The Grassmann Envelope of the Kac Superalgebra sK_{10}

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Abstract

We investigate in detail the structure of the Grassmann envelope $\Gamma(sK_{10})$ of the split Kac superalgebra sK_{10} over an arbitrary ring of scalars, verifying that it is indeed a quadratic Jordan algebra, generically algebraic of degree at most 9, and obtain the bizarre result that in characteristic 5 (but not otherwise) it is the Jordan algebra of a sharped cubic form over Γ_0.[1]

A quadratic Jordan superalgebra over an arbitrary ring of scalars Φ is a $\mathbb{Z}_2$-graded space $J = J_0 \oplus J_1 = A \oplus M$ carrying certain graded products. *All elements will be assumed homogeneous,* in either J_0 or J_1. We follow the convention [10] of denoting even elements of $J_0 = A$ by letters $a, b, c, d, e, f, g, u, v$ and odd elements of $J_1 = M$ by m, n, p, while general homogeneous elements of J of will be denoted by x, y, z (of degree deg(x) etc.) and elements of the Grasmann envelope by tildes. The graded bilinear and trilinear products are denoted $\langle x, y\rangle = V_x(y)$, $\langle x, y, z\rangle = U_{x,z}(y) = V_{x,y}(z)$. The even quadratic products are $U_a x, a^2$ quadratic in a and linear in x, such that $\langle a, y, b\rangle = U_{a,b}y$ is the linearization of the U-operator, and similarly $\langle a, b\rangle$ is the linearization of the square. We denote Jordan bilinear and trilinear products in $\Gamma(sK_{10})$ by braces $\{\widetilde{x}, \widetilde{y}\}$, $\{\widetilde{x}, \widetilde{y}, \widetilde{z}\}$.

The quadratic super-Jordan axioms are that the Grassmann envelope $\Gamma(J) = (\Gamma_0 \otimes J_0) \oplus (\Gamma_1 \otimes J_1)$ becomes a unital quadratic Jordan algebra under the quadratic product

$$U_{\widetilde{x}}\widetilde{y} = \big(\textstyle\sum_{i,j} \alpha_i^2 \beta_j U_{a_i} b_j + \sum_{i<i',\,j} \langle a_i, b_j, a_{i'}\rangle + \sum_{k<k'} \langle x_k, b_j, m_{k'}\rangle + \sum_{i,j,k} \langle a_i, y_\ell, m_k\rangle\big)$$
$$\oplus \big(\textstyle\sum_{i<i',\,k} \langle a_i, n_k, a_{i'}\rangle + \sum_{i,j,k} \langle a_i, b_j, m_k\rangle + \sum_{k<k'} \langle m_k, n_\ell, m_{k'}\rangle\big)$$

for $\widetilde{x} = \sum_i \alpha_i a_i + \sum \gamma_k m_k$, $\widetilde{y} = \sum_k \beta_j b_j + \sum \eta_\ell n_\ell$ $(\alpha, \beta \in \Gamma_0,\ \gamma, \eta \in \Gamma_1)$. As usual we write $(-1)^x$ for $(-1)^{deg(x)}$, $(-1)^{xy}$ for $(-1)^{deg(x)deg(y)}$ [-1 if both x, y are odd, $+1$ otherwise] $(-1)^{xyz}$ for $(-1)^{xy+yz+zx)}$ ["majority rule': -1 if the majority are odd, $+1$ if the majority are even]. A criterion due to K. Meyberg guarantees that a quadratic algebra without 2-torsion will be a quadratic Jordan algebra, and (through its Grassmann envelope) a quadratic superalgebra will be a quadratic Jordan superalgebra, as soon as it satisfies the linear Jordan identity. We apply this to show the free Φ-module $\Gamma(sK_{10}(\Phi))$ is a quadratic Jordan algebra, hence $sK_{10}(\Phi)$ is a quadratic Jordan superalgebra.

For $m, n, p \in M, a, b \in A$, homogeneous $x, y, z \in J$

(0.1.1) Outer SuperSymmetry $\langle x, y\rangle = (-1)^{xy}\langle y, x\rangle$, $\langle x, y, z\rangle = (-1)^{xyz}\langle z, y, x\rangle$,
(0.1.2) Odd Alternation $\langle m, m\rangle = \langle m, n, m\rangle = 0$, $\langle m, n\rangle = -\langle n, m\rangle$, $\langle m, n, p\rangle = -\langle p, n, m\rangle$,
(0.1.3) Peirce Orthogonality $\langle J_{ij}, J_{k\ell}\rangle = \langle J_{ij}, J_{k\ell}, J_{mn}\rangle = 0$ unless indices can be linked

if $J = \bigoplus_{i\leq j} J_{ij}$ is the Peirce decomposition of J relative to a supplementary sum of orthogonal idempotents $1 = \sum_{i=1}^n e_i$.

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1 The Quadratic Criterion

It is in general an onerous task to show that a particular quadratic structure satisfies the axioms (especially the degree-7 Fundamental Formula) for a quadratic Jordan algebra. In particular, to this day nobody has written out the complete set of graded axioms for a quadratic Jordan superalgebra. There is no problem for special algebras and superalgebras, only exceptional algebras present a problem. We now want to emphasize that for a quadratic scheme which is free as a module and defined over $\mathbb{Z}$ (such as the Kac superalgebra), in order to verify strictly the quadratic Jordan axioms for an algebra (or pair/triple)[5, 9, 4]

$$\text{(1.1)}\qquad \begin{aligned}&\text{(QJA1) } U_1 = \mathbf{1}_J,\ \text{(QJA2) } V_{x,y}U_x = U_x V_{y,x},\ \text{(QJA3) } U_{U_x y} = U_x U_y U_x,\\ &\text{(JT1) } P_{P_x y, y} = P_{x, P_y x},\ \text{(JT2) } L_{x,y}P_x = P_x L_{y,x},\ \text{(JT3) } P_{P_x y} = P_x P_y P_x\end{aligned}$$

it suffices to verify the linear Jordan axioms:

$$\text{(1.2)}\qquad \begin{aligned}&\text{(LJA)}\quad 2U_x = V_x^2 - V_{x^2},\ [V_x, U_x] = 0\ (\mathit{or}\ \ [V_x, V_{x^2}] = 0\ \mathit{or}\ \ [V_x, V_{x,x}] = 0)\\ &\text{(LJT)}\quad L_{x,u}P_y + P_y L_{u,x} = P_{\{x,u,y\},y}.\end{aligned}$$

In 1970 K. Meyberg showed [8] that the linear axioms imply the Fundamental Formula (QJA3) in the absence of 2-torsion (along the way he established (QJA2), although he did not state his result for quadratic Jordan algebras), but he did not mention (JT1) as quadratic triples weren't axiomatized until 1972 [9]. Note (LJT) implies (JT1) in the absence of 2-torsion since linearizing $z \to z, x$ yields $0 = \big(L_{x,y}P^{\blacktriangle}_{x,z} + P_{x,z}L_{y,x} - P_{\{x,y,x\},z} - P^{\blacktriangle}_{\{x,y,z\},x}\big)y = 2\big(\{x, P_y x, z\} - \{P_x y, y, z\}\big)$, and (LJT),(JT1) imply (JT2) in the absence of 3-torsion since $0 = \big(L_{x,y}P_x + P_x L_{y,x} - P_{\{x,y,x\},x}\big)z = \big(-\{x, z, P_x y\} + \{x, P_{y,z}x, x\}\big) + P_x\{y, x, z\} - 2P_{P_x y, x}z$ [linearizing $y \to y, z$ in (JT1)] $= -3\{x, z, P_x y\} + 3P_x\{y, x, z\} = 3\big(P_x L_{z,x} - L_{x,z}P_x\big)y$. In the absence of 2-torsion, (LJT),(JT1),(JT2) together imply (JT3) since, as Meyberg showed, $2P_{P_x y}z = L_{P_x y, z}P_x y = P_{\{P_x y, z, x\},x}y - P_x L_{z, P_x y}$ [by (LJT)] $= L_{x,y}L_{x,z}P_x y - P_x\{y, P_x y, z\} = P_x\big(L_{y,x}L_{z,x} - P_{y,z}P_x\big)y$ [using (JT2) twice] $= P_x\big(\{\{z, x, y\}, x, y\} - \{z, P_x y, y\}\big) = P_x\{y, P_x z, y\}$ [by linearized (JT1)] $= 2P_x P_y P_x z$.

It was explicitly proved and stressed by O. Loos [4, 2.2 p,15] that (JT0) implied the axioms for quadratic Jordan pairs, triples, and algebras in the absence of 6-torsion. In the presence of 3-torsion something must be added (usually (JT2)) to the basic 5-linear axiom (LJT). But Meyberg showed [8] that for *algebras* (QJAT2) could be derived from the linear axioms in the absence of 2-torsion.[2]

The Quadratic Criterion states that the quadratic triple axioms for a quadratic system over Φ hold as long as there is no 2- or 3-torsion, and the quadratic algebra axioms hold as long as there is no 2-torsion; in particular, if the system is a scheme defined over $\mathbb{Z}$ on a free module, then there is no torsion whatsoever, and the system over $\mathbb{Z}$ is quadratic Jordan, hence the scalar extension to an arbitrary Φ is quadratic Jordan.

The result for quadratic Jordan systems implies the result for quadratic Jordan supersystems, since the Grassman algebra $\Gamma_k = \bigoplus_{i_1 < \cdots < i_n,\, n \geq 0,\, \bar{n} = \bar{k} \bmod 2} \Phi\gamma_{i_1} \wedge \cdots \wedge \gamma_{i_n}$ is free as Φ-module and

[2] (QJA2) and (LJT) follow from (i) Lie-structurality (JT0) $V_x U_z + U_z V_x = U_{\{x,z\},z}$ of V_x, (ii) $V_{x,y} + V_{y,x} = V_{\{x,y\}}$, (iii) $D_{x,y} := [V_x, V_y] = V_{x,y} - V_{y,x}$ a derivation, and linearized (LJA): $0 = [V_{\{x,y\}}, U_x] + [V_x, U_{x,\{x,y\}}] = [V_{x,y} + V_{y,x}, U_x] + [V_x, V_y U_x + U_x V_y]$ [by (i),(JT0)] $= V_{x,y}U_x^{(1)} + V_{y,x}U_x^{(2)} - U_x V_{y,x}^{(3)} - U_x V_{x,y}^{(4)} + [V_x, V_y]U_x^{(5)} + U_x[V_x, V_y]^{(6)}$ [using (LJA)] $= \big(V_{x,y}U_x^{(1)} - U_x V_{y,x}^{(3)}\big) + \big([V_y, V_x]^{(2a)\blacktriangle} + V_{x,y}^{(2b)}\big)U_x - U_x\big([V_x, V_y]^{(4a)\blacktriangledown} + V_{y,x}^{(4b)}\big) + [V_x, V_y]U_x^{(5)\,\blacktriangle} + U_x[V_x, V_y]^{(6)\blacktriangledown}$ [using (ii) on (2),(4)] $= 2\big(V_{x,y}U_x^{(1,2b)} - U_x V_{y,x}^{(3,4b)}\big)$ yields (QJA2). These also imply (LJT) that $V_{x,y}$ is Lie-structural since $2V_{x,y} = V_{\{x,y\}} + D_{x,y}$ is structural by (JT0) and D a derivation by (iii).

One way to see (i)-(iii) from scratch is to note that $2U_x = V_x^2 - V_{x^2} \Rightarrow 2(V_{x,x} - V_{x^2}) = 0 \Rightarrow V_{x,y} + V_{y,x} = V_{\{x,y\}}$ [as in (ii)] $\Rightarrow V_{z,y} = V_z V_y - U_{z,y} \Rightarrow V_{x,y} - V_{y,x} = D_{x,y} := [V_x, V_y]$, which is a derivation as in (iii) of the square (hence also the quadratic structure in the absence of 2-torsion) by $D(z^2) - \{D(z), z\} = \big(V_x V_{z^2} + V_z V_{\{x,z\}}\big)(y) - \big(V_y V_{z^2}(x) + V_z V_{\{y,z\}}\big)(x) = \big(V_{z^2}V_x^{\blacktriangle} + V_{\{x,z\}}V_z^{\blacktriangledown}\big)(y) - \big(V_{z^2}V_y^{\blacktriangle} + V_{\{y,z\}}V_z^{\blacktriangledown}\big)(x)$ [by linearized (LJA)] $= 0$. Then $D = D_{x,y} = 2V_{x,y} - V_{\{x,y\}}$ a derivation and (LJA) imply Lie-structurality (JT0) as in (i) since $0 = \big(D(\{x, z\}) - \{D(x), z\} - \{x, D(z)\}\big) - \big([V_z, V_{x^2}] + [V_x, V_{\{x,z\}}]\big) = \big(2\{x, y, \{x, z\}\}^{(1)} - V_{\{x,y\}}(\{x, z\})^{\blacktriangle}\big) - V_z\big(\{x, y, x\}^{(2)} - \{y, x, x\}^{\blacktriangledown}\big) - V_x\big(V_x V_y z^{(3)} - V_y V_x z^{\blacktriangleright}\big) + \big(-V_z V_{x^2}^{\blacktriangledown} + V_{x^2}V_z^{(4)} - V_x V_{\{x,z\}}^{\blacktriangleright} + V_{\{x,z\}}V_x^{\blacktriangle}\big)(y) = 2\big(U_{\{x,z\},x}^{(1)} - V_z U_x^{(2)} - U_x V_z^{(3,4)}\big)(y)$.

hence so is the Grassmann envelope $\Gamma(J) := \big(\Gamma_0 \otimes_\Phi J_0\big) \oplus \big(\Gamma_1 \otimes_\Phi J_1\big)$ of any J. The quadratic Jordan axioms concern only the ring structure (the structure of J as $\mathbb{Z}$-algebra), so the Γ_0-algebra $\Gamma(J)$ will be quadratic Jordan as as soon as it is quadratic Jordan as a Φ-algebra.

One frequently said "when $\frac{1}{2} \in \Phi$" when dealing with linear Jordan algebras, whereas the crucial condition was absence of 2-torsion. Of course, in the 2-torsion-free case one can imbed J in $\widetilde{J} = J \otimes_\Phi \Phi[\frac{1}{2}]$ to get an algebra $\widetilde{J}$ with $\frac{1}{2} \in \widetilde{\Phi}$, though one must be careful that the U-operators have been defined in J without the help of $\frac{1}{2}$. It was precisely this creation of $\frac{1}{2}$ and reduction to the linear case in the proof by G. Benkart and A. Elduque [1] that K_{10} was a linear Jordan superalgebra which rekindled the use of the Quadratic Criterion for systems with free module structure. We will use this criterion to verify that the Kac superalgebra over an arbitrary ring of scalars is a quadratic Jordan superalgebra.

2 The Split Kac Superalgebra $sK_{10}(\Phi)$

We make use of the split form of the Kac quadratic Jordan superalgebra $sK_{10}(\Phi) = A \oplus M$, which is a free module of dimension (rank) 10 over an arbitrary ring of scalars introduced in [7]. Here the even Jordan algebra $A = B \boxplus \Phi e_3,\ B = \Phi e_1 \oplus \Phi e_2 \oplus V$ is the direct sum of a 5-dimensional algebra $B = Jord(Q, u)$ of a nondegenerate split quadratic form and a 1-dimensional ideal Φe_3, and the odd part M is a 4-dimensional bimodule with graded bilinear and trilinear products. The superalgebra has split basis $\{e_1, e_2, c_{12}, d_{12}, q_{12}\}$ for $B = J(Q, u)$ for the quadratic form

$$(2.1) \quad Q(b) = \beta_1\beta_2 - \beta_3\beta_4 - \beta_5^2, \quad T(b) = \beta_1 + \beta_2 \quad \textit{for} \quad b = \beta_1 e_1 + \beta_2 e_2 + \beta_3 c_{12} + \beta_4 d_{12} + \beta_5 q_{12},$$

Φe_3 has obvious basis $\{e_3\}$, and M has ordered basis $\{m_{13}, n_{13}, m_{23}, n_{23}\}$, where the subscripts indicate which Peirce space the elements belong to. Using the abbreviation $g_i := 2e_i - 3e_3 \quad (i = 1, 2)$, the bilinear products are given by [7, (5.6)]

(2.2) Bilinear Products $\langle J, J\rangle$

	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	m_{13}	n_{13}	m_{23}	n_{23}
e_1	$2e_1$	0	c_{12}	d_{12}	q_{12}	0	m_{13}	n_{13}	0	0
e_2	0	$2e_2$	c_{12}	d_{12}	q_{12}	0	0	0	m_{23}	n_{23}
c_{12}	c_{12}	c_{12}	0	u	0	0	0	m_{23}	0	$-m_{13}$
d_{12}	d_{12}	d_{12}	u	0	0	0	$-n_{23}$	0	n_{13}	0
q_{12}	q_{12}	q_{12}	0	0	$2u$	0	m_{23}	n_{23}	m_{13}	n_{23}
e_3	0	0	0	0	0	$2e_3$	m_{13}	n_{13}	m_{23}	n_{23}

	m_{13}	n_{13}	m_{23}	n_{23}
m_{13}	0	g_1	$2c_{12}$	q_{12}
n_{13}	$-g_1$	0	$-q_{12}$	$2d_{12}$
m_{23}	$-2c_{12}$	q_{12}	0	g_2
n_{23}	$-q_{12}$	$-2d_{12}$	$-g_2$	0

(2.3) Odd U-Operators $U_{M,M}J$

	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	m_{13}	n_{13}	m_{23}	n_{23}
$U_{m_{13},n_{13}}$	$-3e_3$	0	0	0	0	$2e_1$	$-m_{13}$	$-n_{13}$	0	0
$U_{m_{23},n_{23}}$	0	$-3e_3$	0	0	0	$2e_2$	0	0	$-m_{23}$	$-n_{23}$
$U_{m_{13},m_{23}}$	0	0	0	$-3e_3$	0	$2c_{12}$	0	$-m_{23}$	0	m_{13}
$U_{n_{13},n_{23}}$	0	0	$3e_3$	0	0	$2d_{12}$	n_{23}	0	$-n_{13}$	0
$U_{n_{13},m_{23}}$	0	0	0	0	$3e_3$	$-q_{12}$	$2m_{23}$	$-n_{23}$	$-m_{13}$	$2n_{13}$
$U_{m_{13},n_{23}}$	0	0	0	0	$-3e_3$	q_{12}	m_{23}	$-2n_{23}$	$-2m_{13}$	n_{13}

