Generalized Jordan Polynomials and Bergmann Structions

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Abstract

The Martinez construction of fractions from a Jordan algebra requires a Jordan derivation involving certain quadratic multiplications on the original algebra. We study a general Bergmann construction of such structural transformations (structions) in the context of Jordan pairs, whose natural setting is a universal polynomial envelope (with a universal representation of polynomial operators) generalizing the universal quadratic envelope (with its universal representation of linear operators). The Bergmann structions corresponding to fractions are defined only on a subpair determined by a sesqui-principal inner ideal determined by an element s and an element n dominated by s. We study these inner ideals and the criterion for a creating structions on them, which will be applied to the creation of Jordan algebras of fractions. The methods should have future application to the problem of creating fractions for Jordan pairs.

Throughout, we consider algebraic systems over an arbitrary ring of scalars Φ. A *Jordan pair*, a pair $\mathcal{V} = (V^+, V^-)$ of Φ-modules with compositions $(x,a) \mapsto Q_x(a) \in V^\tau$ for $(x,a) \in V^\tau \times V^{-\tau}$, $\tau = \pm$,[1] which are quadratic in x and linear in a, and satisfy the following axioms *strictly* (in all scalar extensions, equivalently, all their linearizations hold in $\mathcal{V}$ itself): for all $x, y \in V^\tau, a, b \in V^{-\tau}$

$$\text{(JP1)} \quad D_{x,a}Q_x = Q_x D_{a,x}, \quad \text{(JP2)} \quad D_{Q_x a,a} = D_{x,Q_a(x)}, \quad \text{(JP3)} \quad Q_{Q_x a} = Q_x Q_a Q_x,$$

where as usual we set $Q_{x,y} := Q_{x+y} - Q_x - Q_y$, which gives the trilinear product $\{x,a,y\} := Q_{x,y}(a) =: D_{x,a}(y)$ with $\{V^\tau V^{-\tau} V^\tau\} \subseteq V^\tau$.

We will try to economize on superscripts and use typography instead, denoting, for a fixed $\tau = \pm$, elements of V^τ by x, y, z, w and elements of $V^{-\tau}$ by a, b, c. Every

[1] We will use τ instead of the usual σ as our generic superscript $\pm$, since we are especially interested in multiplications by an incipient element $\tilde{q}$ of a fixed degree σ and focus on the important space $V^{-\sigma}$ where $Q_{\tilde{q}}$ is defined. Instead of saying "in our earlier formulas replace all σ's by $-\sigma$" we will say "set $\tau = -\sigma$".

Jordan pair $\mathcal{V} = (V^+, V^-)$ has a **dual** or **opposite** pair $\widetilde{\mathcal{V}} = (\widetilde{V}^+, \widetilde{V}^-)$ for $\widetilde{V}^\tau := V^{-\tau}$ and operations $\widetilde{Q}_{\tilde{x}}\tilde{a} := Q_a x$, $\widetilde{D}_{\tilde{x},\tilde{a}}\tilde{y} := \{a, x, b\}$ for $\tilde{x} = a, \tilde{y} = b \in \widetilde{V}^\tau$, $\widetilde{a} = x \in \widetilde{V}^{-\tau}$ [Loos, p.3]. We could avoid all superscripts by formulating only positive results for $x \in V^+, a \in V^-$, and applying duality for the corresponding negative results, but we won't be quite this parsimonious.

We will use [Loos] as reference bible for all results about Jordan pairs. The following formulas are used frequently enough in the paper for us to display them:

(0.1.1) $D_{x,a}Q_y + Q_y D_{a,x} = Q_{\{x,a,y\},y}$,

(0.1.2) $D_{x,Q_a y} = D_{\{x,a,y\},a} - D_{y,Q_a x} = D_{x,a}D_{y,a} - Q_{x,y}Q_a$,
$D_{Q_a y,x} = D_{a,\{y,a,x\}} - D_{Q_a x,y} = D_{a,y}D_{a,x} - Q_a Q_{y,x}$,

(0.1.3) $Q_{Q_x a,y} = Q_{x,y}D_{a,x} - D_{y,a}Q_x = D_{x,a}Q_{x,y} - Q_x D_{a,y}$,

(0.1.4) $Q_{\{x,a,y\}} + Q_{Q_x Q_a y,y} = Q_x Q_a Q_y + Q_y Q_a Q_x + D_{x,a}Q_y D_{a,x}$,

(0.1.5) $Q_{Q_x Q_a y, D_{x,a} y} = Q_x Q_a Q_y D_{a,x} + D_{x,a}Q_y Q_a Q_x$,

(0.1.6) $Q_{\alpha x + Q_x a} = B_{\alpha,x,a}Q_x = Q_x B_{\alpha,a,x}$,
$(B_{\alpha,x,a} := \alpha^2 \mathbf{1} + \alpha D_{x,a} + Q_x Q_a)$.

1 The Universal Quadratic Envelope

For a subpair $\mathcal{V} \subseteq \widetilde{\mathcal{V}}$, the unital outer multiplication algebra of $\mathcal{V}$ on $\widetilde{\mathcal{V}}$ is denoted by $\mathcal{M}(\mathcal{V}|\widetilde{\mathcal{V}})$; it is generated over Φ by the identity operator $\mathbf{1}$ and all operators of the form $D_{x,a}, Q_x$ for $x, a \in \mathcal{V}$; when $\mathcal{V} = \widetilde{\mathcal{V}}$ we get the full outer multiplication algebra $\mathcal{M}(\widetilde{\mathcal{V}})$.

Though seldom mentioned in polite company, Jordan pairs have a universal gadget for quadratic representations. If $\mathcal{A}$ is a unital associative algebra with 2×2 *matrix grading*, i.e., a decomposition $\mathcal{A} = \bigoplus_{\tau,\sigma\in\{\pm\}} \mathcal{A}^{\tau,\sigma}$ satisfying the matrix relations $\mathcal{A}^{\tau,\sigma}\mathcal{A}^{\rho,\nu} \subseteq \delta_{\sigma,\rho}\mathcal{A}^{\tau,\nu}$ [equivalently, with Peirce decomposition $\mathcal{A}^{\tau,\sigma} = e^\tau \mathcal{A} e^\sigma$ relative to $e^+ \in \mathcal{A}^{+,+}, e^- \in \mathcal{A}^{-,-}$ where $1 = e^+ + e^-$], then a **quadratic specialization**[2] of a Jordan pair $\mathcal{V}$ in $\mathcal{A}$ is a pair of maps $\mathcal{Q} = (q, d) = ((q^{+,-}, q^{-,+}), (d^{+,+}, d^{-,-}))$ for $m^{\tau,\sigma} : \mathcal{V} \to \mathcal{A}^{\tau,\sigma}$ strictly satisfying the *quadratic specialization relations* for pairs: for all $\tau = \pm, x, y \in V^\tau, a, b \in V^{-\tau}$

[2]This was called an *associative representation* in [Loos 2.4 p.16-17], leaving out (QS2) since it follows from (QS4,4*) via $d_{Q_x a,a} - d_{x,Q_a x} = \left(d^2_{x,a} - q_x q_{a,a}\right) - \left(d^2_{x,a} - q_{x,x} q_a\right) = 0$. (QS4) in turn usually follows by applying (QS5) with y, a replaced by m, b, acting on a, and reading the result as an operator on m. But due to the asymmetry between the pair elements x, y and a, b we cannot derive (QS4) this way and must assume it as an axiom. This contrasts with the Jordan algebra case [me, p.282] where $\mu_1 = 1, \mu_{U_x y} = \mu_x \mu_y \mu_x, \mu_{U_x y,x} = \mu_x \nu_{y,x} = \nu_{x,y}\mu_x$ suffice to define quadratic specializations.

(QS1) $d^{\tau,\tau}_{x,a} q^{\tau,-\tau}_x = q^{\tau,-\tau}_{Q_x a,x} = q^{\tau,-\tau}_x d^{-\tau,-\tau}_{a,x}$,
(QS2) $d^{\tau,\tau}_{x,Q_a x} = d^{\tau,\tau}_{Q_x a,a}$,
(QS3) $q^{\tau,-\tau}_{Q_x a} = q^{\tau,-\tau}_x q^{-\tau,\tau}_a q^{\tau,-\tau}_x$,
(QS4) $d^{-\tau,-\tau}_{b,x} d^{-\tau,-\tau}_{a,x} = d^{-\tau,-\tau}_{b,Q_x a} + q^{-\tau,\tau}_{b,a} q^{\tau,-\tau}_x$,
(QS4)* $d^{\tau,\tau}_{x,a} d^{\tau,\tau}_{x,b} = d^{\tau,\tau}_{Q_x a,b} + q^{\tau,-\tau}_x q^{-\tau,\tau}_{a,b}$,
(QS5) $d^{\tau,\tau}_{y,a} q^{\tau,-\tau}_x + q^{\tau,-\tau}_x d^{-\tau,-\tau}_{a,y} = q^{\tau,-\tau}_{\{y,a,x\},x}$.

These relations imply

(QS6) $d^{\tau,\tau}_{Q_x b,a} q^{\tau,-\tau}_x = q^{\tau,-\tau}_x d^{-\tau,-\tau}_{b,Q_x a}$,
(QS7) $d^{\tau,\tau}_{Q_x b,a} d^{\tau,\tau}_{x,b} = q^{\tau,-\tau}_x q^{-\tau,\tau}_b d^{\tau,\tau}_{x,a} = d^{\tau,\tau}_{x,Q_b Q_x a}$,
(QS8) $q^{\tau,-\tau}_{x,y} d^{-\tau,-\tau}_{a,x} = d^{\tau,\tau}_{y,a} q^{\tau,-\tau}_x + q^{\tau,-\tau}_{Q_x a,y}$,
(QS9) $q^{\tau,-\tau}_x q^{-\tau,\tau}_{a,b} + d^{\tau,\tau}_{x,\{a,x,b\}} = d^{\tau,\tau}_{Q_x b,a} + d^{\tau,\tau}_{x,a} d^{\tau,\tau}_{x,b}$.

Here (6),(7),(8) are Lemma 2.6 (4),(5),(2) in [Loos, p.16] ; (9) is JP6, which was not derived for specializations in Lemma 2.6, but is equivalent to (QS4) since (QS9) + (QS4) $= \big(q_x q_{a,b} + d_{x,\{a,x,b\}} - d_{Q_x b,a} - d_{x,a} d_{x,b}\big) + \big(d_{x,a} d_{x,b} - d_{Q_x a,b} - q_x q_{b,a}\big) = d_{x,\{a,x,b\}} - d_{Q_x b,a} - d_{Q_x a,b}$ vanishes as a linearization of (QS2).[3]

The archetypal example of a quadratic specialization is an **outer multiplication representation**, i.e., a quadratic specialization $\mathcal{V} \to \mathcal{M}(\mathcal{V}|\widetilde{\mathcal{V}})|_{\mathcal{M}}$ by outer multiplication operators

$$q^{\tau,-\tau}_x := Q_x|_{M^{-\tau}}, \quad d^{\tau,\tau}_{x,a} := D_{x,a}|_{M^\tau}$$

for $\mathcal{M} = (M^+, M^-)$ a $\mathcal{V}$-invariant subspace of a Jordan pair $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$. The *regular outer representation* is the outer multiplication representation of $\mathcal{V}$ on itself ($\mathcal{M} = \widetilde{V} = \mathcal{V}$). By restriction we obtain a multiplication representation on any *outer ideal* $\mathcal{I} \subseteq \mathcal{V}$.

