Hilbert C*-modules are JB*-triples.

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Abstract

We show that every Hilbert C*-module E is a JB*-triple in a canonical way and establish an explicit expression for the holomorphic automorphisms of the unit ball of E.

1.1 Introduction

Hilbert C*-modules first appeared in 1953 in a work of Kaplansky [4] who worked only with modules over commutative unital C*-algebras. In 1973 Paschke [11] proved that most of the properties of Hilbert C*-modules were valid for modules over an arbitrary C*-algebra. About the same time Reiffel independently developed much of the same theory and used it to study representations of C*-algebras. Since then the subject has grown and spread rapidly and now there is an extensive literature on the topic (see [10] for a systematic introduction). Many interesting developments have been made by Kasparov, who used Hilbert C*-modules as the framework for K-theory. More recently Hilbert C*-modules have been a useful tool in the C*-algebraic approach to quantum groups. The geometry of Hilbert C*-modules has been investigated by Solel in [12], where the isometries of these Banach spaces have been characterized. See also [1].

On the other hand Kaup, searching for a metric-algebraic setting in which he could make the study of bounded symmetric domains in complex Banach spaces, introduced a class of complex Banach spaces called JB*-triples. In 1983 he proved that, except for a biholomorphic bijection, every such a domain is the open unit ball of a JB*-triple [5]. In 1981 he made the complete analytic classification of bonded symmetric domains in reflexive Banach spaces [6]. Since then the study of JB*-triples has grown and spread considerably.

These two theories have developed independently from one another. Here we show that every Hilbert C*-module is, in a canonical way, a JB*-triple, a bridge between the two theories that may be useful in the study of the geometry of Hilbert C*-modules. We also establish an explicit expression of the holomorphic automorphisms of the unit ball of a Hilbert C*-module.

1.2 Hilbert C*-modules

We now introduce formally the objects we shall be studying. Let A be a C*-algebra (not necessarily unital or commutative) where the product is denoted by juxtaposition xy, the norm is $\|\cdot\|_A$ and the $x \mapsto x^*$ stands for the conjugation. An *inner product* A-module is a complex linear space E with two laws of composition $E \times A \to E$ (denoted by $(x, a) \mapsto x.a$) and $E \times E \to A$ (denoted by $(x, y) \mapsto \langle x, y\rangle$) such that the following properties hold:

1. With respect to the operation $(x, a) \mapsto x.a$, E is a right A-module with a compatible scalar multiplication, that is, $\lambda(x.a) = (\lambda x).a = x.(\lambda a)$ for all $x \in E$, $a \in A$ and $\lambda \in \mathbb{C}$.

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2. The inner product $(x, y) \mapsto \langle x, y\rangle$ satisfies

$$\begin{aligned} \langle x, \alpha y + \beta z\rangle &= \alpha\langle x, y\rangle + \beta\langle x, z\rangle, & (1)\\ \langle x, y.a\rangle &= \langle x, y\rangle a & (2)\\ \langle y, x\rangle &= \langle x, y\rangle^* & (3)\\ \langle x, x\rangle &\leq 0, \qquad \text{if} \quad \langle x, x\rangle = 0 \ \text{ then } \ x = 0. & (4) \end{aligned}$$

for all $x, yz \in E$, all $\alpha, \beta \in \mathbb{C}$ and all $a \in A$.

Note that in particular, the inner product is complex linear in the second variable while it is conjugate linear in the first. This convention is in line with the recent research literature. Let E be an inner product A-module; then the Cauchy-Schwarz inequality

$$\langle y, x\rangle\, \langle x, y\rangle \leq \|\langle x, x\rangle\|\, \langle y, y\rangle,$$

holds, hence $\|x\|^2_E \colon = \|\langle x, x\rangle\|_A$ is a norm in E with respect to which the inner product and the module product are continuous, that is

$$\|\langle x, y\rangle\|_A \leq \|x\|_E\, \|y\|_E, \qquad \|x.a\| \leq \|x\|_E\, \|a\|_A.$$

To simplify the notation we shall use the same symbol $\|\cdot\|$ to denote the norms on A and E. We can also define an A-valued *norm* by $|x| \colon = \langle x, x\rangle^{\frac{1}{2}}$ for $x \in E$ and we have

$$|\langle x, y\rangle| \leq \|x\|\, |y|, \qquad |\langle x, y\rangle| \leq |x|\, \|y\|,$$

and the module product is also continuous with respect to the new norm on A since

$$\|x.a\| \leq \|x\|\, |a|.$$

However the A-valued norm in an inner A-module E need to be handled with care. For example it need not be the case $|x + y| \leq |x| + |y|$. An inner product A-module E which is a Banach space with respect to the norm $\|\cdot\|$ is called a *Hilbert C*-algebra module.* Every C*-algebra can be converted into a Hilbert C*-algebra module by taking $E \colon = A$ with the natural module operation $x.a \colon = xa$ and the inner product $\langle a, b\rangle \colon = a^*\, b$ for $x, a, b \in A$.

