H. BOHR'S THEOREM FOR BOUNDED SYMMETRIC DOMAINS

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Abstract. A theorem of H. Bohr (1914) states that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is a holomorphic map from the unit disc $D \subset \mathbb{C}$ into itself, then $\sum_{k=0}^{\infty} \left| a_k z^k \right| \leq 1$ for $|z| \leq \frac{1}{3}$; the value $\frac{1}{3}$ is optimal. This result has been extended by Liu Taishun and Wang Jianfei (2007) to the bounded symmetric domains of the four classical series, and to polydiscs. The result of Liu and Wang may be generalized to all bounded symmetric domains, with a proof which does not depend on classification.

Introduction

The following theorem was proved by Harald Bohr [Bohr 1914] for $|z| < \frac{1}{6}$, then soon improved to $|z| < \frac{1}{3}$ from remarks by M. Riesz, I. Schur, F. Wiener[1]:

Theorem 1. *Let $f : \Delta \to \Delta$ be a holomorphic function from the unit disc $\Delta \subset \mathbb{C}$ into itself. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then*

$$\sum_{k=0}^{\infty} \left| a_k z^k \right| < 1 \tag{0.1}$$

for $|z| < \frac{1}{3}$. The value $\frac{1}{3}$ is optimal.

This result has been extended by Liu Taishun and Wang Jianfei [Liu–Wang 2007] to the bounded symmetric domains of the four classical series, and to polydiscs, using a case-by-case analysis, as follows:

Theorem 2. *Let Ω be an irreducible bounded symmetric domain of classical type in the sense of Hua Luokeng* [Hua 1963]*, or a polydisc. Denote by $\| \ \|_\Omega$ the Minkowski norm associated to Ω. Let $f : \Omega \to \Omega$ be a holomorphic map and let*

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

be its Taylor expansion in k-homogeneous polynomials f_k. Let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(f(0)) = 0$. Then

$$\sum_{k=0}^{\infty} \frac{\|D\phi(f(0)) \cdot f_k(Z)\|_\Omega}{\|D\phi(f(0))\|_\Omega} < 1 \tag{0.2}$$

for all Z such that $\|Z\|_\Omega < \frac{1}{3}$.

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[1]See [Boas–Khavinson 2000] for biographical elements about Friedrich Wiener, not to be confused with Norbert Wiener.

For $\|Z\|_\Omega > \frac{1}{3}$, there exists a holomorphic map $f : \Omega \to \Omega$ such that (0.2) is not true.

Actually, the proof in [Liu–Wang 2007] depends on the following result, which is proved by the authors for classical domains of type I and IV, using *ad hoc* computations:

Theorem 3. *Let $\Omega \subset V$ be a bounded circled symmetric domain. Let $\| \ \|_\Omega$ be the associated spectral norm on V. Let $u \in \Omega$ and let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$. Then the operator norm of the derivative of ϕ at u is*

$$\|\mathrm{d}\,\phi(u)\|_\Omega = \frac{1}{1 - \|u\|_\Omega^2}.$$

In Section 2, we give a classification independent proof of this result and of Theorem 2, which is valid for any bounded circled symmetric domain.

In the above generalization of Bohr's theorem, one considers the Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

of a bounded holomorphic function or map, and one asks for which z the inequality (0.2) holds. One may ask the same type of question for other decompositions of f. For example, the following problem has been considered recently by several authors:

Problem 1. Let Ω be the unit ball of the ℓ^p norm:

$$\Omega = \left\{ (z_1, \cdots, z_n) \in \mathbb{C}^n \mid \sum_{k=1}^{n} |z_k|^p < 1 \right\}.$$

Let $f : \Omega \to \Delta$ be a holomorphic function and consider the expansion of f in monomials

$$f(z) = \sum_{k_1,\ldots,k_n=0}^{\infty} a_{k_1\cdots k_n} z_1^{k_1} \ldots z_n^{k_n}.$$

Determine the best constant K such that $z \in K\Omega$ ensures

$$\sum_{k_1,\ldots,k_n=0}^{\infty} \left| a_{k_1\cdots k_n} z_1^{k_1} \ldots z_n^{k_n} \right| < 1$$

for all $f : \Omega \to \Delta$.

For results about this type of problem, see [Dineen–Timoney 1989], [Dineen–Timoney 1991], [Boas–Khavinson 1997], [Aizenberg 2000], [Boas 2000], [Defant et al. 2003]

1. H. Bohr's theorem

1.1. We first recall Bohr's theorem in dimension 1. We present here a rather elementary proof of this theorem, based on an inequality of Carathéodory or on an improvement of this inequality using a lemma of F. Wiener. This lemma has been generalized in [Liu–Wang 2007] to some classes of bounded symmetric domains. There are other proofs of Bohr's theorem in the literature, for example by S. Sidon [Sidon 1927], following L. Fejér's method of positive kernels [Fejér 1925]; the same proof was published later by M. Tomić [Tomić 1962], who added the case $a_0 = f(0) = 0$ (see Section 1.2).