Further general facts we need about these products are the following.

$$\textit{On } J \textit{ we have } V_{b,c} = V_b V_c - U_{b,c},\ V_{b,b} = V_{b^2} = V_b^2,\ V_{b,b^2} = V_{b^2,b} = V_{b^3},$$

$$(2.4.1) \qquad V_{e_i,e_j} = V_{c_{12},c_{12}} = V_{d_{12},d_{12}} = V_{e_3,B} = V_{B,e_3} = 0\ V_{e_i,e_i} = V_{e_i},\ V_{m,n}a_i = E_i V_m V_n a_i,$$

$$U_{m,m} = 0, \quad U_{n,m} = -U_{m,n}, \quad U_{b,b} = 2U_b;$$

$$\textit{On } M \textit{ we have } V_{b',b} = V_{b'} V_{b^*}\ U_{e_3} = U_b = U_{b,b'} = 0\ U_{e_3,b} = U_{u,b} = V_b.$$

From [7, (7.5)] (and $D_{x,1} = 0$) we get $D_{m,b} = D_{\langle m,b\rangle,u} = -D_{\langle m,b\rangle,e_3} = D_{e_3,\langle b,m\rangle}$,

$$D_{b,m} = D_{\{b,m\},e_3},\ D_{m,b} = D_{e_3,\{b,m\}},\ ,\ D_{B,e_3} = 0 \tag{2.4.2}$$
$$D_{e_i,e_j} = 0,\ D_{m,n} = D_{n,m},\ D_{a,x} = -D_{x,a}$$

From [7, (5.6), (5.4)] we have

$$\begin{aligned} &\langle m_{i3}, n_{i3}\rangle = 2e_i - 3e_3,\ \langle m_{i3}, n_{j3}\rangle = q_{12} = -\langle n_{j3}, m_{i3}\rangle, \\ &\langle m_{i3}, m_{j3}\rangle = (-1)^i 2c_{12},\ \langle n_{i3}, n_{j3}\rangle = (-1)^j 2d_{12}, \\ &\langle c_{12}, n_{i3}\rangle = (-1)^j m_{i3},\ \langle d_{12}, m_{i3}\rangle = (-1)^i n_{j3},\ \langle c_{12}, m_{i3}\rangle = \langle d_{12}, n_{i3}\rangle = 0, \\ &\langle q_{12}, m_{i3}\rangle = m_{j3}, \langle q_{12}, n_{i3}\rangle = n_{j3}. \end{aligned} \tag{2.4.3}$$

From [7, (7.6.5), (7.6.1)] we have

(2.4.4) $D_{m_{13},m_{23}} = 2D'_1 = 2D_{e_1,c_{12}} = -2D_{e_2,c_{12}}$,

(2.4.5) $D_{n_{13},n_{23}} = 2D'_2 = 2D_{e_1,d_{12}} = -2D_{e_2,d_{12}}$,

(2.4.6) $D_{m_{23},n_{13}} = -D_{m_{13},n_{23}} = D'_3 = -D_{e_1,q_{12}} = D_{e_2,q_{12}}$,

(2.4.7) $D_{m_{13},n_{13}} = D_{m_{23},n_{23}} = D_3 = D_{c_{12},d_{12}}$,

(2.4.8) $2D_{m_{13}} = -2D_{m_{23}} = 2D_1 = D_{c_{12},q_{12}} = -D_{q_{12},c_{12}}$,

(2.4.9) $2D_{n_{23}} = -2D_{n_{13}} = 2D_2 = D_{q_{12},d_{12}} = -D_{d_{12},q_{12}}$.

3 $sK_{10}(\Phi)$ is Jordan

Now we are equipped to prove that the Kac superalgebra is a linear Jordan superalgebra and a free module, hence a quadratic Jordan superalgebra. The fact that K_{10} satisfies the linear Jordan super-identity over scalars with $\frac{1}{2}$ was proven in 1996 by Shestakov using his basis [7] which clusters elements into trivial subalgebras; Benkert and Elduque [1] in 2002 gave a proof based on their general formula for the product and left multiplication operator. It seems impossible to extend the description in [1], with its parameter $\frac{3}{4}$, to cover the quadratic case, but Shestakov's proof of $[L_{x^2}, L_x] = 0$ carries over once it is formulated in quadratic terms $[V_{x^2}, V_x] = 0$ and the basis adjusted to fit characteristic 2. We will instead give a proof in the form $D_{\tilde{x}^2,\tilde{x}} = V_{\tilde{x}^2,\tilde{x}} - V_{\tilde{x},\tilde{x}^2} = 0$ in terms of the inner derivations.

Theorem 3.1 *The split superalgebra $sK_{10}(\Phi)$ is a quadratic Jordan superalgebra for all scalar rings Φ.*

PROOF: By the Quadratic Criterion we only need to establish that the Grassmann envelope $\widetilde{J} = \Gamma(J)$ of the Kac superalgebra $J = sK_{10}$ satisfies the linear Jordan identity over $\Phi = \mathbb{Z}$, but since the proof is independent of scalars we establish it for a general ring of scalars Φ. The linear Jordan axiom is equivalent to the vanishing $\widetilde{D}_{\tilde{x}^2,\tilde{x}} = 0$ in terms of the standard inner Jordan derivation $\widetilde{D}_{\tilde{x},\tilde{y}} = \widetilde{V}_{\tilde{x},\tilde{y}} - \widetilde{V}_{\tilde{y},\tilde{x}} = -\widetilde{D}_{\tilde{y},\tilde{x}}$ of $\widetilde{J}$. In the superalgebra J itself, the standard inner derivations $D_{x,y} := V_{x,y} - (-1)^{xy} V_{y,x}$ and $D_m := V_{m,m}$ have

$$D_{a,x} = V_{a,x} - V_{x,a} = -D_{x,a}, \quad D_{m,n} = V_{m,n} + V_{n,m} = D_{n,m}, \quad D_{m,m} = 2D_m.$$

The connection between inner derivations in the Grassmann envelope and the superalgebra is $\widetilde{D}_{\tilde{a},\tilde{x}} = \widetilde{D}_{\alpha\otimes a,\delta\otimes x} = \alpha\delta \otimes D_{a,x}$ and $\widetilde{D}_{\tilde{m},\tilde{n}} = \widetilde{D}_{\gamma\otimes m,\eta\otimes n} = \gamma\eta \otimes V_{m,n} - \eta\gamma \otimes V_{n,m} = \gamma\eta \otimes (V_{m,n} + V_{n,m}) = \gamma\eta \otimes D_{m,n}$. Because so much of the action will take place in the subscripts $D_{x,y}$, we will elevate them to legibility by writing $D(x, y)$.

Writing a general $\widetilde{x}$ as $\widetilde{a} \oplus \widetilde{m} = (\widetilde{b} + \beta e_3) \oplus \widetilde{m}$, the linear Jordan identity $D(\widetilde{x}^2, \widetilde{x}) = 0$ becomes the vanishing of

$$\begin{aligned}
&\widetilde{D}\big((b^2 + \beta^2 e_3 + \widetilde{m}^2) \oplus (\{b, \widetilde{m}\} + \beta\widetilde{m})\,,\,(b + \beta e_3) \oplus \widetilde{m})\big) \\
&= \big(\widetilde{D}(b^2, b)^{<1>} + \beta\widetilde{D}(b^2, e_3)^{\bullet} + \widetilde{D}(b^2, \widetilde{m})^{<2>}\big) + \beta^2\big(\widetilde{D}(e_3, b)^{\bullet} + \beta\widetilde{D}(e_3, e_3)^{\bullet} + \widetilde{D}(e_3, \widetilde{m})^{\blacktriangledown}\big) \\
&\quad + \big(\widetilde{D}(\widetilde{m}^2, b)^{<3>} + \beta\widetilde{D}(\widetilde{m}^2, e_3)^{\bullet} + \widetilde{D}(\widetilde{m}^2, \widetilde{m})^{<4>}\big) + \big(\widetilde{D}(\{b, \widetilde{m}\}, b)^{<5>} + \beta\{b, \widetilde{m}\}, e_3)^{<6>} \\
&\quad + \widetilde{D}(\{b, \widetilde{m}\}, \widetilde{m})^{<7>}\big) + \beta\big(\widetilde{D}(\widetilde{m}, b)^{<8>} + \beta\widetilde{D}(\widetilde{m}, e_3)^{\blacktriangledown} + D(\widetilde{m}, \widetilde{m})^{\blacktriangle}\big) \\
&= \big(\widetilde{D}(b^2, b)\big)^{<1>} + \big(\widetilde{D}(b^2, \widetilde{m}) + \widetilde{D}(\{b, \widetilde{m}\}, b)\big)^{<2,5>} + \big(\widetilde{D}(\widetilde{m}^2, b) + \widetilde{D}(\{b, \widetilde{m}\}, \widetilde{m})\big)^{<3,7>} \\
&\quad + \big(\widetilde{D}(\widetilde{m}^2, \widetilde{m})\big)^{<4>} + \beta\big(\widetilde{D}(\{b, \widetilde{m}\}, e_3) + \widetilde{D}(b, \widetilde{m})\big)^{<6,8>}
\end{aligned}$$

[recall for $\bullet$ that $\widetilde{D}(A, e_3) = 0$, for $\blacktriangle$ that $\widetilde{D}(\widetilde{x}, \widetilde{x}) = \widetilde{D}(\widetilde{x}, \widetilde{y}) + \widetilde{D}(\widetilde{x}, \widetilde{y}) = 0$]. For (4) note that when $\widetilde{m} = \sum \gamma_i \otimes m_i$ we have $\widetilde{D}(\widetilde{m}^2, \widetilde{m}) = \sum_t \sum_{r<s} \widetilde{D}(\gamma_r\gamma_s \otimes \{m_r, m_s\}, \gamma_t \otimes m_y) = \sum_{r<s, t\neq r,s} \gamma_r\gamma_s\gamma_t \otimes D(\{m_r, m_s\}, m_t) = \sum_{i<j<k} \gamma_i\gamma_j\gamma_k \otimes \sum_{cyclic} D(\{m_i, m_j\}, m_k)$ [note $\gamma_i\gamma_j\gamma_k = \gamma_j\gamma_k\gamma_i = \gamma_k\gamma_i\gamma_j$ is invariant under cyclic permutations]. Thus the Grassmann vanishing on $\widetilde{J}$ reduces to 5 homogeneous Kac vanishings on J:

$$\begin{aligned}
&(KV.1) \quad D(b^2, b) = 0, \\
&(KV.2) \quad D(b^2, m) + D(\langle b, m\rangle, b) = 0, \\
&(KV.3) \quad D(\langle m, n\rangle, b) + D(\langle b, m\rangle, n) - D(\langle b, n\rangle, m) = 0, \\
&(KV.4) \quad \textstyle\sum_{cyclic} D(\langle m_i, m_j\rangle, m_k) = 0, \\
&(KV.5) \quad D(\langle b, m\rangle, e_3) - D(b, m) = 0.
\end{aligned}$$

We will establish the 5 parts of this individually. The vanishing (KV.1) results from the fact that the split null extension M is a quadratic Jordan bimodule for A, or directly from (2.4.1). (KV.5) is precisely (2.4.2), and (KV.2) follows from (2.4.2) since $D(b^2, m) + D(\langle b, m\rangle, b) = D(\langle b^2, m\rangle, e_3) + D(e_3, \langle b, \langle b, m\rangle\rangle) = D(e_3, (V_b^2 - V_{b^2})m) = D(e_3, 2U_b m) = 0$. To establish the trilinear identity (KV.4) it suffices to establish it for all triples from the ordered basis $m_{13}, n_{13}, m_{23}, n_{23}$; in each case some index 1 or 2 must be repeated, say m_{i3}, n_{i3}, p_{j3} where $p = m$ or n. If $j = i$ then some basic $m \in M_{i3}$ is repeated, and (KV.4) holds trivially for m, m, n by $\langle m, m\rangle = \langle m, n\rangle + \langle n, m\rangle = 0$. If $j \neq i$ then the formula (KV.4) for $p = m$ becomes $D(\langle m_{i3}, n_{i3}\rangle, m_{j3}) + D(\langle n_{i3}, m_{j3}\rangle, m_{i3}) + D(\langle m_{j3}, m_{i3}\rangle, n_{i3}) = D(2e_i - 3e_3, m_{j3}) - D(q_{12}, m_{i3}) - 2(-1)^j D(c_{12}, n_{i3})$ [by (2.4.3)] $= -3D(e_3, m_{j3}) - D(\langle q_{12}, m_{i3}\rangle, e_3) - 2(-1)^j D(\langle c_{12}, n_{i3}\rangle, e_3)$ [by (2.4.2)] $= D([3 - 1 - 2(-1)^j(-1)^j]m_{j3}, e_3)$ [by (2.4.3)] $= 0$, while (KV.4) for $p = n$ becomes $D(\langle m_{i3}, n_{i3}\rangle, n_{j3}) + D(\langle n_{i3}, n_{j3}\rangle, m_{i3}) + D(\langle n_{j3}, m_{i3}\rangle, n_{i3}) = D(2e_i - 3e_3, n_{j3}) + 2(-1)^j D(d_{12}, m_{i3}) - D(q_{12}, n_{i3})$ [by (2.4.3)] $= -3D(e_3, n_{j3}) + 2(-1)^j(-1)^i D(n_{j3}, e_3) - D(n_{j3}, e_3)$ [by (2.4.2-3)] $= D([3 - 2 - 1]n_{j3}, e_3) = 0$.

The messiest calculation is (KV.3). Certainly this is alternating in m, $D(\langle m, m\rangle, b) + D(\langle b, m\rangle, m) - D(\langle b, m\rangle, m) = 0$, so it suffices to establish it for the 6 ordered pairs (1) m_{13}, n_{13}, (2) m_{13}, m_{23}, (3) m_{13}, n_{23}, (4) n_{13}, m_{23}, (5) n_{13}, n_{23}, (6) m_{23}, n_{23}, which vanish as follows (making heavy use of all parts of (2.4)):

$$\begin{aligned}
(1) \quad & D(\langle m_{13}, n_{13}\rangle, b) + D(\langle b, m_{13}\rangle, n_{13}) - D(\langle b, n_{13}\rangle, m_{13}) \\
= \quad & 2D(2e_1 - e_3^{\bullet}, [\beta_3 c_{12} + \beta_4 d_{12} + \beta_5 q_{12}]) + D([\beta_1 m_{13}^{\blacktriangle} - \beta_4 n_{23} + \beta_5 m_{23}], n_{13}) \\
& \quad D([\beta_1 n_{13}^{\blacktriangle} + \beta_3 m_{23} + \beta_5 n_{23}], m_{13}) \qquad \text{[by (2.2)]} \\
= \quad & \beta_3\big(2D(e_1, c_{12}) - D(m_{13}, m_{23})\big) + \beta_4\big(2D(e_1, d_{12}) - D(n_{13}, n_{23})\big) + \beta_5\big(2D(e_1, q_{12}) \\
& \quad + D(n_{13}, m_{23}) - D(m_{13}, n_{23})\big) \\
= \quad & 0 \qquad \text{[by (2.4) (2), (4), (5), (6)].}
\end{aligned}$$

(2) $D(\langle m_{13}, m_{23}\rangle, b) + D(\langle b, m_{13}\rangle, m_{23}) - D(\langle b, m_{23}\rangle, m_{13})$

$$= D(2c_{12},\, [\beta_1 e_1 + \beta_2 e_2 + \beta_4 d_{12} + \beta_5 q_{12}]) + D([\beta_1 m_{13} - \beta_4 n_{23} + \beta_5 m_{23}],\, m_{23})$$
$$-D([\beta_2 m_{23} + \beta_4 n_{13} + \beta_5 m_{13}],\, m_{13}) \qquad \text{[by (2.2)]}$$
$$= \beta_1\big(-2D(e_1, c_{12}) + D(m_{13}, m_{23})\big) + \beta_2\big(-2D(e_2, c_{12}) - D(m_{13}, m_{23})\big)$$
$$+\beta_4\big(2D(c_{12}, d_{12}) - D(m_{23}, n_{23}) - D(m_{13}, n_{13})\big)$$
$$+\beta_5\big(2D(c_{12}, q_{12}) + 2D(m_{23}) - 2D(m_{13})\big) = 0 \qquad \text{[by (2.4) (2), (4), (4), (7), (8)].}$$

(3) $D(\langle m_{13}, n_{23}\rangle, b) + D(\langle b, m_{13}\rangle, n_{23}) - D(\langle b, n_{23}\rangle, m_{13})$

$$= D(q_{12},\, [\beta_1 e_1 + \beta_2 e_2 + \beta_3 c_{12} + \beta_4 d_{12}]) + D([\beta_1 m_{13} - \beta_4 n_{23} + \beta_5 m_{23}],\, n_{23})$$
$$-D([\beta_2 n_{23} - \beta_3 m_{13} + \beta_5 n_{13}],\, m_{13}) \qquad \text{[by (2.2)]}$$
$$= \beta_1\big(-D(e_1, q_{12}) + D(m_{13}, n_{23})\big) + \beta_2\big(-D(e_2, q_{12}) - D(m_{13}, n_{23})\big)$$
$$+\beta_3\big(-D(c_{12}, q_{12}) + 2D(m_{13})\big) + \beta_4\big(D(q_{12}, d_{12}) - 2D(n_{23})\big)$$
$$+\beta_5\big(D(m_{23}, n_{23}) - D(m_{13}, n_{13})\big) = 0 \qquad \text{[by (2.4) (2), (6), (6), (8), (9), (7)].}$$