A linear map $f \colon E \to E$ is called an A-map if $f(x.a) = f(x).a$ holds for all $x \in E$ and $a \in A$, and we say that f is *adjointable* if there exists an A-map $f^* \colon E \to E$ such that

$$\langle f(x), y\rangle = \langle x, f^*(y)\rangle, \qquad x, y \in E.$$

In such a case f is continuous (though the converse is not true!), f^* is adjointable and $(f^*)^* = f$. We let $\mathcal{A}(E) \subset \mathcal{L}(E)$ denote the vector space of all adjointable A-module maps on E. In fact $\mathcal{A}(E)$ is a C*-algebra in the operator norm since $\|f^*f\| = \|f\|^2$ holds for all $f \in \mathcal{A}(E)$. For $x,\ y \in E$ we define $\theta_{x,y}$ (also denoted by $x \otimes y^*$) by

$$\theta_{x,y}(z) \colon = x.\, \langle y, z\rangle, \qquad z \in E,$$

Then $\theta_{x,y}$ is adjointable and $\theta^*_{x,y} = \theta_{y,x}$ (see [10] p. 9). For later reference we state the following

Lemma 1.1 *Let E be a Hilbert C*-module and let $f \colon E \to E$ be a bounded A-module map. Then f is a positive element in the C*-algebra $\mathcal{A}(E)$ if and only if $\langle x, f(x)\rangle \geq o$ for all $x \in E$.*

We refer to [10] for background on Hilbert C*-modules and for the proofs of the above results.

1.3 JB*-triples.

For a complex Banach space X denote by $\mathcal{L}(X)$ the Banach algebra of all bounded complex-linear operators on X. A complex Banach space Z with a continuous mapping $(a, b, c) \mapsto \{a, b, c\}$ from $Z \times Z \times Z$ to Z is called a *JB*-triple* if the following conditions are satisfied for all $a, b, c, d \in Z$, where the operator $a\Box b \in \mathcal{L}(Z)$ is defined by $z \mapsto \{abz\}$ and $[\,,\,]$ is the commutator product:

1. $\{abc\}$ is symmetric complex linear in a, c and conjugate linear in b.

2. $[a\Box b,\ c\Box d] = \{a, b, c\}\Box d - c\Box\{d, a, b\}$ (called the Jordan identity.)

3. $a\Box a$ is hermitian and has spectrum ≥ 0.

4. $\|\{a,a,a\}\| = \|a\|^3$.

If a complex vector space Z admits a JB*-triple structure, then the norm and the triple product determine each other. An *automorphism* is a linear bijection $\phi \in \mathcal{L}(Z)$ such that $\phi\{z,z,z\} = \{(\phi z),(\phi z),(\phi z)\}$ for $z \in Z$, which occurs if and only if ϕ is a surjective linear isometry of Z.

Recall that every C*-algebra Z is a JB*-triple with respect to the triple product $2\{abc\} := (ab^*c + cb^*a)$. In that case, every projection in Z is a tripotent and more generally the tripotents are precisely the partial isometries in Z. C*-algebra derivations and C*-automorphisms are derivations and automorphisms of Z as a JB*-triple though the converse is not true.

We refer to [5], [13] and the references therein for the background of JB*-triples theory.

1.4 Hilbert C^*-modules are JB*-triples.

For $a \in A$ fixed, we denote by $R_a \in \mathcal{L}(E)$ the operator $x \mapsto x.a$ of right multiplication by a.