A $\mathcal{V}$-**bimodule** consists of a pair $\mathcal{M} = (M^+, M^-)$ of Φ-modules and a **birepresentation** of $\mathcal{V}$ on $\mathcal{M}$, i.e., a quadratic specialization of $\mathcal{V}$ in $\mathrm{End}(\mathcal{M}) = \bigoplus_{\tau,\sigma} e_\tau E e_\sigma$ for e_τ the projection of $\mathcal{M}$ on M^σ. In fact, *all* $\mathcal{V}$-bimodules are invariant subspaces of some $\mathcal{E} \supseteq \mathcal{V}$, and *all* birepresentations $\mathcal{V} \to \mathrm{End}(\mathcal{M})$ are outer multiplication representations $\mathcal{V} \to \mathcal{M}(\mathcal{V}|\mathcal{E})|_{\mathcal{M}}$ on a split null extension.

Bimodule Theorem 1.1 *Any $\mathcal{V}$-bimodule $\mathcal{M}$ gives rise to a* **split null extension** $\mathcal{E} = \mathcal{V} \oplus \mathcal{M} = (V^+ \oplus M^+, V^- \oplus M^-)$, *which is a Jordan pair under the operations*

$$\widetilde{Q}_{x\oplus m}(a \oplus p) := Q_x a \oplus \big(q_x(p) + d_{x,a}(m)\big),$$
$$\widetilde{D}_{x\oplus m,a\oplus p}(y \oplus n) := D_{x,a}(y) \oplus \big(d_{x,a}(n) + q_{x,y}(p) + d_{y,a}(m)\big)$$

[3] Similarly, (QS8) is equivalent to (QS5) since (QS8)+(QS5) equals the linearization $x \to x, y$ in (QS1) $q_x d_{a,x} = q_{Q_x a,x}$. Note that (QS1-3) are (JP1-3), (QS4) is (0.1.2), (QS5) is (0.1.1), (QS8) is (0.1.3). The Bimodule Theorem below shows that (QS8),(QS9) are more directly involved than (QS4),(QS5) in capturing bimodule structure, but we prefer (QS5) as a basic result ($d_{x,a}$ is a Lie struction), and (QS4)* since it is the dual of (QS4).

for all $x, y \in V^\tau$, $m, n \in M^\tau$, $a \in V^{-\tau}$, $p \in M^{-\tau}$, and the original birepresentation is the restriction of the regular outer representation of $\mathcal{E}$ to $\mathcal{V}$ and $\mathcal{M}$.

PROOF: We must verify that the axioms (JP1-3) hold in all extensions of $\mathcal{V}$. Since the relations (QS1-5) hold strictly, it suffices to show they imply (JP1-3) in $\mathcal{V}$ itself.

For (JP1), the LHS is $\widetilde{D}_{x+m,a+p}\widetilde{Q}_{x+m}(b+r) = \{x, a, Q_x b\} \oplus \big(d_{x,a}q_x(r) + q_{Q_x b,x}(p) + (d_{x,a}d_{x,b} + d_{Q_x b,a})(m)\big)$, and the RHS is $\widetilde{Q}_{x+m}\widetilde{D}_{a+p,x+m}(b+r) = Q_x\{a, x, b\} \oplus \big(q_x d_{a,x}(r) + q_x d_{b,x}(p) + (q_x q_{a,b} + d_{x,\{a,x,b\}})(m)\big)$, which agree by (JP1) in $\mathcal{V}$, (QS1) on r and p, and (QS9) on m.

For (JP2), the LHS is $\widetilde{D}_{\widetilde{Q}_{x+m}(a+p),a+p}(y+n) = \{Q_x a, a, y\} \oplus \big(d_{Q_x a,a}(n) + d_{y,a}d_{x,a}(m) + (q_{Q_x a,y} + d_{y,a}q_x)(p)\big)$, and the RHS is $\widetilde{D}_{x+m,\widetilde{Q}_{a+p}(x+m)}(y+n) = \{x, Q_a x, y\} \oplus \big(d_{x,Q_a x}(n) + q_{x,y}d_{a,x}(p) + (d_{y,Q_a x} + q_{x,y}q_a\big)(m)$, which agree by (JP2) in $\mathcal{V}$, (QS2) on n, (QS8) on p, and (QS4)* on m.

Finally, for (JP3) the LHS is $\widetilde{Q}_{\widetilde{Q}_{x+m}(b+r)}(a+p) = Q_{Q_x b}(a) \oplus \big(q_{Q_x b}(p) + d_{Q_x b,a}q_x(r) + d_{Q_x b,a}d_{x,b}(m)\big)$, and the RHS is $\widetilde{Q}_{x+m}\widetilde{Q}_{b+r}\widetilde{Q}_{x+m}(a+p) = Q_x Q_b Q_x(a) \oplus \big(q_x q_b q_x(p) + q_x d_{b,Q_x a}(r) + d_{x,Q_b Q_x a}(m)$, which agree by (JP3) in $\mathcal{V}$, (QS3) on p, (QS6) on r, and (QS7) on m. ■

Bimodules are inherently *outer* modules for $\mathcal{V}$, they have no inner multiplications ($\cap_V(M) = Q_M V = 0$). Thus they can reflect only outer multiplicative properties of a Jordan pair.

The universal gadget for quadratic specializations is the **universal quadratic envelope** $\mathcal{UQE}(\mathcal{V})$ (cf. [me, p. 289-290] for the algebra case), a unital associative $\mathcal{U}$ with 2×2 matrix grading, together with a **universal quadratic specialization** $\mathcal{Q}_u : \mathcal{V} \to \mathcal{U}$, having the universal property that every quadratic specialization $\mathcal{V} \xrightarrow{\mathcal{Q}} \mathcal{A}$ factors through the universal one

(1.2)
$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\mathcal{Q}} & \mathcal{A} \\ \mathcal{Q}_u \searrow & & \nearrow \widehat{\mathcal{Q}} \\ & \mathcal{UQE}(\mathcal{V}) & \end{array}$$

via a unique homomorphism $\widehat{\mathcal{Q}}$ of unital 2×2-graded associative algebras. This implies, in particular, that $\mathcal{UQE}$ is unique up to isomorphism and is generated by the universal elements e^+, e^-, $q_x^{\tau,-\tau} \in \mathcal{U}^{\tau,-\tau}$, $d_{x,a}^{\tau,\tau} \in \mathcal{U}^{\tau,\tau}$ for $x \in V^\tau, a \in V^{-\tau}$. The standard model of $\mathcal{UQE}$ is F/I for F the free unital associative Φ-algebra generated by all e^τ ($e^+ + e^- = 1$), $q_x^{\tau,-\tau}$, $d_{x,a}^{\tau,\tau}$ and I is the ideal generated by $(e^+)^2 = e^+$ and all elements needed to make d linear in x, a and q quadratic in x [all $d_{\alpha x+x',a} - \alpha d_{x,a} - d_{x',a}$, $d_{x,\alpha a+a'} - \alpha d_{x,a} - d_{x,a'}$, $q_{\alpha x} - \alpha^2 q_x$, $q_{\alpha x+x',a} - \alpha q_{x,a} - q_{x',a}$], and insure that (QS1-5), hence also (QS6-9), and their linearizations hold [all elements $LHS - RHS$ in (QS1-5), plus the x-linearizations of the cubic relations (QS1),(QS6),(QS7) and the quartic relation (QS3)]. Since the set of generators for both $\mathcal{U}$ and I are homogeneous and invariant under the **reversal involution** of F (determined by $(q_x^{\tau,-\tau})^* := q_x^{\tau,-\tau}, (d_{x,a}^{\tau,\tau})^* := d_{a,x}^{-\tau,-\tau}$), the quotient $\mathcal{UQE}$ inherits the matrix grading and involution. This leads to the **Duality Principle**: if a Jordan pair operator $\omega \in \mathcal{UQE}(\mathcal{V})$ is an identity, $\omega = 0$ in $\mathcal{UQE}$, then its reversal ω^* is also an identity, $\omega^* = 0$ in $\mathcal{UQE}$.

We will rapidly get tired of writing $q_x^{\tau,-\tau}, d_{x,a}^{\tau,\tau}$ for the generators of $\mathcal{U}$ and simply write $q_x, d_{x,a}$ when the indices are understood. In fact, we will often just write $Q_x, D_{x,a}$ in place of their preimages ($\widehat{\mathcal{Q}}_r(q_x) = Q_x$, $\widehat{\mathcal{Q}}_r(d_{x,a}) = D_{x,a}$) under the regular representation $\mathcal{Q}_r$, and say "in the universal envelope", "in $\mathcal{U}$", or just "universally". The defining relations (JP1-3), (0.1.1-6) show that if $\{x_i\}$ is a set of graded generators $x_i \in V^{\tau(i)}$ for $\mathcal{V}$, then the operators $q_{x_i}, q_{x_i,x_j}, d_{x_i,x_j}$ is a set of generators for $\mathcal{UQE}(\mathcal{V})$.

Any quadratic specialization $\mathcal{V} \xrightarrow{\mathcal{Q}} \mathcal{A}$ turns $\mathcal{A}$ into a $\mathcal{V}$-bimodule $\mathcal{M}(\mathcal{A},\mathcal{Q}) = M^+ \oplus M^-$ ($M^\tau := \mathcal{A}e_\tau = \mathcal{A}^{\tau,\tau} \oplus \mathcal{A}^{-\tau,\tau}$) (which is cyclic with generator $1_{\mathcal{A}} = e_+ \oplus e_-$ if $\mathcal{Q}(\mathcal{V})$ generates $\mathcal{A}$) via the left regular representation $\mathcal{A} \to \mathrm{End}(\mathcal{A})$, so for $\omega \in \mathcal{UQE}(\mathcal{V})$ we have $\omega(a) = \widehat{\mathcal{Q}}(\omega)a$ as in (1.2). The universal envelope $\mathcal{U}$ becomes a **universal cyclic bimodule** $\mathcal{M}(\mathcal{U},\mathcal{Q}_u)$; every cyclic $\mathcal{V}$-bimodule is a homomorphic image of $\mathcal{M}(\mathcal{U},\mathcal{Q}_u)$.

If $\mathcal{V}$ is a subalgebra of $\widetilde{\mathcal{V}}$, we denote by $\mathcal{UQE}(\mathcal{V}|\widetilde{\mathcal{V}})$ the subalgebra of $\mathcal{UQE}(\widetilde{\mathcal{V}})$ generated by 1 and all $\widetilde{d}_{x,a}, \widetilde{q}_x$ for $x, a \in \mathcal{V}$, and we have natural epimorphisms $\mathcal{UQE}(\mathcal{V}) \to \mathcal{UQE}(\mathcal{V}|\widetilde{\mathcal{V}}) \to \mathcal{M}(\mathcal{V}|\widetilde{\mathcal{V}})$ via $d_{x,a}, q_x \to \widetilde{d}_{x,a}, \widetilde{q}_x \to \widetilde{D}_{x,a}, \widetilde{Q}_x \in \mathrm{End}(\widetilde{\mathcal{V}})$. In particular, $\widetilde{\mathcal{V}}$ becomes a left $\mathcal{UQE}(\mathcal{V})$-module, and we can form $\omega(\tilde{x})$ for any $\omega \in \mathcal{UQE}(\mathcal{V})$ and any $\tilde{x} \in \widetilde{\mathcal{V}}$. We also have the **Action Principle**: If a Jordan pair operator $\omega \in \mathcal{UQE}(\mathcal{V})$ is zero as a bimodule operator, $\omega = 0 \in \mathrm{End}(\mathcal{M})$ for all $\mathcal{V}$-bimodules $\mathcal{M}$, then $\omega = 0$ in $\mathcal{UQE}(\mathcal{V})$; indeed if ω is zero on the universal cyclic module $\mathcal{M}(\mathcal{U},\mathcal{Q}_u)$ then $0 = \omega(1_{\mathcal{U}}) = \widehat{\mathcal{Q}_u}(\omega)1_{\mathcal{U}} = \omega$ implies $\omega = 0$ in $\mathcal{U}$ [note that $\widehat{\mathcal{Q}_u} = \mathbf{1}_{\mathcal{U}}$ by uniqueness in (1.2)].