(4) $D(\langle n_{13}, m_{23}\rangle, b) + D(\langle b, n_{13}\rangle, m_{23}) - D(\langle b, m_{23}\rangle, n_{13})$

$$= -D(q_{12},\, [\beta_1 e_1 + \beta_2 e_2 + \beta_3 c_{12} + \beta_4 d_{12}]) + D([\beta_1 n_{13} + \beta_3 m_{23} + \beta_5 n_{23}],\, m_{23})$$
$$-D([\beta_2 m_{23} + \beta_4 n_{13} + \beta_5 m_{13}],\, n_{13}) \qquad \text{[by (2.2)]}$$
$$= \beta_1\big(D(e_1, q_{12}) + D(n_{13}, m_{23})\big) + \beta_2\big(D(e_2, q_{12}) - D(n_{13}, m_{23})$$
$$+\beta_3\big(D(c_{12}, q_{12}) + 2D(m_{23})\big) + \beta_4\big(D(d_{12}, q_{12}) - 2D(n_{13})\big)$$
$$+\beta_5\big(D(m_{23}, n_{23}) - D(m_{13}, n_{13})\big) = 0 \qquad \text{[by (2.4) (2), (6), (6), (8), (9), (7)].}$$

(5) $D(\langle n_{13}, n_{23}\rangle, b) + D(\langle b, n_{13}\rangle, n_{23}) - D(\langle b, n_{23}\rangle, n_{13})$

$$= D(2d_{12},\, [\beta_1 e_1 + \beta_2 e_2 + \beta_3 c_{12} + \beta_5 q_{12}]) + D(\beta_1 n_{13} + \beta_3 m_{23} + \beta_5 n_{23}\ n_{23})$$
$$-D([\beta_2 n_{23} - \beta_3 m_{13} + \beta_5 n_{13}],\, n_{13}) \qquad \text{[by (2.2)]}$$
$$= \beta_1\big(-2D(e_1, d_{12}) + D(n_{13}, n_{23})\big) + \beta_2\big(-2D(e_2, d_{12}) - D(n_{13}, n_{23})\big)$$
$$+\beta_3\big(-2D(c_{12}, d_{12}) + D(m_{23}, n_{23}) + D(m_{13}, n_{13})\big)$$
$$+\beta_5\big(2D(d_{12}, q_{12}) + 2D(n_{23}) - 2D(n_{13})\big) = 0 \qquad \text{[by (2.4) (2), (5), (5), (7), (9)].}$$

(6) $D(\langle m_{23}, n_{23}\rangle, b) + D(\langle b, m_{23}\rangle, n_{23}) - D(\langle b, n_{23}\rangle, m_{23})$

$$= D(2e_2 - 3e_3^{\bullet},\, [\beta_3 c_{12} + \beta_4 d_{12} + \beta_5 q_{12}]) + D([\beta_2 m_{23}^{\blacktriangle} + \beta_4 n_{13} + \beta_5 m_{13}],\, n_{23})$$
$$-D([\beta_2 n_{23}^{\blacktriangle} - \beta_3 m_{13} + \beta_5 n_{13}],\, m_{23}) \qquad \text{[by (2.2)]}$$
$$= \beta_3\big(2D(e_2, c_{12}) + D(m_{13}, n_{23})\big) + \beta_4\big(2D(e_2, d_{12}) + D(n_{13}, n_{23})\big)$$
$$+\beta_5\big(2D(e_2, q_{12}) + D(m_{13}, n_{23}) - D(n_{13}, m_{23})\big) = 0 \qquad \text{[by (2.4) (2), (4), (5), (6)].}$$

This completes the verification of the linear Jordan identities, establishing that $sK_{10}(\Phi)$ is a quadratic Jordan superalgebra. ■

4 The Grassmann Envelope

We now investigate the Grassmann envelope $\widetilde{J} = \Gamma(J) = \widetilde{A} \oplus \widetilde{M}$ in a quest to prove that in characteristic 5 it is the Jordan algebra of a cubic form. We have $\widetilde{A} = \big(\Gamma_0 \otimes Jord(Q,u)\big) \boxplus \big(\Gamma_0 \otimes e_3\big) = Jord(Q_{\Gamma_0}, 1 \otimes u) \boxplus \big(\Gamma_0 \otimes e_3\big) = Jord(Q_{\Gamma_0}, e) \boxplus \Gamma_0 e_3$ for the natural scalar extension Q_{Γ_0} of the quadratic form Q, where we brazenly identify A with $1 \otimes A$ and write $\widetilde{A} = \Gamma_0 A$. In this section we will consistently use β, η to denote even elements in Γ_0 (we will keep α for elements of Φ), and γ, δ to denote odd elements of Γ_1. It will sometimes be convenient to label our ordered basis for $\widetilde{J}$ as

$$b_1 := e_1, \quad b_2 := e_2, \quad b_3 := c_{12}, \quad b_4 := d_{12}, \quad b_5 := q_{12}, \quad e_3,$$
$$m_1 := m_{13}, \quad m_2 := n_{13}, \quad m_3 := m_{23}, \quad m_4 := n_{23}$$

so that we can write the general element of $\widetilde{J}$ as $\sum_{i=1}^{6} \beta_i b_i + \beta e_3 + \sum_{i=1}^{4} \gamma_i m_i$. Because e_3 plays a special role we will not call it b_6, nor its coefficient β_6, but use an unsubscripted β to emphasize its distinctiveness. Remember that in sK_{10} we use $u = e_1 + e_2$ for the unit element of B (the old unit e of K_{10} has metamorphosed into q_{12}), so $1 = u + e_3$.

Grassmann Theorem 4.1 *The Grassmann envelope $\widetilde{J} := \Gamma(J) := (\Gamma_0 \otimes A) \oplus (\Gamma_1 \otimes M)$ of $J = sK_{10}(\Phi)$ is generically algebraic of degree ≤ 9. Writing*

$$\text{(4.1.1)} \qquad \begin{aligned} &\widetilde{x} := \widetilde{a} \oplus \widetilde{m}, \quad \widetilde{m} := \gamma_1 m_1 + \gamma_2 m_2 + \gamma_3 m_3 + \gamma_4 m_4, \\ &\widetilde{a} := \widetilde{b} \boxplus \beta e_3, \ \widetilde{b} := \textstyle\sum_{i=1}^{5} \beta_i b_i = \beta_1 e_1 + \beta_2 e_2 + b_{12}, \\ &b_{12} = \beta_3 c_{12} + \beta_4 c_{12} + \beta_5 q_{12}, \quad 1 := u + e_3 = e_1 + e_2 + e_3 \end{aligned}$$

for Grassmann variables $\beta_i \in \Gamma_0, \gamma_i \in \Gamma_1$, then $\widetilde{J}$ satisfies over any Φ the generic relation

$$\text{(4.1.2)} \qquad \widetilde{x}^3 - T(\widetilde{x})\widetilde{x}^2 + S(\widetilde{x})x - N(\widetilde{x})1 = 5\Big([M(\widetilde{x}) - \mu(\widetilde{m})\beta]e_3 - [\mu(\widetilde{m})\widetilde{m}]\Big)$$

with generic coefficients[3]

$$\text{(4.1.3)} \qquad \begin{aligned} T(\widetilde{x}) \quad &:= T(\widetilde{b}) + \beta = \beta_1 + \beta_2 + \beta, \\ S(\widetilde{x}) \quad &:= \beta T(\widetilde{b}) + Q(\widetilde{b}) - 2\mu(\widetilde{m}) = \beta\beta_1 + \beta\beta_1 + \beta_1\beta_2 - \beta_3\beta_4 - \beta_5^2 - 2\mu(\widetilde{m}), \\ N(\widetilde{x}) \quad &:= Q(\widetilde{b})\beta - 2M(\widetilde{x}) \text{ for} \\ M(\widetilde{x}) \quad &:= M(\widetilde{b};\widetilde{m}) := T(\widetilde{b})\mu(\widetilde{m}) - \nu(\widetilde{b},\widetilde{m}) \\ &= \beta_2\gamma_1\gamma_2 + \beta_1\gamma_3\gamma_4 - \beta_3\gamma_2\gamma_4 - \beta_4\gamma_1\gamma_3 - \beta_5(\gamma_1\gamma_4 - \gamma_2\gamma_3), \\ \nu(\widetilde{b},\widetilde{m}) \quad &:= \beta_1\gamma_1\gamma_2 + \beta_2\gamma_3\gamma_4 + \beta_3\gamma_2\gamma_4 + \beta_4\gamma_1\gamma_3 + \beta_5(\gamma_1\gamma_4 - \gamma_2\gamma_3), \\ \mu(\widetilde{m}) \quad &:= \gamma_1\gamma_2 + \gamma_3\gamma_4 = \nu(u,\widetilde{m}) = M(u;\widetilde{m}) = \omega(u,\widetilde{x}) = \omega(1,\widetilde{x}), \\ \omega(\widetilde{m}) \quad &:= 2\gamma_1\gamma_2 e_1 + 2\gamma_3\gamma_4 e_2 + 2\gamma_1\gamma_3 c_{12} + 2\gamma_2\gamma_4 d_{12} + (\gamma_1\gamma_4 - \gamma_2\gamma_3)q_{12} \\ &=: 2\gamma_1\gamma_2 e_1 + 2\gamma_3\gamma_4 e_2 + w_{12}. \end{aligned}$$

Thus in characteristic 5 the Grassmann envelope is generically algebraic of degree 3, while in general it is generically algebraic of degree ≤ 9:

$$\text{(4.1.2}'\text{)} \qquad \big(\widetilde{x}^3 - T(\widetilde{x})\widetilde{x}^2 + S(\widetilde{x})x - N(\widetilde{x})1\big)^3 = 0.$$

We have explicit formulas

[3] If $\frac{1}{2} \in \Phi$ and we set $b(\widetilde{m}) := \widehat{\gamma}_1 e_1 + \widehat{\gamma}_2 e_2 + \widehat{\gamma}_3 c_{12} + \widehat{\gamma}_4 d_{12} + \widehat{\gamma}_5 q_{12}$ for $\widehat{\gamma}_1 := \gamma_1\gamma_2$, $\widehat{\gamma}_2 := \gamma_3\gamma_4$, $\widehat{\gamma}_3 := \gamma_1\gamma_3$, $\widehat{\gamma}_4 := \gamma_2\gamma_4$, $\widehat{\gamma}_5 := \frac{1}{2}(\gamma_1\gamma_4 - \gamma_2\gamma_3)$ in Γ_0, then we can write $M(\widetilde{x}) = Q(\widetilde{b}, b(\widetilde{m})) = \beta_2\widehat{\gamma}_1 + \beta_1\widehat{\gamma}_2 - \beta_3\widehat{\gamma}_4 - \beta_4\widehat{\gamma}_3 - \beta_5\widehat{\gamma}_5$ and $w_{12} = 2\widehat{\gamma}_3 c_{12} + 2\widehat{\gamma}_4 d_{12} + 2\widehat{\gamma}_5 q_{12} = 2E_{12}(b(\widetilde{m}))$.

$$\begin{aligned}
(4.1.4)\quad \widetilde{a}^2 &= \big(T(\widetilde{b})\widetilde{b} - Q(\widetilde{b})u\big) \boxplus \beta^2 e_3 \\
&= \big(\beta_1^2 + \beta_3\beta_4 + \beta_5^2\big)e_1 + \big(\beta_2^2 + \beta_3\beta_4 + \beta_5^2\big)e_2 + \big(\beta_1 + \beta_2\big)\beta_3 c_{12} \\
&\quad + \big(\beta_1 + \beta_2\big)\beta_4 d_{12} + + \big(\beta_1 + \beta_2\big)\beta_5 q_{12} \boxplus \beta^2 e_3,
\end{aligned}$$

$$\begin{aligned}
(4.1.5)\quad \widetilde{a}^3 &= \big([T(\widetilde{b})^2 - Q(\widetilde{b})]\widetilde{b} - T(\widetilde{b})Q(\widetilde{b})u\big) \boxplus \beta^3 e_3 \\
&= \big(\beta_1^3 + (2\beta_1 + \beta_2)(\beta_3\beta_4 + 2\beta_5^2)\big)e_1 + \big(\beta_2^3 + (2\beta_2 + \beta_1)(\beta_3\beta_4 + 2\beta_5^2)\big)e_2 \\
&\quad + \big(\beta_1^2 + \beta_2^2 + \beta_5^2 + \beta_1\beta_2 + \beta_3\beta_4\big)(\beta_3 c_{12} + \beta_4 d_{12} + \beta_5 q_{12}) \boxplus \beta^3 e_3,
\end{aligned}$$

$$\begin{aligned}
(4.1.6)\quad \{\widetilde{a}, \widetilde{m}\} &= \{\widetilde{b}, \widetilde{m}\} + \beta\widetilde{m} \\
&= \big((\beta_1 + \beta)\gamma_1 - \beta_3\gamma_4 + \beta_5\gamma_3\big)m_{13} + \big((\beta_1 + \beta)\gamma_2 + \beta_4\gamma_3 + \beta_5\gamma_4\big)n_{13} \\
&\quad \big((\beta_2 + \beta)\gamma_3 + \beta_3\gamma_2 + \beta_5\gamma_1\big)m_{23} + \big((\beta_2 + \beta)\gamma_4 - \beta_4\gamma_1 + \beta_5\gamma_2\big)n_{23},
\end{aligned}$$

$$(4.1.7)\quad \{\widetilde{a}^2, \widetilde{m}\} = T(\widetilde{b})\{\widetilde{b}, \widetilde{m}\} + [\beta^2 - Q(\widetilde{b})]\widetilde{m},$$

$$(4.1.8)\quad U_{\widetilde{a}}\widetilde{m} = \beta\{\widetilde{b}, \widetilde{m}\},$$

$$(4.1.9)\quad \widetilde{m}^2 = \omega(\widetilde{m}) \boxplus \big(-3\mu(\widetilde{m})\big)e_3,$$

$$(4.1.10)\quad \widetilde{m}^3 = -3\mu(\widetilde{m})\,\widetilde{m} = -3\big(\gamma_1\gamma_3\gamma_4 m_1 + \gamma_2\gamma_3\gamma_4 m_2 + \gamma_1\gamma_2\gamma_3 m_3 + \gamma_1\gamma_2\gamma_4 m_4\big),$$

$$(4.1.11)\quad U_{\widetilde{m}}\widetilde{a} = \beta\omega(\widetilde{m}) \boxplus \big(-3\nu(\widetilde{b}, \widetilde{m})\big)e_3,$$

$$(4.1.12)\quad \{\widetilde{a}, \widetilde{m}^2\} = \big(2\mu(\widetilde{m})\widetilde{b} + T(\widetilde{b})\omega(\widetilde{m}) - 2M(\widetilde{b}; \widetilde{m})u\big) \boxplus \big(-6\mu(\widetilde{m})\beta\big)e_3.$$

PROOF: We begin calculating the ingredients of (4-12) of the generic polynomial. For (4) we have $\widetilde{a}^2 = \widetilde{b}^2 \boxplus \beta^2 e_3 = [T(\widetilde{b})\widetilde{b} - Q(\widetilde{b})u] \boxplus \beta^2 e_3$ [by the usual rules in the Jordan algebra of a quadratic form], and similarly for (5) $\widetilde{a}^3 = \widetilde{b}^3 \boxplus \beta^3 e_3 = \big(Q(\widetilde{b}, \bar{\widetilde{b}})\widetilde{b} - Q(\widetilde{b})\bar{\widetilde{b}}\big) \boxplus \beta^3 e_3 = \big([T(\widetilde{b})^2\widetilde{b} - Q(\widetilde{b})]\widetilde{b} - T(\widetilde{b})Q(\widetilde{b})u\big) \boxplus \beta^3 e_3$. Then for (6) clearly $\{\widetilde{a}, \widetilde{m}\} = \{\widetilde{b}, \widetilde{m}\} + \beta\widetilde{m}$, and for (7) $\{\widetilde{a}^2, \widetilde{m}\} = \{\big(T(\widetilde{b})\widetilde{b} - Q(\widetilde{b})u \boxplus \beta^2 e_3\big), \widetilde{m}\} = T(\widetilde{b})\{\widetilde{b}, \widetilde{m}\} + [\beta^2 - Q(\widetilde{b})]\widetilde{m}$ [since $\{u, \widetilde{m}\} = \{e_3, \widetilde{m}\} = \widetilde{m}$]. For (8) we have $U_{\widetilde{a}}\widetilde{m} = \big(U_{\beta e_3} + U_{\widetilde{b}} + U_{\beta e_3, \widetilde{b}}\big)(\widetilde{m}) = 0 + 0 + \beta\{e_3, \widetilde{m}, \widetilde{b}\}$ [by Peirce relations (0.2.4)] $= \beta\{\widetilde{m}, \widetilde{b}\}$ [by (2.4.1)].