Theorem 4 ([Bohr 1914]). *Let $f : \Delta \to \Delta$ be a holomorphic function from the unit disc $\Delta \subset \mathbb{C}$ into itself. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then*

$$\sum_{k=0}^{\infty} \left| a_k z^k \right| \le 1 \tag{1.1}$$

for $|z| \le \frac{1}{3}$. The value $\frac{1}{3}$ is optimal: for $\frac{1}{3} < r < 1$, there exists a holomorphic function $f : D \to D$ such that

$$\sum_{k=0}^{\infty} |a_k| \, r^k > 1.$$

For a holomorphic function $f : \Delta \to \Delta$ with Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, denote by $\mathfrak{M}_f(r)$ the sum of absolute values:

$$\mathfrak{M}_f(r) = \sum_{k=0}^{\infty} |a_k| \, r^k.$$

Let us first prove that the bound $\frac{1}{3}$ cannot be improved. For $0 < \alpha < 1$, let f be defined by

$$f(z) = \frac{\alpha - z}{1 - \alpha z}.$$

Then

$$f(z) = \alpha + \sum_{k=1}^{\infty} \left(\alpha^{k+1} - \alpha^{k-1} \right) z^k,$$

so that $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $a_0 = \alpha$, $a_k = \alpha^{k+1} - \alpha^{k-1}$. As a_k is negative for $k > 0$,

$$\mathfrak{M}(r) = \alpha - \sum_{k=1}^{\infty} \left(\alpha^{k+1} - \alpha^{k-1} \right) r^k = 2\alpha - f(r).$$

Then $\mathfrak{M}(r) > 1$ when $r > \frac{1}{1+2\alpha}$; as $\frac{1}{1+2\alpha} \to \frac{1}{3} + 0$ when $\alpha \to 1 - 0$, there exists for each $r > \frac{1}{3}$ a holomorphic function $f : \Delta \to \Delta$ such that $\mathfrak{M}_f(r) > 1$.

To prove the direct part of Theorem 4, one may use one of the following lemmas.

Lemma 1. *Let $f : \Delta \to \Delta$ be a holomorphic function, with $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then for all $k > 0$,*

$$|a_k| \le 2(1 - |a_0|). \tag{1.2}$$

Lemma 2. *Let $f : \Delta \to \Delta$ be a holomorphic function, with $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then for all $k > 0$,*

$$|a_k| \le 1 - |a_0|^2 \,. \tag{1.3}$$

The inequalities (1.2) result directly from inequalities, due to C. Carathéodory, for holomorphic functions in Δ with positive real part; they can also be used to prove Bohr's theorem. Note that (1.3) is slightly sharper than (1.2), as $(1 - t^2) \le 2(1 - t)$ for all $t \in \mathbb{R}$. We will prove directly the second lemma.

First we prove the special case:

$$|a_1| \le 1 - |a_0|^2 \,. \tag{1.4}$$

Consider the automorphism ϕ of Δ defined by

$$\phi(z) = \frac{z - a_0}{1 - \overline{a_0} z};$$

we have

$$\phi(a_0) = 0, \qquad \phi'(a_0) = \frac{1}{1 - a_0\overline{a_0}},$$
$$(\phi \circ f)'(0) = \phi'(a_0) f'(0) = \frac{a_1}{1 - a_0\overline{a_0}}.$$

As $\phi \circ f$ maps Δ into Δ and $(\phi \circ f)(0) = 0$, we have $\left|(\phi \circ f)'(0)\right| \leq 1$, which implies

$$|a_1| \leq 1 - a_0\overline{a_0}.$$

From (1.4), we will now deduce (1.2) for all $k > 0$. For $k > 0$, define

$$g_k(z) = \frac{1}{k} \sum_{j=1}^{k} f(\mathrm{e}^{2\,\mathrm{i}\,\pi j/k}\, z) = \sum_{m=0}^{\infty} a_{km} z^{km}.$$

From the first equality, it results that g_k maps Δ into Δ, and so does also h_k defined by

$$h_k(z) = \sum_{m=0}^{\infty} a_{km} z^m.$$

We have $h_k(0) = a_0$ and $h'_k(0) = a_k$; applying the inequality (1.4) to the function h_k yields

$$|a_k| \leq 1 - a_0\overline{a_0}.$$

Proof of Bohr's theorem. Using (1.3), we deduce

$$\mathfrak{M}(r) = \sum_{k=0}^{\infty} |a_k|\, r^k \leq |a_0| + (1 - a_0\overline{a_0})\, \frac{r}{1-r}$$

and, for $r \leq \frac{1}{3}$,

$$\mathfrak{M}(r) \leq |a_0| + \frac{1}{2}(1 - a_0\overline{a_0}) = 1 - \frac{1}{2}\left(1 - |a_0|^2\right) < 1,$$

as $|a_0| = |f(0)| < 1$. □

1.2. Another Bohr type theorem holds in dimension 1 for the class of holomorphic functions $f : \Delta \to \Delta$ such that $f(0) = 0$. It was proved in [Tomić 1962] using an analogue of the argument in [Sidon 1927]. An elementary proof for this, which uses another result of L. Fejér [Fejér 1914], can also be found in [Landau 1925].

This type of problem was generalized and partially solved in [Ricci 1955] and [Bombieri 1962] by G. Ricci and E. Bombieri.