Another formulation of the Action Principle is that an operator $\omega \in \mathcal{U}$ is zero iff it is zero as an operator in all extensions $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$: if $\omega = 0$ on $\widetilde{\mathcal{V}} = \mathcal{V} \oplus \mathcal{M}(\mathcal{U},\mathcal{Q}_u)$ then $\omega = 0$ on $\mathcal{M}(\mathcal{U},\mathcal{Q}_u)$, so $\omega = 0$ in $\mathcal{U}$. Conversely, if ω vanishes in $\mathcal{U}$ then it vanishes on all $\widetilde{m} \in \widetilde{\mathcal{V}}$ since $\mathcal{M} = \mathcal{M}(V|\widetilde{\mathcal{V}})\widetilde{m}$ is a Jordan bimodule, and $\omega = 0$ on $\mathcal{M}$ implies $\omega = 0$ on $\widetilde{m}$.

2 Universal Polynomial Envelope

Jordan algebras and pairs have linear *outer* multiplications $\cup_x : a \to U_x a, Q_x a$ and $V_{x,a}, D_{x,a} : y \to \{x,a,y\}$ which are linear operators, but they also have *inner* multiplications $\cap_x : a \to Q_a x$ mapping $V^{-\tau} \to V^{-\tau}$ which are quadratic rather than linear operators. We can interpret these as mappings on the associated polarized Jordan triple system $\mathcal{V}^p := V^+ \oplus V^-$ by setting $\cap_{V^\tau}(V^\tau) = \{V^\tau, V^\tau, V\} = 0$. The full **polynomial multiplication algebra** $\mathcal{PM}(\mathcal{V}) \subseteq \mathrm{Pol}(\mathcal{V})$ is the associative algebra of polynomial maps in several variables on $\mathcal{V}$ generated by the $Q_x, D_{x,a}, \cap_x$.[4] The easiest approach to these polynomials is through the free product of $\mathcal{V}$ with a free pair.

Recall from your subconscious that the free Jordan pair $\mathcal{FJP}[X]$ on a nonempty sets $X = X^+ \uplus X^-$ of graded generators is the free pair monad $\mathcal{FPM}[X]$ on the generators divided out by the ideal $\mathcal{I}(X)$ generated by the Jordan pair identities (JP1-JP3) as well as their linearizations. The free pair monad is the free module spanned spanned by all pair monomials in the generators, constructed recursively by taking in degree 1 the generators $x_i^\pm$, and if $p^\tau, q^\tau, a^{-\tau}$ of degrees d, e, f have been constructed, then $m^\tau = Q_p a, \{p, a, q\} = \{q, a, p\}$ are monomials of degrees $2d + f, d + e + f$. The ideal $\mathcal{I}(X)$ is generated by the following relations[5]

$$\begin{array}{ll} (\mathrm{JP1})' & D_{x,a}Q_{x,y} + D_{y,a}Qx = Q_{Q_x a,y} + Q_{Q_{x,y}a,x} = Q_{x,y}D_{a,x} + Q_x D_{a,y}, \\ (\mathrm{JP3})' & Q_{Q_x a, Q_{x,y}a} = Q_x Q_a Q_{x,y} + Q_{x,y}Q_a Q_{x,y}, \\ (\mathrm{JP3})'' & Q_{Q_x a, Q_y a} + Q_{\{x,a,y\}} = Q_x Q_a Q_y + Q_y Q_a Q_x + Q_{x,y}Q_a Q_{x,y}. \end{array}$$

Remember that quadratic identities linearize automatically, so it is only identities of degree 3 or more in a variable that must be assumed to hold, and they hold automatically if the ring of scalars Φ has sufficiently many invertible elements. The quotient Φ-module $\mathcal{FJP}[X] := \mathcal{FPM}[X]/\mathcal{I}(X)$ becomes a Jordan pair by defining $Q_{\sum_i p_i}(\sum_j a_j) = \sum_{i,j} Q_{p_i} a_j + \sum_{i<k,j}\{p_i, a_j, p_k\}$. We call it the *free Jordan pair* $\Phi\langle X\rangle$ *on the free graded variables* $X^+ \uplus X^-$ *over* Φ*,* using pointy brackets to distinguish it from the scalar polynomial ring $\Phi[X]$ in ungraded *scalar variables.* We will speak of *THE free Jordan pair* $\Phi\langle X\rangle$ over Φ when $X^\tau = \{x_1^\tau, x_2^\tau, \ldots\}$ $(\tau = \pm)$ are both countably infinite sets of indeterminates; its elements may be thought of as universal Jordan pair polynomials in any (necessarily finite) number of variables.

The free pair on X is graded by degree in each variable, and agrees with the free monad up to degree 4 (the lowest-degree Jordan identities are (JP1), (JP2) of degree 5), in particular has a natural imbedding $X \xrightarrow{\mathrm{in}} \mathcal{FJP}[X]$. It enjoys the usual universal property, that every set-theoretic map $X^\pm \xrightarrow{\varphi} V^\pm$ extends uniquely to a homomorphism $\mathcal{FJP}[X] \xrightarrow{\tilde{\varphi}} \mathcal{V}$ of Jordan pairs. The universal property leads by universal nonsense to the usual properties of the free object and yields a functor from sets to Jordan pairs.

For generalized Jordan polynomials on a particular Jordan pair $\mathcal{V}$ we must take the free product of $\mathcal{V}$ with $\mathcal{FJP}[X]$ to get $\mathcal{F}[\mathcal{V}, X]$, the polynomials with coefficients in $\mathcal{V}$ in the graded *free variables* $X^+ \uplus X^-$, not to be confused with the scalar polynomials $\mathcal{V} \otimes_\Phi \Phi[X]$ in ungraded *scalar variables.* There is no transparent way to view

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[4]It is not true that $\mathcal{PQE}(\mathcal{V})$ is generated by $q_{x_i}, q_{x_i,x_j}, d_{x_i,x_j}, \cap_{x_i}$ for generators $\{x_k\}$ of $\mathcal{V}$ ($\cap_{Q_{x_i}x_j}$: $a \to Q_a Q_{xi} x_j$ is not directly expressible in terms of these).

[5]If we were willing to add the complete linearization of these, namely (JP1)$''$ linearizing $x \to x, z$ in (JP1)$'$, (JP2)$'$, $''$, $'''$ linearizing $x \to x, z$ and $a \to a, b$ and both in (JP2), and (JP3) linearizations $x \to x, z$ and $x \to x, z, w$ in (JP3)$'$ and $x \to x, z, y \to y, w$, and both in (JP3)$''$, plus the linearizations $a \to a, b$ in all of these, it would suffice to consider only those relations where x, y, a are themselves monomials. But that's too steep a price to pay!

this algebra as there is in the category of associative algebras, where the elements of $\mathcal{F}[A, X]$ are just linear combinations of strings $a_0 x^{(1)} a_1 x^{(2)} \cdots a_n x^{(n)} a_{n+1},\ n \geq 0$, for nonzero $a_i \in A$ (allowing a_0, a_{n+1} to be absent) and nontrivial free noncommutative monomials $x^{(i)}$ in the free associative algebra $\mathcal{F}[X]$ (with the obvious multiplication and linearity in the variables a_i.)[6] The elements of $\mathcal{F}[\mathcal{V}, X]$ can be thought of as **generalized polynomials** in the sense of Martindale (see, for example, [**?**, p.111f]): noncommutative nonassociative Jordan polynomials in indeterminates x_i with coefficients from $\mathcal{V}$. In this paper we will not be concerned with *generalized polynomial identities, nonzero elements* of $\mathcal{F}[\mathcal{V}, X]$ which vanish on $\mathcal{V}$ or related pairs, but rather the *zero elements* themselves, polynomials $p(a_1, \ldots a_n, x_1, \ldots x_m) = 0 \in \mathcal{F}[\mathcal{V}, X]$ where $p(y_1, \ldots y_n, x_1, \ldots x_m) \neq 0 \in \Phi\langle Y \uplus X\rangle$ is a nontrivial Jordan polynomial which vanishes for the particular substitutions $y_i \to a_i \in \mathcal{V}$ *and all possible substitutions* $x_j \to \widetilde{b}_j$ for all pairs $\widetilde{\mathcal{V}}$ containing a homomorphic image of $\mathcal{V}$.

The easiest way to form this free Jordan product is to present $\mathcal{V}$ in the most egregious way (take indeterminates $Y^\tau = V^\tau$ and write $\mathcal{V} \cong \mathcal{FJP}[Y]/K$ induced from the natural inclusion $Y \xrightarrow{\text{in}} \mathcal{V}$, and then form $\mathcal{F}[\mathcal{V}, X] := \mathcal{FJP}[X \uplus Y]/K$ (dividing out by the relations K in the variables Y defining $\mathcal{V}$, but no further relations in the variables X other than those $\mathcal{I}(X)$ imposed in the formation of $\mathcal{FJP}[X]$). There are natural inclusions $X \xrightarrow{\sigma_u} \mathcal{F}[\mathcal{V}, X],\ \mathcal{V} \xrightarrow{\iota_u} \mathcal{F}[\mathcal{V}, X]$. This has the universal property that any set-theoretic map $X \xrightarrow{\sigma} \widetilde{V}$ together with a Jordan pair homomoprhism $\mathcal{V} \xrightarrow{\varphi} \widetilde{\mathcal{V}}$ extends uniquely to a Jordan pair homomorphism $\mathcal{F}[\mathcal{V}, X] \xrightarrow{\widetilde{(\varphi,\sigma)}} \widetilde{\mathcal{V}}$. (We will leave it to the TeXnically proficient reader to construct the corresponding commutative diagram demonstrating the creation and universal diagram for $\mathcal{F}[\mathcal{V}, X]$ from the those for $\mathcal{FJP}[Y], \mathcal{FJP}[X]$.)

