otimes \mathcal{V}_{2\mathcal{T}_2}, \mathcal{W}_{1\mathcal{S}_1} \otimes \mathcal{W}_{2\mathcal{S}_2}).$$

If $\mathcal{W}$ is a submodule of $\mathcal{V}_{\mathcal{T}}$ with linear base $\mathcal{B}$, then

$$\{\pi_{\mathcal{W}}(I) : I \in \mathcal{B}\}$$

is a linear base for a topology $\mathcal{U}$ on $\mathcal{V}/\mathcal{W}$. Again, $\mathcal{U}$ does not depend on the choice of linear base for $\mathcal{T}$ and we write $\mathcal{V}_{\mathcal{T}}/\mathcal{W}$ for $(\mathcal{V}/\mathcal{W})_{\mathcal{U}}$.

If $\mathcal{V}_{\mathcal{T}}$ has a linear base $\mathcal{B}$ such that $\mathcal{V}/I$ is a finitely generated projective module for each $I \in \mathcal{B}$, we say that $\mathcal{V}_{\mathcal{T}}$ is a *residually finitely generated projective module* (or simply an *r.f.g.p. module*). We can now generalize the previous isomorphism result.

Lemma 1 *If $\mathcal{V}_{i\mathcal{T}_i}$ are residually finitely generated projective modules, then so is $\mathcal{V}_{1\mathcal{T}_1} \otimes \mathcal{V}_{2\mathcal{T}_2}$ and*

$$\begin{aligned} \chi \quad : \quad & Hom(\mathcal{V}_{1\mathcal{T}_1}, \mathcal{W}_1) \otimes Hom(\mathcal{V}_{2\mathcal{T}_2}, \mathcal{W}_2) \to \\ & Hom(\mathcal{V}_{1\mathcal{T}_1} \otimes \mathcal{V}_{2\mathcal{T}_2}, \mathcal{W}_1 \otimes \mathcal{W}_2). \end{aligned}$$

is an isomorphism.

Proof. Let $\mathcal{B}_i$ be linear base for $\mathcal{V}_{i\mathcal{T}_i}$ such that $\mathcal{V}_i/I$ is a finitely generated projective module for each $I \in \mathcal{B}_i$. If $I_i \in \mathcal{B}_i$, then

$$(\mathcal{V}_1 \otimes \mathcal{V}_2)/(I_1 * I_2) \cong \mathcal{V}_1/I_1 \otimes \mathcal{V}_2/I_2$$

is a finitely generated projective module. Thus, $\mathcal{V}_{1\mathcal{T}_1} \otimes \mathcal{V}_{2\mathcal{T}_2}$ is residually finitely generated projective. We also have by (1) that

$$Hom(\pi_{I_1} \otimes' \pi_{I_2}, Id) \circ \chi = \chi \circ (Hom(\pi_{I_1}, Id) \otimes' Hom(\pi_{I_2}, Id) \qquad (2)$$

on $Hom(\mathcal{V}_1/I_1, \mathcal{W}_1) \otimes Hom(\mathcal{V}_2/I_2, \mathcal{W}_2)$. If

$$\phi = \sum_j f_j \otimes g_j \in Hom(\mathcal{V}_{1\mathcal{T}_1}, \mathcal{W}_1) \otimes Hom(\mathcal{V}_{2\mathcal{T}_2}, \mathcal{W}_2),$$

then there are $I_i \in \mathcal{B}_i$ with $f_j(I_1) = g_j(I_2) = 0$ for all j. Writing $f_j = f_{jI_1} \circ \pi_{I_1}$ and $g_j = g_{jI_2} \circ \pi_{I_2}$, with $f_{jI_1} \in Hom(\mathcal{V}_1/I_1, \mathcal{W}_1)$ and $g_{jI_2} \in Hom(\mathcal{V}_2/I_2, \mathcal{W}_2)$, we have

$$\phi = (Hom(\pi_{I_1}, Id) \otimes' Hom(\pi_{I_2}, Id)(\bar{\phi})$$

where $\bar{\phi} = \sum_j f_{jI_1} \otimes g_{jI_2}$. The isomorphism result for finitely generated projective modules shows the left side of (2) is injective. Thus, $\chi(\phi) = 0$ implies $\bar{\phi} = 0$ and $\phi = 0$, so χ is injective on $Hom(\mathcal{V}_{1\mathcal{T}_1}, \mathcal{W}_1) \otimes Hom(\mathcal{V}_{2\mathcal{T}_2}, \mathcal{W}_2)$.

If $\theta \in Hom(\mathcal{V}_{1\mathcal{T}_1} \otimes \mathcal{V}_{2\mathcal{T}_2}, \mathcal{W}_1 \otimes \mathcal{W}_2)$, then $\theta(I_1 * I_2) = 0$ for some $I_i \in \mathcal{B}_i$ and we can write

$$\theta = \bar{\theta} \circ (\pi_{I_1} \otimes' \pi_{I_2}) = Hom(\pi_{I_1} \otimes' \pi_{I_2}, Id)(\bar{\theta})$$

with $\bar{\theta} \in Hom(\mathcal{V}_1/I_1 \otimes \mathcal{V}_2/I_2, \mathcal{W}_1 \otimes \mathcal{W}_2)$. Again, the result for finitely generated projective modules shows that θ is in the image of the left side of (2) for some $I_i \in \mathcal{B}_i$. Thus, χ is surjective to $Hom(\mathcal{V}_{1\mathcal{T}_1} \otimes \mathcal{V}_{2\mathcal{T}_2}, \mathcal{W}_1 \otimes \mathcal{W}_2)$. ∎

We need to consider a formal expression $P(\alpha_1, \ldots, \alpha_n)$ which can be evaluated for various choices of the homomorphisms $\alpha_1, \ldots, \alpha_n$. To this end, we consider variables $\mathcal{M}_i$ for modules and variables α_j for homomorphisms with each α_j mapping a m_j-fold tensor product of the $\mathcal{M}_i$ to an l_j-fold tensor product of the $\mathcal{M}_i$. We can formally build expressions $\alpha_{i_1} \otimes' \ldots \otimes' \alpha_{i_r}$ and formally compose them provided the appropriate tensor products match. We say that a resulting formal expression $P(\alpha_1, \ldots, \alpha_n)$ from a m-fold tensor product to an l-fold tensor product has *type* (m, l). Let β_j be the homomorphism variable obtained by reversing the domain and codomain of α_j and let $P^{op}(\beta_1, \ldots, \beta_n)$ denote the expression formed by replacing α_j by β_j and composing in the reverse order. Clearly, P^{op} has type (l, m). Let

$$\chi_n : \alpha_1 \otimes \ldots \otimes \alpha_n \rightarrow \alpha_1 \otimes' \ldots \otimes' \alpha_n,$$

so $\chi_1 = Id$ and $\chi_2 = \chi$.

Lemma 2 *If P is a formal expression of type (m, l), if θ_j, ϕ_j, and ω_j are homomorphisms such that $P(\theta_1, \ldots, \theta_n)$ and $P^{op}(\phi_1, \ldots, \phi_n)$ make sense, and if*

$$Hom(\phi_j, \theta_j) \circ \chi_{m_j} = \chi_{l_j} \circ \omega_j,$$

then $P(\omega_1, \ldots, \omega_n)$ makes sense and

$$Hom(P^{op}(\phi_1, \ldots, \phi_n), P(\theta_1, \ldots, \theta_n)) \circ \chi_m = \chi_l \circ P(\omega_1, \ldots, \omega_n).$$

Proof. If $\theta_j \in Hom(\bigotimes_{k=1}^{m_j} \mathcal{V}_{i_k}, \bigotimes_{k=1}^{l_j} \mathcal{V}_{j_k})$ and $\phi_j \in Hom(\bigotimes_{k=1}^{l_j} \mathcal{W}_{j_k}, \bigotimes_{k=1}^{m_j} \mathcal{W}_{i_k})$, then

$$\omega_j \in Hom(\bigotimes_{k=1}^{m_j} Hom(\mathcal{W}_{i_k}, \mathcal{V}_{i_k}), \bigotimes_{k=1}^{l_j} Hom(\mathcal{W}_{j_k}, \mathcal{V}_{j_k}))$$

so $P(\omega_1, \ldots, \omega_n)$ makes sense. For the second claim, it suffices to consider a single factor of

$$Hom(P^{op}(\phi_1, \ldots, \phi_n), P(\theta_1, \ldots, \theta_n)).$$

To simplify notation, we write this factor as

$$Hom(\phi_1 \otimes' \ldots \otimes' \phi_n, \theta_1 \otimes' \ldots \otimes' \theta_n)$$

with $m = \sum m_i$ and $l = \sum l_i$. Since χ is a morphism, so is χ_n; i.e.,

$$\begin{aligned} & Hom(\phi_1 \otimes' \ldots \otimes' \phi_n, \theta_1 \otimes' \ldots \otimes' \theta_n) \circ \chi_n \\ = \; & \chi_n \circ (Hom(\phi_1, \theta_1) \otimes' \ldots \otimes' Hom(\phi_n, \theta_n)). \end{aligned} \tag{3}$$

Since

$$\begin{aligned} \chi_m &= \chi_n \circ (\chi_{m_1} \otimes' \ldots \otimes' \chi_{m_n}), \\ \chi_l &= \chi_n \circ (\chi_{l_1} \otimes' \ldots \otimes' \chi_{l_n}), \end{aligned}$$

we see that

$$\begin{aligned} & Hom(\phi_1 \otimes' \ldots \otimes' \phi_n, \theta_1 \otimes' \ldots \otimes' \theta_n) \circ \chi_m \\ = \; & \chi_n \circ (Hom(\phi_1, \theta_1) \circ \chi_{m_1} \otimes' \ldots \otimes' Hom(\phi_n, \theta_n) \circ \chi_{m_n}) \\ = \; & \chi_n \circ (\chi_{l_1} \circ \omega_1 \otimes' \ldots \otimes' \chi_{l_n} \circ \omega_n) \\ = \; & \chi_l \circ (\omega_1 \otimes' \ldots \otimes' \omega_n). \end{aligned}$$

■

If $\mathcal{A}$ is a unital (associative) algebra over k, the multiplication map defined by $\mu(a \otimes b) = ab$ and the unit map defined by $\eta(t) = t1$ satisfy the associative and unit conditions

$$\mu \circ (Id \otimes' \mu) = \mu \circ (\mu \otimes' Id), \tag{A}$$
$$\mu \circ (Id \otimes' \eta) = \rho, \tag{R}$$
$$\mu \circ (\eta \otimes' Id) = \lambda. \tag{L}$$

Similarly, a *coalgebra* $\mathcal{C}$ has linear maps $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ (coproduct) and $\varepsilon : \mathcal{C} \to k$ (counit) satisfying

$$(Id \otimes' \Delta) \circ \Delta = (\Delta \otimes' Id) \circ \Delta, \tag{A*}$$
$$(Id \otimes' \varepsilon) \circ \Delta = \rho^{-1}, \tag{R*}$$
$$(\varepsilon \otimes' Id) \circ \Delta = \lambda^{-1}. \tag{L*}$$

$\mathcal{A}$ is commutative if

$$\mu \circ \tau = \mu \tag{C}$$

and $\mathcal{C}$ is *cocommutative* if

$$\tau \circ \Delta = \Delta \qquad \text{(C*)}$$

where $\tau(x \otimes y) = y \otimes x$. A *Hopf algebra* $\mathcal{H}$ is both an algebra and a coalgebra such that Δ and ε are algebra homomorphisms; i.e.,

$$\begin{aligned}
\Delta \circ \mu &= (\mu \otimes' \mu) \circ (Id \otimes' \tau \otimes' Id) \circ (\Delta \otimes' \Delta), && \text{(PP*)} \\
\varepsilon \circ \mu &= \mu_k \circ (\varepsilon \otimes' \varepsilon), && \text{(PU*)} \\
\Delta \circ \eta &= (\eta \otimes' \eta) \circ \mu_k^{-1}, && \text{(UP*)} \\
\varepsilon \circ \eta &= Id && \text{(UU*)}
\end{aligned}$$

and $S : \mathcal{H} \to \mathcal{H}$ is an *antipode*; i.e.,

$$\mu \circ (S \otimes' Id) \circ \Delta = \mu \circ (Id \otimes' S) \circ \Delta = \eta \circ \varepsilon. \qquad \text{(S)}$$

An algebra, a coalgebra, or a Hopf algebra is *linear topological* if it has a linear topology such that the defining maps are continuous. Here we take k with discrete topology. An *r.f.g.p. algebra* is a linear topological algebra which is a residually finitely generated projective module, and similarly for a coalgebra or a Hopf algebra.