Definition 1. *Consider the following classes of analytic functions:*

$$\mathcal{F}_0 = \{f : \Delta \to \Delta\},$$
$$\mathcal{F}_{0,\alpha} = \{f : \Delta \to \Delta \mid f(0) = \alpha\} \qquad (0 \leq \alpha < 1),$$
$$\mathcal{F}_m = \left\{ f : \Delta \to \Delta \mid f(z) = \sum_{k=m}^{\infty} a_k z^k \right\} \qquad (m \in \mathbb{N}),$$
$$\mathcal{F}_{m,\alpha} = \left\{ f : \Delta \to \Delta \mid f(z) = \alpha z^m + \sum_{k=m+1}^{\infty} a_k z^k \right\} \qquad (m \in \mathbb{N}, \quad 0 \leq \alpha < 1).$$

Define the corresponding Bohr numbers *by*

$$B_0 = \sup\{r \mid \mathfrak{M}_f(r) < 1 \text{ for all } f \in \mathcal{F}_0\},$$

$$
\begin{aligned}
B_{0,\alpha} &= \sup\left\{ r \mid \mathfrak{M}_f(r) < 1 \text{ for all } f \in \mathcal{F}_{0,\alpha} \right\}, \\
B_m &= \sup\left\{ r \mid \mathfrak{M}_f(r) < 1 \text{ for all } f \in \mathcal{F}_m \right\}, \\
B_{m,\alpha} &= \sup\left\{ r \mid \mathfrak{M}_f(r) < 1 \text{ for all } f \in \mathcal{F}_{m,\alpha} \right\}.
\end{aligned}
$$

The original Bohr theorem means that $B_0 = \frac{1}{3}$.

The result in [Tomić 1962] (already proved in [Landau 1925]) says that $B_1 > \frac{1}{2}$. In [Ricci 1955], the author proves, among other results, that $\frac{3}{5} < B_1 \leq \frac{1}{\sqrt{2}}$. The optimal result $B_1 = \frac{1}{\sqrt{2}}$ was obtained by [Bombieri 1962].

Let us prove that Ricci's result $B_1 > \frac{3}{5}$ easily follows from Wiener's lemma 2.

Proposition 1. *Let $f : \Delta \to \Delta$ be a holomorphic function such that $f(0) = 0$, $f(z) = \sum_{k=1}^{\infty} a_k z^k$. Then*

$$\sum_{k=0}^{\infty} \left| a_k z^k \right| \leq 1$$

for $|z| \leq \frac{3}{5}$.

Proof. Consider the function

$$g(z) = \frac{f(z)}{z} = \sum_{k=0}^{\infty} a_{k+1} z^k.$$

By the Schwarz lemma, g maps Δ into Δ. Lemma 2 applied to g gives

$$|a_k| \leq 1 - |a_1|^2 \qquad (k > 1).$$

Hence

$$
\begin{aligned}
\mathfrak{M}_f(r) &= \sum_{k=0}^{\infty} |a_k| \, r^k \leq |a_1| \, r + \left(1 - |a_1|^2\right) \frac{r^2}{1-r}, \\
\mathfrak{M}_f\left(\frac{3}{5}\right) &\leq \frac{3}{5}\left(|a_1| + \frac{3}{2}\left(1 - |a_1|^2\right)\right) = 1 - \frac{9}{10}\left(\frac{1}{3} - |a_1|\right)^2 \leq 1.
\end{aligned}
$$

□

2. H. Bohr's theorem for bounded symmetric domains

2.1. Let $\Omega \subset V$ be a bounded circled homogeneous domain in a finite dimensional complex vector space V. See the appendix for notations and general results. Denote by $\| \ \|_\Omega$ the spectral norm associated to Ω.

The main result in [Liu–Wang 2007] is the following theorem, which is proved there for domains of type I (rectangular matrices) and for polydiscs.

Theorem 5. *Let Ω be a bounded circled symmetric domain. Denote by $\| \ \|_\Omega$ the spectral norm associated to Ω. Let $f : \Omega \to \Omega$ be a holomorphic map and let*

$$f = \sum_{k=0}^{\infty} f_k$$

be its Taylor expansion in k-homogeneous polynomials f_k. Let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(f(0)) = 0$. Then

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\,\phi(f(0)) \cdot f_k(z)\|_\Omega}{\|\mathrm{d}\,\phi(f(0))\|_\Omega} < 1 \tag{2.1}$$

for all z such that $\|z\|_\Omega \leq \frac{1}{3}$.

The bound $\frac{1}{3}$ is optimal: for each $z \in \Omega$ with $\|z\|_\Omega > \frac{1}{3}$, there exists a holomorphic map $f = \sum_{k=0}^{\infty} f_k : \Omega \to \Omega$ such that

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\,\phi(f(0)) \cdot f_k(z)\|_\Omega}{\|\mathrm{d}\,\phi(f(0))\|_\Omega} > 1.$$

This theorem will be proved below for all bounded symmetric domains.

2.2. The following properties generalize to all bounded circled symmetric domains results in [Liu–Wang 2007] for the differential at $u \in \Omega$ of an automorphism $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$, and may be of independent interest.

For $u \in V$, denote by τ_u the translation $z \mapsto z + u$ and by $\widetilde{\tau}_u$ the rational map

$$\widetilde{\tau}_u(z) = z^u,$$

where z^u is the *quasi-inverse*

$$z^u = B(z,u)^{-1}\left(z - Q(z)u\right),$$

defined in the open set of points z such that $B(z,u) = \mathrm{id}_V - D(z,u) + Q(x)Q(u)$ is invertible.

Recall that for $u \in \Omega$, the operator $B(u,u)$ is positive (with respect to the Hermitian scalar product on V: $(x \mid y) = \operatorname{tr} D(x,y)$), so that $B(u,u)^t$ is well defined for $t \in \mathbb{R}$. Let $u \in \Omega$ and consider a spectral decomposition

$$u = \lambda_1 e_1 + \cdots + \lambda_r e_r,$$
$$1 > \lambda_1 \geq \cdots \geq \lambda_r \geq 0,$$

where $\mathbf{e} = (e_1, \ldots, e_r)$ is a frame. Then, by (A11),

$$B(u,u) = \sum_{0 \leq i \leq j \leq r} \left(1 - \lambda_i^2\right)\left(1 - \lambda_j^2\right) p_{ij}$$

and $B(u,u)^t$ is given by

$$B(u,u)^t = \sum_{0 \leq i \leq j \leq r} \left(1 - \lambda_i^2\right)^t \left(1 - \lambda_j^2\right)^t p_{ij}, \tag{2.2}$$

where (p_{ij}) is the family of orthogonal projectors onto the subspaces of the simultaneous Peirce decomposition with respect to the frame $\mathbf{e} = (e_1, \ldots, e_r)$.