Turning to odd elements, for (9) [using abbreviations ν, μ, w_{12} for $\nu(\widetilde{b}, \widetilde{m}), \mu(\widetilde{m}), w_{12}(\widetilde{m})$]

$$\begin{aligned}
\widetilde{m}^2 &= \textstyle\sum_{i<j}\langle \gamma_i \otimes m_i, \gamma_j \otimes m_j\rangle \\
&:= \gamma_1\gamma_2 \otimes \langle m_{13}, n_{13}\rangle + \gamma_1\gamma_3 \otimes \langle m_{13}, m_{23}\rangle + \gamma_1\gamma_4 \otimes \langle m_{13}, n_{23}\rangle \\
&\quad + \gamma_2\gamma_3 \otimes \langle n_{13}, m_{23}\rangle + \gamma_2\gamma_4 \otimes \langle n_{13}, n_{23}\rangle + \gamma_3\gamma_4 \otimes \langle m_{23}, n_{23}\rangle \\
&= \gamma_1\gamma_2(g_1) + \gamma_1\gamma_3(2c_{12}) + \gamma_1\gamma_4(q_{12}) + \gamma_2\gamma_3(-q_{12}) + \gamma_2\gamma_4(2d_{12}) + \gamma_3\gamma_4(g_2) \qquad \text{[by (2.2)]} \\
&= 2\gamma_1\gamma_2 e_1 + 2\gamma_3\gamma_4 e_2 - 3(\gamma_1\gamma_2 + \gamma_3\gamma_4)e_3 + (\gamma_1\gamma_4 - \gamma_2\gamma_3)q_{12} + 2\gamma_1\gamma_3 c_{12} + 2\gamma_2\gamma_4 d_{12} \\
&= \omega(\widetilde{m}) \boxplus (-3\mu)e_3.
\end{aligned}$$

A longer calculation gives (10) [recalling $\langle m, n, m\rangle = 0, \langle m, n, p\rangle = -\langle p, n, m\rangle$],

$$\begin{aligned}
\widetilde{m}^3 &= U_{\widetilde{m}}\widetilde{m} = \textstyle\sum_{i<j}\langle \gamma_i \otimes m_i, \widetilde{m}, \gamma_j \otimes m_j\rangle \\
&= \gamma_1\langle m_{13}, \widetilde{m}, n_{13}\rangle\gamma_2 + \gamma_1\langle m_{13}, \widetilde{m}, m_{23}\rangle\gamma_3 + \gamma_1\langle m_{13}, \widetilde{m}, n_{23}\rangle\gamma_4 \\
&\quad + \gamma_2\langle n_{13}, \widetilde{m}, m_{23}\rangle\gamma_3 + \gamma_2\langle n_{13}, \widetilde{m}, n_{43}\rangle\gamma_4 + \gamma_3\langle m_{23}, \widetilde{m}, n_{23}\rangle\gamma_4 \\
&= \gamma_1\langle m_{13}, (\gamma_1 m_{13} + \gamma_2 n_{13} + \gamma_3 m_{23} + \gamma_4 n_{23}), n_{13}\rangle\gamma_2 \\
&\quad + \gamma_1\langle m_{13}, (\gamma_1 m_{13} + \gamma_2 n_{13} + \gamma_3 m_{23} + \gamma_4 n_{23}), m_{23}\rangle\gamma_3 \\
&\quad + \gamma_1\langle m_{13}, (\gamma_1 m_{13} + \gamma_2 n_{13} + \gamma_3 m_{23} + \gamma_4 n_{23}), n_{23}\rangle\gamma_4
\end{aligned}$$

$$
\begin{aligned}
&+\gamma_2\langle n_{13},(\gamma_1 m_{13}+\gamma_2 n_{13}+\gamma_3 m_{23}+\gamma_4 n_{23}),m_{23}\rangle\gamma_3\\
&+\gamma_2\langle n_{13},(\gamma_1 m_{13}+\gamma_2 n_{13}+\gamma_3 m_{23}+\gamma_4 n_{23}),n_{23}\rangle\gamma_4\\
&+\gamma_3\langle m_{23},(\gamma_1 m_{13}+\gamma_2 n_{13}+\gamma_3 m_{23}+\gamma_4 n_{23}),n_{23}\rangle\gamma_4\\
&=\gamma_1(-\gamma_1 m_{13}-\gamma_2 n_{13}+0+0)\gamma_2+\gamma_1(0-\gamma_2 m_{23}+0+\gamma_4 m_{13})\gamma_3\\
&\quad+\gamma_1(\gamma_1 m_{23}-2\gamma_2 n_{23}-2\gamma_3 m_{13}+\gamma_4 n_{13})\gamma_4+\gamma_2(2\gamma_1 m_{23}-\gamma_2 n_{23}-\gamma_3 m_{13}+2\gamma_4 n_{13})\gamma_3\\
&\quad+\gamma_2(\gamma_1 n_{23}+0-\gamma_3 n_{13}+0)\gamma_4+\gamma_3(0+0-\gamma_3 m_{23}-\gamma_4 n_{23})\gamma_4 \qquad \text{[by (2.3)]}\\
&=(0)+(-\gamma_1\gamma_2\gamma_3 m_{23}+\gamma_1\gamma_4\gamma_3 m_{13})+(-2\gamma_1\gamma_2\gamma_4 n_{23}-2\gamma_1\gamma_3\gamma_4 m_{13})\\
&\quad+(2\gamma_2\gamma_1\gamma_3 m_{23}+2\gamma_2\gamma_4\gamma_3 n_{13})+(\gamma_2\gamma_1\gamma_4 n_{23}-\gamma_2\gamma_3\gamma_4 n_{13})+(0) \qquad \text{[by } \gamma_i^2=0\text{]}\\
&=\big(\gamma_1\gamma_4\gamma_3-2\gamma_1\gamma_3\gamma_4\big)m_{13}+\big(2\gamma_2\gamma_4\gamma_3-\gamma_2\gamma_3\gamma_4\big)n_{13}+\big(-\gamma_1\gamma_2\gamma_3+2\gamma_2\gamma_1\gamma_3\big)m_{23}\\
&\quad+\big(-2\gamma_1\gamma_2\gamma_4 n_{23}+\gamma_2\gamma_1\gamma_4\big)n_{23}\\
&=-3(\gamma_1\gamma_3\gamma_4)m_{13}-3(\gamma_2\gamma_3\gamma_4)n_{13}-3(\gamma_1\gamma_2\gamma_3)m_{23}-3(\gamma_1\gamma_2\gamma_4)n_{23}\\
&=-3(\gamma_1\gamma_2+\gamma_3\gamma_4)\gamma_1 m_{13}-3(\gamma_1\gamma_2+\gamma_3\gamma_4)\gamma_2 n_{13}-3(\gamma_1\gamma_2+\gamma_3\gamma_4)\gamma_3 m_{23}\\
&\quad-3(\gamma_1\gamma_2+\gamma_3\gamma_4)\gamma_4 n_{23} \qquad \text{[by } \gamma_i^2=0\text{]}\\
&=-3\mu(\gamma_1 m_{13}+\gamma_2 n_{13}+\gamma_3 m_{23}+\gamma_4 n_{23})=-3\mu\widetilde{m}.
\end{aligned}
$$

For (11) we compute

$$
\begin{aligned}
U_{\widetilde{m}}\widetilde{a}&=\beta_1(U_{\widetilde{m}}e_1)+\beta_2(U_{\widetilde{m}}e_2)+\beta_3(U_{\widetilde{m}}c_{12})+\beta_4(U_{\widetilde{m}}d_{12})+\beta_5(U_{\widetilde{m}}q_{12})+\beta(U_{\widetilde{m}}e_3)\\
&=-3\big(\beta_1\gamma_1\gamma_2+\beta_2\gamma_3\gamma_4+\beta_3\gamma_2\gamma_4+\beta_4\gamma_1\gamma_3+\beta_5[\gamma_1\gamma_4-\gamma_2\gamma_3]\big)e_3\\
&\quad+\beta E_{11+12+22}\big(\gamma_1\gamma_2(2e_1)+\gamma_1\gamma_3(2c_{12})+\gamma_1\gamma_4(q_{12})+\gamma_2\gamma_3(-q_{12})+\gamma_2\gamma_4(2d_{12})+\gamma_3\gamma_4(2e_2)\big)
\end{aligned}
$$

[From (2.3), on e_1, e_2 only $U_{m_{13},n_{13}}, U_{m_{23},n_{23}}$ respectively contribute a term $-3e_3$, on q_{12} only $-U_{n_{13},m_{23}}, U_{m_{13},n_{23}}$ contribute, on c_{12} only $-U_{n_{13},n_{23}}$, and on d_{12} only $U_{m_{13},m_{23}}$.]

$$
\begin{aligned}
&=\beta\big(2\gamma_1\gamma_3 c_{12}+2\gamma_2\gamma_4 d_{12}+[\gamma_1\gamma_4-\gamma_2\gamma_3]q_{12}+2\gamma_1\gamma_2 e_1+2\gamma_3\gamma_4 e_2\big)\\
&\quad-3\big(\beta_1\gamma_1\gamma_2+\beta_2\gamma_3\gamma_4+\beta_3\gamma_1\gamma_4+\beta_4\gamma_1\gamma_3+\beta_5[\gamma_1\gamma_4-\gamma_2\gamma_3]\big)e_3\\
&=\beta w_{12}+2\beta[\gamma_1\gamma_2 e_1+\gamma_3\gamma_4 e_2]-3\nu e_3.
\end{aligned}
$$

Before attacking (12) we first show

$$\{b_{12},\omega\}=2[\nu-\beta_1\gamma_1\gamma_2-\beta_2\gamma_3\gamma_4]u+2\mu b_{12}.$$

Here $T(b_{12})=0$, $T(\omega)=2\gamma_1\gamma_2+2\gamma_3\gamma_4=2\mu$, and $Q(q_{12})=-1=Q(c_{12},d_{12})$ implies $Q(b_{12},\omega)=Q(b_{12},w_{12})=Q(\beta_3 c_{12}+\beta_4 d_{12}+\beta_5 q_{12},\ 2\gamma_1\gamma_3 c_{12}+2\gamma_2\gamma_4 d_{12}+(\gamma_1\gamma_4-\gamma_2\gamma_3)q_{12})=-2\big(\beta_3\gamma_2\gamma_4+\beta_4\gamma_1\gamma_3+\beta_5(\gamma_1\gamma_4-\gamma_2\gamma_3)\big)=-2\big(\nu-\beta_1\gamma_1\gamma_2-\beta_2\gamma_3\gamma_4\big)$. These together imply that $\{b_{12},\omega\}=T(b_{12})\omega+T(\omega)b_{12}-Q(b_{12},\omega)=0+2\mu b_{12}+2[\nu-\beta_1\gamma_1\gamma_2-\beta_2\gamma-3\gamma_4]u$.

For the direct assault on (12), by (2.2) we have

$$
\begin{aligned}
\{\widetilde{a},\widetilde{m}^2\}&=\{\big([\beta_1 e_1+\beta_2 e_2+b_{12}]\boxplus\beta e_3\big),\ \big(\omega\boxplus[-3\mu e_3]\big)\}\\
&=\big(4\beta_1\gamma_1\gamma_2 e_1+4\beta_2\gamma_3\gamma_4 e_2+(\beta_1+\beta_2)w_{12}+\{b_{12},\omega\}\big)\boxplus\big([-6\beta\mu]e_3\big)=:b'\boxplus[-6\beta\mu]e_3
\end{aligned}
$$

where by the above and $\beta_1+\beta_2=T(\widetilde{b})$ the element b' becomes

$= 4\beta_1\gamma_1\gamma_2 e_1 + 4\beta_2\gamma_3\gamma_4 e_2 + T(\widetilde{b})\big(\omega - 2\beta_1\gamma_1\gamma_2 e_1 - 2\beta_2\gamma_3\gamma_4 e_2\big) + 2[\nu - \beta_1\gamma_1\gamma_2 - \beta_2\gamma_3\gamma_4]u + 2\mu b_{12}$

$= \big[4\beta_1\gamma_1\gamma_2 - 2T(\widetilde{b})\gamma_1\gamma_2 - 2[\beta_1\gamma_1\gamma_2 + \beta_2\gamma_3\gamma_4]\big]e_1 + \big[4\beta_2\gamma_3\gamma_4 - 2T(\widetilde{b})\gamma_3\gamma_4 - 2[\beta_1\gamma_1\gamma_2 + \beta_2\gamma_3\gamma_4]\big]e_2$
$\quad + T(\widetilde{b})\omega + 2\nu u + 2\mu b_{12}.$

Now we have a general scalar identity

$$4\beta\gamma^\bullet - 2(\beta^\bullet + \beta'^\blacktriangle)\gamma - 2(\beta^\bullet\gamma + \beta'^\blacktriangle\gamma') = 2\beta^\blacktriangledown(\gamma + \gamma') - 2(\beta^\blacktriangledown + \beta'^\blacktriangle)(\gamma + \gamma').$$

If we replace $\beta,\ \beta',\ \gamma,\ \gamma' \to \beta_1,\ \beta_2,\ \gamma_1\gamma_2,\ \gamma_3\gamma_4$ then this becomes $4\beta_1\gamma_1\gamma_2 - 2T(b)\gamma_1\gamma_2 - 2(\beta_1\gamma_1\gamma_2 + \beta_2\gamma_3\gamma_4) = 2\mu\beta_1 - 2T(b)\mu$, and the analogous substitution yields $4\beta_2\gamma_3\gamma_4 - 2T(b)\gamma_3\gamma_4 - 2(\beta_1\gamma_1\gamma_2 + \beta_2\gamma_3\gamma_4) = 2\mu\beta_2 - 2T(b)\mu$. Thus b' reduces to $[2\mu\beta_1^\triangle - 2T(b)\mu^\triangledown]e_1 + [2\mu\beta_2^\triangle - 2T(b)\mu^\triangledown]e_2 + T(\widetilde{b})\omega + 2\nu u + 2\mu b_{12}^\triangle = 2\mu b^\triangle + T(b)\omega + 2[\nu - T(b)\mu^\triangledown]u = 2\mu b + T(b)\omega - 2M(\widetilde{b};\widetilde{m})u$ [by (4.1.3)] as claimed in (12).

Summing up our results, we have the following table of ingredients:

Term	$B = \Phi u + V$	$\boxplus\Phi e_3$	$\oplus M$
$\widetilde{a}^3$	$[T(\widetilde{b})^{2<1>} - Q(\widetilde{b})^{<2>}]\widetilde{b}$ $-Q(\widetilde{b})T(\widetilde{b})u^{<3>}$	$[\beta^3]e_3^{<1>}$	
$\{\widetilde{a},\widetilde{m}^2\}$	$T(\widetilde{b})\omega(\widetilde{m})^{<4>} + 2\mu\widetilde{b}^{<5>}$ $-2M(\widetilde{b};\widetilde{m})u^{<6>}$	$[-6\beta\mu]e_3^{<a>}$	
$U_{\widetilde{m}}\widetilde{a}$	$\beta\omega(\widetilde{m})^{<7>}$	$[-3\nu]e_3^{<b>}$	
$\widetilde{m}^3$			$-3\mu\widetilde{m}^{<a>}$
$\{\widetilde{a}^2,\widetilde{m}\}$			$[\beta^{2<1>} - Q(\widetilde{b})^{<2>}]\widetilde{m}$ $+T(\widetilde{b})\{\widetilde{b},\widetilde{m}\}^{<3>}$
$U_{\widetilde{a}}\widetilde{m}$			$\beta\{\widetilde{b},\widetilde{m}\}^{<4>}$
$-T(\widetilde{x})\widetilde{a}^2$	$-[\beta^{<8>} + T(\widetilde{b})^{<1>}]T(\widetilde{b})\widetilde{b}$ $+\beta Q(\widetilde{b})u^{<9>} + T(\widetilde{b})Q(\widetilde{b})u^{<3>}$	$[-\beta^{3<1>} - \beta^2 T(\widetilde{b})^{<2>}]e_3$	
$-T(\widetilde{x})\widetilde{m}^2$	$-\beta\omega(\widetilde{m})^{<7>} - T(b)\omega(\widetilde{m})^{<4>}$	$[3\beta\mu^{<a>} + 3T(\widetilde{b})\mu^{<c>}]e_3$	
$-T(\widetilde{x})\{\widetilde{a},\widetilde{m}\}$			$-\beta^2\widetilde{m}^{<1>} - \beta\{\widetilde{b},\widetilde{m}\}^{<4>}$ $-\beta T(\widetilde{b})\widetilde{m}^{<5>}$ $-T(\widetilde{b})\{\widetilde{b},\widetilde{m}\}^{<3>}$
$+S(\widetilde{x})\widetilde{a}$	$\beta T(\widetilde{b})\widetilde{b}^{<8>} + Q(\widetilde{b})\widetilde{b}^{<2>}$ $-2\mu\widetilde{b}^{<5>}$	$[\beta^2 T(\widetilde{b})^{<2>} + \beta Q(\widetilde{b})^{<3>}$ $- 2\beta\mu^{<a>}]e_3$	
$+S(\widetilde{x})\widetilde{m}$			$\beta T(\widetilde{b})\widetilde{m}^{<5>}$ $+Q(\widetilde{b})\widetilde{m}^{<2>} - 2\mu\widetilde{m}^{<a>}$
$-N(\widetilde{x})1$	$-\beta Q(\widetilde{b})u^{<9>} + 2M(\widetilde{x})u^{<6>}$	$[-\beta Q(\widetilde{b})^{<3>} + 2M(\widetilde{x})^{<d>}]e_3$	
Total	0	$5[M(\widetilde{x})^{<b,c,d>} - \mu\beta^{<a>}]e_3$	$-5\mu\widetilde{m}^{<a>}$

[The terms in each column are superscripted **¡1¿, ¡2¿** etc. to show how they cancel out, or **¡a¿, ¡b¿**

etc. if they remain.