Theorem 3 *The Hom functor for* $LTopMod_k$ *induces a functor*

$$Hom : \mathfrak{C}^{op} \times \mathfrak{D} \to \mathfrak{D}$$

for each of the following pairs of categories:

(i) $\mathfrak{C}$ *= linear topological k-coalgebras,* $\mathfrak{D}$ *= k-algebras,*

(ii) $\mathfrak{C}$ *= r.f.g.p. k-algebras,* $\mathfrak{D}$ *= k-coalgebras,*

(iii) $\mathfrak{C}$ *= r.f.g.p. k-Hopf algebras,* $\mathfrak{D}$ *= k-Hopf algebras,*

(iv) the (co)commutative subcategories of $\mathfrak{C}$ *and* $\mathfrak{D}$ *in (i), (ii), or (iii).*

The maps on $\mathcal{Z} = Hom(\mathcal{X}_\mathcal{T}, \mathcal{Y})$ *are (where applicable)*

$$\begin{aligned}
\mu_\mathcal{Z} &= Hom(\Delta_\mathcal{X}, \mu_\mathcal{Y}) \circ \chi, \\
\eta_\mathcal{Z} &= Hom(\varepsilon_\mathcal{X}, \eta_\mathcal{Y}) \circ \xi_k^{-1}, \\
\Delta_\mathcal{Z} &= \chi^{-1} \circ Hom(\mu_\mathcal{X}, \Delta_\mathcal{Y}), \\
\varepsilon_\mathcal{Z} &= \xi_k \circ Hom(\eta_\mathcal{X}, \varepsilon_\mathcal{Y}), \\
S_\mathcal{Z} &= Hom(S_\mathcal{X}, S_\mathcal{Y}).
\end{aligned}$$

Proof. Each identity I has the form

$$P(\theta_1, \dots, \theta_n) = Q(\theta_1, \dots, \theta_n)$$

where P and Q are formal expressions of type (m, l) and each θ_i is one of the maps

$$\mu, \eta, \Delta, \varepsilon, S, Id, \tau, \rho, \rho^{-1}, \lambda, \lambda^{-1}, \mu_k, \mu_k^{-1}.$$

Each θ_i for $\mathcal{Y}$ is paired with a "dual" map ϕ_i for $\mathcal{X}_\mathcal{T}$ and we can take

$$\omega_i = \chi_{l_i}^{-1} \circ Hom(\phi_i, \theta_i) \circ \chi_{m_i}$$

for all pairs with χ_{l_i} invertible. This covers all pairs in (ii) and (iii) by Lemma 1 and any pair with $l_i = 1$. The only remaining pair is $(\tau_\mathcal{X}, \tau_\mathcal{Y})$ for which we take $\omega_i = \tau_\mathcal{Z}$. If $\mathcal{Y}$ satisfies I and $\mathcal{X}_\mathcal{T}$ satisfies the "dual" identity

$$P^{op}(\phi_1, \dots, \phi_n) = Q^{op}(\phi_1, \dots, \phi_n),$$

then Lemma 2 shows that

$$P(\omega_1, \dots, \omega_n) = Q(\omega_1, \dots, \omega_n)$$

since χ_l is invertible in each case. This gives I for $\mathcal{Z}$ after replacing $Hom(k, k)$ with k and making suitable adjustments to ω_i using ξ_k. Similarly, we can express the conditions that $\theta \in Hom(\mathcal{Y}, \mathcal{Y}')$ is a homomorphism in $\mathfrak{D}$ in the form

$$P(\theta, \theta_1, \dots, \theta_n) = Q(\theta, \theta_1, \dots, \theta_n)$$

for suitable P, Q. Moreover, $\phi \in Hom(\mathcal{X}'_{\mathcal{T}'}, \mathcal{X}_\mathcal{T})$ is a homomorphism in $\mathfrak{C}$ provided

$$P^{op}(\phi, \phi_1, \dots, \phi_n) = Q^{op}(\phi, \phi_1, \dots, \phi_n).$$

As before we get

$$P(Hom(\phi, \theta), \omega_1, \dots, \omega_n) = Q(Hom(\phi, \theta), \omega_1, \dots, \omega_n),$$

so $Hom(\phi, \theta)$ is a homomorphism in $\mathfrak{D}$. ■

Corollary 4 *For $\mathfrak{C}$ and $\mathfrak{D}$ as in Theorem 3, χ gives a morphism of the functor $Hom(-,-) \otimes Hom(-,-)$ to the functor $Hom(- \otimes -, - \otimes -)$ which is an equivalence in (ii) and (iii).*

Proof. The maps on $\tilde{\mathcal{Y}} = \mathcal{Y} \otimes \mathcal{Y}'$ are given by

$$\begin{aligned}
\tilde{\mu} &= (\mu \otimes' \mu') \circ (Id \otimes' \tau \otimes' Id), \\
\tilde{\eta} &= (\eta \otimes' \eta') \circ \mu_k^{-1}, \\
\tilde{\Delta} &= (Id \otimes' \tau \otimes' Id) \circ (\Delta \otimes' \Delta'), \\
\tilde{\varepsilon} &= \mu_k \circ (\varepsilon \otimes \varepsilon'), \\
\tilde{S} &= \bar{S} = S \otimes S'.
\end{aligned}$$

Let $\bar{\theta} = \tilde{\theta}$ except $\bar{\eta} = \eta \otimes' \eta'$ and $\bar{\varepsilon} = \varepsilon \otimes \varepsilon'$. Each $\bar{\theta}$ is of the form $\bar{\theta} = P(\theta, \theta', \dots)$ where P is a formal expression of type $(2m, 2l)$ if θ has type (m, l). Also, the corresponding map for $\tilde{\mathcal{X}}_{\tilde{\mathcal{T}}} = \mathcal{X}_\mathcal{T} \otimes \mathcal{X}'_{\mathcal{T}'}$ is $\tilde{\phi}$ with $\bar{\phi} = P^{op}(\phi, \phi', \dots)$. Choosing ω (and ω') as in the proof of Theorem 3, we can use Lemma 2 to get

$$Hom(\bar{\phi}, \bar{\theta}) \circ \chi_{2m} = \chi_{2l} \circ P(\omega, \omega', \dots).$$

Since χ_l is invertible in these cases, we can set $\hat{\omega} = \chi_l^{-1} \circ Hom(\bar{\phi}, \bar{\theta}) \circ \chi_m$ and $\bar{\omega} = P(\omega, \omega', \dots)$ to get

$$\hat{\omega} \circ \overbrace{(\chi \otimes' \dots \otimes' \chi)}^{m} = \overbrace{(\chi \otimes' \dots \otimes' \chi)}^{l} \circ \bar{\omega}.$$

After adjusting $Hom(k,k)$ and $k \otimes k$ to k, $\hat{\omega}$ gives the map for $Hom(\mathcal{X}_\mathcal{T} \otimes \mathcal{X}'_{\mathcal{T}'}, \mathcal{Y} \otimes \mathcal{Y}')$ and $\bar{\omega}$ gives the map for $Hom(\mathcal{X}_\mathcal{T}, \mathcal{Y}) \otimes Hom(\mathcal{X}'_{\mathcal{T}'}, \mathcal{Y}')$. Thus, χ is a homomorphism in $\mathfrak{D}$ and an isomorphism in the cases (ii) and (iii). ∎

We now consider some special cases of Theorem 3.

Example 1. If $\mathcal{A}$ is an algebra and $\mathcal{C}$ is a coalgebra, then we have the well-known result that $Hom(\mathcal{C}, \mathcal{A})$ is an algebra with product

$$\phi\theta = \mu \circ (\phi \otimes' \theta) \circ \Delta$$

and unit $\eta \circ \varepsilon$. More generally, if $\mathcal{C}_\mathcal{T}$ is a linear topological coalgebra, then $Hom(\mathcal{C}_\mathcal{T}, \mathcal{A})$ is a subalgebra of $Hom(\mathcal{C}, \mathcal{A})$.

Example 2. We can view k as an object in $\mathfrak{D}$ in Theorem 3 with $\Delta = \mu_k^{-1}$ and $\varepsilon = S = Id$. If $\mathcal{X}_\mathcal{T}$ is in $\mathfrak{C}$, then the *continuous dual*

$$\mathcal{X}_\mathcal{T}^* = Hom(\mathcal{X}_\mathcal{T}, k)$$

is in $\mathfrak{D}$.

Corollary 5 *For each of the pairs of categories in Theorem 3 (ii) or (iii), the maps*

$$\zeta_{\mathcal{X},\mathcal{Y}} : \mathcal{X}_\mathcal{T}^* \otimes \mathcal{Y} \to Hom(\mathcal{X}_\mathcal{T}, \mathcal{Y})$$

with $\zeta_{\mathcal{X},\mathcal{Y}}(f \otimes y)(x) = f(x)y$ *gives an equivalence of the functor* $(\mathcal{X}_\mathcal{T}, \mathcal{Y}) \to \mathcal{X}_\mathcal{T}^* \otimes \mathcal{Y}$ *with Hom.*

Proof. By Corollary 4,

$$\chi : Hom(\mathcal{X}_\mathcal{T}, k) \otimes Hom(k, \mathcal{Y}) \to Hom(\mathcal{X}_\mathcal{T} \otimes k, k \otimes \mathcal{Y})$$

and hence

$$\zeta_{\mathcal{X},\mathcal{Y}} = Hom(\rho_\mathcal{X}^{-1}, \lambda_\mathcal{Y}) \circ \chi \circ Id_{\mathcal{X}_\mathcal{T}^*} \otimes \xi_\mathcal{Y}^{-1}$$

are isomorphisms in $\mathfrak{D}$. ∎

Example 3. If I is an ideal in an algebra $\mathcal{A}$, then

$$\pi_I \circ \mu_\mathcal{A} = \mu_{\mathcal{A}/I} \circ (\pi_I \otimes' \pi_I)$$

shows that $\mu_\mathcal{A}(I * I) \subset I$. Thus, if $\mathcal{A}$ has a linear topology $\mathcal{T}$ with a linear base $\mathcal{B}$ consisting of ideals, then $\mu_\mathcal{A}$ is continuous and $\mathcal{A}_\mathcal{T}$ is a linear topological algebra. If $\mathcal{A} = \mathcal{H}$ is a Hopf algebra, we now determine additional conditions on $\mathcal{B}$ to make $\mathcal{H}_\mathcal{T}$ a linear topological Hopf algebra. If I, J are submodules, let

$$I \wedge J = \Delta^{-1}(I * J) = \ker((\pi_I \otimes' \pi_J) \circ \Delta).$$

The maps ε, Δ, and S for $\mathcal{H}_{\mathcal{T}}$ are continuous provided

$$\text{there is } K \in \mathcal{B} \text{ with } K \subset \ker(\varepsilon), \tag{4}$$
$$\text{for } I, J \in \mathcal{B}, \text{ there is } K \in \mathcal{B} \text{ with } K \subset I \wedge J, \tag{5}$$
$$\text{for } I \in \mathcal{B}, \text{ there is } J \in \mathcal{B} \text{ with } J \subset S^{-1}(I). \tag{6}$$

We say that a linear base $\mathcal{B}$ of ideals satisfying the above conditions and with each $\mathcal{H}/I$ a finitely generated projective module is a *Hopf dualizing base.* Clearly, in this case, $\mathcal{H}_{\mathcal{T}}$ is an r.f.g.p. Hopf algebra.