We will adopt the shorthand notation $\mathcal{V}\langle X\rangle$ for the free pair $\mathcal{F}[\mathcal{V}, X]$, again using pointy brackets to distinguish it from the scalar polynomial extension $\mathcal{V}[X] = \mathcal{V} \otimes_\Phi \Phi[X]$. When X is countably infinite we call $\mathcal{V}\langle X\rangle$ the **free polynomial algebra** $\mathcal{F}\langle\mathcal{V}\rangle$ or **universal polynomial envelope** $\mathcal{UPE}(\mathcal{V})$ of $\mathcal{V}$ over Φ. The universal property leads by universal nonsense to standard properties of the free object: it determines a functor from Jordan-pairs-and-sets to Jordan pairs, distinct variables can be adjoined one-by-one or in one fell swoop,

$$\mathcal{V}\langle X\rangle\langle Y\rangle \cong \mathcal{V}\langle X \uplus Y\rangle, \tag{2.1}$$

that a bijection of sets induces an isomorphism of free pairs

$$X_1 \cong X_2 \implies \mathcal{V}\langle X_1\rangle \cong \mathcal{V}\langle X_2\rangle, \tag{2.2}$$

[6]As pair theorists, we can blithely ignore the complications in the category of unital algebras, where we would want $1 \in A$ to remain the unit in $\mathcal{F}[A, X]$ and therefore must face the sort of collapse $ax^{(1)}1x^{(2)}b - a(x^{(1)}x^{(2)})b$ familiar from the case of free groups. Indeed, associative ring theory wants the entire center C of A to remain the center of $\mathcal{F}[A, X]$, forming the free product over C instead of Φ.

in particular that the free polynomial algebra is indifferent to countable extensions,

$$\mathcal{F}\langle\mathcal{V}\rangle \cong \mathcal{F}\langle\mathcal{V}\rangle\langle Y\rangle \qquad (Y \textit{ countable}). \tag{2.3}$$

We have the **Action Principle** that $p = 0$ in $\mathcal{F}\langle\mathcal{V}\rangle$ iff the map induced by p vanishes on all Jordan algebras $\widetilde{\mathcal{V}}$ with homomorphism (not necessarily an imbedding) $\mathcal{V} \xrightarrow{\varphi} \widetilde{\mathcal{V}}$. Certainly if $p(x_1, \ldots, x_n) = 0$ in $\mathcal{V}\langle X\rangle$ then for any $\tilde{x}_1, \ldots \tilde{x}_n \in \widetilde{\mathcal{V}}$ and $\sigma(x_i) = \tilde{x}$ we have $0 = \widetilde{(\varphi, \sigma)}(p) = p(\tilde{x}_1, \ldots, \tilde{x}_n)$ and p vanishes on $\widetilde{\mathcal{V}}$. Conversely, if p vanishes on all pairs $\widetilde{\mathcal{V}}$ it certainly vanishes on the pair $\mathcal{F}\langle\mathcal{V}\rangle$ itself, so $p = \widetilde{(\sigma_u, \iota_u)}(p) = 0$.

Any free polynomial algebra $\mathcal{V}\langle X\rangle$ is again X-graded, with the elements in degree 0 being precisely $\mathcal{V}$. We have a graded decomposition $\mathcal{V}\langle X\rangle = \mathcal{V} \oplus \bigoplus_{x\in X} \mathcal{V}_x \oplus \mathcal{V}_2$ into homogeneous parts of degree 0,1, and ≥ 2. Importantly, *the homogeneous polynomials of degree 1 are naturally isomorphic to the universal multiplications of the universal quadratic envelope.*

Quadratic Envelope Imbedding 2.4 *Fix an even and odd variable $x_0^\pm \in X^\pm$, set $x_0 := x_0^+ \oplus x_0^-$. Then the cyclic $\mathcal{V}$-sub-bimodule $M = \mathcal{M}(\mathcal{V})x_0 = \mathcal{V}_{x_0^+} \oplus \mathcal{V}_{x_0^-} \subseteq \mathcal{V}\langle X\rangle$ is naturally isomorphic to the universal cyclic bimodule $\mathcal{M}(\mathcal{U}, \mathcal{Q}_u)) = \mathcal{U}(1_\mathcal{U}) = \mathcal{U}$ via the inverse linear maps $\mathcal{U} \xrightarrow{\psi} M$ given by $\omega \to eval_{x_0}(\omega) = \omega(x_0)$ and $M \xrightarrow{\varphi_0} \mathcal{U}$ by $p(x_0) \to p$. Under this isomorphism $\mathcal{UQE}(\mathcal{V}) \cong \mathcal{V}_{x_0^+} \oplus \mathcal{V}_{x_0^-}$ and $\mathcal{UQE}(\mathcal{V})^{\pm,\tau} \cong \mathcal{V}_{x_0^\tau}$ as spaces.*

Proof: We have a multiplication representation $\mathcal{V} \to \mathcal{M}(\mathcal{V}|\mathcal{V}_{x_0})$, so by the universal property of $\mathcal{U}$ this induces an algebra homomorphism $\mathcal{U} \to \mathcal{M}(\mathcal{V}|\mathcal{V}_{x_0})$, which can be followed by the evaluation map $eval_{x_0}$. Since evaluation is a $\mathcal{V}$-bimodule map, the resulting composite $\psi : \omega \mapsto \omega(x_0)$ is a homomorphism of cyclic $\mathcal{V}$-bimodules.

The specializations $x_i \to 0, x_0 \to 1_\mathcal{U}$ (i.e., $x_0^\tau \to e^\tau$) induce a homomorphism $\mathcal{V}\langle X\rangle \xrightarrow{\varphi} \mathcal{V} \oplus M$ by the univeral property, which restricts to a $\mathcal{V}$-bimodule homomorphism $\mathcal{V}_{x_0} \xrightarrow{\varphi_0} \mathcal{U}1_\mathcal{U} = \mathcal{U}$. The recursive construction shows the only polynomials in $\mathcal{V}_{x_0}$ have the form $p(x_0) = \omega(x_0)$ for a muliplication operator ω: in degree 1 there is just $x_0^\tau = e^\tau(x_0)$, if true for degrees less than n then in degree n any homogeneous degree 1 monomial must be $Q_p q$ (where p must be constant and by recursion $q = \omega(x_0)$ for a multiplication operator ω) or $\{p, q, r\}$ (where we must have two constant factors and one an operator on x_0 by recursion, so $\{p, q, \omega(x_0)\} = D_{p,q}\omega(x_0)$ or $\{p, q(x_0), r\} = Q_{p,r}\omega(x_0)$.

These two homomorphisms are inverses since $(\varphi_0 \circ \psi)(\omega) = \varphi_0(\omega(x_0)) = \omega(1_\mathcal{U}) = \omega$ and $(\psi \circ \varphi_0)(\omega(x_0)) = \psi(\omega(1_\mathcal{U})) = \psi(\omega) = \omega(x_0)$. Thus the two bimodules are isomorphic. It is clear that under this bimodule isomorphism $\mathcal{UQE}(\mathcal{V})^{\sigma,\tau} = e^\sigma \mathcal{UQE}(\mathcal{V}) e^\tau = e^\sigma \mathcal{M} e^\tau$ corresponds to $e^\sigma \mathcal{V}_{x_0^\tau}$ and $\mathcal{UQE}(\mathcal{V})^{\pm,\tau}$ to $\mathcal{V}_{x_0^\tau}$ as spaces. ∎

We remark that $\mathcal{V}\langle X\rangle$ has no involution corresponding to the powerful reversal involution on $\mathcal{UQE}(\mathcal{V})$. Nevertheless some traces of duality remain. For example, if x^+, x^-, y^+ are distinct variables and for some elements z, w, a, b, c in a pair $\mathcal{V}$ the quadratic polynomial $D^+_{z,a}Q^+_wQ^-_bQ_{x^+}c$ vanishes universally (as a function of x^+ on all $\widetilde{\mathcal{V}}$ over $\mathcal{V}$, equivalently in $\mathcal{F}(\mathcal{V})$), then its linearization $D^+_{z,a}Q^+_wQ^-_bQ_{x^+,y^+}c$ vanishes universally as a bilinear function of x^+, y^+, so $\big(D^+_{z,a}Q^+_wQ^-_bD^+_{\tilde{y}^+,c}\big)(x^+) = 0$ in $\mathcal{F}(\mathcal{V})_{x^+}$, and under the isomorphism $D^+_{z,a}Q^+_wQ^-_bD^+_{\tilde{y}^+,c} = 0$ in $\mathcal{UQE}(\mathcal{V})^{\pm,+}$. But then its reverse $D^-_{c,\tilde{y}^+}Q^-_bQ^+_wD^-_{a,z}$ also vanishes, leading (via the isomorphism, this time of $\mathcal{UQE}(\mathcal{V})^{\pm,-}$ with $\mathcal{V}_{x^-}$) to an unexpected relation $D_{c,y^+}Q_bQ_w\{a,z,x^-\} = 0$ back in $\mathcal{V}\langle X\rangle$. Notice that vanishing of a function of x^+, y^+ has led to vanishing of a function of x^-, y^+ (which is exactly what happens in the universal quadratic envelope, where $d_{x,a} = 0$ as a universal map on modules M^τ leads to $d_{a,x} = 0$ universally on $M^{-\tau}$). One suspects the original quadratic function of x vanishing implies some "dual" quadratic function vanishes, but I have been unable to find examples. At any rate, universal vanishing of a generalized Jordan pair polynomial has powerful unexpected consequences.

3 Dominions

An **inner ideal** $I^\tau \lhd_{in} V^\tau$ is a subspace closed under inner multiplication, $Q_{I^\tau}V^{-\tau} \subseteq I^\tau$; then $\mathcal{V}(I^\tau) := (I^\tau, V^{-\tau})$ forms a subpair of $\mathcal{V}$. By (JP3)and (0.1.6), every element $s^\tau \in V^\tau$ determine closed and open *principal inner ideals* $K^\tau_s := \Phi s + Q^\tau_sV^{-\tau}$ and $I^\tau_s := Q^\tau_sV^{-\tau}$. In the theory of Jordan fractions an important role is played by a *sesqui-principal inner ideal* determined by a dominating pair. We say that an element s **dominates** the element n if there are pairs $\mathcal{N}_{s,n} = (N^\tau, N^{-\tau})$, $\mathcal{S}_{s,n} = (S^\tau, S^{-\tau})$ of operators $M^\tau \in \mathrm{End}(V^\tau)$, $\tau = \pm$, such that

(3.1) $$Q^\tau_n = N^\tau Q^\tau_s = Q^\tau_sN^{-\tau}, \qquad Q^\tau_{n,s} = S^\tau Q^\tau_s = Q^\tau_sS^{-\tau}.$$

Note that s automatically dominates all $n = \alpha s + Q_sa$ in the principal inner ideal K_s by (0.1.6), (JP1), and we will see that any y dominated n is already halfway in K_s. Such a pair (s, n) of dominator and dominatee determines in inner ideal which is almost principal.

Dominion Theorem 3.2 *If the element s dominates n, then the* **dominion**

(3.2.1) $$K^\tau_{s\succ n} := \Phi n + \Phi s + Q^\tau_sV^{-\tau}$$

is an inner ideal satisfying

(3.2.2) $$Q_{K_{s\succ n}}V^{-\tau} \subseteq Q^\tau_sV^{-\tau} = I^\tau_s \subseteq K^\tau_s \subseteq K_{s\succ n}.$$

The elements $x := \gamma n + \alpha s + Q_sa$, $y := \alpha s + Q_sa$, $z := Q_sa$ of the dominion have Q-operators which can be "divided by Q_s",

(3.2.3) $Q^\tau_z = Q^\tau_sQ^{-\tau}_aQ^\tau_s$,

(3.2.4) $Q^\tau_{n,z} = D^\tau_aQ^\tau_s = Q^\tau_sD^{-\tau}_a \quad \big((D^\tau_a, D^{-\tau}_a) := (S^\tau D_{s,a} - D_{n,a},\ D_{a,s}S^{-\tau} - D_{a,n})\big)$,

(3.2.5) $Q^\tau_{n,y} = G^\tau Q^\tau_s = Q^\tau_sG^{-\tau} \quad (G^\sigma = \alpha S^\sigma + D^\sigma_a)$

(3.2.6) $Q^\tau_y = B^\tau Q^\tau_s = Q^\tau_sB^{-\tau} \quad \big((B^\tau, B^{-\tau}) := (B_{\alpha,s,a}, B_{\alpha,a,s})\big)$,

(3.2.7) $Q^\tau_x = X^\tau Q^\tau_s = Q^\tau_sX^{-\tau} \quad (X^\sigma = \gamma^2N^\sigma + \gamma G^\sigma + B^\sigma)$.

where $\sigma = \pm\tau$.