Theorem 1.2 *Every Hilbert C*-module E is a a JB*-triple in a canonical way*

PROOF.
Let E be a Hilbert C*-module over the C*-algebra A and define a triple product in E by

$$2\,\{x,y,z\} := x.\langle y,z\rangle + z.\langle y,x\rangle, \qquad x,y,z \in E. \tag{5}$$

It is clear that $\{\cdot,\cdot,\cdot\}$ symmetric complex linear in the external variables, and complex conjugate linear in the middle variable. It is a matter of routine calculation to check that the triple product satisfies the Jordan identity. On the other hand, for fixed $x \in E$ we have

$$2\,(x\Box x)\,z = x.\langle x,z\rangle + z.\langle x,x\rangle, \qquad z \in E$$

which can be written in the form $x\Box x = \frac{1}{2}(\theta_{x,x} + R_{|x|^2})$. We show that the summands in the right hand side of the latter are hermitian elements in the algebra $\mathcal{L}(E)$. Since $\mathcal{A}(E)$ is a closed complex subalgebra of $\mathcal{L}(E)$ and contains the unit element, it suffices to consider the numerical range of $\theta_{x,x}$ and $R_{|x|^2}$ viewed as elements in the C*-algebra $\mathcal{A}(E)$, and we have seen before that $\theta_{x,x}$ is selfadjoint. Clearly

$$(\exp\, itR_{|x|^2})\,(w) = w.(\exp\, it|x|^2) \qquad w \in E,$$

and as $\exp\, it|x|^2$ is a unitary element in A, the operator $\exp\, itR_{|x|^2}$ is an isometry of E for all $t \in \mathbb{R}$, which shows that $R_{|x|^2}$ is hermitian. For $y \in E$ we have

$$\langle y,\ \theta_{x,x}(y)\rangle = \langle y,\ x.\langle x,y\rangle\rangle = \langle y,x\rangle\langle x,y\rangle \geq 0$$

which by (1.1) proves that $\theta_{x,x} \geq 0$ in $\mathcal{A}(E)$ hence also in $\mathcal{L}(E)$. Clearly $|x|^2 \geq 0$ in A, hence its spectrum satisfies $\sigma_A(|x|^2) \subset [0,\infty)$ and therefore

$$\sigma_{\mathcal{L}(A)}(R_{|x|^2}) \subset \sigma_A(|x|^2) \subset [0,\infty)$$

Since the numerical range is the convex hull of the spectrum, $R_{|x|^2} \geq 0$ as we wanted to check.

Let us set $y := \langle x,x\rangle \in A$ for every $x \in E$. The definition of the norm in E and the properties of the norm in the C*-algebra A yield

$$\begin{aligned} \|\{x,x,x\}\|^2 &= \|x.\langle x,x\rangle\|^2 = \|\langle x.\langle x,x\rangle,\ x.\langle x,x\rangle\rangle\| = \\ \|\langle x,x\rangle\,\langle x,x\rangle\,\langle x,x\rangle\| &= \|\{y,y,y\}\| = \|y\|^3 = \|\langle x,x\rangle\|^3 = \|x\|^6 \end{aligned}$$

which shows property 4. Finally, this is the unique JB*-triple structure on E since the triple product is determined by the norm of E. $\square$

1.5 Holomorphic automorphisms of the unit ball.

Motivated by the deep formal analogy between Hilbert C*-modules E and Hilbert spaces H, we shall establish an explicit formula for the holomorphic automorphisms of the unit ball of E. Recall [5] that, for $c \in E$, the *Bergmann operator* of E is given by

$$B(c,c)(x) \colon = x - 2(c\Box c)(x) + Q_c^2(x), \qquad x \in E.$$

In our case

$$\begin{aligned} 2(c\Box c)(x) &= 2\{c,c,x\} = c.\langle x,x\rangle + x.|c|^2 = c\otimes c^*(x) + x.|c|^2, \\ Q_c^2(x) &= \{c,\, Q_c(x),\, c\} = \{c,\ c.\langle x,c\rangle,\, c\} = \\ c.\langle c.\langle x,c\rangle,\, c\rangle &= c.\langle c,x\rangle|c|^2 = (c\otimes c^*)(x.|c|^2). \end{aligned}$$

Therefore

$$\begin{aligned} B(c,c)(x) &= x.(\mathbf{1} - |c|^2) + (c\otimes c^*)\left(x.(\mathbf{1} - |c|^2)\right) = \\ &= (\mathbf{1} - c\otimes c^*)\left(x.(\mathbf{1} - |c|^2)\right). \end{aligned}$$