For later use, we record some basic properties of $\wedge$.

Lemma 6 *If I, J, K are submodules of a Hopf algebra $\mathcal{H}$, then*

(i) $(I \wedge J) \wedge K = I \wedge (J \wedge K)$,
(ii) $S^{-1}(I \wedge J) = S^{-1}(J) \wedge S^{-1}(I)$,
(iii) $I \wedge J \subset I \cap J$, *if* $\varepsilon(I) = \varepsilon(J) = 0$.

If I, J are algebra ideals, then $I \wedge J$ is an ideal.

Proof. Using (A*), we have (i). Using the Hopf algebra identity (see [1], Theorem 2.1.4)

$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$$

we see

$$\ker((\pi_I \otimes' \pi_J) \circ \Delta \circ S) = \ker(\tau \circ ((\pi_J \circ S) \otimes' (\pi_I \circ S)) \circ \Delta)$$

showing (ii). If $\varepsilon(I) = \varepsilon(J) = 0$, then by (R*)

$$\begin{aligned} I \wedge J &= (\rho \circ (Id \otimes' \varepsilon) \circ \Delta)(I \wedge J) \\ &\subset (\rho \circ (Id \otimes' \varepsilon))(I * J) \\ &\subset I\varepsilon(\mathcal{H}) + \mathcal{H}\varepsilon(J) = I \end{aligned}$$

and similarly, $I \wedge J \subset J$. If I, J are algebra ideals, then $I \wedge J$ is an ideal, since Δ is an algebra homomorphism. ■

Corollary 7 *If $\mathcal{B}$ is a family of algebra ideals of a Hopf algebra $\mathcal{H}$ such that for $I, J \in \mathcal{B}$,*

(i) there is $K \in \mathcal{B}$ with $K \subset I \wedge J$,
(ii) there is $K \in \mathcal{B}$ with $K \subset S^{-1}(I)$,
(iii) $\varepsilon(I) = 0$,
(iv) $\mathcal{H}/I$ is a finitely generated module,

then $\mathcal{B}$ is a Hopf dualizing base.

Proof. Clearly, (4) holds and $\mathcal{B}$ is a linear base since $I \wedge J \subset I \cap J$. ■

The conditions of Corollary 7 are similar to Kostant's definition of an admissible family of ideals in a Hopf algebra over $\mathbb{Z}$ ([4], p. 91).

Example 4. If k is a field, it is easy to verify that $\mathcal{B}$ consisting of all ideals of finite codimension is a Hopf dualizing base with topology $\mathcal{T}$. We call

$$\mathcal{H}^*_{\mathcal{T}} = \{\phi \in \mathcal{H}^* : \phi(I) = 0 \text{ for some ideal of finite codimension}\},$$

the *finite dual* of $\mathcal{H}$.

Example 5. Let

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

be a $\mathbb{Z}$-graded Hopf algebra with each $\mathcal{H}_n$ a finitely generated projective module over k. Let $I_m = \bigoplus_{n=m+1}^{\infty} \mathcal{H}_n$ so

$$\mathcal{H}/I_m \cong \bigoplus_{n=0}^{m} \mathcal{H}_n.$$

It is easy to see that

$$\begin{aligned} I_m &\subset \ker(\varepsilon), \\ I_{m+n} &\subset I_m \wedge I_n, \\ I_m &\subset S^{-1}(I_m). \end{aligned}$$

Thus, by Corollary 7, $\mathcal{B} = \{I_m : m > 0\}$ is a Hopf dualizing base. In this case, the continuous dual $\mathcal{H}^*_{\mathcal{T}}$ is also the *graded dual* $\mathcal{H}^g$. In general, the graded dual of $\mathcal{V} = \bigoplus_i \mathcal{V}_i$ is $\mathcal{V}^g = \bigoplus_i \mathcal{V}_i^*$, where we view $\mathcal{V}_i^* \subset \mathcal{V}^*$ with $\mathcal{V}_i^*(\mathcal{V}_j) = 0$ for $j \neq i$. A direct calculation shows that the identification $(\mathcal{H}_i^*)^* = \mathcal{H}_i$ gives $(\mathcal{H}^g)^g = \mathcal{H}$ as Hopf algebras.

Example 6. Let I be a Hopf ideal of the Hopf algebra $\mathcal{H}$. We have $\varepsilon(I) = 0$, $S(I) \subset I$, and $\Delta(I) \subset I * I$. Since S is an antihomomorphism and Δ is a homomorphism, we see that

$$\begin{aligned} S(I^n) &\subset I^n, \\ \Delta(I^{n+m-1}) &\subset (I * I)^{n+m-1} \subset I^n * I^m. \end{aligned}$$

Thus, if $\mathcal{H}/I^n$ is a finitely generated projective module for each $n > 0$, then $\mathcal{B} = \{I^n : n > 0\}$ is a Hopf dualizing base.

In view of Lemma 6, we can define

$$\wedge^n I = \overbrace{I \wedge \ldots \wedge I}^{n}.$$

Theorem 8 *Let $\mathcal{F}$ be a family of algebra ideals in a Hopf algebra $\mathcal{H}$ such that for all $J, K \in \mathcal{F}$, we have $J \wedge K \in \mathcal{F}$ and the image of $(\pi_J \otimes' \pi_K) \circ \Delta$ is a direct summand of $\mathcal{H}/J \otimes \mathcal{H}/K$. If $I \in \mathcal{F}$ is such that $\varepsilon(I) = 0$, $S(I) \subset I$, and $\mathcal{H}/I$ is a finitely generated projective module, then*

$$\mathcal{B}(I) = \{\wedge^n I : n \geq 1\}$$

is a Hopf dualizing base with topology $\mathcal{T}$. Moreover, $\mathcal{H}^*_{\mathcal{T}}$ is generated as an algebra by

$$\mathcal{Z}_{\mathcal{H}^*}(I) = \{f \in \mathcal{H}^* : f(I) = 0\}$$

and is therefore finitely generated as an algebra.

Proof. In general, let $\mathcal{Z}_{\mathcal{H}^*}(J) = \{f \in \mathcal{H}^* : f(J) = 0\}$. We shall first show if $J, K \in \mathcal{F}$ and $\mathcal{H}/J, \mathcal{H}/K$ are finitely generated projective modules, then $\mathcal{H}/(J \wedge K)$ is a finitely generated projective module and

$$\mathcal{Z}_{\mathcal{H}^*}(J)\mathcal{Z}_{\mathcal{H}^*}(K) = \mathcal{Z}_{\mathcal{H}^*}(J \wedge K). \tag{7}$$

Indeed, let $\mathcal{C}$ be the image of $\mathcal{H}$ under $(\pi_J \otimes' \pi_K) \circ \Delta$, so $\mathcal{H}/(J \wedge K) \cong \mathcal{C}$. Since $\mathcal{C}$ is a direct summand of the finitely generated projective module $\mathcal{H}/J \otimes \mathcal{H}/K$, it is also a finitely generated projective module.

For arbitrary $g = g_J \circ \pi_J \in \mathcal{Z}_{\mathcal{H}^*}(J)$ and $h = h_K \circ \pi_K \in \mathcal{Z}_{\mathcal{H}^*}(K)$, we see that

$$\begin{aligned} gh &= \mu_k \circ (g \otimes' h) \circ \Delta \\ &= \mu_k \circ (g_J \otimes' h_K) \circ (\pi_J \otimes' \pi_K) \circ \Delta \end{aligned}$$

vanishes on $J \wedge K$, so $\mathcal{Z}_{\mathcal{H}^*}(J)\mathcal{Z}_{\mathcal{H}^*}(K) \subset \mathcal{Z}_{\mathcal{H}^*}(J \wedge K)$. Conversely, we can write $f \in \mathcal{Z}_{\mathcal{H}^*}(J \wedge K)$ as $f = f' \circ ((\pi_J \otimes' \pi_K) \circ \Delta)$ with $f' \in \mathcal{C}^*$, and then f' as the restriction to $\mathcal{C}$ of $f'' \in (\mathcal{H}/J \otimes \mathcal{H}/K)^*$. Since $(\mathcal{H}/J \otimes \mathcal{H}/K)^* \cong (\mathcal{H}/J)^* \otimes' (\mathcal{H}/K)^*$, we can write

$$f'' = \mu_k \circ \sum g_{iJ} \otimes' h_{iK}$$

with $g_{iJ} \in (\mathcal{H}/J)^*$ and $h_{iK} \in (\mathcal{H}/K)^*$. For $g_i = g_{iJ} \circ \pi_J$ and $h_i = h_{iK} \circ \pi_K$, we have

$$\begin{aligned} f &= \mu_k \circ \sum (g_{iJ} \otimes' h_{iK}) \circ (\pi_J \otimes' \pi_K) \circ \Delta \\ &= \sum g_i h_i \in \mathcal{Z}_{\mathcal{H}^*}(J)\mathcal{Z}_{\mathcal{H}^*}(K), \end{aligned}$$

showing (7).

We have $\mathcal{H}/(\wedge^n I)$ is a finitely generated projective module,

$$\begin{aligned} \varepsilon(\wedge^n I) &\subset \varepsilon(I) = 0, \\ \wedge^n I &\subset \wedge^n S^{-1}(I) = S^{-1}(\wedge^n I), \end{aligned}$$

by Lemma 6(ii), so $\mathcal{B}(I)$ is a Hopf dualizing base by Corollary 7. Moreover, (7) shows that $\mathcal{Z}_{\mathcal{H}^*}(\wedge^n I) = (\mathcal{Z}_{\mathcal{H}^*}(I))^n$, so $\mathcal{H}^*_{\mathcal{T}}$ is generated as an algebra by $\mathcal{Z}_{\mathcal{H}^*}(I) \cong (\mathcal{H}/I)^*$, and hence by a finite number of elements of $\mathcal{Z}_{\mathcal{H}^*}(I)$. ■

For later use, we now describe a double dual if k is a field.

Lemma 9 *If $\mathcal{H}_{\mathcal{T}}$ is an r.f.g.p. Hopf algebra over a field k, then the closure $\overline{\{0\}}$ of $\{0\}$ in $\mathcal{H}_{\mathcal{T}}$ is a Hopf ideal. If $\mathcal{S}$ is the topology on $\mathcal{H}^*_{\mathcal{T}}$ with linear base consisting of all*

$$\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}) = \{f \in \mathcal{H}^*_{\mathcal{T}} : f(\mathcal{W}) = 0\}$$

where $\mathcal{W}$ is a finite dimensional subspace of $\mathcal{H}$, then $(\mathcal{H}^*_{\mathcal{T}})_{\mathcal{S}}$ is an r.f.g.p. Hopf algebra, and $(\mathcal{H}^*_{\mathcal{T}})^*_{\mathcal{S}} \cong \mathcal{H}/\overline{\{0\}}$ as Hopf algebras.

Proof. Let $\mathcal{B}$ be a linear base for $\mathcal{H}_{\mathcal{T}}$ with $\mathcal{H}/I$ finite dimensional for each $I \in \mathcal{B}$. It is easy to see that

$$\overline{\{0\}} = \bigcap_{I \in \mathcal{B}} I.$$

We also note that $\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_1 + \mathcal{W}_2) = \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_1) \cap \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_2)$, so the $\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W})$ form a linear base.