From the generic relation $\widetilde{x}^3 - T(\widetilde{x})\widetilde{x}^2 + S(\widetilde{x})x - N(\widetilde{x})1 = 5\big([M(\widetilde{x}) - \mu(\widetilde{m})\beta]e_3 - [\mu(\widetilde{m})\widetilde{m}\big) = 5R(\widetilde{x})$ we see $\widetilde{J}$ is generically algebraic of degree 3 in characteristic 5, and that in general $\big(\widetilde{x}^3 - T(\widetilde{x})\widetilde{x}^2 + S(\widetilde{x})x - N(\widetilde{x})1\big)^3 = 5^3 R(\widetilde{x})^3 = 0$ since $R(\widetilde{x})^3$ is homogeneous of degree $3 \cdot 2 = 6$ in the variables $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and any polynomial function of degree ≥ 5 in 4 odd variables must have a repeated variable in Γ_1 and hence vanish. ■

5 Characteristic 5

In characteristic 5 the Grassmann envelope is not merely degree 3, it is actually the Jordan algebra of the cubic form N and sharp mapping $\#$. Recall the General Construction of a quadratic Jordan algebra from a sharped cubic form when $\frac{1}{2} \in \Phi$ [6, p. 182, C.2 pp. 480-481]:

$$U_x y := T(x,y)x - x^{\#}\# y, \qquad x^2 := U_x 1 \tag{5.1}$$

determined by a *sharped cubic form* $(N, \#, c)$, that is, a cubic form with basepoint c where $N(c) = 1$ and a sharp mapping related by

$$\begin{array}{ll} \text{(SC1)} & c\#y = T(y)c - y, \quad (c\textit{-Sharp Identity}), \\ \text{(SC2)} & T(x^{\#}, y) = \partial_y N|_x \quad (\textit{Trace-Sharp Identity}) \\ \text{(SC3)} & x^{\#\#} = N(x)x \quad (\textit{Adjoint Identity}) \\ & \qquad\qquad \textit{which imply} \\ \text{(SC4)} & x^3 - T(x)x^2 + S(x)x - N(x)1 = 0, \\ \text{(SC5)} & x^2 - T(x)x + S(x)1 = x^{\#}, \end{array} \tag{5.2}$$

where the trace and spur mappings are defined by

$$\begin{array}{lll} T(x) := \partial_x N|_c, & \textit{hence} & T(c) = S(c) = 3, \\ S(x) := \partial_c N|_x, & \textit{hence} & S(x,y) := \partial_y S|_x = \partial_x \partial_y N|_c \\ T(x,y) := T(x)T(y) - S(x,y), & \textit{hence} & T(x,c) = T(x),\ S(x,1) = 2T(x), \\ x\#y := \partial_y \#|_x, & \textit{hence} & x\#x = 2x^{\#}, \quad T(x^{\#}) = S(x). \end{array} \tag{5.3}$$

We will verify our three axioms on $\Gamma(sK_{10}(\Phi))$ in characteristic 5 for the sharped cubic form N with basepoint $1 = e_1 + e_2 + e_3$ and trace T

$$\begin{array}{l} N(\widetilde{x}) := Q(\widetilde{b})\beta - 2M(\widetilde{b};\widetilde{m}), \quad T(\widetilde{x}) = \beta + T(\widetilde{b}), \quad M(u;\widetilde{m}) = \mu(\widetilde{m}), \\ M(\widetilde{b};\widetilde{m}) := \beta_2\gamma_1\gamma_2 + \beta_1\gamma_3\gamma_4 - \beta_3\gamma_2\gamma_4 - \beta_4\gamma_1\gamma_3 - \beta_5(\gamma_1\gamma_4 - \gamma_2\gamma_3), \\ \textit{for} \quad \widetilde{x} := \widetilde{b} \boxplus \beta e_3 \oplus \widetilde{m} \\ \qquad = \big(\beta_1 e_1 + \beta_2 e_2 + \beta_3 c_{12} + \beta_4 c_{12} + \beta_5 q_{12}\big) \boxplus \beta e_3 \oplus \big(\gamma_1 m_{13} + \gamma_2 n_{13} + \gamma_3 m_{23} + \gamma_4 n_{23}\big), \end{array} \tag{5.4}$$

as in (4.1.3) [where by abuse of notation $Q(\widetilde{b})$ denotes the extension of Q to $\widetilde{b} \in \widetilde{B} = \Gamma_0 \otimes B$], together with the sharp mapping

$$\widetilde{x}^{\#} := \big(\overline{\beta\widetilde{b} - \omega(\widetilde{m})}\big) \boxplus \big(Q(\widetilde{b}) - 5\mu(\widetilde{m})\big)e_3 \oplus \big(-\{\overline{\widetilde{b}}, m\}\big). \tag{5.5}$$

On the Peirce components this sharp mapping becomes

$$\begin{array}{ll} \widetilde{b}^{\#} & = Q(\widetilde{b})e_3, \quad \widetilde{b}\#e_3 = \overline{\widetilde{b}}, \quad e_3^{\#} = e_3\#\widetilde{m} = 0, \quad \widetilde{b}\#\widetilde{m} = -\{\overline{\widetilde{b}}, \widetilde{m}\}, \\ \widetilde{m}^{\#} & = -2\gamma_3\gamma_4 e_1 - 2\gamma_1\gamma_2 e_2 + 2\gamma_1\gamma_3 c_{12} + 2\gamma_2\gamma_4 d_{12} + (\gamma_1\gamma_4 - \gamma_2\gamma_3)q_{12} - 5\mu(\widetilde{m})e_3 \\ & = -\overline{\omega(\widetilde{m})} - 5\mu(\widetilde{m})e_3. \end{array} \tag{5.6}$$

If we set

$$\widetilde{x}^{\#} := \Big(\sum_{i=1}^{5} \beta_i^{\#} b_i\Big) \boxplus \beta^{\#} e_3 \oplus \Big(\sum_{i=1}^{4} \gamma_i^{\#} m_i\Big)$$

where $(b_1, b_2, b_3, b_4, b_5) := (e_1, e_2, c_{12}, d_{12}, q_{12})$, e_3, $(m_1, m_2, m_3, m_4) := (m_{13}, n_{13}, m_{23}, n_{23})$ is the ordered split basis, then the sharp coefficients are

$$\text{(5.7)}\qquad \begin{array}{ll} \beta_1^{\#} := \beta\beta_2 - 2\gamma_3\gamma_4, & \gamma_1^{\#} := -\beta_2\gamma_1 + \beta_5\gamma_3 - \beta_3\gamma_4, \\ \beta_2^{\#} := \beta\beta_1 - 2\gamma_1\gamma_2, & \gamma_2^{\#} := -\beta_2\gamma_2 + \beta_4\gamma_3 + \beta_5\gamma_4, \\ \beta_3^{\#} := -\beta\beta_3 + 2\gamma_1\gamma_3, & \gamma_3^{\#} := -\beta_1\gamma_3 + \beta_5\gamma_1 + \beta_3\gamma_2, \\ \beta_4^{\#} := -\beta\beta_4 + 2\gamma_2\gamma_4, & \gamma_4^{\#} := -\beta_1\gamma_4 - \beta_4\gamma_1 + \beta_5\gamma_2, \\ \beta_5^{\#} := -\beta\beta_5 + \gamma_1\gamma_4 - \gamma_2\gamma_3, & \\ \beta^{\#} := Q(\widetilde{b}) - 5\mu(\widetilde{m}), & \end{array}$$

Lemma 5.8 *The associated mappings* (5.3) *for the cubic norm form of* (5.4) *with basepoint* $c = 1$ *at elements* $\widetilde{x} = \widetilde{b} \boxplus \beta e_3 \oplus \widetilde{m}$, $\widetilde{y} = \widetilde{d} \boxplus \delta e_3 \oplus \widetilde{n}$, *are*

$$\text{(5.8}*\text{)}\qquad \begin{array}{ll} \partial_{\widetilde{y}} N|_{\widetilde{x}} & = \delta Q(\widetilde{b}) + \beta Q(\widetilde{b}, \widetilde{d}) - 2M(\widetilde{d}; \widetilde{m}) - 2M(\widetilde{b}; \widetilde{m}, \widetilde{n}), \\ T(\widetilde{x}) & = \partial_{\widetilde{x}} N|_1 = \beta_1 + \beta_2 + \beta, \\ S(\widetilde{x}) & = \partial_1 N|_{\widetilde{x}} = Q(\widetilde{b}) + \beta T(\widetilde{b}) - 2\mu(\widetilde{m}), \\ S(\widetilde{x}, \widetilde{y}) & = \beta T(\widetilde{d}) + \delta T(\widetilde{b}) + Q(\widetilde{b}, \widetilde{d}) - 2\mu(\widetilde{m}, \widetilde{n}), \quad S(\widetilde{x}, 1) = 2T(\widetilde{x}),\ T(\widetilde{x}, 1) = T(\widetilde{x}), \\ T(\widetilde{x}, \widetilde{y}) & = \beta\delta + T(\widetilde{b})T(\widetilde{d}) - Q(\widetilde{b}, \widetilde{d}) + 2\mu(\widetilde{m}, \widetilde{n}) = \beta\delta + Q(\overline{\widetilde{b}}, d) + 2\mu(\widetilde{m}, \widetilde{n}). \end{array}$$

PROOF: From the definition of the norm in (5.4), its general partial derivative is

$$\partial_{\widetilde{y}} N|_{\widetilde{x}} = \beta Q(\widetilde{b}, \widetilde{d}) + \delta Q(\widetilde{b}) - 2M(\widetilde{d}; \widetilde{m}) - 2M(\widetilde{b}; \widetilde{m}, \widetilde{n})$$

as in (*). In particular, when $\widetilde{x} = 1$ ($\widetilde{b} = u, \beta = 1, \widetilde{m} = 0$) the trace is $\partial_{\widetilde{y}} N|_1 = Q(u, \widetilde{d}) + \delta + 0 + 0 = \delta_1 + \delta_2 + \delta = T(\widetilde{y})$ as in (*), and when $\widetilde{y} = 1$ ($\widetilde{d} = u, \delta = 1, \widetilde{n} = 0$) the spur is $\partial_1 N|_{\widetilde{x}} = \beta Q(\widetilde{b}, u) + Q(\widetilde{b}) - 2M(u; \widetilde{m}) + 0 = \beta T(\widetilde{b}) + Q(\widetilde{b}) - 2\mu(\widetilde{m})$ [by (5.4)] as in (*). Moreover, the linearized spur also take the indicated form since when $\widetilde{y} = 1$ we have $S(\widetilde{x}, 1) = \beta T(u) + T(\widetilde{b}) + Q(\widetilde{b}, u) + 0 = 2\beta + 2T(\widetilde{b}) = 2T(\widetilde{x})$. Turning to the trace bilinear form, for general $\widetilde{y}$ we have $T(\widetilde{x}, \widetilde{y}) := T(\widetilde{x})T(\widetilde{y}) - S(\widetilde{x}, \widetilde{y}) = (\beta + T(\widetilde{b}))(\delta + T(\widetilde{d})) - \big(\delta T(\widetilde{b}) + \beta T(\widetilde{d}) + Q(\widetilde{b}, \widetilde{d}) + 2\mu(\widetilde{m}, \widetilde{n})\big) = \beta\delta + T(\widetilde{b})T(\widetilde{d}) - Q(\widetilde{b}, \widetilde{d}) + 2\mu(\widetilde{m}, \widetilde{n})$, with $T(\widetilde{x}, 1) = T(\widetilde{x})T(1) - S(\widetilde{x}, 1) = T(\widetilde{x})3 - 2T(\widetilde{x})$ [using the above] $= T(\widetilde{x})$. ■

Thus the spur S and trace T derived from the norm in (*) agree with (4.1.3). We know by (4.1.2) that (SC4) holds only up to a multiple involving 5μ. Both the Sharp Condition (SC5) and the c-Sharp Identity (SC1) *automatically hold in general*, but the Trace-Sharp Identity (SC2) again has a similar remainder 5μ.

Lemma 5.9 *For arbitrary* Φ *we have*

(5.9.1) $\widetilde{x}^{\#} = \widetilde{x}^2 - T(\widetilde{x})\widetilde{x} + S(\widetilde{x})1$,

(5.9.2) $\widetilde{x} \# 1 = T(\widetilde{x})1 - \widetilde{x}$,

(5.9.3) $2M(\widetilde{b}; \widetilde{m}) = Q(\widetilde{b}, \omega(\widetilde{m}))$, $\quad M(\widetilde{b}; \widetilde{m}, \widetilde{n}) = \mu(\langle \overline{\widetilde{b}}, \widetilde{m}\rangle, \widetilde{n})$,

(5.9.4) $T(\widetilde{x}^{\#}, \widetilde{y}) = \partial_{\widetilde{y}} N|_{\widetilde{x}} - 5\delta\mu(\widetilde{x}) \qquad (\widetilde{y} = \widetilde{d} + \delta e_3 + \widetilde{n})$,

(5.9.5) $T(\widetilde{x}^{\#}) = S(\widetilde{x}) - 5\mu(\widetilde{x})$.

PROOF: With our usual notation for the element $\widetilde{x}$ and the above maps, for (1) we calculate

$$
\begin{aligned}
&\widetilde{x}^2 - T(\widetilde{x})\widetilde{x} + S(\widetilde{x})1 = [\widetilde{a}^2 + \{\widetilde{a},\widetilde{m}\} + \widetilde{m}^2] - T(\widetilde{x})[\widetilde{a} + \widetilde{m}] + S(\widetilde{x})1 \\
&= [T(\widetilde{b})\widetilde{b} - Q(\widetilde{b})u + \beta^2 e_3] + [\{\widetilde{b},\widetilde{m}\} + \beta\widetilde{m}] + [\omega - 3\mu e_3] - (T(\widetilde{b}) + \beta)[\widetilde{b} + \beta e_3 + \widetilde{m}] \\
&\quad + [Q(\widetilde{b}) + \beta T(\widetilde{b}) - 2\mu](u + e_3) \qquad \text{[by (4.1.6,9) for } \{\widetilde{a},\widetilde{m}\}, \widetilde{m}^2] \\
&= [T(\widetilde{b})\widetilde{b}^{\blacktriangle} + \omega - T(\widetilde{b})\widetilde{b}^{\blacktriangle} - \beta(\textstyle\sum_{i=1}^5 \beta_i b_i)] + [-Q(\widetilde{b})u^{\blacktriangleleft} + Q(\widetilde{b})u^{\blacktriangleleft} + \beta(\beta_1 + \beta_2)u - 2\mu\, u] \\
&\quad + [\beta^{2\blacktriangledown} - 3\mu - T(\widetilde{b})\beta^{\bullet} - \beta^{2\blacktriangledown} + Q(\widetilde{b}) + \beta T(\widetilde{b})^{\bullet} - 2\mu]e_3 + [\{\widetilde{b},\widetilde{m}\} + \beta\widetilde{m}^{\blacktriangleright} - T(\widetilde{b})\widetilde{m} - \beta\widetilde{m}^{\blacktriangleright}] \\
&= [\beta(\beta_1^{\bullet} + \beta_2) - \beta\beta_1^{\bullet} + 2\gamma_1\gamma_2^{\bullet} - 2(\gamma_1\gamma_2^{\bullet} + \gamma_3\gamma_4)]e_1 + [\beta(\beta_1 + \beta_2^{\bullet}) - \beta\beta_2^{\bullet} + 2\gamma_3\gamma_4^{\bullet} - 2(\gamma_1\gamma_2 + \gamma_3\gamma_4^{\bullet})]e_2 \\
&\quad + (2\gamma_1\gamma_3 - \beta\beta_3)c_{12} + (2\gamma_2\gamma_4 - \beta\beta_4)d_{12} + (\gamma_1\gamma_4 - \gamma_2\gamma_3 - \beta\beta_5)q_{12}] \\
&\quad + [Q(\widetilde{b}) - 5\mu]e_3 + [-\{\overline{\widetilde{b}}, \widetilde{m}\}] \qquad \text{[by (4.1.3) for } \omega] \\
&= [(\beta_1^{\#})e_1 + (\beta_2^{\#})e_2 + (\beta_1^{\#})c_{12} + (\beta_4^{\#})d_{12} + (\beta_5^{\#})q_{12}] + [\beta^{\#}]e_3 + [-\{\overline{\widetilde{b}}, \widetilde{m}\}]
\end{aligned}
$$

by (5.7), where by (4.1.6) [with $\beta = 0$] the component in M is

$$
\begin{aligned}
-\{\overline{\widetilde{b}}, \widetilde{m}\} &= [-\beta_1\gamma_1 + \beta_3\gamma_4 - \beta_5\gamma_3]m_{13} + [-\beta_1\gamma_2 - \beta_4\gamma_3 - \beta_5\gamma_4]n_{13} \\
&\quad + [\beta_2\gamma_3 - \beta_3\gamma_2 - \beta_5\gamma_1]m_{23} + [-\beta_2\gamma_4 + \beta_4\gamma_1 - \beta_5\gamma_2]n_{23}, \\
&= \gamma_1^{\#} m_{13} + \gamma_2^{\#} n_{13} + \gamma_3^{\#} m_{23} + \gamma_4^{\#} n_{23}.
\end{aligned} \tag{5.9.6}
$$

These are precisely the coefficients $\beta_i^{\#}$, $\gamma_i^{\#}$ of $\widetilde{x}^{\#}$ by (5.7).

(2) follows from (1) by linearizing $\widetilde{x} \to \widetilde{x}, 1$ to get $\widetilde{x}\#1 = \{\widetilde{x}, 1\} - T(\widetilde{x})1 - T(1)\widetilde{x} + S(\widetilde{x}, 1)1 = 2\widetilde{x} - T(\widetilde{x})1 - 3\widetilde{x} + 2T(\widetilde{x})1 = T(\widetilde{x})1 - \widetilde{x}$. Alternately, it follows directly from linearizing the definition (5.5): $\widetilde{x}\#1 = \big(\overline{\beta u + 1b - 0}\big) \boxplus \big(T(b) - 0\big)e_3 \oplus \big(-\langle \overline{u}, m\rangle\big)$ [again ω, μ involve only M so $\partial_1\omega = \partial_1\mu = 0$] $= \big(\beta u + (T(b)u - b)\big) \boxplus \big(T(\widetilde{x}) - \beta\big)e_3 \oplus \big(-m\big) = T(\widetilde{x})(u + e_3) - b - \beta e_3 - m = T(\widetilde{x})1 - \widetilde{x}$.

In the proof of the Trace-Sharp Identity we will need two other relations between μ, ω, and M. For the first part of (3) we have $Q(\widetilde{b}, \omega(\widetilde{m})) = Q\big([\beta_1 e_1 + \beta_2 e_2 + \beta_3 c_{12} + \beta_4 c_{12} + \beta_5 q_{12}],\ [2\gamma_1\gamma_2 e_1 + 2\gamma_3\gamma_4 e_2 + 2\gamma_1\gamma_3 c_{12} + 2\gamma_2\gamma_4 d_{12} + (\gamma_1\gamma_4 - \gamma_2\gamma_3)q_{12}]\big) = (\beta_1)(2\gamma_3\gamma_4) + (\beta_2)(2\gamma_1\gamma_2) - (\beta_3)(2\gamma_2\gamma_4) - (\beta_4)(2\gamma_1\gamma_3) - (\beta_5)(\gamma_1\gamma_4 - \gamma_2\gamma_3)(2)$ [using (2.1)] $= 2\big(\beta_1\gamma_3\gamma_4 + \beta_2\gamma_1\gamma_2 - \beta_3\gamma_2\gamma_4 - \beta_4\gamma_1\gamma_3 - \beta_5(\gamma_1\gamma_4 - \gamma_2\gamma_3)\big) = 2M(\widetilde{b}; \widetilde{m})$ [by (4.1.3)].