The following result is well known.

Lemma 3. *For $u \in \Omega$, the map*

$$\phi_u = \widetilde{\tau}_u \circ B(u,u)^{-1/2} \circ \tau_{-u} \tag{2.3}$$

is an automorphism of Ω which sends u to 0. The derivative of ϕ_u at u is

$$\mathrm{d}\,\phi_u(u) = B(u,u)^{-1/2}. \tag{2.4}$$

Proof. Let

$$\psi_u = \tau_u \circ B(u,u)^{1/2} \circ \widetilde{\tau}_{-u}.$$

Then, by [Loos 1977, Proposition 9.8 (1)], ψ_u is an automorphism of Ω. As $\widetilde{\tau}_{-u}(0) = 0$ and $B(u,u)^{1/2}$ is linear, we have $\psi_u(0) = u$. The inverse of ψ_u is

$$\phi_u = \widetilde{\tau}_u \circ B(u,u)^{-1/2} \circ \tau_{-u}$$

and $\phi_u(u) = 0$. The derivative of $\widetilde{\tau}_u$ is

$$\mathrm{d}\,\widetilde{\tau}_u(z) = B(z,u)^{-1}$$

(see [Roos 2000, Proposition III.4.1 (ii)]). As $B(0,u) = \mathrm{id}_V$, it follows that

$$\mathrm{d}\,\phi_u(u) = B(u,u)^{-1/2}.$$

□

Denote by $\| \ \|$ the Hermitian norm on V ($\|z\|^2 = \operatorname{tr} D(z,z)$) and by $\| \ \|$ the associated operator norm for linear endomorphisms of V. The *spectral norm* $\| \ \|_\Omega$ on V is given by

$$\|z\|_\Omega^2 = \frac{1}{2}\|D(z,z)\| = \|Q(z)\|\,;$$

the unit ball of this norm is Ω (see the appendix). We denote also by $\| \ \|_\Omega$ the associated operator norm for linear endomorphisms of V.

Lemma 4. *For $u \in \Omega$, $t \in \mathbb{R}$,*

$$\left\|B(u,u)^t\right\|_\Omega = \left\|B(u,u)^t\right\|. \tag{2.5}$$

That is, the operator $B(u,u)^t$ has the same operator norm with respect to the spectral norm or to the Hermitian norm.

Proof. Let $u \in \Omega$ and consider a spectral decomposition

$$u = \lambda_1 e_1 + \cdots + \lambda_r e_r,$$
$$1 > \lambda_1 \geq \cdots \geq \lambda_r \geq 0,$$

where $\mathbf{e} = (e_1, \ldots, e_r)$ is a frame. Then, by (A11),

$$B(u,u) = \sum_{0\leq i\leq j\leq r} \left(1-\lambda_i^2\right)\left(1-\lambda_j^2\right) p_{ij}$$

and

$$B(u,u)^t = \sum_{0\leq i\leq j\leq r} \left(1-\lambda_i^2\right)^t\left(1-\lambda_j^2\right)^t p_{ij},$$

where (p_{ij}) is the family of orthogonal projectors onto the subspaces of the simultaneous Peirce decomposition with respect to the frame $\mathbf{e} = (e_1, \ldots, e_r)$.

For $t \geq 0$, let $\mu_j \geq 0$ be defined by

$$\left(1-\lambda_i^2\right)^t = 1-\mu_j^2$$

and define

$$w = \mu_1 e_1 + \cdots + \mu_r e_r.$$

Then, again by (A11),

$$B(u,u)^t = B(w,w).$$

Using (A5), we get for $t \geq 0$ and all $z \in V$

$$Q(B(u,u)^t z) = B(u,u)^t Q(z) B(u,u)^t. \tag{2.6}$$

This implies

$$Q(B(u,u)^{-t} z) = B(u,u)^{-t} Q(z) B(u,u)^{-t}, \tag{2.7}$$

so that (2.6) is finally valid for all $t \in \mathbb{R}$.

Now, by (A12), for $t \in \mathbb{R}$, $z \in V$,

$$\left\|B(u,u)^t z\right\|_\Omega^2 = \left\|Q(B(u,u)^t z)\right\| \leq \left\|B(u,u)^t\right\|^2 \|Q(z)\| = \left\|B(u,u)^t\right\|^2 \|z\|_\Omega^2 .$$

This implies

$$\left\|B(u,u)^t\right\|_\Omega \leq \left\|B(u,u)^t\right\|.$$

Recall that

$$B(u,u)^t = \sum_{0\leq i\leq j\leq r} \left(1-\lambda_i^2\right)^t \left(1-\lambda_j^2\right)^t p_{ij},$$

where the p_{ij}'s are orthogonal projections with respect to the Hermitian metric. Then $\|B(u,u)^t\|$ is the greatest eigenvalue of $B(u,u)^t$, and $\|B(u,u)^t\|_\Omega \geq \|B(u,u)^t\|$. □

For $u \in \Omega$ with spectral decomposition $u = \lambda_1 e_1 + \cdots + \lambda_r e_r$, $1 > \lambda_1 \geq \cdots \geq \lambda_r \geq 0$, denote by $\beta(u)$ the greatest eigenvalue of $B(u,u)$.