If $\mathcal{W}$ is a finite dimensional subspace, we can write $\mathcal{W} = \mathcal{W}' \oplus (\mathcal{W} \cap \overline{\{0\}})$ and get $\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}) = \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}')$, since $f(\overline{\{0\}}) = 0$ for $f \in \mathcal{H}^*_{\mathcal{T}}$. Thus, we may assume that $\mathcal{W} \cap \overline{\{0\}} = 0$. Let $I \in \mathcal{B}$ with minimal $\dim(\mathcal{W} \cap I)$. If $0 \neq x \in \mathcal{W} \cap I$, we can choose $J \subset I$ with $x \notin J$ to get $\dim(\mathcal{W} \cap I) > \dim(\mathcal{W} \cap J)$. Thus, $\mathcal{W} \cap I = 0$. Write $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}$ with $I \subset \mathcal{V}$. We can view $\mathcal{W}^* \subset \mathcal{H}^*$ with $\mathcal{W}^*(\mathcal{V}) = 0$. Since $\mathcal{W}^*(I) = 0$, we have $\mathcal{W}^* \subset \mathcal{H}^*_{\mathcal{T}}$ and $\mathcal{H}^*_{\mathcal{T}} = \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}) \oplus \mathcal{W}^*$. In particular, $\mathcal{H}^*_{\mathcal{T}}/\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}) \cong \mathcal{W}^*$ is finite dimensional. To show that $(\mathcal{H}^*_{\mathcal{T}})_{\mathcal{S}}$ is an r.f.g.p. Hopf algebra, it remains to show that the operations are continuous relative to $\mathcal{S}$.

If $\mathcal{W}_i$ is a subspace of $\mathcal{H}$, we can identify $\mathcal{W}_1 \otimes \mathcal{W}_2$ with its image in $\mathcal{H} \otimes \mathcal{H}$, since k is a field. Given a finite dimensional subspace $\mathcal{W}$, there are finite dimensional $\mathcal{W}_i$ with $\Delta(\mathcal{W}) \subset \mathcal{W}_1 \otimes \mathcal{W}_2$. Thus,

$$\begin{aligned} \chi(\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_1) * \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_2))(\Delta(\mathcal{W})) &= 0, \\ \mu_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_1) * \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_2)) &\subset \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}), \end{aligned}$$

so $\mu_{\mathcal{H}^*_{\mathcal{T}}}$ is continuous. Trivially, $\eta_{\mathcal{H}^*_{\mathcal{T}}}$ is continuous. Given finite dimensional $\mathcal{W}_i$ with $\mathcal{H}^*_{\mathcal{T}} = \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_i) \oplus \mathcal{W}^*_i$ as above, we have

$$\mathcal{H}^*_{\mathcal{T}} \otimes \mathcal{H}^*_{\mathcal{T}} = \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_1) * \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_2) \oplus \mathcal{W}^*_1 \otimes \mathcal{W}^*_2$$

so $u \in \mathcal{H}^*_{\mathcal{T}} \otimes \mathcal{H}^*_{\mathcal{T}}$ is in $\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_1) * \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_2)$ if and only if $\chi(u)$ vanishes on $\mathcal{W}_1 \otimes \mathcal{W}_2$. Let $\mathcal{W} = \mathcal{W}_1\mathcal{W}_2$, a finite dimensional space. We see that

$$\begin{aligned} \chi(\Delta_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W})))(\mathcal{W}_1 \otimes \mathcal{W}_2) &= \mu_k^{-1}(\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W})(\mu(\mathcal{W}_1 \otimes \mathcal{W}_2))) = 0, \\ \Delta_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W})) &\subset \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_1) * \mathcal{Z}_{\mathcal{H}^*_{\mathcal{T}}}(\mathcal{W}_2), \end{aligned}$$

so $\Delta_{\mathcal{H}^*_\mathcal{T}}$ is continuous. Since $\varepsilon_{\mathcal{H}^*_\mathcal{T}}(\mathcal{Z}_{\mathcal{H}^*_\mathcal{T}}(k1)) = \mathcal{Z}_{\mathcal{H}^*_\mathcal{T}}(k1)(1) = 0$, we see that $\varepsilon_{\mathcal{H}^*_\mathcal{T}}$ is continuous. Finally,

$$S_{\mathcal{H}^*_\mathcal{T}}(\mathcal{Z}_{\mathcal{H}^*_\mathcal{T}}(S(\mathcal{W}))) = \mathcal{Z}_{\mathcal{H}^*_\mathcal{T}}(S(\mathcal{W})) \circ S \subset \mathcal{Z}_{\mathcal{H}^*_\mathcal{T}}(\mathcal{W})$$

shows that $S_{\mathcal{H}^*_\mathcal{T}}$ is continuous.

For $x \in \mathcal{H}$, define $\sigma_x \in (\mathcal{H}^*_\mathcal{T})^*$ by $\sigma_x(f) = f(x)$. Since $\sigma_x^{-1}(0) = \mathcal{Z}_{\mathcal{H}^*_\mathcal{T}}(kx)$, we see that σ_x is continuous; i.e. $\sigma : \mathcal{H} \to (\mathcal{H}^*_\mathcal{T})^*_\mathcal{S}$. Now $x \in \ker(\sigma)$ if and only if $f_I(\pi_I(x)) = 0$ for all $f_I \in (\mathcal{H}/I)^*$ and all $I \in \mathcal{B}$. Equivalently, $\pi_I(x) = 0$ for all $I \in \mathcal{B}$. Thus, $\ker(\sigma) = \overline{\{0\}}$. Also, a straightforward calculation shows that σ is a Hopf algebra homomorphism. Hence, it suffices to show that σ is surjective.

If $\phi \in (\mathcal{H}^*_\mathcal{T})^*_\mathcal{S}$, there is a finite dimensional subspace $\mathcal{W}$ of $\mathcal{H}$ with $\mathcal{Z}_{\mathcal{H}^*_\mathcal{T}}(\mathcal{W}) \subset \phi^{-1}(0)$ and $\mathcal{H}^*_\mathcal{T} = \mathcal{Z}_{\mathcal{H}^*_\mathcal{T}}(\mathcal{W}) \oplus \mathcal{W}^*$. Now ϕ restricted to $\mathcal{W}^*$ is in $\mathcal{W}^{**}$, so there is $x \in \mathcal{W}$ with $\phi(g) = g(x)$ for all $g \in \mathcal{W}^*$. If $f \in \mathcal{Z}_{\mathcal{H}^*_\mathcal{T}}(\mathcal{W})$, then

$$\phi(f+g) = \phi(g) = (f+g)(x)$$

so $\phi = \sigma_x$. ∎

3 Group schemes

We recall a few facts about affine schemes, particularly k-group schemes. See [3] for details. A functor on commutative k-algebras is a *k-functor*. A k-functor X is an *affine scheme* if there is a commutative k-algebra $k[X]$ such that X is equivalent to $Alg(k[X], -)$, the subfunctor of $Hom(k[X], -)$ of algebra homomorphisms. We generally identify X with $Alg(k[X], -)$. An affine scheme X is *algebraic* if $k[X] \cong k[T_1, \ldots, T_n]/I$ where I is a finitely generated ideal in the polynomial ring $k[T_1, \ldots, T_n]$. In particular, if k is a field, X is algebraic if $k[X]$ is finitely generated as an algebra. A k-functor to groups is a *k-group functor*. A *k-group scheme* is a k-group functor and an affine scheme. An *algebraic k-group* is a k-group scheme which is algebraic as an affine scheme.

If G is a k-group scheme, then $k[G]$ is a Hopf algebra and the product on $G(K)$ is given by

$$fg = \mu_K \circ (f \otimes' g) \circ \Delta;$$

i.e., $G(K)$ is a subgroup of the group of units of the k-algebra $Hom(k[G], K)$. Clearly, any commutative Hopf algebra gives a k-group scheme in this way. If $\mathcal{H}_\mathcal{T}$ is a cocommutative r.f.g.p. Hopf algebra, then Theorem 3 shows that $\mathcal{H}^*_\mathcal{T}$ is a commutative Hopf algebra, so that $G_{\mathcal{H}_\mathcal{T}} = Alg(\mathcal{H}^*_\mathcal{T}, -)$ is a k-group scheme. For $\mathcal{H}$, $\mathcal{F}$, and I as in Theorem 8, we denote $G_{\mathcal{H}_\mathcal{T}}$ by $G_{\mathcal{H},I}$. In particular, if k is a field, then we can take $\mathcal{F}$ to be all ideals in $\mathcal{H}$.

Lemma 10 *If I is an algebra ideal of finite codimension in a cocommutative Hopf algebra $\mathcal{H}$ over a field k with $I \subset \ker(\varepsilon)$ and $S(I) \subset I$, then $G_{\mathcal{H},I}$ is an algebraic k-group. If $\mathcal{H}'$ is a Hopf subalgebra of $\mathcal{H}$ and $I' = I \cap \mathcal{H}'$, then $G_{\mathcal{H}',I'}$ is an algebraic k-subgroup of $G_{\mathcal{H},I}$.*

Proof. The first statement is clear by Theorem 8. For any ideal J of $\mathcal{H}$ of finite codimension, let $J' = J \cap \mathcal{H}'$. The monomorphism $\mathcal{H}'/J' \to \mathcal{H}/J$ shows J' has finite codimension. Also, $\mathcal{H}'/J' \otimes \mathcal{H}'/K' \to \mathcal{H}/J \otimes \mathcal{H}/K$ is a monomorphism, so $J' \wedge K'$ in $\mathcal{H}'$ coincides with $(J \wedge K)'$. This shows that $\mathcal{B}(I') = \{J' : J \in \mathcal{B}(I)\}$. Now the inclusion $\iota : \mathcal{H}'_{\mathcal{T}'} \to \mathcal{H}_{\mathcal{T}}$ is continuous, so the restriction map

$$\iota^* = Hom(\iota, Id) : \mathcal{H}^*_{\mathcal{T}} \to \mathcal{H}'^*_{\mathcal{T}'}$$

is a Hopf algebra homomorphism by Theorem 3. It is easy to see that ι^* is surjective, so $Hom(\iota^*, Id)$ gives a monomorphism of $G_{\mathcal{H}',I'}(K)$ into $G_{\mathcal{H},I}(K)$. ■

Lemma 11 *If $\mathcal{H}$ in Theorem 8 is cocommutative and K is a commutative k-algebra, then $G_{\mathcal{H},I}(K)$ is isomorphic to a subgroup of the group of units of $\mathcal{H}/I \otimes K$.*

Proof. Since $\pi_I : \mathcal{H}_{\mathcal{T}} \to \mathcal{H}/I$ is continuous, Theorem 3 shows that

$$\pi_I^* = Hom(\pi_I, Id) : (\mathcal{H}/I)^* \to \mathcal{H}^*_{\mathcal{T}}$$

is a coalgebra homomorphism and that

$$Hom(\pi_I^*, Id) : Hom(\mathcal{H}^*_{\mathcal{T}}, K) \to Hom((\mathcal{H}/I)^*, K)$$

is an algebra homomorphism for any commutative k-algebra K. Since $\mathcal{H}/I$ is a finitely generated projective module, we can identify $((\mathcal{H}/I)^*)^*$ with $\mathcal{H}/I$ and use Corollary 5 to see that

$$\zeta : \mathcal{H}/I \otimes K \to Hom((\mathcal{H}/I)^*, K)$$

is an algebra isomorphism. The algebra homomorphism

$$\zeta^{-1} \circ Hom(\pi_I^*, Id) : Hom(\mathcal{H}^*_{\mathcal{T}}, K) \to \mathcal{H}/I \otimes K$$

restricts to a group homomorphism ψ of $G_{\mathcal{H},I}(K)$ into the group of units of $\mathcal{H}/I \otimes K$.