Recall that $\mathbf{1} - c\otimes c^*$ and $\mathtt{1} - |c|^2$ are selfadjoint elements in the C*-algebras $\mathcal{A}(E)$ and A respectively, hence they have well defined square roots. We show the operator

$$B_c(x) \colon = (\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\left(x.(\mathtt{1} - |c|^2)^{\frac{1}{2}}\right), \qquad x \in E,$$

satisfies $B_c^2 = B(c,c)$. Indeed, since $\mathbf{1} - c\otimes c^*$ is an A-linear map so is its square root and we have

$$\begin{aligned} &B_c(B_c(x)) = (\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\left(B_c(x).\,(\mathtt{1} - |c|^2)^{\frac{1}{2}}\right) = \\ &\left((\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\, B_c(x)\right).(\mathtt{1} - |c|^2)^{\frac{1}{2}}) = \\ &\left((\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\left[(\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\, x.(\mathtt{1} - |c|^2)^{\frac{1}{2}}\right]\right).\,(\mathtt{1} - |c|^2)^{\frac{1}{2}} = \\ &(\mathbf{1} - c\otimes c^*)x(\mathtt{1} - |c|^2) = B(c,c)(x) \end{aligned}$$

as we wanted to check.

Now we prove that for c and x in the open unit ball of E, $\mathtt{1} + \langle c,x\rangle$ is an invertible element in A. Indeed, let us denote by $\sigma(a)$ and $v(a)$ the spectrum and the numerical range of a in the unital algebra A. Then

$$\sigma(\mathtt{1} + \langle c,x\rangle) \subset v(\mathtt{1} + \langle c,x\rangle) \subset 1 + v(\langle c,x\rangle).$$

Since $\|\langle c.x\rangle\| \le \|c\|\,\|x\| < 1$, the numerical range $v(\langle c,x\rangle)$ is contained in the open unit disc of $\mathbb{C}$, therefore $-1 \notin v(\langle c,x\rangle)$ and by the above $0 \notin \sigma(\mathtt{1} + \langle c,x\rangle)$. In particular,

$$x.(\mathtt{1} + \langle c,x\rangle)^{-1}$$

is well defined in A. Recall [5] that for c in the open unit ball of E, the *transvection* g_c is the holomorphic automorphism of the open ball of E given by

$$g_c(x) \colon = c + B_c\big(x(\mathbf{1} + c\Box x)^{-1}\big), \qquad \|x\| < 1.$$

Replacing the expression of B_c and $\mathbf{1} + c\Box x$ we get

$$g_c(x) = c + (\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\left[x.(\mathtt{1} + \langle c,x\rangle)^{-1}\,(\mathtt{1} - |c|^2)^{\frac{1}{2}}\right]$$

an expression that can be restated in terms of projections and coincides with the well known formula for the transvections of the ball in a Hilbert space see ([3] p. 21). By [5] every holomorphic automorphism h of the unit ball of E can be represented in the form $h = L\circ g_c$ for some surjective linear isometry of E and some $c \in E$ with $\|c\| < 1$.

1.6 Extreme points of the unit ball.

For a complex Banach spce E, the set $\mathsf{Extr}\, B_E$ of extreme points in the unit ball B_E of E palys an important role in the study of the geometry of E. Obviously, we can replace a HIlbert C*-module E with its associated JB*-triple in order to study the extreme points of the ball B_E. By ([7] prop. 3.5) we have

$$\mathsf{Extr}\, B_E = \{c \in E\colon B(c,c) = 0\}$$

that is, for $c \in E$ the condition $c \in \mathsf{Extr}\, B_E$ is equivalent to $(\mathbf{1} - c \otimes c^*)\left(x.(\mathbf{1} - |c|^2)\right)$ for all $x \in E$. Therefore we have two obvious families of extreme points given by

$$\begin{aligned} E.(\mathsf{1} - |c|) &= \{0\} \Longrightarrow c \in \mathsf{Extr}\, B_E, \\ (\mathsf{Id} - c \otimes c^*)\, E &= \{0\} \Longrightarrow c \in \mathsf{Extr}\, B_E. \end{aligned}$$

These two families may coincide (as it occurs when E is Hilbert space) but in general they are different. We do not know whter every extreme point lies in one of the above families. Every extreme point is a tripotent, that is, it satisfies $c = \{c,c,c\} = c.\langle c, c\rangle$.

It might be interesting to characterize Hilbert C*-triples within the category of JB*-triples.

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