Lemma 5. *Let Ω be a bounded* irreducible *symmetric domain with invariants a, b, r. For $u \in \Omega$ with spectral decomposition $u = \lambda_1 e_1 + \cdots + \lambda_r e_r$, $1 > \lambda_1 \geq \cdots \geq \lambda_r \geq 0$, the value of $\beta(u)$ is*

(1) *if Ω is the unit disc of $\mathbb{C}$ ($b = 0$, $r = 1$),*

$$\beta(u) = \left(1 - \|u\|_\Omega^2\right)^2;$$

(2) *if Ω is the Hermitian ball of dimension $n > 1$ ($b = n-1$, $r = 1$),*

$$\beta(u) = 1 - \|u\|_\Omega^2;$$

(3) *if Ω is of tube type ($b = 0$),*

$$\beta(u) = \left(1-\lambda_r^2\right)^2;$$

(4) *if Ω is of non tube type ($b > 0$),*

$$\beta(u) = 1 - \lambda_r^2.$$

Proof. The lemma follows from the fact that the eigenvalues of $B(u,u)$ are: $(1-\lambda_i^2)^2$ $(1 \leq i \leq r)$, $(1-\lambda_i^2)(1-\lambda_j^2)$ $(1 \leq i < j \leq r)$ and *only in the non tube case* $1 - \lambda_i^2$ $(1 \leq i \leq r)$. □

Note that $\lambda_1 = \lambda_r$ occurs if and only if u is a scalar multiple of a *maximal tripotent* (an element of the Shilov boundary of Ω). In this case, one has $\beta(u) = \left(1 - \|u\|_\Omega^2\right)^2$ or $\beta(u) = 1 - \|u\|_\Omega^2$, depending on whether the domain Ω is of tube type or not.

Proposition 2. *Let Ω be a bounded circled symmetric domain. For $u \in \Omega$ we have*

$$\left\|B(u,u)^{-1/2}\right\|_\Omega = \frac{1}{1-\|u\|_\Omega^2}, \tag{2.8}$$

$$\left\|B(u,u)^{-1/2}u\right\|_\Omega = \frac{\|u\|_\Omega}{1-\|u\|_\Omega^2}, \tag{2.9}$$

$$\left\|B(u,u)^{1/2}\right\|_\Omega = \beta(u)^{1/2}. \tag{2.10}$$

For any automorphism $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$,

$$\|\mathrm{d}\,\phi(u)\|_\Omega = \frac{1}{1-\|u\|_\Omega^2}, \tag{2.11}$$

$$\|\mathrm{d}\,\phi(u)u\|_\Omega = \frac{\|u\|_\Omega}{1-\|u\|_\Omega^2} \tag{2.12}$$

$$\left\| (\mathrm{d}\,\phi(u))^{-1} \right\|_\Omega = \beta(u)^{1/2}. \tag{2.13}$$

Proof. Let $u = \lambda_1 e_1 + \cdots + \lambda_r e_r$, $1 > \lambda_1 \geq \cdots \geq \lambda_r \geq 0$ be a spectral decomposition of u. The operator norm $\left\| B(u,u)^{-1/2} \right\|$ with respect to the Hermitian product is its greatest eigenvalue $\left(1-\lambda_1^2\right)^{-1} = \left(1 - \|u\|_\Omega^2\right)^{-1}$; then Lemma 4 implies (2.8). In the same way, Lemma 4 implies (2.10). We have

$$B(u,u)^{-1/2} u = \sum_{i=1}^{r} \frac{\lambda_i}{\left(1-\lambda_i^2\right)^{1/2}} e_i,$$

hence

$$\left\| B(u,u)^{-1/2} u \right\|_\Omega = \frac{\lambda_1}{\left(1-\lambda_1^2\right)^{1/2}}$$

and (2.9).

Let ϕ be any automorphism $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$. Then $\phi = k \circ \phi_u$, where k is a linear automorphism of Ω; the spectral norm is invariant by k and

$$\begin{aligned}
\|\mathrm{d}\,\phi(u)\|_\Omega &= \left\| k \circ B(u,u)^{-1/2} \right\|_\Omega = \left\| B(u,u)^{-1/2} \right\|_\Omega, \\
\|\mathrm{d}\,\phi(u) u\|_\Omega &= \left\| k \circ B(u,u)^{-1/2} u \right\|_\Omega = \left\| B(u,u)^{-1/2} u \right\|_\Omega, \\
\left\| (\mathrm{d}\,\phi(u))^{-1} \right\|_\Omega &= \left\| B(u,u)^{1/2} \right\|_\Omega.
\end{aligned}$$

□

2.3. Let Ω be a bounded circled symmetric domain. If $f : \Omega \to \Omega$ is a holomorphic map, denote by

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

its Taylor expansion in k-homogeneous polynomials f_k. The following lemma generalizes Lemma 2 to bounded symmetric domains.