If $\phi \in \ker(\psi)$, then

$$Hom(\pi_I^*, Id)(\phi) = Hom(\pi_I^*, Id)(e)$$

where $e = \eta_K \circ \varepsilon_{\mathcal{H}^*_{\mathcal{T}}}$ is the identity of $G_{\mathcal{H},I}(K)$. Since $Hom(\pi_I^*, Id)(\phi) = \phi \circ \pi_I^*$ and $\mathcal{Z}_{\mathcal{H}^*}(I) = \pi_I^*((\mathcal{H}/I)^*)$, we see that ϕ agrees with e on $\mathcal{Z}_{\mathcal{H}^*}(I)$. Now $\phi = e$ follows from the fact that $\mathcal{H}^*_{\mathcal{T}}$ is generated as an algebra by $\mathcal{Z}_{\mathcal{H}^*}(I)$ (see Theorem 8). ■

We now consider a dual of the Hopf algebra $k[G]$ for a k-group scheme G. Let $I = \ker(\varepsilon)$ and assume that each $k[G]/I^n$, $n > 0$, is a finitely generated projective (equivalently, finitely presented flat) module. By **Example 6**, $\mathcal{B} = \{I^n : n > 0\}$ is a Hopf dualizing base. Following [3] p. 113, we say that

G is *infinitesimally flat,* and we write $Dist(G) = k[G]^*_{\mathcal{T}}$, the *distribution* or *hyperalgebra* of G. It is easy to see that the *Lie algebra of* G

$$Lie(G) = \{f \in Dist(G) : f(1) = f(I^2) = 0\}.$$

coincides with the set of *primitive* elements in $Dist(G)$, namely

$$\{f \in Dist(G) : \Delta(f) = f \otimes 1 + 1 \otimes f\}.$$

We remark that any algebraic k-group G over a field k is infinitesimally flat. Indeed, since $k[G] = k1 \oplus I$ is finitely generated as an algebra, we can choose generators $1, x_1, \ldots, x_m$ with $x_i \in I$. Now 1 and the monomials in x_i span $k[G]$. Moreover, I is spanned by all monomials, so I^n is spanned by all monomials of length at least n. Thus, $k[G]/I^n$ is finite dimensional.

4 Vector groups, G-modules, and toral actions

If $\mathcal{V}$ is a k-module, let $\mathcal{V}_a$ denote the k-group functor with $\mathcal{V}_a(K)$ being the additive group $\mathcal{V} \otimes K$. More generally, if $\mathcal{V}_{\mathcal{T}}$ is a linear topological module, $(\mathcal{V}_{\mathcal{T}})_a$ is the topological k-group functor with $(\mathcal{V}_{\mathcal{T}})_a(K) = \mathcal{V}_{\mathcal{T}} \otimes K$. We say that $(\mathcal{V}_{\mathcal{T}})_a$ is a *topological vector group.* The symmetric algebra $\mathcal{S}(\mathcal{V}^*)$ is a commutative and cocommutative Hopf algebra with

$$\begin{aligned} \Delta(f) &= f \otimes 1 + 1 \otimes f, \\ \varepsilon(f) &= 0, \\ S(f) &= -f, \end{aligned}$$

for $f \in \mathcal{V}^*$. Moreover, $\mathcal{S}(\mathcal{V}^*)$ is graded with $\mathcal{S}_0(\mathcal{V}^*) = k$ and $\mathcal{S}_1(\mathcal{V}^*) = \mathcal{V}^*$. The k-group scheme $Alg(\mathcal{S}(\mathcal{V}^*), -)$ has $Alg(\mathcal{S}(\mathcal{V}^*), K) \cong Hom(\mathcal{V}^*, K)$ via the restriction map. If $\mathcal{V}$ is finitely generated projective, the isomorphism $Hom(\mathcal{V}^*, K) \cong \mathcal{V} \otimes K$ shows that $\mathcal{V}_a$ is an algebraic k-group with $k[\mathcal{V}_a] = \mathcal{S}(\mathcal{V}^*)$.

We recall the following definition from [2]. If $\mathcal{V}$ is a k-module and $\mathcal{A}$ is a k-algebra, then $\rho_n : \mathcal{V} \to \mathcal{A}$, $n \geq 0$, is a sequence of *binomial divided power* (b.d.p.) maps if

- ρ_n is homogeneous of degree n,
- $\rho_0(v) = 1$,
- $(u, v) \to \rho_i(u)\rho_j(v)$ is the (i, j)-linearization of ρ_n for $n = i + j$.

Given $\mathcal{V}$, we can form the unital, associative and commutative algebra $\mathcal{V}^{(\infty)}$ generated by symbols $v^{(n)}$ for $v \in \mathcal{V}$ and $n \geq 0$ subject to the relations

$$\begin{aligned} v^{(0)} &= 1, \\ (av)^{(n)} &= a^n v^{(n)}, \\ (v + w)^{(n)} &= \sum_{i+j=n} v^{(i)} w^{(j)}. \end{aligned}$$

It is easy to see that $\mathcal{V}^{(\infty)}$ is a universal object for sequences of b.d.p. maps; i.e., $v \to v^{(n)}$ is a sequence of b.d.p. maps and given a sequence $\rho : \mathcal{V} \to \mathcal{A}$ of b.d.p. maps there is a unique algebra homomorphism $\hat{\rho} : \mathcal{V}^{(\infty)} \to \mathcal{A}$ with $\hat{\rho}(v^{(n)}) = \rho_n(v)$. The algebra $\mathcal{V}^{(\infty)}$ has an obvious $\mathbb{Z}$-grading with $v^{(n)} \in \mathcal{V}^{(\infty)}_n = \mathcal{V}^{(n)}$. Moreover, $\mathcal{V}^{(\infty)}$ is a $\mathbb{Z}$-graded Hopf algebra with coproduct $\Delta = \hat{\rho}$ for $\rho : \mathcal{V} \to \mathcal{V}^{(\infty)} \otimes \mathcal{V}^{(\infty)}$ defined by

$$\rho_n(v) = \sum_{i+j=n} v^{(i)} \otimes v^{(j)},$$

counit $\varepsilon = \hat{\rho}$ for $\rho : \mathcal{V} \to k$ defined by $\rho_0(v) = 1$, $\rho_n(v) = 0$ for $n > 0$, and antipode $S = \hat{\rho}$ for $\rho : \mathcal{V} \to \mathcal{V}^{(\infty)}$ defined by $\rho_n(v) = (-v)^{(n)}$. Recall that a *divided power sequence* $1 = x_0, x_1, \ldots$ of elements in a coalgebra $\mathcal{C}$ is a sequence satisfying

$$\Delta(x_n) = \sum_{i+j=n} x_i \otimes x_j,$$

and that the sequence is *homogeneous* if $\mathcal{C}$ is $\mathbb{Z}$-graded and $x_i \in \mathcal{C}_i$. Clearly, $v^{(0)}, v^{(1)}, \ldots$ is a homogeneous divided power sequence in $\mathcal{V}^{(\infty)}$.

Lemma 12 *If $\mathcal{V}$ is a finitely generated projective module over k, then $\mathcal{V}_a$ is infinitesimally flat and $Dist(\mathcal{V}_a) = S(\mathcal{V}^*)^g$ is isomorphic to $\mathcal{V}^{(\infty)}$ as $\mathbb{Z}$-graded Hopf algebras.*

Proof. We eventually will show that $\mathcal{S}(\mathcal{V}^*)^g$ is a universal object for sequences of b.d.p. maps. In anticipation of this, for $v \in \mathcal{V}$, we define $v^{(0)} = Id \in \mathcal{S}_0(\mathcal{V}^*)^*$ and $v^{(n)} \in \mathcal{S}_n(\mathcal{V}^*)^*$ by

$$v^{(n)}(g_1 \ldots g_n) = g_1(v) \ldots g_n(v).$$

Note that by the definition of $S(\mathcal{V}^*)^g$, we have $v^{(n)}(\mathcal{S}_m(\mathcal{V}^*)) = 0$ for $m \neq n$. Let $\{v_i : 1 \leq i \leq m\}$ and $\{f_i : 1 \leq i \leq m\}$ be dual generating sets for $\mathcal{V}$ and $\mathcal{V}^*$ so

$$g = \sum_{i=1}^{m} g(v_i) f_i$$

for $g \in \mathcal{V}^*$. We can find a set dual to the generating set $\{f_1^{n_1} \ldots f_m^{n_m} : \sum n_i = n\}$ of $\mathcal{S}_n(\mathcal{V}^*)$ as follows. The m-fold coproduct for $\mathcal{S}(\mathcal{V}^*)$ has

$$g_1 \ldots g_n \to \sum_{P(n,m)} \left(\prod_{i \in P_1} g_i \right) \otimes \ldots \otimes \left(\prod_{i \in P_m} g_i \right),$$

for $g_i \in \mathcal{V}^*$, where $P(n,m)$ is the set of ordered partitions $(P_1, \ldots, P_m)$ of $\{1, \ldots, n\}$ with $P_i = \emptyset$ allowed. Thus,

$$\begin{aligned} (v_1^{(n_1)} \ldots v_m^{(n_m)})(g_1 \ldots g_n) &= \sum_{P(n,m)} v_1^{(n_1)}(\prod_{i \in P_1} g_i) \ldots v_m^{(n_m)}(\prod_{i \in P_m} g_i) \\ &= \sum_{\substack{P(n,m) \\ |P_j| = n_j}} \prod_{i \in P_1} g_i(v_1) \ldots \prod_{i \in P_m} g_i(v_m) \end{aligned}$$

and

$$\begin{aligned} g_1 \dots g_n &= \sum_{\lambda \in M^n} g_1(v_{\lambda_1}) \dots g_n(v_{\lambda_n}) f_{\lambda_1} \dots f_{\lambda_n} \\ &= \sum_{P(n,m)} \left(\prod_{i \in P_1} g_i(v_1) \dots \prod_{i \in P_m} g_i(v_m) \right) f_1^{|P_1|} \dots f_m^{|P_m|} \\ &= \sum_N (v_1^{(n_1)} \dots v_m^{(n_m)})(g_1 \dots g_n) f_1^{n_1} \dots f_m^{n_m} \end{aligned}$$

where $M = \{1, \dots, m\}$ and where

$$N = \{(n_1, \dots, n_m) : \sum n_i = n\}.$$

This shows that $\{v_1^{(n_1)} \dots v_m^{(n_m)} : \sum n_i = n\}$ is dual to the generating set $\{f_1^{n_1} \dots f_m^{n_m} : \sum n_i = n\}$ of $\mathcal{S}_n(\mathcal{V}^*)$, and that $\mathcal{S}_n(\mathcal{V}^*)$ is a finitely generated projective module.

Since $\mathcal{S}_i(\mathcal{V}^*)\mathcal{S}_j(\mathcal{V}^*) = \mathcal{S}_{i+j}(\mathcal{V}^*)$, we see that

$$I^m = \bigoplus_{n=m}^{\infty} \mathcal{S}_n(\mathcal{V}^*),$$

for $I := \ker(\varepsilon)$; i.e., $I^m = I_{m-1}$ as in **Example 5**. Thus, $\mathcal{V}_a$ is infinitesimally flat and $Dist(\mathcal{V}_a) = \mathcal{S}(\mathcal{V}^*)^g$ as Hopf algebras. Moreover, $\mathcal{S}(\mathcal{V}^*)^g$ is a $\mathbb{Z}$-graded as Hopf algebra and generated as an algebra by all $v^{(n)}$, $v \in \mathcal{V}$.