Proof: We will omit all indices in the following arguments, since they are clear by context from the statements in the theorem. (2) will show that the dominion is indeed an inner ideal, and (2) will follow from (7) since $Q_x V^{-\tau} = Q_s^\tau X^{-\tau} V^{-\tau} \subseteq Q_s^\tau V^{-\tau}$. So all that remains is to establish the formulas (3)-(7). The fundamental formula (3) is just (JP3), the Bergmann formula (6) is (0.1.6). For (4), we have $Q_{n,z} = Q_{n,Q_s a} = D_{s,a} Q_{s,n} - Q_s D_{a,n}$ [by (0.1.3)] $= D_{s,a}(Q_s S^{-\tau}) - Q_s D_{a,n}$ [by (3.1)] $= Q_s(D_{a,s} S^{-\tau} - D_{a,n})$ [by (JP1)] $= Q_s D_a^{-\tau}$, and dually $Q_{n,Q_s a} = Q_{s,n} D_{a,s} - D_{n,a} Q_s = S^\tau Q_s D_{a,s} - D_{n,a} Q_s = (S^\tau D_{s,a} - D_{n,a}) Q_s = D_a^\tau Q_s$. Then (5) follows immediately since $Q_{n,y} = Q_{n,\alpha s} + Q_{n,z} = Q_s^\tau(\alpha S^{-\tau} + D_a^\tau)$ [by (3.1),(3.4)]. The formula for Q_y follows from (0.1.6). For (7) we have $Q_x = Q_{\gamma n + y} = \gamma^2 Q_n + \gamma Q_{n,y} + Q_y = Q_s\big(\gamma^2 N^{-\tau} + \gamma G^{-\tau} + B_{\alpha,a,s}^{-\tau}\big) = Q_s X^{-\tau}$ by (3.1), (3.6), and dually. ■

This inner ideal is not bi-principal, since the formulas indicate that n is already "half in K_s", so a fraction $Q_s^{-1} n$ is really of degree -1 in s, not -2. We will see that the operator G provides important "glue" binding the two structural transformations N and B into a new structural X.

We will denote the operator pairs by $\mathcal{D}_{a,s,n} = (D_a^\tau, D_a^{-\tau})$, $\mathcal{G}_{\alpha,a,s,n} = \alpha \mathcal{S}_{s,n} + \mathcal{D}_{a,s,n} = (G^\tau, G^{-\tau})$, $\mathcal{B}_{\alpha,a,s} := (B_{\alpha,s,a}, B_{\alpha,a,s})$, $\mathcal{X}_{\gamma,\alpha,a,s,n} := \gamma^2 \mathcal{N}_{s,n} + \gamma \mathcal{G}_{\alpha,a,s,n} + \mathcal{B}_{\alpha,a,s}$. The individual operators $N^\pm, S^\pm, D_a^\pm$ also depend on s, n, but we will always omit these subscripts from the notation.

We will see that life gets easier the smaller our dominions get. It is easy to construct subdominions inside a given dominion $K_{s \succ n}$.

Subdominion Theorem 3.3 *If s dominates n in V^τ, then also for any $c \in V^{-\tau}$ the element $s' := Q_s c$ dominates $n' := Q_s Q_x n$, inducing a* **subdominion**

$$(3.3.1) \quad K^\tau_{s' \succ n'} = \Phi n' + \Phi s' + Q_{s'} V^{-\tau} = Q_s\big(\Phi c + Q_c(\Phi n + Q_s V^{-\tau})\big) \subseteq Q_s K_c^{-\tau}.$$

For any element $x' := \gamma n' + \alpha s' + Q_{s'} a = Q_s(\alpha c + Q_c y') \in K^\tau_{s' \succ n'}$, $y' := \gamma n + Q_s a \in V^\tau$ we have the relations

$$(3.3.2) \quad \{s, c, y'\} = Q_s^\tau a'' \quad (a'' := -\gamma \Delta_0(c) + \{a, s, c\}),$$

$$(3.3.3) \quad Q_{y'}^\tau = Q_s^\tau A^{-\tau} = A^\tau Q_s^\tau, \quad (A^\tau := \gamma^2 N^\tau + \gamma D_a^\tau + Q_s^\tau Q_a^{-\tau}, \quad A^{-\tau} := \gamma^2 N^{-\tau} + \gamma D_a^{-\tau} + Q_a^\tau Q_s^{-\tau}),$$

$$(3.3.4) \quad Q_{n'}^\tau = Q_{s'}^\tau N'^{-\tau} = N'^\tau Q_{s'}^\tau, \quad (N'^\tau := Q_s^\tau Q_c^{-\tau} N^\tau,\ N'^{-\tau} := N^{-\tau} Q_c^\tau Q_s^{-\tau}),$$

$$(3.3.5) \quad Q_{n',s'}^\tau = Q_{s'}^\tau (\Delta_0')^{-\tau} = (\Delta_0')^\tau Q_{s'}^\tau, \quad (\Delta_0'^\tau := -D_{s,\Delta_0(c)} - D_{n,c},\ \Delta_0'^{-\tau} := -D_{\Delta_0(c),c} - D_{c,n}),$$

$$(3.3.6) \quad Q_{n',Q_{s'}a}^\tau = Q_{s'}^\tau D_a'^{-\tau} = D_a'^\tau Q_{s'}^\tau, \ (D_a'^\tau := Q_s Q_c D_a^\tau,\ D_a'^{-\tau} := D_a^{-\tau} Q_c Q_s),$$

$$(3.3.7) \quad Q_{x'} = Q_{s'}^\tau X'^{-\tau} = X'^\tau Q_{s'}^\tau, \quad X'^\tau :== \alpha^2 I + \alpha\big(D_{s,a''} - D_{y',c}\big) + Q_s Q_c A^\tau, \quad X'^{-\tau} := \alpha^2 I + \alpha\big(D_{a'',s} - D_{c,y'}\big) + A^{-\tau} Q_c Q_s.$$

Thus the sub-derivation $\Delta'_0 \in D_{\mathcal{V},\mathcal{V}}$ is always an inner derivation. If $\mathcal{N}$ is a struction, so is $\mathcal{N}'$.

PROOF: (1) is clear. For (2), $\{s,c,y'\} = \gamma Q_{s,n}c + Q_{s,Q_s a}c = Q_s\big(-\gamma\Delta_0(c) + D_{a,s}(c)$ [by (5.1.4), (JP1)] $= Q_s a''$. For (3), by (3.2.7) [with $\alpha = 0$] $Q_{y'} = Q_{\gamma n + Q_s a} = Q_s A^{-\tau} = A^\tau Q_s$ for $A^\sigma = \gamma^2 N^\sigma + \gamma D_a^\sigma + B^\sigma$ where $B_{0,a,s} = Q_a Q_s, B_{0,s,a} = Q_s Q_a$. (4) follows from (JP3), (5.1.3) by $Q_{n'} = Q_{Q_s Q_c n} = Q_s Q_c Q_n Q_c Q_s = Q_s Q_c (Q_s N^{-\tau}) Q_c Q_s = Q_{s'}\big(N^{-\tau} Q_c Q_s\big)$, so $N'^{-\tau} = N^{-\tau} Q_c Q_s$, and dually $N'^{\tau} = Q_s Q_c N^{-\tau}$. For (5) we first note

$$D_{n,c}Q_s = -Q_s\Delta_0'^{-\tau}, \quad Q_s D_{c,n} = -\Delta_0'^{\tau} Q_s,$$
$$\Delta_0'^{\tau} = -D_{s,\Delta_0(c)} - D_{n,c}, \quad \Delta_0'^{-\tau} = -D_{\Delta_0(c),s} - D_{c,n},$$

since $Q_s D_{c,n} + D_{n,c} Q_s = Q_{\{n,c,s\},s}$ [by (0.1.1)] $= -Q_{Q_s\Delta_0(c),s}$ [by (5.1.4)] $= -Q_s D_{\Delta_0(c),s} = -D_{s,\Delta_0(c)} Q_s$ [(by (JP1)]. Then $Q_{n',s'} = Q_{Q_s Q_c n, Q_s c} = Q_s Q_{Q_c n, c} Q_s = Q_s\big(Q_c D_{n,c}\big) Q_s = Q_s Q_c Q_s \Delta_0'^{-\tau}$ [by (JP1) and the above] $= Q_{s'}\Delta_0'^{-\tau}$ [by (JP3)], and dually, yielding (5).

For the operators D'_a of (6), we compute $Q_{n', Q_{s'} a} = Q_{Q_s Q_c n, Q_s Q_c Q_s a} = Q_s Q_c Q_{n, Q_s a} Q_c Q_s = Q_s Q_c\big(Q_s D_a^{-\tau}\big) Q_c Q_s$ [by (3.2.4)] $= Q_{s'}\big(D_a^{-\tau} Q_c Q_s\big) = Q_{s'} D_a'^{-\tau}$, and dually.

For (7) we need another result,

$$Q_s B_{\alpha,c,y'} = X^\tau Q_s, \quad B_{\alpha,y',c} Q_s = Q_s X^{-\tau},$$
$$X^\tau = \alpha^2 + \alpha\big(D_{s,a''} - D_{y',c}\big) + \big(Q_s Q_c A^\tau\big),$$
$$X^{-\tau} = \alpha^2 I + \alpha\big(D_{a'',s} - D_{c,y'}\big) + \big(A^{-\tau} Q_c Q_s\big),$$

which follows from immediately from the separate pieces of the Bergmann operator: $Q_s = I Q_s = Q_s I$, $D_{y',c} Q_s + Q_s D_{c,y'} = Q_{\{y',c,s\},s}$ [by (0.1.1)] $= Q_{Q_s a'',s}$ [by (2)] $= Q_s D_{a'',s}$, and $Q_{y'} Q_c Q_s = (Q_s A^{-\tau}) Q_c Q_s$ [by (3)] and dually. Then $Q_{x'} = Q_{Q_s(\alpha c + Q_c y')} = Q_s Q_{\alpha c + Q_c y'} Q_s = Q_s(Q_c B_{\alpha,y',c}) Q_s$ [by (0.1.6)] $= Q_s Q_c Q_s Q_s X^{-\tau}$, [by the above] $= Q_{s'} X^{-\tau}$ as in (7), and dually.

Automatically Δ'_0 is inner in $\delta_{\mathcal{V},\mathcal{V}}$ by (5). If $\mathcal{N}$ is already an inner multiplication, so is $\mathcal{N}'$. We will see later (Remark 6.2) that $\mathcal{N}$ will be a (complicated) inner multiplication if one small part of the struction condition is satisfied. In case $\mathcal{N}$ is already principal, so is $\mathcal{N}'$: if $N^\tau = Q_s Q_q$ (with $Q_s q = n$) then $N'^{\tau} = Q_s Q_c (Q_s Q_q) = Q_{Q_s c} Q_q = Q_{s'} Q_q$ and dually. ∎

4 Bergmann Triples and Pairs

A **structural pair** $\mathcal{T} = (T^+, T^-)$, or pair of **structural transformations (struction)**, on a Jordan pair $\mathcal{V}$ consists of two linear transformations $T^\tau \in \mathrm{End}(V^\tau)$ (the superscript indicates the domain and range) satisfying

$$Q^\tau_{T^\tau(x)} = T^\tau Q^\tau_x T^{-\tau}. \tag{4.1}$$

for all $x \in V^\tau$ and $\tau = \pm$. The structural pairs form a submonoid of $\mathrm{End}(V^+) \times \mathrm{End}(V^-)$ under $\mathcal{T}_1\mathcal{T}_2 := (T_1^+ T_2^+, T_2^- T_1^-)$.