Lemma 6. *Let $u = f(0)$ and let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$. Then*

$$\|\mathrm{d}\,\phi(u) \cdot f_k(z)\|_\Omega \leq \|z\|_\Omega^k. \tag{2.14}$$

Proof. For $k > 1$, consider $g_k : \Omega \to \Omega$ defined by

$$g_k(z) = \frac{1}{k} \sum_{j=1}^{k} f\left(\mathrm{e}^{2\,\mathrm{i}\,\pi j/k}\, z\right) = \sum_{m=0}^{\infty} f_{mk}(z).$$

Let $h_k = \phi \circ g_k$. Then

$$\begin{aligned}
h_k(z) &= \phi\left(\sum_{m=0}^{\infty} f_{mk}(z)\right) \\
&= \phi(u) + \mathrm{d}\,\phi(u) \cdot \left(\sum_{m=1}^{\infty} f_{mk}(z)\right) + \cdots = \mathrm{d}\,\phi(u) \cdot f_k(z) + \cdots
\end{aligned}$$

and, for $z \in \Omega$,

$$\mathrm{d}\,\phi(u) \cdot f_k(z) = \int_0^1 \phi \circ g_k\left(\mathrm{e}^{2\,\mathrm{i}\,\pi t}\, z\right) \mathrm{e}^{-2\,\mathrm{i}\,\pi k t}\, \mathrm{d}\,t.$$

This implies

$$\left\|\mathrm{d}\,\phi(u)\cdot f_k(z)\right\|_\Omega \leq 1$$

for $z \in \Omega$, and by homogeneity of f_k,

$$\left\|\mathrm{d}\,\phi(u)\cdot f_k(z)\right\|_\Omega \leq \|z\|_\Omega^k$$

for $z \in V$. □

Proposition 3. *Let Ω be a bounded circled symmetric domain. Let $u \in \Omega$ and let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$.*

Then

$$\sum_{k=0}^{\infty} \frac{\left\|\mathrm{d}\,\phi(u)\cdot f_k(z)\right\|_\Omega}{\left\|\mathrm{d}\,\phi(u)\right\|_\Omega} < 1 \tag{2.15}$$

for all $f : \Omega \to \Omega$ such that $f(0) = u$ and for all z such that $\|z\|_\Omega \leq \frac{1}{2+\|u\|_\Omega}$.

Proof. The proof goes along the same lines as in [Liu–Wang 2007]. For $z \in V$ we have by Lemma 6

$$\sum_{k=0}^{\infty} \frac{\left\|\mathrm{d}\,\phi(f(0))\cdot f_k(z)\right\|_\Omega}{\left\|\mathrm{d}\,\phi(f(0))\right\|_\Omega} \leq \|u\|_\Omega + \frac{1}{\left\|\mathrm{d}\,\phi(u)\right\|_\Omega}\sum_{k=1}^{\infty} \|z\|_\Omega^k .$$

Using $\left\|\mathrm{d}\,\phi(u)\right\|_\Omega = \frac{1}{1-\|u\|_\Omega^2}$ from Proposition 2, we get for $\|z\|_\Omega \leq r < 1$

$$\sum_{k=0}^{\infty} \frac{\left\|\mathrm{d}\,\phi(f(0))\cdot f_k(z)\right\|_\Omega}{\left\|\mathrm{d}\,\phi(f(0))\right\|_\Omega} \leq \|u\|_\Omega + \left(1-\|u\|_\Omega^2\right)\frac{r}{1-r}.$$

For fixed u, the right hand side is less than 1 if and only if $r < \frac{1}{2+\|u\|_\Omega}$. □

Proposition 4. *Let Ω be a bounded circled symmetric domain. Let $u \in \Omega$ and let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$. Assume $\|u\|_\Omega > \frac{1}{3}$ and $\frac{1}{1+2\|u\|_\Omega} < a < 1$.*

Then there exist a holomorphic map $f : \Omega \to \Omega$ with $f(0) = u$ and $z \in \Omega$ with $\|z\|_\Omega = a$, such that

$$\sum_{k=0}^{\infty} \frac{\left\|\mathrm{d}\,\phi(u)\cdot f_k(z)\right\|_\Omega}{\left\|\mathrm{d}\,\phi(u)\right\|_\Omega} > 1.$$

Proof. Consider $u \in \Omega$ with spectral decomposition

$$\begin{aligned} u &= \lambda_1 e_1 + \cdots + \lambda_r e_r, \\ 1 > \|u\|_\Omega &= \lambda_1 \geq \cdots \geq \lambda_r \geq 0. \end{aligned}$$

Define $f : \Omega \to \Omega$ by

$$f(z) = \sum_{i=1}^{r} \frac{\lambda_i - (z \mid e_i)}{1 - \lambda_i\,(z \mid e_i)} e_i.$$

This is a well defined holomorphic map, as $|(z \mid e_i)| < 1$ for all $z \in \Omega$, due to the convexity of Ω. From

$$\frac{\lambda - \zeta}{1 - \lambda\zeta} = \lambda + \sum_{k=1}^{\infty}\left(\lambda^{k+1} - \lambda^{k-1}\right)\zeta^k,$$

we get the Taylor expansion of f:

$$f = \sum_{k=0}^{\infty} f_k,$$
$$u = f(0) = f_0(z),$$
$$f_k(z) = \sum_{i=1}^{r} \left(\lambda_i^{k+1} - \lambda_i^{k-1}\right) (z \mid e_i)^k \, e_i \qquad (k \geq 1).$$

In particular, taking $z_0 = ae_1$, we have

$$f_k(z_0) = \left(\lambda_1^{k+1} - \lambda_1^{k-1}\right) a^k e_1 \qquad (k \geq 1).$$

Take ϕ_u as in Lemma 3. Then

$$\mathrm{d}\,\phi_u(u) = B(u,u)^{-1/2},$$
$$\|\mathrm{d}\,\phi_u(u)\|_\Omega = \frac{1}{1 - \|u\|_\Omega^2}.$$