Clearly, $(av)^{(n)} = a^n v^{(n)}$ and

$$\begin{aligned} (w_1 + w_2)^{(n)}(g_1 \dots g_n) &= \sum_{\lambda \in \{1,2\}^n} g_1(w_{\lambda_1}) \dots g_n(w_{\lambda_n}) \\ &= \sum_{P(n,2)} w_1^{(|P_1|)}(\prod_{i \in P_1} g_i) w_2^{(|P_2|)}(\prod_{j \in P_2} g_j) \end{aligned}$$

so

$$(w_1 + w_2)^{(n)} = \sum_{i+j=n} w_1^{(i)} w_2^{(j)}.$$

Thus, $v \to v^{(n)}$ is a sequence of b.d.p. maps. Given a sequence of b.d.p. maps $\rho : \mathcal{V} \to \mathcal{A}$, we define $\hat{\rho}$ by

$$\hat{\rho}(\phi) = \sum_N \phi(f_1^{n_1} \dots f_m^{n_m}) \rho_{n_1}(v_1) \dots \rho_{n_m}(v_m)$$

for $\phi \in \mathcal{S}_n(\mathcal{V}^*)^*$. If $\phi' \in \mathcal{S}_{n'}(\mathcal{V}^*)^*$ and $\tilde{n} = n + n'$, we can use

$$\Delta(f_1^{\tilde{n}_1} \dots f_m^{\tilde{n}_m}) = \sum_{n_i + n_i' = \tilde{n}_i} \binom{\tilde{n}_1}{n_1} \dots \binom{\tilde{n}_m}{n_m} f_1^{n_1} \dots f_m^{n_m} \otimes f_1^{n_1'} \dots f_m^{n_m'}$$

and

$$\rho_k(v)\rho_l(v) = \binom{k+l}{k}\rho_{k+l}(v)$$

to compute

$$\begin{aligned}\hat{\rho}(\phi\phi') &= \sum_{\tilde{N}}(\phi\phi')(f_1^{\tilde{n}_1}\dots f_m^{\tilde{n}_m})\rho_{\tilde{n}_1}(v_1)\dots\rho_{\tilde{n}_m}(v_m)\\ &= \hat{\rho}(\phi)\hat{\rho}(\phi').\end{aligned}$$

Also,

$$\begin{aligned}\hat{\rho}(v^{(n)}) &= \sum_N f_1(v)^{n_1}\dots f_m(v)^{n_m}\rho_{n_1}(v_1)\dots\rho_{n_m}(v_m)\\ &= \rho_n(\sum_{i=0}^m f_i(v)v_i) = \rho_n(v).\end{aligned}$$

Thus, $\mathcal{S}(\mathcal{V}^*)^g$ is a universal object for b.d.p. maps, so $\mathcal{S}(\mathcal{V}^*)^g \cong \mathcal{V}^{(\infty)}$ as $\mathbb{Z}$-graded algebras. Since for $i+j=n$

$$\mu_k((v^{(i)}\otimes' v^{(j)})(g_1\dots g_i\otimes g_{i+1}\dots g_n)) = v^{(n)}(g_1\dots g_n),$$

we have

$$\Delta_{\mathcal{S}(\mathcal{V}^*)^g}(v^{(n)}) = \sum_{i+j=n} v^{(i)}\otimes v^{(j)},$$

so the isomorphism respects coproducts. Similarly,

$$\varepsilon_{\mathcal{S}(\mathcal{V}^*)^g}(v^{(n)}) = v^{(n)}(1) = 0$$

for $n>0$ and

$$S_{\mathcal{S}(\mathcal{V}^*)^g}(v^{(n)})(f_1\dots f_n) = (v^{(n)}(S_{\mathcal{S}(\mathcal{V}^*)}(f_1\dots f_n)) = (-1)^n v^{(n)}(f_1\dots f_n)$$

show that $\mathcal{S}(\mathcal{V}^*)^g \cong \mathcal{V}^{(\infty)}$ as $\mathbb{Z}$-graded Hopf algebras. ■

We now turn our attention to group actions. A k-group functor G has a *right action* on a k-functor X if there is a morphism $X\times G\to X$ giving a group action of each group $G(K)$ on $X(K)$. If G is a k-group scheme and X is an affine scheme, there is a corresponding coproduct $\delta: k[X]\to k[X]\otimes k[G]$ with $x\cdot g = \mu_K\circ(x\otimes' g)\circ\delta$ for $g\in G(K)$ and $x\in X(K)$. Moreover, $k[X]$ is a right $k[G]$-comodule and δ is an algebra homomorphism.

If G is a k-group functor, we say that $\mathcal{V}_\mathcal{T}$ is a *G-module* if G acts on $(\mathcal{V}_\mathcal{T})_a$ and each $G(K)$ acts via maps in $Hom_K(\mathcal{V}_\mathcal{T}\otimes K, \mathcal{V}_\mathcal{T}\otimes K)$. In other words, each $g\in G(K)$ acts as a continuous K-linear map. If G is a k-group scheme and $\mathcal{V}_\mathcal{T}$ is a left G-module, then there is a continuous coproduct $\delta_\mathcal{V}:\mathcal{V}_\mathcal{T}\to\mathcal{V}_\mathcal{T}\otimes k[G]$ with

$$g\cdot(v\otimes 1_K) = (Id_\mathcal{V}\otimes' g)(\delta_\mathcal{V}(v))$$

for $g \in G(K)$ and $v \in \mathcal{V}_{\mathcal{T}}$. Moreover, $\mathcal{V}_{\mathcal{T}}$ is a right $k[G]$-comodule.

If G is a k-group scheme with a right action on an affine scheme X, then the coproduct δ makes $k[X]$ a left G-module. Moreover, since δ is an algebra homomorphism, each $G(K)$ acts by automorphisms of the K-algebra $k[X] \otimes K$. If X is itself a k-group scheme and each $G(K)$ acts by group automorphisms on $X(K)$, then the multiplication and inversion maps commute with the action of G. Thus, the coproduct and counit maps for $k[X]$ are G-module homomorphisms; i.e., G acts on $k[X]$ by Hopf algebra automorphisms.

The *k-torus* k_m is the k-group scheme with $k[k_m] = k[T, T^{-1}]$ and

$$\begin{aligned} \Delta(T) &= T \otimes T, \\ \varepsilon(T) &= 1, \\ S(T) &= T^{-1}, \end{aligned}$$

so $k_m(K)$ is isomorphic to the group of units of K. We shall be interested in toral actions.

Lemma 13 *A k-module $\mathcal{V}$ is a left k_m-module if and only if $\mathcal{V}$ is $\mathbb{Z}$-graded with*

$$g \cdot (v_i \otimes 1_K) = t^i (v_i \otimes 1_K)$$

for $g \in k_m(K)$ with $g(T) = t$ and $v_i \in \mathcal{V}_i$. Moreover, k_m-homomorphisms coincide with graded homomorphisms. A Hopf algebra $\mathcal{H}$ is $\mathbb{Z}$-graded as a Hopf algebra if and only if k_m acts by automorphisms on $\mathcal{H}$. Also, k_m acts as automorphisms of a k-group scheme G if and only if $k[G]$ is $\mathbb{Z}$-graded as a Hopf algebra.

Proof. We know that $\mathcal{V}$ is a left k_m-module if and only if $\mathcal{V}$ is a right $k[T, T^{-1}]$-comodule. If $\delta \in Hom(\mathcal{V}, \mathcal{V} \otimes k[T, T^{-1}])$, define $e_i \in Hom(\mathcal{V}, \mathcal{V})$ by

$$\delta(v) = \sum_i e_i(v) \otimes T^i.$$

The coassociative condition for δ is

$$(\delta \otimes' Id_{k[T,T^{-1}]}) \circ \delta = (Id_{\mathcal{V}} \otimes' \Delta) \circ \delta$$

or

$$\sum_{i,j} e_j(e_i(v)) \otimes T^j \otimes T^i = \sum_i e_i(v) \otimes T^i \otimes T^i$$

i.e., the e_i are orthogonal idempotent maps. The counit condition

$$\rho_{\mathcal{V}} \circ (Id_{\mathcal{V}} \otimes' \varepsilon_{k[T,T^{-1}]}) \circ \delta = Id_{\mathcal{V}}$$

is just

$$\sum_i e_i(v) = v.$$

Thus, $\mathcal{V}$ is a comodule via δ if and only if $\mathcal{V}$ is $\mathbb{Z}$-graded with

$$\delta(v_i) = v_i \otimes T^i$$

for $v_i \in \mathcal{V}_i$. This corresponds to $g \cdot (v_i \otimes 1_K) = v_i \otimes t^i$. Clearly, $\phi \in Hom(\mathcal{V}, \mathcal{V}')$ commutes with the action of k_m if and only if $\phi(\mathcal{V}_i) \subset \mathcal{V}'_i$.

The torus k_m acts by automorphisms on $\mathcal{H}$ if and only if the defining maps are graded homomorphisms; i.e., $\mathcal{H}$ is $\mathbb{Z}$-graded as a Hopf algebra. We have already noted that k_m acts on G by automorphisms if and only if k_m acts on $k[G]$ by automorphisms. ∎

For later use, we now examine the action of k_m on $\mathcal{V}_a$ given by the grading on $\mathcal{S}(\mathcal{V}^*)$. We have

$$\begin{aligned}(u \cdot \phi)(f) &= (\mu_K \circ (u \otimes' \phi) \circ \delta)(f) \\ &= u(f)\phi(T)\end{aligned}$$

for $\phi \in k_m(K)$, $u \in Alg(\mathcal{V}, K)$, and $f \in \mathcal{V}^*$, so ϕ acts by multiplication by $t = \phi(T) \in K$.

We now look at the transfer of group actions to duals.

Theorem 14 *If G is a k-group scheme, then $Hom(-, k)$ induces a functor from r.f.g.p. left G-modules to right G-modules with the action given by*

$$\gamma_K((f \otimes t) \cdot g)(v \otimes s) = \gamma_K(f \otimes t)(g \cdot (v \otimes s))$$

for $f \in \mathcal{V}^*_{\mathcal{T}}$, $v \in \mathcal{V}$, $g \in G(K)$, and $s, t \in K$ where

$$\gamma_K(f \otimes t)(v \otimes s) = f(v)ts.$$

Moreover,

$$Hom(Id_{\mathcal{V}\otimes\mathcal{W}}, \mu_k) \circ \chi : \mathcal{V}^*_{\mathcal{T}} \otimes \mathcal{W}^*_{\mathcal{S}} \to (\mathcal{V}_{\mathcal{T}} \otimes \mathcal{W}_{\mathcal{S}})^*$$

is a G-module isomorphism.

Proof. We shall show that the three K-modules $\mathcal{V}^*_{\mathcal{T}} \otimes K$, $(\mathcal{V}_{\mathcal{T}} \otimes K)^* = Hom_K(\mathcal{V}_{\mathcal{T}} \otimes K, K)$, and $Hom(\mathcal{V}_{\mathcal{T}}, K)$ are isomorphic. It is easiest to verify the group action for $(\mathcal{V}_{\mathcal{T}} \otimes K)^*$ and the other properties for $Hom(\mathcal{V}_{\mathcal{T}}, K)$ and transfer the results to $\mathcal{V}^*_{\mathcal{T}} \otimes K$ via the isomorphisms. We first get the isomorphisms. We note that

$$\alpha_K : Hom(\mathcal{V}_{\mathcal{T}}, K) \to (\mathcal{V}_{\mathcal{T}} \otimes K)^*$$

given by $\alpha_K(f)(v \otimes s) = f(v)s$ has inverse $\alpha_K^{-1}(h)(v) = h(v \otimes 1_K)$. Also, by Corollary 5, $\zeta_{\mathcal{V},K}(f \otimes t)(v) = f(v)t$ gives an equivalence $(\mathcal{V}^*_{\mathcal{T}})_a \to Hom(\mathcal{V}_{\mathcal{T}}, -)$ of k-functors. We see that $\gamma_K = \alpha_K \circ \zeta_{\mathcal{V},K}$.