An **oddstruction** $T^\tau \in \mathrm{Hom}(V^{-\tau}, V^\tau)$ (the superscript indicates the range) is a linear transformation satisfying

$$Q^\tau_{T^\tau(a)} = T^\tau Q^{-\tau}_a T^\tau. \tag{4.2}$$

for all $a \in V^{-\tau}$. The product of two oddstructions T^+, T^- gives a struction (T^+T^-, T^-T^+). Each structural pair induces a **homotope** Jordan pair $\mathcal{V}^{(\mathcal{T})} = (V^+, V^-)$ under $Q^{\mathcal{T}\tau}_x := Q^\tau_x T^{-\tau}$, $D^{\mathcal{T}\tau}_{x,a} := D^\sigma_{x,T^{-\tau}(a)}$, any struction $\mathcal{S}$ induces a homomorphism $\mathcal{V}^{(\mathcal{S}\mathcal{T}\mathcal{S})} \to \mathcal{V}^{(\mathcal{T})}$, and $(I^+, I^-) = (T^+(V^+), T^-(V^-))$ is always a pair of inner ideals $Q^\tau_{I^\tau} V^{-\tau} \subseteq I^\tau$. Any oddstruction $T^{-\tau}$ induces a Jordan triple system $V^{\tau(T^{-\tau})}$ via $P_x y := Q^\tau_x T^{-\tau}(y)$ (and if $T^{-\tau} = Q^{-\tau}_t$ for $t \in V^{-\tau}$, then $V^{\tau(T^\tau)}$ becomes a Jordan algebra via $x^{2(T^\tau)} := Q^\tau_x t^{-\tau}$), and $I^\tau = T^\tau(V^{-\tau})$ is always an inner ideal.

An **inner** struction (**innstruction**) is one which is built internally out of multiplications, not just accidentally, but universally: $T^\tau \in \mathcal{U}$, $T^{-\tau} = (T^\tau)^*$ and (4.1) holds in $\mathcal{U}$ (i.e. with Q replaced by $q^{\tau,-\tau}$). Similarly, an inner oddstruction (**innoddstruction**) satisfies (4.2) in $\mathcal{U}$. The basic examples of innoddstructions are the **principal oddstructions** Q^τ_x, and the basic examples of innstructions are the **principal structions**

$$\mathcal{T}_{x,a} = (Q_x Q_a, Q_a Q_x) \quad \textit{for} \quad x \in V^\tau, a \in V^{-\tau}$$

(strictly speaking we should write, more cumbersomely, $\mathcal{T}_{x,a} = (q^{\tau,-\tau}_x q^{-\tau,\tau}_a, q^{-\tau,\tau}_a q^{\tau,-\tau}_x)$, but we won't).

We say $\mathcal{G} = (G^+, G^-)$ consisting of two linear transformations $G^\tau : V^\tau \to V^\tau$ is **structural glue** for two structural pairs $\mathcal{T}_1, \mathcal{T}_2$, and call $(\mathcal{T}_1, \mathcal{G}, \mathcal{T}_2)$ a **Bergmann triple**, if the following two relations hold for all $x \in V^\tau$:

$$T^\tau_i Q_x G^{-\tau} + G^\tau Q_x T^{-\tau}_i = Q_{T^\tau_i(x), G^\tau(x)} \quad (i = 1, 2), \tag{4.3.1}$$

$$T^\tau_1 Q_x T^{-\tau}_2 + T^\tau_2 Q_x T^{-\tau}_1 + G^\tau Q_x G^{-\tau} = Q_{G^\tau(x)} + Q_{T^\tau_1(x), T^\tau_2(x)}. \tag{4.3.2}$$

In this case we can glue the two pairs together via $\mathcal{G}$ and create a **Bergmann struction**

$$\mathcal{X}_{\mathcal{T}_1,\mathcal{G},\mathcal{T}_2} : \quad X^\tau := T^\tau_1 + G^\tau + T^\tau_2. \tag{4.4}$$

Indeed, structurality comes from $Q_{X^\tau(x)} = Q_{T^\tau_1(x)} + Q_{T^\tau_1(x), G^\tau(x)} + (Q_{G^\tau(x)} + Q_{T^\tau_1(x), T^\tau_2(x)}) + Q_{T^\tau_2(x), G^\tau(x)} + Q_{T^\tau_2(x)} = T^\tau_1 Q_x T^{-\tau}_1 + (T^\tau_1 Q_x G^{-\tau} + G^\tau Q_x T^{-\tau}_1) + (T^\tau_1 Q_x T^{-\tau}_2 + T^\tau_2 Q_x T^{-\tau}_1 + G^\tau Q_x G^{-\tau}) + (T^\tau_2 Q_x G^{-\tau} + G^\tau Q_x T^{-\tau}_2) + T^\tau_2 Q_x T^{-\tau}_2 = X^\tau Q_x X^{-\tau}$. Notice that for any $\alpha_1, \alpha_2 \in \Phi$ the triple $(\alpha_1^2 \mathcal{T}_1, \alpha_1\alpha_2 \mathcal{G}, \alpha_2^2 \mathcal{T}_2)$ is again a Bergmann triple with $X^\tau = \alpha_1^2 T^\tau_1 + \alpha_1\alpha_2 G^\tau + \alpha_2^2 T^\tau_2$.

For the special case $\mathcal{I} = (Id, Id)$, we say $(\mathcal{G}, \mathcal{T})$ is a **Bergmann pair** if $(\mathcal{I}, \mathcal{G}, \mathcal{T})$ is a Bergmann triple, (4.3.1′) $G^\tau Q_x + Q_x G^{-\tau} = Q_{G^\tau(x),x}$,

(4.3.1″) $T^\tau Q_x G^{-\tau} + G^\tau Q_x T^{-\tau} = Q_{T^\tau(x),G(x)}$,

(4.3.2′) $T^\tau Q_x + Q_x T^{-\tau} + G^\tau Q_x G^{-\tau} = Q_{G^\tau(x)} + Q_{T^\tau(x),x}$.

Thus $\mathcal{T}$ is a structural pair and $\mathcal{G}$ is a **Lie structural pair** or **Lie struction** by (2.3.1′), and for any α we obtain a Bergmann struction

$$B_{\alpha,\mathcal{G},\mathcal{T}} = \mathcal{X}_{\alpha^2\mathcal{I},\alpha\mathcal{G},\mathcal{T}} := \alpha^2\mathcal{I} + \alpha\mathcal{G} + \mathcal{T}. \tag{4.4′}$$

The Lie structions form the *structure Lie subalgebra* $Strl(\mathcal{V})$ of $\mathrm{End}(V^+) \times \mathrm{End}(V^-)$ under $[\mathcal{G}_1, \mathcal{G}_2] := ([G_1^+, G_2^+], [G_2^-, G_1^-])$.

We have obvious universal notions of *innstructural glue* (with $(G^\tau)^* = G^{-\tau}$), *Lie innstruction,* and *inner Bergmann triple or pair,* which produce a *Bergmann innstruction* satisfying (2.3-4) in $\mathcal{U}$.

Principal Example 4.5 *The archetypal example of an inner Bergmann triple is the principal triple* $(\mathcal{T}_1, \mathcal{G}, \mathcal{T}_2) = (\mathcal{T}_{x_1,a}, \mathcal{G}_{x_1,x_2;a}, \mathcal{T}_{x_2,a})$ *with glue* $\mathcal{G}_{x_1,x_2;a} := (Q_{x_1,x_2}Q_a, Q_aQ_{x_1,x_2})$ *for* $x_i \in V^\tau, a \in V^{-\tau}$. *Here the resulting Bergmann operators* $\mathcal{B}_{(\mathcal{T}_1,\mathcal{G},\mathcal{T}_2)} = \mathcal{T}_{x_1+x_2,a}$ *and* $\mathcal{B}_{\alpha_1^2\mathcal{T}_1,\alpha_1\alpha_2\mathcal{G},\alpha_2^2\mathcal{T}_2} = \mathcal{T}_{\alpha_1x_1+\alpha_2x_2,a}$ *are again principal structions.*

The archetypal example of an inner Bergmann pair is, of course, the principal pair $\mathcal{D}_{x,a} = (D_{x,a}, D_{a,x})$, $\mathcal{T}_{x,a} = (Q_xQ_a, Q_aQ_x)$, *with* $\mathcal{B}_{\alpha,x,a} = (B_{\alpha,x,a}, B_{\alpha,a,x})$ *the usual Bergmann innstruction. If x happens to be invertible, then* $\mathcal{B}_{\alpha,x,a}$ *reduces to a principal struction* $\mathcal{T}_{x,\tilde{a}} = (Q_xQ_{\alpha x^{-1}+a}, Q_{\alpha x^{-1}+a}Q_x)$.

Proof: In the principal triple clearly $\mathcal{T}_1, \mathcal{T}_2$ are structions, and $\mathcal{G}_{x_1,x_2;a}$ is structural glue since for $\sigma = \pm\tau$ and all $z \in V^\sigma$ we have universally in $\mathcal{U}$ that

(4.5.1) $T_i^\sigma Q_x G^{-\sigma} + G^\sigma Q_x T_i^{-\sigma} = Q_{T_i^\sigma(x),G^\sigma(x)}$,

(4.5.2) $T_i^\sigma Q_x T_j^{-\sigma} + T_j^\sigma Q_x T_i^{-\sigma} + G^\sigma Q_x G^{-\sigma} = Q_{G^\sigma(x)} + Q_{T_i^\sigma(x),T_j^\sigma(x)}$

which follow directly from the linearizations

(4.5.1′) $Q_{x_i}Q_bQ_{x_i,x_j} + Q_{x_i,x_j}Q_bQ_{x_i} = Q_{Q_{x_i,x_j}(b),Q_{x_i}(b)}$,

(4.5.2′) $Q_{x_i}Q_bQ_{x_j} + Q_{x_j}Q_bQ_{x_i} + Q_{x_i,x_j}Q_bQ_{x_i,x_j} = Q_{Q_{x_i,x_j}(b)} + Q_{Q_{x_i}(b),Q_{x_j}(b)}$

of (JP3) for all $b = Q_ax$ (when $\sigma = \tau$) and $b = x$ (when $\sigma = -\tau$). ■

It is clear from the definitions that the restriction to an invariant subpair $\mathcal{V} \subseteq \widetilde{\mathcal{V}}$ of a struction, oddstruction, Bergmann triple, or pair on $\widetilde{\mathcal{V}}$ remains such on $\mathcal{V}$. More generally, if the oddstruction does not leave the subpair invariant, we can sometimes shove it down into the subpair. We say $\widetilde{T}^\sigma \in \mathrm{Hom}(\widetilde{V}^{-\sigma}, \widetilde{V}^\sigma)$[7] has *denominator* $s \in V^{-\sigma}$ if $Q_s^{-\sigma}$ shoves $\widetilde{T}$ down to an endomorphism on $\mathcal{V}$ in the sense that $S^{-\sigma} := Q_s^{-\sigma}\widetilde{T}^\sigma \in \mathrm{End}(V^{-\sigma})$, $S^\sigma := \widetilde{T}^\sigma Q_s^{-\sigma} \in \mathrm{End}(V^\sigma)$ [more precisely, $Q_s^{-\sigma}\widetilde{T}^\sigma$, $\widetilde{T}^\sigma Q_s^{-\sigma}$ leave $\mathcal{V}$ invariant, and $S^{-\sigma}, S^\sigma$ are their restrictions to $\mathcal{V}$]. Then $T^{-\sigma} := Q_s^{-\sigma}\widetilde{T}^\sigma Q_s^{-\sigma} =$

[7]Here we begin our convention that our denominators s and our inner ideals K will always belong to $V^{-\sigma}$.