From (2.2) we get

$$B(u,u)^{-1/2} e_1 = (1 - \|u\|_\Omega^2)^{-1} e_1,$$
$$\|\mathrm{d}\,\phi_u(u) \cdot u\| = (1 - \|u\|_\Omega^2)^{-1} \|u\|_\Omega\,,$$
$$\mathrm{d}\,\phi_u(u) \cdot f_k(z_0) = -a^k \lambda_1^{k-1} e_1,$$
$$\|\mathrm{d}\,\phi_u(u) \cdot f_k(z_0)\|_\Omega = a^k \lambda_1^{k-1} = a^k \|u\|_\Omega^{k-1}.$$

Hence

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\,\phi(u) \cdot f_k(z_0)\|_\Omega}{\|\mathrm{d}\,\phi(u)\|_\Omega} = \|u\|_\Omega + (1 - \|u\|_\Omega^2) \sum_{k=1}^{\infty} a^k \|u\|_\Omega^{k-1}$$
$$= \|u\|_\Omega + (1 - \|u\|_\Omega^2) \frac{a}{1 - a\|u\|_\Omega}.$$

An elementary computation shows that this is greater than 1 if and only if

$$a > \frac{1}{1 + 2\|u\|_\Omega}.$$

□

Proof of Theorem 5. Proposition 3 shows that (2.15) is satisfied for all maps $f : \Omega \to \Omega$ and all z such that $\|z\|_\Omega < \frac{1}{3}$. Moreover, for $\|z\|_\Omega = \frac{1}{3}$,we have

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\,\phi(f(0)) \cdot f_k(z)\|_\Omega}{\|\mathrm{d}\,\phi(f(0))\|_\Omega} \leq \|f(0)\|_\Omega + \left(1 - \|f(0)\|_\Omega^2\right) \sum_{k=1}^{\infty} \frac{1}{3^k}$$
$$= \|f(0)\|_\Omega + \frac{1 - \|f(0)\|_\Omega^2}{2} = 1 - \frac{(1 - \|f(0)\|_\Omega)^2}{2} < 1.$$

As $\frac{1}{1+2\|u\|_\Omega} \to \frac{1}{3}$ as $\|u\|_\Omega \to 1 - 0$, proposition 4 implies that $\frac{1}{3}$ is the optimal bound. □

2.4. Open questions.

Problem 2. Let $f : \Omega \to \Omega$ be a holomorphic map.

(1) With the assumption $f(0) = 0$, Theorem 5 gives

$$\sum_{k=0}^{\infty} \|f_k(z)\|_\Omega < 1 \tag{2.16}$$

for all z such that $\|z\|_\Omega < \frac{1}{2}$. Is the optimal bound equal to $\frac{1}{\sqrt{2}}$, as proved by E. Bombieri [Bombieri 1962] in the one dimensional case?

(2) What is the same optimal bound for all maps f satisfying $f(0) = u$, with $u \in \Omega$ fixed? Propositions 3 and 4 show that this optimal bound belongs to $\left[\frac{1}{2+\|u\|_\Omega}, \frac{1}{1+2\|u\|_\Omega}\right]$. In the one dimensional case, this is Ricci's estimate (see [Ricci 1955]). But Bombieri's results for the one dimensional case show that this estimate may be sharpened: in particular, the optimal bound is $\frac{1}{\sqrt{2}}$ when $u = 0$, $\frac{1}{1+2\|u\|_\Omega}$ when $\frac{1}{2} < \|u\|_\Omega < 1$.

For Ω a bounded circled symmetric domain, a natural problem generalizing H. Bohr's problem would be the following.

Problem 3. Let $\Omega \subset V$ be a bounded circled symmetric domain of rank r. Let $f : \Omega \to \Omega$ be a holomorphic map and consider the *Schmidt decomposition* of its Taylor expansion

$$f(z) = \sum_{k_1 \geq \ldots k_r \geq 0} f_{k_1 \cdots k_r}(z)$$

(where the $f_{k_1 \cdots k_r}$'s are polynomials in the irreducible K-modules for the linear subgroup K of $\operatorname{Aut} \Omega$). Determine the best constant C such that $z \in C\Omega$ ensures

$$\sum_{k_1 \geq \ldots k_r \geq 0} \|f_{k_1 \cdots k_r}(z)\|_\Omega < 1$$

for all holomorphic maps $f : \Omega \to \Omega$.

Appendix

We recall here some notations and results about complex bounded symmetric domains and their associated Jordan triple structure (see [Loos 1977], [Roos 2000]).

Bounded symmetric domains and Jordan triples. Let Ω be an irreducible bounded circled homogeneous domain in a complex vector space V. This circled realization is unique up to a linear isomorphism. Let K be the identity component of the (compact) Lie group of (linear) automorphisms of Ω leaving 0 fixed. Let ω be a volume form on V, invariant by K and by translations. Let $\mathcal{K}$ be the Bergman kernel of Ω with respect to ω, that is, the reproducing kernel of the Hilbert space $H^2(\Omega, \omega) = \operatorname{Hol}(\Omega) \cap L^2(\Omega, \omega)$. The Bergman metric at $z \in \Omega$ is defined by

$$h_z(u, v) = \partial_u \overline{\partial}_v \log \mathcal{K}(z).$$

The *Jordan triple product* on V is characterized by

$$h_0(\{uvw\}, t) = \partial_u \overline{\partial}_v \partial_w \overline{\partial}_t \log \mathcal{K}(z) \,|_{z=0} \,.$$

The triple product $(x, y, z) \mapsto \{xyz\}$ is complex bilinear and symmetric with respect to (x, z), complex antilinear with respect to y. It satisfies the *Jordan identity*

$$\{xy\{uvw\}\} - \{uv\{xyw\}\} = \{\{xyu\}vw\} - \{u\{vxy\}w\}. \tag{J}$$

The space V endowed with the triple product $\{xyz\}$ is called a *(Hermitian) Jordan triple system.* For $x, y, z \in V$, denote by $D(x, y)$ and $Q(x, z)$ the operators defined by

$$\{xyz\} = D(x, y)z = Q(x, z)y. \tag{A1}$$

The Bergman metric at 0 is related to D by

$$h_0(u, v) = \operatorname{tr} D(u, v). \tag{A2}$$

A Jordan triple system is called *Hermitian positive* if $(u|v) = \operatorname{tr} D(u, v)$ is positive definite. As the Bergman metric of a bounded domain is always definite positive, the Jordan triple system associated to a bounded symmetric domain is Hermitian positive.