Clearly, $G(K)$ acts on $(\mathcal{V}_{\mathcal{T}} \otimes K)^*$ with $(h \cdot g)(u) = h(g \cdot u)$ for $h \in (\mathcal{V}_{\mathcal{T}} \otimes K)^*$, $g \in G(K)$, and $u \in \mathcal{V}_{\mathcal{T}} \otimes K$. Note $h \cdot g$ is continuous since g acts on $\mathcal{V}_{\mathcal{T}} \otimes K$ via a continuous map. The specified action on $\mathcal{V}_{\mathcal{T}}^* \otimes K$ is obtained by transferring the action via γ_K. The action transferred to $Hom(\mathcal{V}_{\mathcal{T}}, K)$ via α_K has an easy description. If $\delta_{\mathcal{V}} : \mathcal{V}_{\mathcal{T}} \to \mathcal{V}_{\mathcal{T}} \otimes k[G]$ is the coproduct for $\mathcal{V}_{\mathcal{T}}$ as a $k[G]$-comodule, then for $f \in Hom(\mathcal{V}_{\mathcal{T}}, K)$, we have

$$\begin{aligned}(\alpha_K(f) \cdot g)(v \otimes 1_K) &= (\alpha_K(f) \circ (Id_{\mathcal{V}} \otimes' g) \circ \delta_{\mathcal{V}})(v) \\ &= (\mu_K \circ (f \otimes' g) \circ \delta_{\mathcal{V}})(v)\end{aligned}$$

so

$$f \cdot g = \mu_K \circ (f \otimes' g) \circ \delta_{\mathcal{V}}. \tag{8}$$

We next show that $Hom \times G \to Hom$ is morphism of functors. If $\phi \in Hom_G(\mathcal{W}_{\mathcal{S}}, \mathcal{V}_{\mathcal{T}})$ and $\theta \in Alg(K, K')$, then ϕ is also a comodule homomorphism, so

$$\begin{aligned}(\theta \circ f \circ \phi) \cdot (\theta \circ g) &= \mu_{K'} \circ ((\theta \circ f \circ \phi) \otimes' (\theta \circ g) \circ \delta_{\mathcal{W}} \\ &= \theta \circ \mu_K \circ (f \otimes' g) \circ (\phi \otimes' Id_{k[G]}) \circ \delta_{\mathcal{W}} \\ &= \theta \circ \mu_K \circ (f \otimes' g) \circ \delta_{\mathcal{V}} \circ \phi \\ &= \theta \circ (f \cdot g) \circ \phi.\end{aligned}$$

In particular, $Hom(\mathcal{V}_{\mathcal{T}}, -) \times G \to Hom(\mathcal{V}_{\mathcal{T}}, -)$ is a morphism, so $(\mathcal{V}_{\mathcal{T}}^*)_a \times G \to (\mathcal{V}_{\mathcal{T}}^*)_a$ is a morphism and $\mathcal{V}_{\mathcal{T}}^*$ is a G-module. Moreover,

$$Hom(\phi, Id_K) : Hom(\mathcal{V}_{\mathcal{T}}, K) \to Hom(\mathcal{W}_{\mathcal{S}}, K)$$

and hence

$$Hom(\phi, Id_k) \otimes' Id_K : \mathcal{V}_{\mathcal{T}}^* \otimes K \to \mathcal{W}_{\mathcal{S}}^* \otimes K$$

are $G(K)$-module homomorphisms. Thus, $Hom(-, k)$ is a functor of G-modules.

Finally, we consider tensor products. The comodule map $\delta_{\mathcal{V} \otimes \mathcal{W}}$ is given by

$$\delta_{\mathcal{V} \otimes \mathcal{W}}(v \otimes w) = \sum_{i,j} v_i \otimes w_j \otimes a_i b_j$$

where

$$\begin{aligned}\delta_{\mathcal{V}}(v) &= \sum_i v_i \otimes a_i \\ \delta_{\mathcal{W}}(w) &= \sum_j w_j \otimes b_j.\end{aligned}$$

Thus, for $f \in Hom(\mathcal{V}_{\mathcal{T}}, K)$ and $h \in Hom(\mathcal{W}_{\mathcal{S}}, K)$, we have

$$\begin{aligned}((\mu_K \circ (f \otimes' h)) \cdot g)(v \otimes w) &= \sum_{i,j} f(v_i) h(w_j) g(a_i b_j) \\ &= (\sum_i f(v_i) g(a_i))(\sum_j h(w_j) g(b_j)) \\ &= (\mu_K \circ (f \cdot g \otimes' h \cdot g))(v \otimes w)\end{aligned}$$

so

$$Hom(Id_{\mathcal{V}\otimes\mathcal{W}},\mu_K)\circ\chi : Hom(\mathcal{V}_{\mathcal{T}},K)\otimes_K Hom(\mathcal{W}_{\mathcal{S}},K)\to Hom(\mathcal{V}_{\mathcal{T}}\otimes\mathcal{W}_{\mathcal{S}},K)$$

is a $G(K)$-module isomorphism. It is easy to see that the corresponding map is

$$(Hom(Id_{\mathcal{V}\otimes\mathcal{W}},\mu_k)\circ\chi)\otimes' Id_K : (\mathcal{V}^*_{\mathcal{T}}\otimes\mathcal{W}^*_{\mathcal{S}})\otimes K\to(\mathcal{V}_{\mathcal{T}}\otimes\mathcal{W}_{\mathcal{S}})^*\otimes K$$

so $Hom(Id_{\mathcal{V}\otimes\mathcal{W}},\mu_k)\circ\chi$ is a G-module isomorphism. ■

Corollary 15 *If G is a k-group scheme acting on itself by the right regular, left regular, or conjugation action and if $\mathcal{T}$ is a r.f.g.p. topology on $k[G]$ compatible with action of G, then*

$$\zeta_{k[G],K} : k[G]^*_{\mathcal{T}}\otimes K\to Hom(k[G]_{\mathcal{T}},K)$$

is a $G(K)$-module isomorphism.

Proof. Since $G(K)=Alg(k[G],K)$ is a subgroup of the group of units in $Hom(k[G],K)$, the right multiplication action of $G(K)$ on $Hom(k[G],K)$ extends the right regular action of G on itself. On the other hand, this action defines a left G-module structure on $k[G]$. If $\mathcal{T}$ is a r.f.g.p. topology on $k[G]$ compatible with the action of G; i.e., $k[G]_{\mathcal{T}}$ is a left G-module, then $k[G]^*_{\mathcal{T}}$ is a right G-module. Now (8) with $\delta_{\mathcal{V}}=\Delta_{k[G]}$ shows that $\zeta_{k[G],K}$ is a $G(K)$-module isomorphism. Similarly, $\zeta_{k[G],K}$ is a $G(K)$-module isomorphism for the left regular action or the conjugation action of G on itself. ■

Corollary 16 *If $\mathcal{V}_{\mathcal{T}}$ is a r.f.g.p. left G-module for a k-group scheme G, if $\mathcal{W}$ is a G-submodule of $\mathcal{V}_{\mathcal{T}}$, and if $\mathcal{V}_{\mathcal{T}}/\mathcal{W}$ is a r.f.g.p. module, then*

$$\mathcal{Z}_{\mathcal{V}^*_{\mathcal{T}}}(\mathcal{W})=\{f\in\mathcal{V}^*_{\mathcal{T}} : f(\mathcal{W})=0\}$$

is a G-submodule of $\mathcal{V}^*_{\mathcal{T}}$.

Proof. It is easy to check that $G(K)$ acts continuously on $(\mathcal{V}_{\mathcal{T}}/\mathcal{W})\otimes K$, so $\mathcal{V}_{\mathcal{T}}/\mathcal{W}$ is a G-module and $\pi_{\mathcal{W}}$ is a homomorphism of r.f.g.p. G-modules. Thus, $\pi^*_{\mathcal{W}}$ is a G-module monomorphism with image $\mathcal{Z}_{\mathcal{V}^*_{\mathcal{T}}}(\mathcal{W})$. ■

Corollary 17 *If G is a k-group scheme acting by automorphisms on an r.f.g.p. Hopf algebra $\mathcal{H}_{\mathcal{T}}$, then G acts by automorphisms on $\mathcal{H}^*_{\mathcal{T}}$.*

Proof. We see that

$$\begin{aligned}
\mu_{\mathcal{H}^*_{\mathcal{T}}} &= Hom(\Delta_{\mathcal{H}},Id_k)\circ Hom(Id_{\mathcal{H}\otimes\mathcal{H}},\mu_k)\circ\chi,\\
\eta_{\mathcal{H}^*_{\mathcal{T}}} &= Hom(\varepsilon_{\mathcal{H}},Id_k)\circ\xi_k^{-1},\\
\Delta_{\mathcal{H}^*_{\mathcal{T}}} &= \chi^{-1}\circ Hom(Id_{\mathcal{H}\otimes\mathcal{H}},\mu_k^{-1})\circ Hom(\mu_{\mathcal{H}},Id_k),\\
\varepsilon_{\mathcal{H}^*_{\mathcal{T}}} &= \xi_k\circ Hom(\eta_{\mathcal{H}},Id_k),\\
S_{\mathcal{H}^*_{\mathcal{T}}} &= Hom(S_{\mathcal{H}},Id_k)
\end{aligned}$$

are G-module homomorphisms. ■

Corollary 18 *If $\mathcal{H}_\mathcal{T}$ is an r.f.g.p. Hopf algebra, if $\mathcal{H}$ is $\mathbb{Z}$-graded as Hopf algebra, and if $\mathcal{T}$ has a linear base of graded submodules, then $\mathcal{H}_\mathcal{T}^*$ is $\mathbb{Z}$-graded as Hopf algebra. If $\mathcal{W}$ is a graded submodule of $\mathcal{H}_\mathcal{T}$, then $\mathcal{Z}_{\mathcal{H}_\mathcal{T}^*}(\mathcal{W})$ is a graded submodule of $\mathcal{H}_\mathcal{T}^*$.*

Proof. By Lemma 13, k_m acts by automorphisms on $\mathcal{H}$. If I is a graded submodule, then $k_m(K)$ maps $I * K$ to itself. Thus, $k_m(K)$ acts on $\mathcal{H}_\mathcal{T} \otimes K$ by continuous maps, so k_m acts by automorphisms on $\mathcal{H}_\mathcal{T}$. Now Corollary 17 and Lemma 13 show that $\mathcal{H}_\mathcal{T}^*$ is $\mathbb{Z}$-graded as Hopf algebra. The last statement follows from Corollary 16. ■

5 Jordan pairs

We shall presently apply the results of the previous sections to Jordan pairs. We will freely use the definitions, notation, and results in [2], but recall a few now. If $v \to v^{(n)}$ is a sequence of b.d.p. maps from $\mathcal{V}$ to $\mathcal{A}$, then $v \to ad_v^{(n)}$ is a sequence of b.d.p. maps from $\mathcal{V}$ to $End(\mathcal{A})$ where

$$ad_v^{(n)}(a) = \sum_{i+j=n} v^{(i)} a(-v)^{(j)}.$$

Let $\mathcal{V} = (\mathcal{V}^+, \mathcal{V}^-)$ be a Jordan pair. A *divided power specialization* of $\mathcal{V}$ is a pair $\rho = (\rho^+, \rho^-)$ of sequences of b.d.p. maps $\rho_n^\sigma : v \to v^{(n)}$ from $\mathcal{V}^\sigma$ to $\mathcal{A}$ such that

$$ad_x^{(i)}(y^{(j)}) = \begin{cases} (Q_x(y))^{(l)} & \text{for } i = 2j, \\ 0 & \text{for } i > 2j. \end{cases}$$

and all linearizations hold for $x \in \mathcal{V}^\sigma, y \in \mathcal{V}^{-\sigma}$. We called this a *divided power representation* in [2] but will now reserve that terminology for a divided power specialization into some $End(\mathcal{W})$. In particular, the *TKK-representation* of $\mathcal{V}$ is a divided power specialization into the endomorphisms of the Tits-Kantor-Koecher Lie algebra $TKK(\mathcal{V}, \mathcal{D}_0)$ where $\mathcal{D}_0 = k(Id_{\mathcal{V}^+}, Id_{\mathcal{V}^-}) + Inder(\mathcal{V})$.

The *universal divided power specialization* $\mathcal{U}(\mathcal{V})$ is a cocommutative $\mathbb{Z}$-graded Hopf algebra with $x^{(n)} \in \mathcal{U}(\mathcal{V})_{\sigma n}$ if $x \in \mathcal{V}^\sigma$. Let $\mathcal{X}$ be the subalgebra of $\mathcal{U}(\mathcal{V})$ generated by all $x^{(n)}$ with $x \in \mathcal{V}^+$ and $\mathcal{Y}$ the subalgebra for $x \in \mathcal{V}^-$. The universal property of $\mathcal{U}(\mathcal{V})$ gives an algebra homomorphism $\mathcal{U}(\mathcal{V}) \to End(TKK(\mathcal{V}, \mathcal{D}_0))$. The kernel J is a graded ideal and $(\mathcal{V}^\sigma)^{(1)} \cap J = \{0\}$.