For the second part, we linearize $\widetilde{m} \to \widetilde{m}, \widetilde{n}$ in the definition (5.4) of M to get $M(\widetilde{b}; \widetilde{m}, \widetilde{n})$

$$
\begin{aligned}
&= \beta_2(\gamma_1\eta_2 + \eta_1\gamma_2) + \beta_1(\gamma_3\eta_4 + \eta_3\gamma_4) - \beta_3(\gamma_2\eta_4 + \eta_2\gamma_4) - \beta_4(\gamma_1\eta_3 + \eta_1\gamma_3) - \beta_5([\gamma_1\eta_4 + \eta_1\gamma_4] - [\gamma_2\eta_3 + \eta_2\gamma_3]) \\
&= \big(\beta_2\gamma_1 + \beta_3\gamma_4 - \beta_5\gamma_3\big)\eta_2 + \eta_1\big(\beta_2\gamma_2 - \beta_4\gamma_3 - \beta_5\gamma_4\big) + \big(\beta_1\gamma_3 - \beta_3\gamma_2 - \beta_5\gamma_1\big)\eta_4 + \eta_3\big(\beta_1\gamma_4 + \beta_4\gamma_1 - \beta_5\gamma_2\big) \\
&= \mu\Big(\ \big([\beta_2\gamma_1 + \beta_3\gamma_4 - \beta_5\gamma_3]m_1 + [\beta_2\gamma_2 - \beta_4\gamma_3 - \beta_5\gamma_4]m_2 \\
&\qquad + [\beta_1\gamma_3 - \beta_3\gamma_2 - \beta_5\gamma_1]m_3 + [\beta_1\gamma_4 + \beta_4\gamma_1 - \beta_5\gamma_2]m_4\big),\ |; \widetilde{n}\Big) \\
&= \mu\big(\langle \bar{b}, \widetilde{m}\rangle, \widetilde{n}\big) \text{ [using (5.9.6) above]. This establishes (3).}
\end{aligned}
$$

Now we can establish the Trace-Sharp Identity (4):

$$
\begin{aligned}
T(\widetilde{x}^{\#}, \widetilde{y}) &= T(\overline{[\beta\widetilde{b} - \omega]} + [Q(\widetilde{b}) - 5\mu]e_3 + [-\langle \overline{\widetilde{b}}, m\rangle],\ \widetilde{d} + \delta_3 + \widetilde{n}) && \text{[by definition (5.5) of \#]} \\
&= [Q(\widetilde{b}) - 5\mu]\delta + Q(\overline{\overline{[\beta\widetilde{b} - \omega]}}, \widetilde{d}) + 2\mu([-\langle \overline{\widetilde{b}}, m\rangle], \widetilde{n}) && \text{[by definition (5.8*) of } T(\widetilde{x}, \widetilde{y})] \\
&= Q(\widetilde{b})\delta + \beta Q(\widetilde{b}, \widetilde{d}) - Q(\widetilde{d}, \omega) - 2\mu(\langle \overline{\widetilde{b}}, \widetilde{m}\rangle, \widetilde{n}) - 5\delta\mu \\
&= \delta Q(\widetilde{b}) + \beta Q(\widetilde{b}, \widetilde{d}) - 2M(\widetilde{d}; \widetilde{m}) - 2M(\widetilde{b}; \widetilde{m}, \widetilde{n}) - 5\delta\mu && \text{[by (3)]} \\
&= \partial_{\widetilde{y}} N|_{\widetilde{x}} - 5\delta\mu && \text{[by (5.8*)]}
\end{aligned}
$$

The identity (5) is the special case $y = 1$ of (4) since $\partial_1 N = S$. It also follows from (5.5) or (5.7) by $T(\widetilde{x}^{\#}) = \beta_1^{\#} + \beta_2^{\#} + \beta^{\#} = (\beta\beta_2 - 2\gamma_3\gamma_4) + (\beta\beta_1 - 2\gamma_1\gamma_2) + (Q(\widetilde{b}) - 5\mu(\widetilde{m})) = \beta T(\widetilde{b}) - 2\mu(\widetilde{m}) + [Q(\widetilde{b}) - 5\mu(\widetilde{m})] = S(\widetilde{x}) - 5\mu(\widetilde{x})$. ■

Remark 5.10 *We can also see how close the Trace-Sharp Identity* (5.9.4) *comes to holding by comparing the two expressions for each basic variable* $\widetilde{y}$. From (5.8.1) we have $T(\widetilde{x}^\#, \widetilde{a}) = Q(\widetilde{b}^\#, \overline{\widetilde{a}}) + \beta^\# \alpha$ and $T(\widetilde{x}^\#, \widetilde{n}) = 2\mu(\widetilde{m}^\#, \widetilde{n}) = 2(\gamma_1\delta_2 + \delta_1\gamma_2 + \gamma_3\delta_4 + \delta_3\gamma_4)$, so

$$
\begin{aligned}
&T(\widetilde{x}^\#, e_1) = Q(b^\#, e_2) = \beta_1^\#,\\
&T(\widetilde{x}^\#, e_2) = Q(b^\#, e_1) = \beta_2^\#,\\
&T(\widetilde{x}^\#, e_3) = \beta^\#,\\
&T(\widetilde{x}^\#, c_{12}) = -Q(b^\#, c_{12}) = \beta_4^\#,\\
&T(\widetilde{x}^\#, d_{12}) = -Q(b^\#, d_{12}) = \beta_3^\#,\\
&T(\widetilde{x}^\#, q_{12}) = -Q(q^\#, q_{12}) = 2\beta_5^\#,\\
&T(\widetilde{x}^\#, \delta_1 m_{13}) = 2\mu(\widetilde{m}^\#, \delta_1 m_{13}) = 2\delta_1\gamma_2^\#,\\
&T(\widetilde{x}^\#, \delta_2 n_{13}) = 2\mu(\widetilde{x}^\#, \delta_2 n_{13}) = 2\gamma_1^\#\delta_2,\\
&T(\widetilde{x}^\#, \delta_3 m_{23}) = 2\mu(\widetilde{x}^\#, \delta_3 m_{23}) = 2\delta_3\gamma_4^\#,\\
&T(\widetilde{x}^\#, \delta_4 n_{23}) = 2\mu(\widetilde{x}^\#, \delta_4 n_{23}) = 2\gamma_3^\#\delta_4.
\end{aligned}
$$

On the other hand, from (5.8*) we have $\partial_{\widetilde{y}} N|_{\widetilde{x}} = \delta Q(\widetilde{b}) + \beta Q(\widetilde{b}, \widetilde{d}) - 2M(\widetilde{d}; \widetilde{m})$ for $\widetilde{y} = \widetilde{d} + \delta e_3 \in \widetilde{A}$, so from (5.4), (5.7)

$$
\begin{aligned}
&\partial_{e_1} N|_{\widetilde{x}} = \beta Q(b, e_1) - 2M(e_1; \widetilde{m}) = \beta\beta_2 - 2\gamma_3\gamma_4 = \beta_1^\#,\\
&\partial_{e_2} N|_{\widetilde{x}} = \beta Q(b, e_2) - 2M(e_2; \widetilde{m}) = \beta\beta_1 - 2\gamma_1\gamma_2 = \beta_2^\#,\\
&\partial_{e_3} N|_{\widetilde{x}} = Q(b) = \beta^\# + 5\mu,\\
&\partial_{c_{12}} N|_{\widetilde{x}} = \beta Q(b, c_{12}) - 2M(c_{12}; \widetilde{m}) = -\beta\beta_4 + 2\gamma_2\gamma_4 = \beta_3^\#,\\
&\partial_{d_{12}} N|_{\widetilde{x}} = \beta Q(b, d_{12}) - 2M(d_{12}; \widetilde{m}) = -\beta\beta_3 + 2\gamma_1\gamma_3 = \beta_4^\#,\\
&\partial_{q_{12}} N|_{\widetilde{x}} = \beta Q(b, q_{12}) - 2M(q_{12}; \widetilde{m}) = -2\beta\beta_5 + 2(\gamma_1\gamma_4 - \gamma_2\gamma_3) = 2\beta_5^\#,
\end{aligned}
$$

and from (5.4) again for an element $\widetilde{y} = \widetilde{n} = \delta_1 m_{13} + \delta_2 n_{13} + \delta_3 m_{23} + \delta_4 n_{23} \in \widetilde{M}$ we have $\partial_{\widetilde{n}} N|_{\widetilde{x}} = -2M(\widetilde{b}; \widetilde{m}, \widetilde{n}) = -2\delta_1[\beta_2\gamma_2 - \beta_4\gamma_3 - \beta_5\gamma_4] - 2[\beta_2\gamma_1 - \beta_5\gamma_3 + \beta_3\gamma_4]\delta_2 - 2\delta_3[\beta_4\gamma_1 - \beta_5\gamma_2 + \beta_1\gamma_4]$ $-2[-\beta_5\gamma_1 - \beta_3\gamma_2 + \beta_1\gamma_3]\delta_4$, so from (5.7)

$$
\begin{aligned}
&\partial_{\delta_1 m_{13}} N|_{\widetilde{x}} = 2\delta_1(-\beta_2\gamma_2 + \beta_4\gamma_3 + \beta_5\gamma_4) = 2\delta_1\gamma_2^\#,\\
&\partial_{\delta_2 n_{13}} N|_{\widetilde{x}} = 2(-\beta_2\gamma_1 + \beta_5\gamma_3 - \beta_3\gamma_4)\delta_2 = 2\gamma_1^\#\delta_2,\\
&\partial_{\delta_3 m_{23}} N|_{\widetilde{x}} = 2\delta_3(-\beta_4\gamma_1 + \beta_5\gamma_2 - \beta_1\gamma_4) = 2\delta_3\gamma_4^\#,\\
&\partial_{\delta_4 n_{23}} N|_{\widetilde{x}} = 2(\beta_5\gamma_1 + \beta_3\gamma_2 - \beta_1\gamma_3)\delta_4 = 2\gamma_3^\#\delta_4.
\end{aligned}
$$

Comparing these, we see they coincide in characteristic 5, but not in general. Only the coefficients with $\beta^\#$ differ by a multiple of 5; we have tried adjusting N and S and # to remedy this, but to no avail. ■

Note that in characteristic 5 we have $\frac{1}{2} = 3 \in \Phi$ so the General Construction [6, C.2] is defined.

Characteristic Five Theorem 5.11 *If* Φ *has characteristic* 5*, then the Grassmann envelope* $\Gamma(sK_{10}(\Phi))$ *is the quadratic Jordan algebra over* Γ_0 *determined by the sharped cubic form* N *with basepoint* $1 = e_1 + e_2 + e_3$ *and trace* T *given by* (5.4) *and the sharp given* (5.5-6).

Proof: We must establish the sharped-cubic axioms (5.2). We have already verified (SC1) in arbitrary characteristic in (5.9.2), so we must verify (SC2) holds in characteristic 5 by (5.9.4). Though they follow from (SC1-3), note that (SC5) holds in general by (5.9.1), and (SC4) holds in characteristic 5 by the Grassmann Theorem 4.1.

The *pièce de résistance* is the Adjoint Identity (SC3). We will show each coefficient $\beta_i^{\#\#}, \gamma_i^{\#\#}$ of $(\widetilde{x}^\#)^\#$ is (in characteristic 5) the scalar $N(\widetilde{x})$ times the original coefficients β_i, γ_i of $\widetilde{x}$.

Coefficient of e_1 (similarly e_2): $\beta_1^{\#\#} = [\beta_2^\#\beta^\# - 2\gamma_3^\#\gamma_4^\#]$

$= [\beta_1\beta - 2\gamma_1\gamma_2][Q(\widetilde{b}) - 5\mu] - 2[\beta_3\gamma_2 + \beta_5\gamma_1 - \beta_1\gamma_3][-\beta_4\gamma_1 + \beta_5\gamma_2 - \beta_1\gamma_4]$

$= [\beta_1\beta Q(\widetilde{b}) - 5R_1] - 2\big[(\beta_1\beta_2^{<\mathbf{1}>} - \beta_3\beta_4^{\blacktriangle} - \beta_5^{2\blacktriangledown})\gamma_1\gamma_2 + (-\beta_3\beta_4\gamma_2\gamma_1^{\blacktriangle} + 0\text{-}\beta_3\beta_1\gamma_2\gamma_4^{<\mathbf{3}>})$

$$+(0+\beta_5\beta_5\gamma_1\gamma_2^{\blacktriangledown}-\beta_5\beta_1\gamma_1\gamma_4^{<\mathbf{5}>})+(\beta_1\beta_4\gamma_3\gamma_1^{<\mathbf{4}>}-\beta_1\beta_5\gamma_3\gamma_2^{<\mathbf{6}>}+\beta_1\beta_1\gamma_3\gamma_4^{<\mathbf{2}>})\big]$$

$$=\big[Q(\widetilde{b})\beta-2\big(\beta_2\gamma_1\gamma_2^{<\mathbf{1}>}+\beta_1\gamma_3\gamma_4^{<\mathbf{2}>}-\beta_3\gamma_2\gamma_4^{<\mathbf{3}>}-\beta_4\gamma_1\gamma_3^{<\mathbf{4}>}-\beta_5(\gamma_1\gamma_4^{<\mathbf{5}>}-\gamma_2\gamma_3^{<\mathbf{6}>})\big)\big]\beta_1-5R_1$$

$$=[Q(\widetilde{b})\beta-2M(\widetilde{x})]\beta_1-5R_1=N(\widetilde{x})\beta_1-5R \text{ [by (4.1.3)}$$

for $R=[\beta\beta_1-2\gamma_1\gamma_2]\mu$.

Coefficient of e_3 : $\quad \beta^{\#\#}+5\mu(m\#)=Q(b^{\#})=[\beta_1^{\#}\beta_2^{\#}-\beta_3^{\#}\beta_4^{\#}-\beta_5^{2\#}]$

$$=[2\gamma_3\gamma_4-\beta_2\beta][2\gamma_1\gamma_2-\beta_1\beta]-[2\gamma_1\gamma_3-\beta_3\beta][2\gamma_2\gamma_4-\beta_4\beta]-[\gamma_1\gamma_4-\gamma_2\gamma_3-\beta_5\beta]^2$$

$$=\beta_2\beta_1\beta^2+2\Big[2\gamma_3\gamma_4\gamma_1\gamma_2-\gamma_3\gamma_4\beta_1\beta-\beta_2\beta\gamma_1\gamma_2]-\beta_3\beta_4\beta^2-2[2\gamma_1\gamma_3\gamma_2\gamma_4-\beta_4\beta\gamma_1\gamma_3-\beta_3\beta\gamma_2\gamma_4]$$

$$-[0-2\gamma_1\gamma_4\gamma_2\gamma_3+0-2\gamma_1\gamma_4\beta_5\beta+2\gamma_2\gamma_3\beta_5\beta+\beta_5^2\beta^2]$$

$$=\beta^2\Big[\beta_1\beta_2^{<\mathbf{1}>}-\beta_3\beta_4^{<\mathbf{2}>}+\beta_5^{2<\mathbf{3}>}\Big]-2\beta\Big[\beta_2\gamma_1\gamma_2^{<\mathbf{4}>}+\beta_1\gamma_3\gamma_4^{<\mathbf{5}>}-\beta_3\gamma_2\gamma_4^{<\mathbf{6}>}$$

$$-\beta_4\gamma_1\gamma_3^{<\mathbf{7}>}-\beta_5(\gamma_1\gamma_4-\gamma_2\gamma_3)^{<\mathbf{8}>}\Big]+\Big[4\gamma_3\gamma_4\gamma_1\gamma_2-4\gamma_1\gamma_3\gamma_2\gamma_4+2\gamma_1\gamma_4\gamma_2\gamma_3\Big]^{<\mathbf{9}>}$$

$$=\beta\Big[\beta Q(b)^{<\mathbf{1,2,3}>}\Big]-2\beta\Big[M(\widetilde{b};\widetilde{x})^{<\mathbf{4,5,6,7,8}>}\Big]+\Big[(4+4+2)\gamma_1\gamma_2\gamma_3\gamma_4^{<\mathbf{9}>}\Big]$$

$$=\beta N(\widetilde{x})+10\gamma_1\gamma_2\gamma_3\gamma_4.$$

A calculation shows $\mu(m\#)=\gamma_1^{\#}\gamma_2^{\#}+\gamma_3^{\#}\gamma_4^{\#}=T(\widetilde{b})M(\widetilde{b};\widetilde{m})-Q(\widetilde{b})\mu(\widetilde{m})$, so the error term $\beta^{\#\#}-N(\widetilde{x})\beta$ is $5R$ for $R=[\mu^2+Q(\widetilde{b})\mu-T(\widetilde{b})M(\widetilde{b};\widetilde{m})]$ since $\mu^2=2\gamma_1\gamma_2\gamma_3\gamma_4$.

Coefficient of c_{12} (similarly d_{12}) : $\quad \beta_3^{\#\#}-5\beta_3^{\#}\mu(\widetilde{m})=2\gamma_1^{\#}\gamma_3^{\#}-\beta_3^{\#}[\beta^{\#}+5\mu(\widetilde{m})]$

$$=2[-\beta_2\gamma_1+\beta_5\gamma_3-\beta_3\gamma_4][\beta_5\gamma_1+\beta_3\gamma_2-\beta_1\gamma_3]-[2\gamma_1\gamma_3-\beta_3\beta]Q(\widetilde{b})$$

$$=2\Big[\big(0-\beta_2\beta_3\gamma_1\gamma_2^{<\mathbf{2}>}+\beta_1\beta_2\gamma_1\gamma_3^{\blacktriangle}\big)+\big(\beta_5^2\gamma_3\gamma_1^{\blacktriangledown}+\beta_5\beta_3\gamma_3\gamma_2^{<\mathbf{5b}>}-(0)\big)+\big(-\beta_3\beta_5\gamma_4\gamma_1^{<\mathbf{5a}>}-\beta_3^2\gamma_4\gamma_2^{<\mathbf{3}>}$$

$$+\beta_1\beta_3\gamma_4\gamma_3^{<\mathbf{1}>}\big)\Big]+\Big[\beta Q(\widetilde{b})\beta_3^{<\mathbf{6}>}-2(\beta_1\beta_2^{\blacktriangle}-\beta_3\beta_4^{<\mathbf{5a}>}-\beta_5^{2\blacktriangledown})\gamma_1\gamma_3\Big]$$

$$=\Big[\beta Q(\widetilde{b})^{<\mathbf{6}>}-2\big(\beta_1\gamma_3\gamma_4^{<\mathbf{1}>}+\beta_2\gamma_1\gamma_2^{<\mathbf{2}>}-\beta_3\gamma_2\gamma_4^{<\mathbf{3}>}-\beta_4\gamma_1\gamma_3^{<\mathbf{4}>}-\beta_5(\gamma_1\gamma_4^{<\mathbf{5a}>}-\gamma_2\gamma_3^{<\mathbf{5b}>})\big)\Big]\beta_3$$

$$=[\beta Q(\widetilde{b})-2M(\widetilde{b};\widetilde{m})]\beta_3=N(\widetilde{x})\beta_3.$$

Thus the error term here is $5R$ for $R=\beta_3^{\#}\mu$.