The *quadratic representation*

$$Q : V \longrightarrow \operatorname{End}_{\mathbb{R}}(V)$$

is defined by $Q(x)y = \frac{1}{2}\{xyx\}$. The *Bergman operator* B is defined by

$$B(x, y) = I - D(x, y) + Q(x)Q(y), \tag{A3}$$

where I denotes the identity operator in V. The quadratic representation and the Bergman operator satisfy to many identities, the most important of which are

$$Q(Q(x)y) = Q(x)Q(y)Q(x), \tag{A4}$$

$$Q(B(x, y)z) = B(x, y)Q(z)B(y, x) \tag{A5}$$

(see [Loos 1975], [Roos 2000]).

The Bergman operator gets its name from the following property:

$$h_z\left(B(z, z)u, v\right) = h_0(u, v) \quad (z \in \Omega;\ u, v \in V).$$

The Bergman kernel of Ω is then given by

$$\mathcal{K}(z) = \frac{1}{\operatorname{vol}\Omega} \frac{1}{\det B(z, z)}.$$

Spectral theory. Let V be an Hermitian positive Jordan triple system. An element $c \in V$ is called *tripotent* if $c \neq 0$ and $\{ccc\} = 2c$.

Two tripotents c_1 and c_2 are called *orthogonal* if $D(c_1, c_2) = 0$. If c_1 and c_2 are orthogonal tripotents, then $D(c_1, c_{1)}$ and $D(c_2, c_2)$ commute and $c_1 + c_2$ is also a tripotent.

A tripotent c is called *primitive* (or *minimal*) if it is not the sum of two orthogonal tripotents. A tripotent c is *maximal* if there is no tripotent orthogonal to c.

A *frame* of V is a maximal sequence $(c_1, \ldots, c_r)$ of pairwise orthogonal primitive tripotents. Then there exist frames for V. All frames have the same number of elements, which is the *rank* r of V. The frames of V form a manifold $\mathcal{F}$, which is called the *Satake-Furstenberg boundary* of Ω.

Let V be a simple Hermitian positive Jordan triple system. Then any $x \in V$ can be written in a unique way

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p, \tag{A6}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$ and $c_1, c_2 \ldots, c_p$ are pairwise orthogonal tripotents. The element x is called *regular* iff $p = r$; then $(c_1, c_2, \ldots, c_r)$ is a frame of V. The decomposition (A6) is called the *spectral decomposition* of x.

Let $\mathbf{c} = (c_1, \ldots, c_r)$ be a frame. For $0 \leq i \leq j \leq r$, let

$$V_{ij}(\mathbf{c}) = \left\{ x \in V \mid D(c_k, c_k)x = (\delta_i^k + \delta_j^k)x,\ 1 \leq k \leq r \right\}. \tag{A7}$$

The decomposition

$$V = \bigoplus_{0 \leq i \leq j \leq r} V_{ij}(\mathbf{c}) \tag{A8}$$

is orthogonal with respect to the Hermitian product (A2) and is called the *simultaneous Peirce decomposition* with respect to the frame $\mathbf{c}$.

Let (p_{ij}) be the family of orthogonal projectors onto the subspaces of the decomposition (A8). Then, for $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r$, $\lambda_i \in \mathbb{R}$ $(1 \leq i \leq r)$ and $\lambda_0 = 0$,

$$D(x, x) = \sum_{0 \leq i \leq j \leq r} \left(\lambda_i^2 + \lambda_j^2 \right) p_{ij}, \tag{A9}$$

$$Q(x)^2 = \sum_{0 \leq i \leq j \leq r} \lambda_i^2 \lambda_j^2 p_{ij}, \tag{A10}$$

$$B(x, x) = \sum_{0 \leq i \leq j \leq r} \left(1 - \lambda_i^2 \right) \left(1 - \lambda_j^2 \right) p_{ij}. \tag{A11}$$

(See [Loos 1977, Corollary 3.15]).

The map $x \mapsto \lambda_1$, where $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p$ is the spectral decomposition of x $(\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0)$ is a norm on V, called the *spectral norm*. We will denote the spectral norm by $\|x\|_\Omega$. It satisfies

$$\|x\|_\Omega^2 = \|Q(x)\| = \frac{1}{2} \|D(x, x)\|, \tag{A12}$$

where $\|u\|$ denotes the operator norm of an $\mathbb{R}$-linear operator $u \in \operatorname{End}_{\mathbb{R}} V$ with respect to the Hermitian norm $\|x\|^2 = \operatorname{tr} D(x, x)$.

If V has the Jordan triple structure associated to the bounded symmetric domain $\Omega \subset V$, then Ω is the unit ball of V for the spectral norm. Conversely, if V is a positive Hermitian Jordan triple, the unit ball of the spectral norm is a bounded circled symmetric domain, whose associated Jordan triple is V.

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