$S^{-\sigma}Q_s^{-\sigma} = Q_s^{-\sigma}S^\sigma \in \mathrm{Hom}(V^\sigma, V^{-\sigma})$, and we say that $S^{-\sigma}, S^\sigma$ result by *cancelling* Q_s from T^σ on the right and left. If $\widetilde{T}^{-\sigma}$ is an oddstruction, then $T^{-\sigma}$ is an oddstruction on $\widetilde{\mathcal{V}}$ leaving $\mathcal{V}$ invariant, and its s-cancellation $(S^{-\sigma}, S^\sigma)$ is a struction on $\mathcal{V}$,

$$(4.6) \qquad Q^{-\sigma}_{S^{-\sigma}(x)} = S^{-\sigma}Q_x^{-\sigma}S^\sigma, \quad Q^\tau_{S^\sigma(a)} = S^\sigma Q_a^\tau S^{-\sigma} \qquad (x \in V^{-\sigma}, a \in V^\sigma),$$

since $Q_{Q_s\widetilde{T}(x)} = Q_sQ_{\widetilde{T}(x)}Q_s = Q_s\widetilde{T}Q_x\widetilde{T}Q_s$ on $\widetilde{\mathcal{V}}$ becomes $Q_{S^{-\sigma}(x)} = S^{-\sigma}Q_xS^\sigma, Q_{S^\sigma(x)} = S^\sigma Q_xS^{-\sigma}$ on $\mathcal{V}$, and dually $Q_{\widetilde{T}Q_sa} = \widetilde{T}Q_sQ_aQ_s\widetilde{T}$ becomes $Q_{S^\sigma(a)} = S^\sigma Q_aS^{-\sigma}$.

Pseudo-Principal Example 4.7 *The example we are more interested in is the restriction to $\mathcal{V}$ of a principal struction on $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$. Let $n^{-\sigma}, y^{-\sigma} = \alpha s^{-\sigma} + Q_s^{-\sigma}a^\sigma \in V^{-\sigma}$ for some particular $\alpha \in \Phi, a \in V^\sigma$. Suppose there are elements $\tilde{q}^\sigma, \tilde{u}^\sigma \in \widetilde{\mathcal{V}}$ such that $Q_s^{-\sigma}(\tilde{q}) = n^{-\sigma}, Q_s^{-\sigma}(\tilde{u}^\sigma) = y$ where $Q_{\tilde{q}}, Q_{\tilde{u}}, Q_{\tilde{q},\tilde{u}}$ have denominator $s \in V^{-\sigma}$. [For example, if $s^{-1} \in \widetilde{V}^\sigma$ we may take $\tilde{q}^\sigma = (Q_s^{-\sigma})^{-1}(n^{-\sigma})$ and $\tilde{u}^\sigma = \alpha s^{-1} + a \in \widetilde{V}^\sigma$.] Then s dominates $x = \gamma n + y$ for any $\gamma \in \Phi : x = Q_s^{-\sigma}(\tilde{v})$ for $\tilde{v} = \gamma\tilde{q} + \tilde{u}$, and*

$$\begin{aligned}
Q_n^{-\sigma} &= N^{-\sigma}Q_s^{-\sigma} = Q_s^{-\sigma}N^\sigma \quad (N^{-\sigma} := Q_s^{-\sigma}Q^\tau_{\tilde{q}}|_V, N^\sigma := Q^\tau_{\tilde{q}}Q_s^{-\sigma}|_V),\\
Q_{n,s}^{-\sigma} &= \Delta_0^{-\sigma}Q_s^{-\sigma} = -Q_s^{-\sigma}\Delta_0^\sigma \quad (\Delta_0^{-\sigma} := D_{s,\tilde{q}}|_V,\ \Delta_0^\sigma := -D_{\tilde{q},s}|_V),\\
Q_{n,y}^{-\sigma} &= G^{-\sigma}Q_s^{-\sigma} = Q_s^{-\sigma}G^\sigma \quad (G^{-\sigma} := Q_s^{-\sigma}Q^\sigma_{\tilde{q},\tilde{u}}|_V, G^\sigma := Q^\sigma_{\tilde{q},\tilde{u}}Q_s^{-\sigma}|_V),\\
Q_y^{-\sigma} &= B^{-\sigma}Q_s^{-\sigma} = Q_s^{-\sigma}B^\sigma \quad (B^{-\sigma} := Q_s^{-\sigma}Q^\sigma_{\tilde{u}}|_V, B^\sigma := Q^\sigma_{\tilde{u}}Q_s^{-\sigma}|_V),\\
Q_x^{-\sigma} &= \gamma^2Q_n^{-\sigma} + \gamma Q_{n,y}^{-\sigma} + Q_y^{-\sigma} = X^{-\sigma}Q_s^{-\sigma} = Q_s^{-\sigma}X^\sigma \quad \textit{for}\\
X^\tau &= \gamma^2N^\tau + \gamma G^\tau + B^\tau \in \mathrm{End}(V^\tau) \quad (X^{-\sigma} = Q_s^{-\sigma}Q^\sigma_{\tilde{v}}|_V, X^\sigma = Q^\sigma_{\tilde{v}}Q_s^{-\sigma}|_V).
\end{aligned}$$

Then $\widetilde{T} = Q_{\tilde{q}}, Q_{\tilde{u}}, Q_{\tilde{v}}$ and $T = Q_n, Q_y, Q_x$ are oddstructions whose s-cancellations $\mathcal{N} := (N^\sigma, N^{-\sigma}) = \mathcal{T}_{\tilde{q},s}|_\mathcal{V}$, $\mathcal{B} := (B^\sigma, B^{-\sigma}) = \mathcal{T}_{\tilde{u},s}|_\mathcal{V}$, $\mathcal{X} := (X^\sigma, X^{-\sigma}) = \mathcal{T}_{\tilde{v},s}|_\mathcal{V}$ are structions induced on $\mathcal{V}$ by restriction from principal structions on $\widetilde{\mathcal{V}}$, and $\mathcal{G} := (G^\sigma, G^{-\sigma}) = \mathcal{T}_{\tilde{q},\tilde{u};s}\ |_\mathcal{V}$ is structural glue, with resulting Bergmann struction $\mathcal{X} := \mathcal{B}_{(\gamma^2N,\gamma\mathcal{G},\mathcal{B})} = \gamma^2\mathcal{N} + \gamma\mathcal{G} + \mathcal{B}$.[8]

Moreover, the following relations with $w_1 := Q_{\tilde{q}}s$, $w_2 := Q_{\tilde{q}}n$, $z_1 := Q_sw_1 = N^\sigma(s)$, $z_2 := Q_sw_2 = N^{-\sigma}(n)$ are satisfied as linear transformations on $\mathcal{W}$ for all $w \in W^\tau$ for all extensions $\mathcal{W} \supseteq \widetilde{\mathcal{V}} \supseteq \mathcal{V}$:

[8]Since already $Q_z = Q_{\alpha s+Q_sa} = Q_sB_{\alpha,a,s} = B_{\alpha,s,a}Q_s$ by (0.1.6), by hypothesis $B^{-\sigma}Q_s^{-\sigma} = B_{\alpha,s,a}Q_s^{-\sigma}$, so *automatically* $B^{-\sigma} = B_{\alpha,s,a}$ if Q_s is surjective on $\widetilde{V}^\sigma$, and $Q_s^{-\sigma}B^\sigma = Q_s^{-\sigma}B_{\alpha,a,s}$, so *automatically* $B^\sigma = B_{\alpha,a,s}$ if Q_s is injective on $\widetilde{V}^\sigma$, and both will hold if s is invertible in $\widetilde{\mathcal{V}}$. If s is merely regular and we take $\tilde{u} = \alpha\tilde{w} + a$, then $B^{-\sigma} - B_{\alpha,s,a} = \alpha^2[Q_sQ_{\tilde{w}} - I] + \alpha[Q_sQ_{\tilde{w},a} - D_{s,a}] + [Q_sQ_a - Q_sQ_a] = \alpha^2[Q_sQ_{\tilde{w}} - I] + \alpha[D_{s,w} - 2I]D_{s,a}$ where $[Q_sQ_{\tilde{w}} - I]Q_s = [D_{s,w} - 2I]Q_s = 0$, and dually.

(4.7.1) $\quad Q_{N^\tau(w)} = N^\tau Q_w N^{-\tau},$

(4.7.2) $\quad N^\tau Q_w + Q_w N^{-\tau} = \Delta_0^\tau Q_w \Delta_0^{-\tau} + Q_{\Delta_0^\tau(w)} + Q_{N^\tau(w),w},$

(4.7.3) $\quad N^\tau Q_w \Delta_0^{-\tau} - \Delta_0^\tau Q_w N^{-\tau} + Q_{N^\tau(w),\Delta_0^\tau(w)} = 0,$

(4.7.4) $\quad \Delta_0^\tau Q_w = Q_{\Delta_0^\tau(w),w} + Q_w \Delta_0^{-\tau},$

(4.7.5) $\quad Q_{n,Q_s a} = D_a^\sigma Q_s = Q_s D_a^\sigma \quad (D_a^\sigma = -D_{a,s}\Delta_0^\sigma - D_{a,n},\ D_a^{-\sigma} = \Delta_0^{-\sigma} D_{s,a} - D_{n,a}),$

(4.7.6) $\quad \Delta_0^{-\sigma} Q_{s,n}^{-\sigma} = -Q_{s,n}^{-\sigma}\Delta_0 = 2Q_n^{-\sigma} + Q_{z_1,s}^{-\sigma},$

(4.7.7) $\quad D_{a,s}N^\sigma + D_{a,n}\Delta_0^\sigma + D_{a,z_1} = 0,$

(4.7.7)* $\quad N^{-\sigma}D_{s,a} - \Delta_0^{-\sigma}D_{n,a} + D_{z_1,a} = 0,$

(4.7.8) $\quad D_{a,n}^\sigma N^\sigma + D_{a,z_1}^\sigma \Delta_0^\sigma + D_{a,z_2}^\sigma = 0,$

(4.7.8)* $\quad N^{-\sigma}D_{n,a}^{-\sigma} - \Delta_0^{-\sigma}D_{z_1,a}^{-\sigma} + D_{z_2,a}^{-\sigma} = 0,$

(4.7.9) $\quad N^\tau \Delta_0^\tau = \Delta_0^\tau N^\tau = M^\tau \quad (M^\sigma := -D_{w_1}^\sigma,\ M^{-\sigma} := D_{w_1}).$

Proof: To see these relations for the restrictions of the principal structions $\mathcal{N} = \mathcal{T}_{\tilde q,s}, \Delta_0 = \mathcal{D}_{\tilde q,s}, \mathcal{B} = \mathcal{T}_{\tilde u,s}$, and glue $\mathcal{G} = \mathcal{T}_{\tilde q,\tilde u;s}$, (1) follows from (JP3), (2) from (0.1.4), (3) from (0.1.5), (4) since $\delta_{\tilde q,s} = (D_{\tilde q,s}, -D_{s,\tilde q})$ is always a derivation of Jordan pairs by (0.1.1), we saw (5) in (3.2.4), and (6) follows from $D_{s,\tilde q}Q_{s,Q_s\tilde q} = Q_{s,Q_s\tilde q}D_{\tilde q,s} = Q_s\big(D_{\tilde q,s}^2\big) = Q_s\big(D_{Q_{\tilde q}s,s} + 2Q_{\tilde q}Q_s\big) = Q_{Q_sQ_{\tilde q}s,s} + 2Q_{Q_s\tilde q} = Q_{z_1,s} + 2Q_n$ for $z_1 := Q_sQ_{\tilde q}s = N^{-\sigma}(s)$. For (7), we have the general Jordan identity $D_{a,s}Q_{\tilde q}Q_s - D_{a,Q_s\tilde q}D_{\tilde q,s} + D_{a,Q_sQ_{\tilde q}s} = V_aU_{\tilde q} - V_{a,\tilde q}V_{\tilde q} + V_{a,\tilde q^2} = 0$ in the Jordan algebra $J = V^{\sigma(s)}$, dually (7*) is a Jordan pair identity (thanks to the involution in $\mathcal{UQE}$, not because of any homotope). In the same way, (8) is a general pair identity $D_{a,Q_s\tilde q}Q_{\tilde q}Q_s - D_{a,Q_sQ_{\tilde q}s}D_{\tilde q,s} + D_{a,Q_sQ_{\tilde q}Q_s\tilde q} = V_{a,\tilde q}U_{\tilde q} - V_{a,\tilde q^2}V_{\tilde q} + V_{a,\tilde q^3} = 0$, and dually for (8*).