Coefficient of q_{12} : $\quad \beta_5^{\#\#}-5\mu(\widetilde{m})=\gamma_1^{\#}\gamma_4^{\#}-\gamma_2^{\#}\gamma_3^{\#}-[\beta^{\#}+5\mu(\widetilde{m})]\beta_5^{\#}$

$$=[-\beta_2\gamma_1+\beta_5\gamma_3-\beta_3\gamma_4][-\beta_4\gamma_1+\beta_5\gamma_2-\beta_1\gamma_4]-[-\beta_2\gamma_2+\beta_4\gamma_3+\beta_5\gamma_4][\beta_5\gamma_1+\beta_3\gamma_2-\beta_1\gamma_3]$$

$$-Q(\widetilde{b})[\gamma_1\gamma_4-\gamma_2\gamma_3-\beta\beta_5]$$

$$=[(0-\beta_2\beta_5\gamma_1\gamma_2^{<\mathbf{2}>}+\beta_2\beta_1\gamma_1\gamma_4^{\blacktriangle})+(-\beta_5\beta_4\gamma_3\gamma_1^{<\mathbf{4}>}-\beta_5^2\gamma_3\gamma_2^{<\mathbf{5b}>}-\beta_5\beta_1\gamma_3\gamma_4^{<\mathbf{1}>})+(\beta_3\beta_4\gamma_4\gamma_1^{\blacktriangleright}-\beta_3\beta_5\gamma_4\gamma_2^{<\mathbf{3}>}+0)]$$

$$+[-(0-\beta_2\beta_5\gamma_2\gamma_1^{<\mathbf{2}>}+\beta_2\beta_1\gamma_2\gamma_3^{\blacktriangleleft})-(\beta_4\beta_5\gamma_3\gamma_1^{<\mathbf{4}>}+\beta_4\beta_3\gamma_3\gamma_2^{\bullet}+0)-(\beta_5^2\gamma_4\gamma_1^{<\mathbf{5a}>}+\beta_5\beta_3\gamma_4\gamma_2^{<\mathbf{3}>}$$

$$-\beta_5\beta_1\gamma_4\gamma_3^{<\mathbf{2}>})]-\big[(\beta_1\beta_2^{\blacktriangle}-\beta_3\beta_4^{\blacktriangleright}-\beta_5^{2<\mathbf{5a}>})\gamma_1\gamma_4-(\beta_1\beta_2^{\blacktriangleleft}-\beta_3\beta_4^{\bullet}-\beta_5^{2<\mathbf{5b}>}\gamma_2\gamma_3)^{<\mathbf{5b}>}\big]$$

$$=\big[\beta Q(\widetilde{b})^{<\mathbf{6}>}-2\big(\beta_1\gamma_3\gamma_4^{<\mathbf{1}>})+\beta_2\gamma_1\gamma_2^{<\mathbf{2}>}-\beta_3\gamma_2\gamma_4^{<\mathbf{3}>}-\beta_4\beta_5\gamma_1\gamma_3^{<\mathbf{4}>}-\beta_5(\gamma_1\gamma_4^{<\mathbf{5a}>}-\gamma_2\gamma_3^{<\mathbf{5b}>})\big)\big]\beta_5$$

$$=\big[\beta Q(\widetilde{b})^{<\mathbf{6}>}-2M(\widetilde{b};\widetilde{m})\big]\beta_5=N(\widetilde{x})\beta_5.$$

Here the error terms is $5R$ for $R=\beta_5^{\#}\ mu$.

Coefficient of m_{13} (similarly for n_{13},m_{23},n_{23}) : $\quad \gamma_1^{\#\#}=-\beta_2^{\#}\gamma_1^{\#}+\beta_5^{\#}\gamma_3^{\#}-\beta_3^{\#}\gamma_4^{\#}$

$$=-[\beta\beta_1-2\gamma_1\gamma_2][-\beta_2\gamma_1+\beta_5\gamma_3-\beta_3\gamma_4]+[\gamma_1\gamma_4-\gamma_2\gamma_3-\beta\beta_5][\beta_5\gamma_1+\beta_3\gamma_2-\beta_1\gamma_3]$$

$$
\begin{aligned}
&-[2\gamma_1\gamma_3-\beta\beta_3][-\beta_4\gamma_1+\beta_5\gamma_2-\beta_1\gamma_4]\\
&=\Big[-\big(-\beta\beta_1\beta_2\gamma_1^{\mathbf{<1>}}+\beta\beta_1\beta_5\gamma_3^{\blacktriangle}-\beta\beta_1\beta_3\gamma_4^{\blacktriangledown}\big)+2\big(0+\beta_5\gamma_1\gamma_2\gamma_3^{\mathbf{<2>}}-\beta_3\gamma_1\gamma_2\gamma_4^{\mathbf{<3>}}\big)\Big]\\
&\quad+\Big[\big(0+\beta_3\gamma_1\gamma_4\gamma_2^{\mathbf{<3>}}-\beta_1\gamma_1\gamma_4\gamma_3^{\mathbf{<4>}}\big)-\big(\beta_5\gamma_2\gamma_3\gamma_1^{\mathbf{<2>}}-0-0\big)+\big(-\beta\beta_5^2\gamma_1^{\mathbf{<6>}}-\beta\beta_5\beta_3\gamma_2^{\blacktriangleright}+\beta\beta_5\beta_1\gamma_3^{\blacktriangle}\big)\Big]\\
&\quad+\Big[-2\big(0+\beta_5\gamma_1\gamma_3\gamma_2^{\mathbf{<2>}}-\beta_1\gamma_1\gamma_3\gamma_4^{\mathbf{<4>}}\big)+\big(-\beta\beta_3\beta_4\gamma_1^{\mathbf{<5>}}+\beta\beta_3\beta_4\gamma_2^{\blacktriangleright}-\beta\beta_3\beta_1\gamma_4^{\blacktriangledown}\big)\Big]\\
&=\Big[\beta\big(\beta_1\beta_2^{\mathbf{<1>}}-\beta_3\beta_4^{\mathbf{<5>}}-\beta_5^{2\,\mathbf{<6>}}\big)+3\big(\beta_1\gamma_3\gamma_4^{\mathbf{<4>}}-\beta_3\gamma_2\gamma_4^{\mathbf{<3>}}+\beta_5\gamma_2\gamma_3^{\mathbf{<2>}}\big)\Big]\gamma_1\\
&=\Big[\beta Q(\widetilde b)^{\mathbf{<1,5,6>}}+3\big(\beta_1\gamma_3\gamma_4+\beta_2\gamma_1^{\bullet}\gamma_2-\beta_3\gamma_2\gamma_4-\beta_4\gamma_1^{\bullet}\gamma_3-\beta_5(\gamma_1^{\bullet}\gamma_4-\gamma_2\gamma_3)\big)\Big]\gamma_1\quad[\text{since } \gamma_1^2=0^{\bullet}]\\
&=[\beta Q(\widetilde b)+3M(\widetilde b;\widetilde m)]\gamma_1=[N(\widetilde x)+5M(\widetilde b;\widetilde m)]\gamma_1.
\end{aligned}
$$

Here the error term is $5R$ for $R=M(\widetilde b;\widetilde m)$.

This completes the verification of the Adjoint Identity (SC3) [for those who don't like the weasel-words "similarly", in the next section we make the symmetries precise]. Thus in characteristic 5 the envelope $\widetilde J$ satisfies the axioms (SC1-3) to be the Jordan algebra of the sharped cubic form $(N,1,\#)$. ■

In particular, we have an explicit formula for the U-operator in $\widetilde J$:

$$U_{\widetilde x}\widetilde y=T(\widetilde x,\widetilde y)\widetilde x-(\widetilde x)^{\#}\#\widetilde y.$$

In turn, this gives us (only in characteristic 5 !) a closed-form expression for the quadratic operators in wJ, resulting in the table

(5.11) Characteristic 5 quadratic products $U_{\widetilde x}\widetilde y$

$\widetilde y\backslash\widetilde x$	$U_{\widetilde b}$	U_{e_3}	$U_{\widetilde m}$	$U_{\widetilde b,e_3}$	$U_{\widetilde b,\widetilde m}$	$U_{\widetilde m,e_3}$
$\widetilde d$	$Q(\widetilde b,\overline{\widetilde d})\widetilde b$-$Q(\widetilde b)\overline{\widetilde d}$	0	$Q(\omega(\widetilde m),\widetilde d)e_3$	0	$Q(\widetilde b,\overline{\widetilde d})\widetilde m$-$\{\overline{\widetilde d},\{\overline{\widetilde b},\widetilde m\}\}$	0
δe_3	0	δe_3	$\delta\omega(\widetilde m)$	0	0	$\delta\widetilde m$
$\widetilde n$	0	0	$2\mu(\widetilde m,\widetilde n)\widetilde m$-$\{\omega(\widetilde m),\widetilde n\}$	$\{\widetilde b,\widetilde m\}$	$\overline{2\mu(\widetilde m,\widetilde n)\widetilde b\text{-}\omega(\{\overline{\widetilde b},\widetilde m\},\widetilde n)}$	$2\mu(\widetilde m,\widetilde n)e_3$

Note that since the characteristic is 5 we have $2\mu=-3\mu$ and $\overline{-\omega(\{\overline{\widetilde b},m\},\widetilde n)}=3\mu(\{\overline{\widetilde b},m\},\widetilde n)u+\omega(\{\overline{\widetilde b},\widetilde m\},\widetilde n)$ [recalling that $T(\omega)=2\mu$].

Over arbitrary scalars we can only say that the quadratic structure is built out of a sharped-cubic-with-remainder.

U Construction 5.12 *The quadratic operator in the Grassmann envelope* $\Gamma(sK_{10})$ *has the form*

$$
\begin{aligned}
U_{\widetilde x}\widetilde y&=T(\widetilde x,\widetilde y)\widetilde x-\widetilde x^{\#}\#\widetilde y+5\mathcal R(\widetilde x,wy)\qquad\textit{with remainder}\\
\mathcal R(\widetilde x,\widetilde y)&:=[-\mu(\widetilde m)\overline{\widetilde d}]\boxplus[M(\widetilde d;\widetilde m)+\mu((\{\widetilde b,\widetilde m\}-\beta\widetilde m),\widetilde n)]e_3\oplus[-\mu(\widetilde m,\widetilde n)].
\end{aligned}
$$

Proof: Since we are working entirely within the Grassmann envelope, we will drop the tildes and denote the elements (no longer segregated into even $\widetilde b$ and odd $\widetilde m$) uniformly by x,y. We make use of the generic polynomial-with-remainder formula (4.1.2) $x^3=T(x)x^2-S(x)x+N(x)1+5R$ for the cube and the degree 2 formula (5.9.1) $x^2=T(x)x-S(x)1+x^{\#}$ without remainder for the square to compute $U_xy=\partial_y(x^3)-U_{x,y}x=\partial_y(x^3)-\{x^2,y\}$, $z\#y=\{z,y\}-T(z)y-T(y)z+S(z,y)1$ to get

$$
\begin{aligned}
&U_x y - T(x,y)x + x^{\#}\#y = \partial_y(x^3) - \{x^2, y\} + x^{\#}\#y - [T(x)T(y) - S(x,y)]x \\
&= \Big(T(y)x^{2<\mathbf{1}>} + T(x)\{x,y\}^{\blacktriangle} - S(x)y^{\vartriangle} - S(x,y)x^{\blacktriangledown} + \partial_y N|_x 1^{<\mathbf{2}>} + 5\partial_y(R)\Big) \\
&-\{[x^{\#\bullet} + T(x)x^{\blacktriangle} - S(x)1^{\vartriangle}],\, y\} + \Big(\{x^{\#}, y\}^{\bullet} - T(x^{\#})y^{<\mathbf{3a}>} - T(y)x^{\#<\mathbf{3b}>} + S(x^{\#}, y)1^{<\mathbf{3c}>}\Big) \\
&\quad - T(x)T(y)x^{<\mathbf{4}>} + S(x,y)x^{\blacktriangledown} \\
&= T(y)[x^2 - x^{\#}]^{<\mathbf{1,3b}>} + [S(x)y^{\vartriangle} - T(x^{\#})^{<\mathbf{3a}>}]y + \big(T(x^{\#}, y)^{<\mathbf{2a}>} + 5R_1\big)1 - T(x)T(y)x^{<\mathbf{4}>} \\
&\quad + 5R \qquad\qquad [\text{by } (5.9.4)] \\
&= T(y)[T(x)x^{\blacktriangleleft} - S(x)1]^{<\mathbf{1,3b}>} + 5R_2^{\vartriangle<\mathbf{1,3a}>} + \Big(T(x^{\#})T(y)^{<\mathbf{2a}>} + 5R_1^{<\mathbf{2b}>} + S(x,y)^{<\mathbf{3c}>}\Big)1 \\
&\quad - T(x)T(y)x^{<\mathbf{4}>\blacktriangleleft} + 5R_1 1 + 5\partial_y(R) \qquad\qquad [\text{by } (5.9.1),\ (5.9.5),\ (5.9.4)] \\
&= T(y)[-S(x)^{<\mathbf{1,3b}>} + T(x^{\#})^{<\mathbf{2a}>}]1 + 5R_2 y + 5R_1 1 + 5\partial_y(R) \\
&= T(y)[-5R_2]1 + 5R_2 y + 5R_1 1 + 5\partial_y(R) = 5\mathcal{R}(x,y).
\end{aligned}
$$

The remainder terms are $R(\widetilde{x}) = [M(\widetilde{b}; \widetilde{m}) - \mu(\widetilde{m})\beta]e_3 - \mu(\widetilde{m})\widetilde{m},\ R_1 = \delta\mu(\widetilde{m}),\ R_2 = \mu(\widetilde{m})$, hence the final expression for $\mathcal{R}$ is $\big((-T(\widetilde{d}) - \delta)\mu(\widetilde{m})^{\blacktriangle}\big)1 + \big(\mu(\widetilde{m})[\widetilde{d} + \delta^{\blacktriangledown} e_3 + \widetilde{n}^{\bullet}]\big) + \big(\delta\mu(\widetilde{m})^{\blacktriangle}\big)1 + \big([M(\widetilde{d}; \widetilde{m}) + M(b; \widetilde{m}, \widetilde{n}) - \mu(\widetilde{m})\delta^{\blacktriangledown} - \mu(\widetilde{m}, \widetilde{n})\beta]e_3 - \mu(\widetilde{m})\widetilde{n}^{\bullet} - \mu(\widetilde{m}, \widetilde{n})\widetilde{m}\big) = [-\mu(\widetilde{m})\overline{\widetilde{d}}] \boxplus [M(\widetilde{d}; \widetilde{m}) + \mu((\{\widetilde{b}, \widetilde{m}\} - \beta\widetilde{m}), \widetilde{n})]e_3 \oplus [-\mu(\widetilde{m}, \widetilde{n})]$. ■

We remark that (5.9.1) shows that the square $\widetilde{x}^2$ in the Grassmann envelope (hence $\{\widetilde{x}, \widetilde{y}\}, 2U_{\widetilde{x}}\widetilde{y}$) can be constructed in terms of the sharped-cubic structure without reference to a characteristic 5 remainder, but the resulting expression doesn't seem to be divisible by 2 and so doesn't lead to an explicit formula for the quadratic structure over $\mathbb{Z}$.

6 Automorphisms

Rather than doing all 10 distinct checks of the Adjoint Identity in 5.4, we can make use of symmetry arguments used by Dan King [2] to reduce the verification to certain "basic" vectors, justifying our use of the words "similarly" for the 5 cases we didn't check explicitly.

King Automorphism Lemma 6.1 *The maps $\varphi_q := U_{q_{12}+e_3}$, $\varphi_{cd} := U_{c_{12}+d_{12}+e_3}$ are involutory automorphisms of $J = sK_{10}$, $\varphi^2 = \mathbf{1}_{\widetilde{J}}$, whose extensions $\widetilde{\varphi}$ to the Grassmann envelope $\widetilde{J} = \Gamma(J)$ for arbitrary Φ preserve the sharped-cubic structure, i.e., the maps $\mu, \nu, \beta, T, Q, S, N, M$ and $\#$:*

(i)	$\beta(\varphi(\widetilde{x})) = \beta(\widetilde{x})$,	*(ii)*	$T(\varphi(\widetilde{x})) = T(\widetilde{x})$,	*(iii)*	$Q(\varphi(\widetilde{x})) = Q(\widetilde{x})$,
(iv)	$\nu(\varphi(\widetilde{x})) = \nu(\widetilde{x})$,	*(v)*	$\varphi(\widetilde{x}^{\#}) = (\varphi(\widetilde{x}))^{\#}$,	*(vi)*	$\mu(\varphi(\widetilde{x})) = \mu(\widetilde{x})$,
(vii)	$S(\varphi(\widetilde{x})) = S(\widetilde{x})$,	*(viii)*	$N(\varphi(\widetilde{x})) = N(\widetilde{x})$,	*(ix)*	$M(\varphi(\widetilde{x})) = M(\widetilde{x})$.

These automorphisms φ have the action table

x	e_1	e_2	c_{12}	d_{12}	q_{12}	e_3	m_{13}	n_{13}	m_{23}	n_{23}
φ_q	e_2	e_1	$-c_{12}$	$-d_{12}$	q_{12}	e_3	m_{23}	n_{23}	m_{13}	n_{13}
φ_{cd}	e_2	e_1	d_{12}	c_{12}	$-q_{12}$	e_3	$-n_{23}$	m_{23}	n_{13}	$-m_{13}$

so that the orbits under the group they generate are $\{e_1, e_2\},\ \{c_{12}, d_{12}\},\ \{q_{12}\},\ \{m_{13}, n_{13}, m_{23}, n_{23}\}$.