If the base ring k is a field, then $\mathcal{X}$ and $\mathcal{Y}$ are Hopf subalgebras with $\mathcal{X} \cong (\mathcal{V}^+)^{(\infty)}$ and $\mathcal{Y} \cong (\mathcal{V}^-)^{(\infty)}$. Moreover, $\mathcal{U}(\mathcal{V}) = \mathcal{Y}\mathcal{H}\mathcal{X}$ where $\mathcal{H}$ is a Hopf subalgebra generated as an algebra by certain elements of degree 0 in the $\mathbb{Z}$-grading.

Theorem 19 *If $\mathcal{V}$ is a finite dimensional Jordan pair over a field k, J is the kernel of the TKK-representation, and*

$$I = \ker(\varepsilon) \cap J \cap S(J),$$

then $G = G_{\mathcal{U}(\mathcal{V}),I}$ *is an algebraic* k*-group with algebraic* k*-subgroups*

$$U^+ = G_{\mathcal{X},I^+}, U^- = G_{\mathcal{Y},I^-}, H = G_{\mathcal{H},I^0}$$

where $I^+ = \mathcal{X} \cap I$*,* $I^- = \mathcal{Y} \cap I$*, and* $I^0 = \mathcal{H} \cap I$*. Moreover,* $U^\sigma \cong V_a^\sigma$ *and* $H(K)$ *acts as automorphisms of the Jordan pair* $\mathcal{V}_K = (\mathcal{V}^+ \otimes K, \mathcal{V}^- \otimes K)$*.*

Proof. Since $TKK(\mathcal{V}, \mathcal{D}_0)$ is finite dimensional, J and I have finite codimension. Since $\varepsilon \circ S = \varepsilon$ in any Hopf algebra and since $S^2 = Id$ if the Hopf algebra is cocommutative, we see that $S(I) = I$. Lemma 10 shows that G is an algebraic k-group with algebraic k-subgroups U^+, U^-, and H. We have $\mathcal{X} \cong (\mathcal{V}^+)^{(\infty)}$ as Hopf algebras. Also, if ρ is the divided power specialization for the TKK-representation, we have $\ker(\rho_1^+) = 0$ and $\rho_n^+ = 0$ for $n > 2$. Thus, $(\mathcal{V}^+)^{(1)} \cap J = \{0\}$ and $(\mathcal{V}^+)^{(n)} \subset J$ for $n > 2$. Since $\varepsilon((\mathcal{V}^+)^{(n)}) = 0$ for $n > 0$, since $S((\mathcal{V}^+)^{(n)}) = (\mathcal{V}^+)^{(n)}$, and since J is a graded ideal, we have

$$I_3 \subset I^+ \subset I_2$$

where $I_m = \bigoplus_{n=m+1}^{\infty} (\mathcal{V}^+)^{(n)}$. It is easy to check that $I_k \wedge I_l = I_{k+l}$, so the linear bases $\{\wedge^n I^+\}$ and $\{I_n\}$ determine the same topology $\mathcal{T}^+$ on $\mathcal{X}$. Thus, using Lemma 12, we see that $\mathcal{X}^*_{\mathcal{T}^+} \cong S(\mathcal{V}^{+*})^{gg} = S(\mathcal{V}^{+*})$, so $U^+ \cong V_a^+$ and similarly $U^- \cong V_a^-$.

Let $\mathcal{T}$ be the topology on $\mathcal{U}(\mathcal{V})$ determined by I and let $\mathcal{S}$ be the topology on $\mathcal{U}(\mathcal{V})^*_{\mathcal{T}}$ as in Lemma 9. Since J is a graded ideal and $\mathcal{U}(\mathcal{V})$ is a graded Hopf algebra, each $\wedge^n I$ is graded and $\mathcal{U}(\mathcal{V})^*_{\mathcal{T}}$ is a graded Hopf algebra by Corollary 18. If $\mathcal{W}$ is a finite dimensional subspace of $\mathcal{U}(\mathcal{V})$, let $\mathcal{W}_i$ be the projection of $\mathcal{W}$ onto $\mathcal{U}(\mathcal{V})_i$. Clearly, $\widehat{\mathcal{W}} = \sum \mathcal{W}_i$ is a finite dimensional graded subspace and $\mathcal{Z}_{\mathcal{U}(\mathcal{V})^*_{\mathcal{T}}}(\widehat{\mathcal{W}})$ is a graded submodule of $\mathcal{U}(\mathcal{V})^*_{\mathcal{T}}$ by Corollary 18. Since $\mathcal{Z}_{\mathcal{U}(\mathcal{V})^*_{\mathcal{T}}}(\widehat{\mathcal{W}}) \subset \mathcal{Z}_{\mathcal{U}(\mathcal{V})^*_{\mathcal{T}}}(\mathcal{W})$, we see that $\mathcal{S}$ has a linear base of graded subspaces. Thus,

$$\widetilde{\mathcal{U}} = \mathcal{U}(\mathcal{V})/\overline{\{0\}} \cong (\mathcal{U}(\mathcal{V})^*_{\mathcal{T}})^*_{\mathcal{S}}$$

is a $\mathbb{Z}$-graded Hopf algebra. The conjugation action $x \to g^{-1}xg$ of G makes $k[G] = \mathcal{U}(\mathcal{V})^*_{\mathcal{T}}$ a right $k[G]$-module with

$$\delta = (Id_{k[G]} \otimes' \mu_{k[G]}) \circ (((Id_{k[G]} \otimes' S) \circ \tau \circ \Delta_{k[G]}) \otimes' Id_{k[G]}) \circ \Delta_{k[G]}.$$

By Lemma 9, $(\mathcal{U}(\mathcal{V})^*_{\mathcal{T}})_{\mathcal{S}}$ is a r.f.g.p. Hopf algebra, so δ is continuous. Thus, $(\mathcal{U}(\mathcal{V})^*_{\mathcal{T}})_{\mathcal{S}}$ is a left G-module and, by Corollary 17, $\widetilde{\mathcal{U}}$ is a right G-module with $G(K)$ acting as Hopf algebra automorphisms. Since $\mathcal{H} \subset \mathcal{U}(\mathcal{V})_0$, we see that k_m fixes $\mathcal{H}$, $\mathcal{H}^*_{\mathcal{T}^0}$, and H. By Lemma 13, k_m acts by automorphisms on G, so the action commutes with the conjugation action of H on G. Thus, k_m and H commute acting on $\widetilde{\mathcal{U}}$; i.e., $H(K)$ acts as graded Hopf algebra automorphisms.

The only properties of $\mathcal{U}(\mathcal{V})$ used in the proofs of the results in [2] from Lemma 24 through Corollary 28 are that $\mathcal{U}(\mathcal{V})$ is a $\mathbb{Z}$-graded Hopf algebra and that the TKK-representation factors through $\mathcal{U}(\mathcal{V})$. Since $J \supset \overline{\{0\}}$, $\widetilde{\mathcal{U}}$ also has these properties. Thus, we may replace $\mathcal{U}(\mathcal{V})$ by $\widetilde{\mathcal{U}}$ in Corollary 28 of [2] to get

$$\mathcal{P}(\widetilde{\mathcal{U}}) = (\mathcal{V}^+)^{(1)} \oplus \mathcal{P}(\widetilde{\mathcal{H}}) \oplus (\mathcal{V}^-)^{(1)}$$

where $\mathcal{P}$ denotes the primitive elements of a Hopf algebra and $\widetilde{\mathcal{H}}$ is the image of $\mathcal{H}$ in $\widetilde{\mathcal{U}}$. Since $H(K)$ acts as graded Hopf algebra automorphisms of $\widetilde{\mathcal{U}} \otimes K$, we see that $H(K)$ stabilizes $(\mathcal{V}^\sigma)^{(1)} \otimes K \cong \mathcal{V}^\sigma \otimes K$. The homogeneous divided power sequence over $x \in (\mathcal{V}^\sigma)^{(1)} \otimes K$ in $\mathcal{U} \otimes K$ gives one in $\widetilde{\mathcal{U}} \otimes K$. Moreover, it is unique by Lemma 4 of [2]. Since $H(K)$ acts as Hopf algebra automorphisms, we see that $(h(x))^{(n)} = h(x^{(n)})$ for $h \in H(K)$. Using Theorem 5 of [2], we see that $H(K)$ acts as automorphisms of the Jordan pair $\mathcal{V}' = ((\mathcal{V}^+)^{(1)} \otimes K, (\mathcal{V}^-)^{(1)} \otimes K)$ with product

$$Q_x(y) = x^{(2)}y - xyx + yx^{(2)}.$$

Using the homomorphism of $\widetilde{\mathcal{U}}$ to $End(TKK(\mathcal{V}, \mathcal{D}_0))$, we see that $\mathcal{V}' \cong \mathcal{V}_K$. ■

In [5], Loos defines an *elementary action* of the torus k_m on a separated k-group sheaf G to be an action of k_m by automorphisms of G with subgroup sheaves H, U^+, U^- such that

(i) H is fixed by k_m.

(ii) U^+ and U^- are vector subgroups on which k_m acts by scalar multiplication (respectively, the inverse of scalar multiplication).

(iii) $\Omega = U^- H U^+$ is open in G.

(iv) G is generated as a k-group sheaf by H, U^+, U^-.

Loos ([5], Theorem 4.1) uses the structure of Ω to define quasi-invertibility leading to a Jordan pair. We get a similar result by our methods.

Theorem 20 *If G is an infinitesimally flat affine algebraic group scheme, then every elementary action of k_m on G gives a $\mathbb{Z}$-grading of $Dist(G)$ as a Hopf algebra such that the induced $\mathbb{Z}$-grading of $Lie(G)$ is*

$$Lie(G) = Lie(U^-) \oplus Lie(H) \oplus Lie(U^+)$$

and there is a homogeneous divided power sequence $\{x^{(n)} : n \geq 0\}$ over every $x \in Lie(U^{\pm})$. Moreover, $(Lie(U^+), Lie(U^-))$ is a Jordan pair with $Q_x(y) = x^{(2)}y - xyx + yx^{(2)}$.

Proof. By Lemma 13, the action of k_m on G by automorphisms corresponds uniquely to a $\mathbb{Z}$-grading of $k[G]$ as a Hopf algebra. Since $I = \ker(\varepsilon)$ is a graded ideal, each I^n is graded and $Dist(G) = k[G]^*_{\mathcal{I}}$ is graded by Corollary 18. Lemma 3.4 of [5] states that the multiplication map

$$U^- \times H \times U^+ \to G$$

is an open imbedding, so

$$Lie(G) = Lie(U^-) \oplus Lie(H) \oplus Lie(U^+).$$

Since H is fixed by k_m, we see that

$$\begin{aligned} k[H] &= k[H]_0, \\ Dist(H) &= Dist(H)_0, \text{ and} \\ Lie(H) &\subset Lie(G)_0. \end{aligned}$$

Since k_m acts on U^+ by scalar multiplication, the grading on

$$k[U^+] = \mathcal{S}((Lie(U^+))^*)$$

is the usual one. Thus, $Lie(U^+) = Dist(U^+)_1$ so $Lie(U^+) \subset Lie(G)_1$. On the other hand, the action on U^- reverses the roles of T and T^{-1}, so

$$k[U^-]_{-n} = \mathcal{S}_n((Lie(U^-)^*)$$

and $Lie(U^-) \subset Lie(G)_{-1}$. Lemma 12 shows there is a homogeneous divided power sequence over each element of $Lie(U^{\pm})$. Finally, Theorem 5 of [2] gives the last statement. ■

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