Proof: The φ are super-automorphisms by Grassmann detour: since $s = q_{12} + e_3$ and $s = c_{12} + d_{12} + e_3$ in A have $s^2 = 1$, their extensions $\widetilde{\varphi} := 1 \otimes \varphi = 1 \otimes U_s = \widetilde{U_{1\otimes s}} = \widetilde{U_{\widetilde{s}}}$ to $\widetilde{J}$ also have $\widetilde{s}^2 = \widetilde{1}$ and hence $\widetilde{U}_{\widetilde{s}}$ is an automorphism of the quadratic Jordan algebra $\widetilde{J}$, so the original U_s is an automorphsm of the quadratic Jordan superalgebra J. To see the action table, note that φ_q is $U_{q_{12}}$ on B, U_{e_3} on Φe_3, and $U_{e_3, q_{12}} = V_{q_{12}}$ on M by (2.4.1). Similarly φ_{cd} is $U_{c_{12}} + U_{c_{12}, d_{12}} + U_{d_{12}}$, U_{e_3}, $V_{c_{12}} + V_{d_{12}}$ on $B, \Phi e_3, M$, so the table follows from (2.2), (2.4.1), and the action $U_b c = Q(b, \bar{c})b - Q(b)\bar{c}$. For the orbit statement note that $m_{13} \xrightarrow{\varphi_q} m_{23} \xrightarrow{\varphi_{cd}} n_{13} \xrightarrow{\varphi_q} n_{23}$.

Just because the φ are automorphisms doesn't mean their extensions $\widetilde{\varphi}$ preserve the sharped-cubic structure, since only in characteristic 5 is this structure tied intimately to the Jordan product. Thus we must verify (i)-(ix) from the action of $\widetilde{\varphi}$ and the definitions of the ingredients $\mu, \nu, \beta, T, Q, S, N, M$. [Remember that the sharped-cubic lives only on $\widetilde{J}$, not on J.] By (4.1.3) $S(\widetilde{x}) = \beta(\widetilde{x})T(\widetilde{x}) + Q(\widetilde{x}) - 2\mu(\widetilde{x})$, $M(\widetilde{x}) = T(\widetilde{x})\mu(\widetilde{x}) - \nu(\widetilde{x})$, $N(\widetilde{x}) = \beta(\widetilde{x})Q(\widetilde{x}) - 2M(\widetilde{x})$, and also $\mu(\widetilde{x}) = \mu(\widetilde{m}) = \nu(u, \widetilde{m})$ where the φ preserve u, e_3, B, M, so (vi-ix) will follow from (i)-(v). We proceed to verify these 5 cases for $\widetilde{\varphi}_q$ and $\widetilde{\varphi}_{cd}$.

We begin with $\widetilde{\varphi} = \widetilde{\varphi}_q$, where

$$\text{(6.1.1)} \qquad \begin{array}{c} \widetilde{x}' := \widetilde{\varphi}_q(\widetilde{x}) = \sum_{i=1}^5 \beta_i' b_i + \beta' e_3 + \sum_{j=1}^4 \gamma_j' \otimes m_j \quad \textit{for} \\ \beta_1' = \beta_2,\ \beta_2' = \beta_1,\ \beta_3' = -\beta_3,\ \beta_4' = -\beta_4,\ \beta_5' = \beta_5,\ \beta' = \beta, \\ \gamma_1' = \gamma_3,\ \gamma_2' = \gamma_4,\ \gamma_3' = \gamma_1,\ \gamma_4' = \gamma_2. \end{array}$$

Then we have

$$\begin{aligned}
\beta(\widetilde{x}') &= \beta = \beta(\widetilde{x}), \\
T(\widetilde{x}') &= \beta_1' + \beta_2' = \beta_2 + \beta_1 = \beta_1 + \beta_2 = T(\widetilde{x}), \\
Q(\widetilde{x}') &= Q(\widetilde{b}') = \beta_1'\beta_2' - \beta_3'\beta_4' - (\beta_5')^2 = \beta_2\beta_1 - (-\beta_3)(-\beta_4) - (\beta_5)^2 = Q(\widetilde{b}) = Q(\widetilde{x}), \\
\nu(\widetilde{x}') &= \beta_1'\gamma_1'\gamma_2' + \beta_2'\gamma_3'\gamma_4' + \beta_3'\gamma_2'\gamma_4' + \beta_4'\gamma_1'\gamma_3' + \beta_5'(\gamma_1'\gamma_4' - \gamma_2'\gamma_3') \\
&= \beta_2\gamma_3\gamma_4 + \beta_1\gamma_1\gamma_2 - \beta_3\gamma_4\gamma_2 - \beta_4\gamma_3\gamma_1 + \beta_5(\gamma_3\gamma_2 - \gamma_4\gamma_1) \\
&= \beta_2\gamma_3\gamma_4 + \beta_1\gamma_1\gamma_2 + \beta_3\gamma_2\gamma_4 + \beta_4\gamma_1\gamma_3 + \beta_5(-\gamma_2\gamma_3 + \gamma_1\gamma_4) = \nu(\widetilde{x}).
\end{aligned}$$

The most delicate calculation is preservation of the adjoint map. Here from (5.4.3) we have

$$\begin{aligned}
&\big(\widetilde{\varphi}(\widetilde{x})\big)^{\#} = \big(\widetilde{x}'\big)^{\#} = [\beta_2'\beta' - 2\gamma_3'\gamma_4']e_1 + [\beta_1'\beta' - 2\gamma_1'\gamma_2']e_2 + [Q(\widetilde{b}') - 5\mu(\widetilde{x}')]e_3 + [2\gamma_1'\gamma_3' - \beta'\beta_3']c_{12} \\
&\quad + [2\gamma_2'\gamma_4' - \beta'\beta_4']d_{12} + [\gamma_1'\gamma_4' - \gamma_2'\gamma_3' - \beta'\beta_5']q_{12} + [-\beta_2'\gamma_1' + \beta_5'\gamma_3' - \beta_3'\gamma_4']m_1 \\
&\quad + [-\beta_2'\gamma_2' + \beta_4'\gamma_3' + \beta_5'\gamma_4']m_2 + [\beta_5'\gamma_1' + \beta_3'\gamma_2' - \beta_1'\gamma_3']m_3 + [-\beta_4'\gamma_1' + \beta_5'\gamma_2' - \beta_1'\gamma_4']m_4 \\
&= [\beta_1\beta - 2\gamma_1\gamma_2]\varphi(e_2) + [\beta_2\beta - 2\gamma_3\gamma_4]\varphi(e_1) + [Q(\widetilde{b}') - 5\mu(\widetilde{x}'))\varphi(e_3) - [2\gamma_3\gamma_1 + \beta\beta_3]\varphi(c_{12}) \\
&\quad - [2\gamma_4\gamma_3 + \beta\beta_4]\varphi(d_{12}) \\
&\quad + [\gamma_3\gamma_2 - \gamma_4\gamma_1 - \beta\beta_5]\varphi(q_{12}) + [-\beta_1\gamma_3 + \beta_5\gamma_1 + \beta_3\gamma_2]\varphi(m_3) \\
&\quad + [-\beta_1\gamma_4 - \beta_4\gamma_1 + \beta_5\gamma_2]\varphi(m_4) + [\beta_5\gamma_3 - \beta_3\gamma_4 - \beta_2\gamma_1]\varphi(m_1) \\
&\quad + [\beta_4\gamma_3 + \beta_5\gamma_4 - \beta_2\gamma_2]\varphi(m_2) \qquad\qquad \text{[by the action table]} \\
&= \widetilde{\varphi}\big([\beta_2\beta - 2\gamma_3\gamma_4]e_1 + [\beta_1\beta - 2\gamma_1\gamma_2]e_2 + [Q(\widetilde{b}) - 5\mu(\widetilde{x})]e_3 + [2\gamma_1\gamma_3 - \beta\beta_3]c_{12} + [2\gamma_3\gamma_4 - \beta\beta_4]d_{12} \\
&\quad + [-\gamma_2\gamma_3 + \gamma_1\gamma_4 - \beta\beta_5]q_{12} + [\beta_5\gamma_3 - \beta_3\gamma_4 - \beta_2\gamma_1]m_1 + [\beta_4\gamma_3 + \beta_5\gamma_4 - \beta_2\gamma_2]m_2 \\
&\quad + [-\beta_1\gamma_3 + \beta_5\gamma_1 + \beta_3\gamma_2]m_3 + [-\beta_1\gamma_4 - \beta_4\gamma_1 + \beta_5\gamma_2]m_4\big) = \widetilde{\varphi}(\widetilde{x}^{\#}).
\end{aligned}$$

This completes the verification that $\widetilde{\varphi}_q$ preserves the norm and sharp structure.

Now we turn to $\widetilde{\varphi} = \widetilde{\varphi}_{cd}$, where with the same notation as above

$$\text{(6.1.1)} \qquad \begin{array}{c} \beta_1' = \beta_2,\ \beta_2' = \beta_1,\ \beta_3' = \beta_4,\ \beta_4' = \beta_3,\ \beta_5' = -\beta_5,\ \beta' = \beta, \\ \gamma_1' = -\gamma_4,\ \gamma_2' = \gamma_3,\ \gamma_3' = \gamma_2,\ \gamma_4' = -\gamma_1, \end{array}$$

so

$$\begin{aligned}
\beta(\widetilde{x}') &= \beta = \beta(\widetilde{x}), \\
T(\widetilde{x}') &= T(\widetilde{b}') = \beta_1' + \beta_2' = \beta_2 + \beta_1 = \beta_1 + \beta_2 = T(\widetilde{b}) = T(\widetilde{x}), \\
Q(\widetilde{x}') &= Q(\widetilde{b}') = \beta_1'\beta_2' - \beta_3'\beta_4' - (\beta_5')^2 = \beta_2\beta_1 - \beta_4\beta_3 - (-\beta_5)^2 = Q(\widetilde{b}) = Q(\widetilde{x}), \\
\nu(\widetilde{x}') &= \beta_1'\gamma_1'\gamma_2' + \beta_2'\gamma_3'\gamma_4' + \beta_3'\gamma_2'\gamma_4' + \beta_4'\gamma_1'\gamma_3' + \beta_5'(\gamma_1'\gamma_4' - \gamma_2'\gamma_3') \\
&= \beta_2(-\gamma_4)\gamma_3 + \beta_1\gamma_2(-\gamma_1) + \beta_4\gamma_3(-\gamma_1) + \beta_3(-\gamma_4)\gamma_2 + (-\beta_5)(-\gamma_4)(-\gamma_1) - \gamma_{43}\gamma_2) \\
&= \beta_2\gamma_3\gamma_4 + \beta_1\gamma_1\gamma_2 + \beta_4\gamma_1\gamma_3 + \beta_3\gamma_2\gamma_4 + \beta_5(-\gamma_2\gamma_3 + \gamma_1\gamma_4) = \nu(\widetilde{x}),
\end{aligned}$$

and

$$\begin{aligned}
&\big(\widetilde{\varphi}(\widetilde{x})\big)^{\#} = \big(\widetilde{x}'\big)^{\#} = [\beta_2'\beta' - 2\gamma_3'\gamma_4']e_1 + [\beta_1'\beta' - 2\gamma_1'\gamma_2']e_2 + [Q(\widetilde{b}') - 5\mu(\widetilde{x}')]e_3 + [2\gamma_1'\gamma_3' - \beta'\beta_3']c_{12} \\
&\quad + [2\gamma_2'\gamma_4' - \beta'\beta_4']d_{12} + [\gamma_1'\gamma_4' - \gamma_2'\gamma_3' - \beta'\beta_5']q_{12} + [-\beta_2'\gamma_1' + \beta_5'\gamma_3' - \beta_3'\gamma_4']m_1 \\
&\quad + [-\beta_2'\gamma_2' + \beta_4'\gamma_3' + \beta_5'\gamma_4']m_2 + [\beta_5'\gamma_1' + \beta_3'\gamma_2' - \beta_1'\gamma_3']m_3 + [-\beta_4'\gamma_1' + \beta_5'\gamma_2' - \beta_1'\gamma_4']m_4 \\
&= [\beta_1\beta - 2\gamma_2(-\gamma_1)]\varphi(e_2) + [\beta_2\beta - 2(-\gamma_4)\gamma_3]\varphi(e_1) + [Q(\widetilde{b}') - 5\mu(\widetilde{x}')])\varphi(e_3) + [2(-\gamma_4)\gamma_2 - \beta\beta_4]\varphi(d_{12})
\end{aligned}$$

$+[2\gamma_3(-\gamma_1) - \beta\beta_3]\varphi(c_{12})$
$+[(-\gamma_4)(-\gamma_1) - \gamma_3\gamma_2 - \beta(-\beta_5)]\varphi(-q_{12}) + [-\beta_1(-\gamma_4) + (-\beta_5)\gamma_2 - \beta_4(-\gamma_1)]\varphi(-m_4)$
$+[-\beta_1\gamma_3 + \beta_3\gamma_2 + (-\beta_5)(-\gamma_1)]\varphi(m_3) + [(-\beta_5)(-\gamma_4) + \beta_4\gamma_3 - \beta_2\gamma_2]\varphi(m_2)$
$+[-\beta_3(-\gamma_4) + (-\beta_5)\gamma_3 - \beta_2(-\gamma_1)]\varphi(-m_1)$ [by the action table]
$= \widetilde{\varphi}\big([\beta_1\beta - 2\gamma_1\gamma_2]e_2 + [\beta_2\beta - 2\gamma_3\gamma_4]e_1 + [Q(\widetilde{b}) - 5\mu(\widetilde{x})]e_3 + [2\gamma_2\gamma_4 - \beta\beta_4]d_{12} + [2\gamma_1\gamma_3 - \beta\beta_3]c_{12}$
$+[\gamma_1\gamma_4 - \gamma_2\gamma_3 - \beta\beta_5]q_{12} + [-\beta_1\gamma_4 + \beta_5\gamma_2 - \beta_4\gamma_1]m_4 + [-\beta_1\gamma_3 + \beta_3\gamma_2 + \beta_5\gamma_1]m_3$
$+[\beta_5\gamma_4 + \beta_4\gamma_3 - \beta_2\gamma_2]m_2 + [-\beta_3\gamma_4 + \beta_5\gamma_3 - \beta_2\gamma_1]m_1\big) = \widetilde{\varphi}(\widetilde{x}^{\#})$.

This completes the verification that $\widetilde{\varphi}_{cd}$ preserves the sharped-cubic structure. ∎

Now we show why checking the coefficients of only one representative $e_1, e_3, c_{12}, q_{12}, m_{13}$ from each orbit under these two sharped-cubic automorphisms suffices to establish the Adjoint Identity (SC3) in 5.4.

Lemma 6.2 *(i) If $\widetilde{\varphi}$ is any Ω-linear map which "scalarly permutes" a decomposition of an Ω-module $V = \bigoplus_{i\in I} V_i$ in the sense that $\widetilde{\varphi}(x_i) = \varepsilon_i x_{\pi(i)}$ for some scalars ε_i and a fixed permutation π of the index set I (where V_i is Ω-linearly identified with $V_{\pi(i)}$), then the coordinate projections $p_k(x) = \alpha_k$ for $x = \sum \alpha_i x_i$ satisfy*

$$p_{\pi(k)}(\widetilde{\varphi}(x)) = \varepsilon_k p_k(x) \text{ for all } x \in J.$$

If a surjective $\widetilde{\varphi}$ preserves a polynomial map $F : V \to V$ (in the sense that $\widetilde{\varphi}(F(x))) = F(\widetilde{\varphi}(x))$, and the k^{th} coordinate of the image of F vanishes, then so does the $\pi(k)^{th}$ coordinate: $p_k(F(V)) = 0 \implies p_{\pi(k)}(F(V)) = 0$.

(ii) If a group of bijective "scalar permutations" $\widetilde{\varphi}_\alpha$ of a decomposition of V preserve a polynomial map F, then $F(J) = 0$ if $p_k(F(V)) = 0$ for at least one k from each orbit of the corresponding permutation group $G = \{\pi_\alpha\}$ on I.

Proof: (i) holds since $\widetilde{\varphi}(x) = \sum \alpha_i\varepsilon_i x_{\pi(i)}$ if $x = \sum \alpha_i x_i$, and thus $p_{\pi(k)}(F(V)) = p_{\pi(k)}(F(\widetilde{\varphi}(V)))$ [by surjectivity] $= p_{\pi(k)}(\widetilde{\varphi}(F(V))) = \varepsilon_k p_k(F(V)) = 0$. For (ii), if $k' \in I$ then $k' = \pi_\alpha(k)$ for some k in the G-orbit of k' where p_k vanishes, so $p_{k'}$ vanishes too by (i). ∎

Applying the lemma to the map $F(x) = x^{\#\#} - N(x)x$ on the decomposition $V := \widetilde{J} = \bigoplus_{i=1}^{10} V_i$ for $V_i := \Gamma_0 b_i$ $(1 \le i \le 5)$, $V_6 := \Gamma_0 e_3$, $V_{i+6} \oplus \Gamma_1 m_i$ $(1 \le i \le 4)$ under the automorphisms $\widetilde{\varphi}_q, \widetilde{\varphi}_{cd}$, we see that we were justified in 5.9 checking the coefficients in the Adjoint Identity (SC3) only for the orbit representatives $1 \otimes e_1, 1 \otimes e_3, 1 \otimes c_{12}, 1 \otimes q_{12}, \gamma \otimes m_{13}$.

Note that we cannot formulate our lemma in terms of a *basis* for $\widetilde{J}$; the b_i form a Γ_0-basis for $\widetilde{A}$, but the m_i $(1 \le i \le 4)$ are a basis for $\widetilde{M}$ only in a metaphorical sense since they don't even belong to $\widetilde{M}$, and while the $\gamma_i m_i$ for $\gamma_i \in \Gamma_1$ truly span $\widetilde{M}$, they are "independent" only in the sense that $\sum \gamma_i m_i = \sum \delta_i m_i$ implies all $\gamma_i = \delta_i$ (not independent in the usual sense over Γ_0, since $(\gamma_3\gamma_2)\gamma_1 m = -(\gamma_3\gamma_1)\gamma_2 m)$. Instead, we have a *decomposition* $\widetilde{J} = \widetilde{B} \oplus \Gamma_0 e_3 \oplus \widetilde{M} = \big(\bigoplus_{i=1}^{5} \Gamma_0 b_i\big) \oplus \Gamma_0 e_3 \oplus \big(\bigoplus_{i=1}^{4} \Gamma_1 m_i\big)$ with natural Γ_0-linear identifications of $\Gamma_0 b_i$ with $\Gamma_0 b_j$ and $\Gamma_1 m_i$ with $\Gamma_1 m_j$.

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