Finally, (9) holds for $\tau = \sigma$ since $Q_{\tilde q}Q_s(-D_{\tilde q\tilde q,s}) = (-D_{\tilde q,s})Q_{\tilde q}Q_s$ [by (JP1) twice] $= -Q_{Q_{\tilde q}s,\tilde q}Q_s$ [by (JP1)] $= -Q_{w_1,\tilde q} = -D_{w_1,s}D_{\tilde q,s} + D_{w_1,Q_s\tilde q}$ [by (0.1.2)] $= D_{w_1,s}\Delta_0^\sigma + D_{w_1,n} = -D_{w_1}^\sigma$, and dually for $\tau = -\sigma$ we have $Q_sQ_{\tilde q}(D_{s,\tilde q}) = (D_{s,\tilde q})Q_sQ_{\tilde q} = Q_sQ_{w_1,\tilde q} = D_{s,\tilde q}D_{s,w_1} - D_{Q_s\tilde q,w_1} = \Delta_0^{-\sigma}D_{s,w_1} - D_{n,w_1} = D_{w_1}^{-\sigma}$. ■

This list of ancillary relations (4.7.1-9) provides the necessary tools in the next two sections to make $\mathcal{N}$ a struction, without the help of $\tilde q$ or $\widetilde{\mathcal{V}}$. The problem of creating "fractions" is the situation where $\tilde q, \tilde u, \tilde v \in \widetilde{\mathcal{V}}$ are merely figments of our imagination, and all that exists is their traces N, G, S on $\mathcal{V}$ and n, z, x in $\mathcal{V}$. The most general problem would be that of creating a "holomorph" $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$ where suitable structions become "inner".

5 Structural Dominance

A central question in fractions ([?],[?]) is whether $\mathcal{N}, \mathcal{X}$ as above are innstructions on $\mathcal{V}$ without the help of $\widetilde{\mathcal{V}}$. Here $\tilde q$ is fictitious, but N, X are often built from multiplications from $\mathcal{V}$, and the question is whether they are intrinsically structural.

We say $s \in V^{-\sigma}$ **structurally dominates** $n \in V^{-\sigma}$ **on** $\mathcal{V}$ if there is a derivation $\Delta_0 = (\Delta_0^+, \Delta_0^-)$ (with corresponding Lie struction $\widehat{\Delta_0} = (-\Delta_0^\sigma, \Delta_0^{-\sigma})$) and a struction

$\mathcal{N} = (N^+, N^-)$ satisfying (for all $x \in V^\tau, a \in V^\sigma$, $\tau = \pm\sigma$):

(5.1.1) $Q_n = N^{-\sigma} Q_s = Q_s N^\sigma$ ($\mathcal{N}$ *results by cancelling* Q_s),

(5.1.2) $Q_{n,s} = \Delta_0^{-\sigma} Q_s = -Q_s \Delta_0^\sigma$, ($\Delta_0$ *results by cancelling* Q_s),

(5.1.3) $Q_{N^\tau(x)} = N^\tau Q_x N^{-\tau}$ ($\mathcal{N}$ *is a struction*),

(5.1.4) $\Delta_0^\tau Q_x = Q_{\Delta_0^\tau(x),x} + Q_x \Delta_0^{-\tau}$ (Δ_0 *is a derivation*),

(5.1.5) $Q_{n,Q_s a} = D_a^\sigma Q_s = Q_s D_a^\sigma$ ($D_a^\sigma = -D_{a,s}\Delta_0^\sigma - D_{a,n},\ D_a^{-\sigma} = \Delta_0^{-\sigma} D_{s,a} - D_{n,a}$),

as in (3.1-2), and in addition that

(5.1.6) $(\mathcal{N}_{s,n}, \mathcal{G}_{\alpha,a,s,n}, \mathcal{B}_{\alpha,a,s})$ *is a Bergmann triple*

for all $\gamma, \alpha \in \Phi$, $a \in V^\sigma$. Thus $\mathcal{N}_{s,n}, \mathcal{B}_{\alpha,a,s}, \mathcal{X}_{\gamma,\alpha,a,s,n}$ as in (3.2) and (4.4) are all structions. (5.1.3) guarantees $\mathcal{N}$ is a structural pair ($\mathcal{B}_{\alpha,a,s}$ always is by (0.1.6)), so (5.1.6) amounts to saying that $\mathcal{G} = \alpha\widehat{\Delta}_0 + \mathcal{D}_a$ is structural glue.

In practice (see [2]) both Δ_0 and N in the pseudo-principal example can be built from multiplications entirely within the original pair $\mathcal{V}$ (not just $\widetilde{\mathcal{V}}$). We say the domination is **inner** if $\Delta_0, \mathcal{N} \in \mathcal{M}(\mathcal{V})$ are given as multiplications, and is **universal** if $\Delta_0^\tau \in \mathcal{U}^{\tau,\tau}$ and $N^\tau \in \mathcal{U}^{\tau,-\tau}$ are given as universal multiplication operators in $\mathcal{UQE}(\mathcal{V})$ with $\Delta_0^- = -(\Delta_0^+)^*$, $N^- = (N^+)^*$ and (5.1.1-6) holding universally in $\mathcal{V}\langle X\rangle$ (i.e., for *generic* x, not just $x \in V^\tau$). In both cases they act as inner multiplications on $\mathcal{V}$ and $\mathcal{N}_{s,n}, \mathcal{G}_{\alpha,a,s,n}, \mathcal{B}_{\alpha,a,s}$ are all innstructions.

Conditions (5.1.1-2) go a long way towards structural domination.

Injectivity Theorem 5.2 *If the operator Q_s is injective on $\mathcal{V}$, then the structural conditions* (5.1.1 − 2) *alone guarantee that s structurally dominates n on the subpair* $(V^\sigma, Q_s V^{-\sigma})$. *Indeed* (5.1.5) *always holds, the structural conditions* (5.1.3 − 4) *and gluing condition* (5.1.6) *always hold on the subpair for $\tau = -\sigma$, and* (5.1.3−4), (5.1.6) *hold hold on the subpair for $\tau = \sigma$ if the map Q_s is injective.*

Proof: We know (5.1.5) is always a consequence of (5.1.1-2) by (3.2.4). It remains to verify the conditions that (5.1.3-4) and gluing (5.1.6), i.e., (4.3.1-2), vanish as maps on V^σ when $\tau = -\sigma$, and as maps on $Q_s V^\sigma$ when $\tau = \sigma$ *and we can cancel* Q_s. Now it is a general fact that whenever T_1, T_2, G result by cancelling Q_s as in (4.5),

(5.2.1) $$Q_s T_i^\sigma = T_i^{-\sigma} Q_s = Q_{t_i}, \quad Q_s G^\sigma = G^{-\sigma} Q_s = Q_{t_1,t_2},$$

that (5.1.3), (4.3.1-2) hold as stated, so (T_1, G, T_2) is a Bergmann triple *on the subpair* $(V^\sigma, Q_s V^\sigma)$ [note that the inner ideal $Q_s V^\sigma$ is invariant under $T = T_i^{-\sigma}, G^{-\sigma}$ since $T^{-\sigma}(Q_s V^\sigma) = Q_s(T^\sigma(V^\sigma)) \subseteq Q_s V^\sigma$], and that in the particular case $T_1 := N, t_1 := n$ [as in (3.1)], $T_2 := B, t_s := \alpha s + Q_s a = y$ [as in (3.2.6)], $G = \alpha\widehat{\Delta_0} + D_a$ [as in (3.2.5)] we also have (5.1.4).

To include G and $\widehat{\Delta_0}$ in the notation, we agree $Q_{T_i} := Q_{t_i}, Q_I := Q_s, Q_G := Q_{t_1,t_2}, Q_{\widehat{\Delta_0}} := Q_{s,n}$ (not quite true Q_x-operators) and note that for $T, T' \in \{T_1, T_2, G, \widehat{\Delta_0}, I\}$ we have that the maps $Q_{T(x)},\ Q_{T(x),\,T'(x)}, T Q_x T'$ satisfy

$$
(5.2.2)\qquad
\begin{gathered}
Q_{T^{-\sigma}(Q_s b)} = Q_{Q_T(b)} = Q_{Q_s T^\sigma(b)} = Q_s Q_{T^\sigma(b)} Q_s,\\
T^{-\sigma} Q_{Q_s b} T'^{\sigma} = T^\sigma Q_s Q_b Q_s T'^{\sigma} = Q_T Q_b Q_{T'},\\
Q_{T^{-\sigma}(Q_s b), T'^{\sigma}(Q_s b)} = Q_{Q_T b, Q_{T'} b}\\
= Q_{Q_s(T^\sigma(b)), Q_s(T'^{\sigma}(b))} = Q_s Q_{T^\sigma(b), T'^{\sigma}(b)} Q_s.
\end{gathered}
$$

First, structionality (5.1.3) of T holds whenever T results by cancelling Q_s from Q_t: setting $F^\tau(x) := Q_{T^\tau(x)} - T^\tau Q_x T^{-\tau}$ we have by (2) that $F^{-\sigma}(Q_s b) = Q_s F^\sigma(b) Q_s = Q_{Q_t b} - Q_t Q_b Q_t = 0$ by (JP3), which shows $F^{-\sigma}(x) = 0$ as map on V^σ when $x = Q_s b \in Q_s V^\sigma$, and $Q_s F^\sigma(x) = 0$ as map on $Q_s V^\sigma$ when $x = b \in V^\sigma$, so that *if we can cancel* Q_s then $F^\sigma(x) = 0$ on $Q_s V^\sigma$.

Similarly, Lie structionality of $G_0 = \widehat{\Delta_0}$ holds when it results from cancelling Q_s from $Q_{s,n}$ as in (5.1.4) since by (2) (with $T = \widehat{dz}, T' = I$) the map $F^\tau(x) := \Delta_0^\tau Q_x - Q_{dz^\tau(x),x} - Q_x dz^{-\tau}$ has $F^{-\sigma}(Q_s b) = -Q_s F^\sigma(b) Q_s = Q_{s,n} Q_b Q_s - Q_{Q_{s,n}(b), Q_s(b)} + Q_s Q_b Q_{s,n} = 0$ by linearized (JP3) [beware the minus sign, since $\widehat{\Delta_0} = (-\Delta_0^\sigma, \Delta_0^{-\sigma})$] Thus again $F^{-\sigma}(x) = 0$ on V^σ for $x = Q_s b \in Q_s V^\sigma$ and $Q_s F^\sigma(b) Q_s = 0$ implies $F(b) = 0$ on $Q_s V^\sigma$ as long as we can cancel Q_s.

The gluing condition (4.3.1-2) can similarly be formulated in terms of $F_i^{1,\tau}(x) := T_i^\tau Q_x G^{-\tau} + G^\tau Q_x T_i^{-\sigma} - Q_{T^\tau(x), G^\tau(x)}$, $F^{2,\tau}(x) := T_1 Q_x T_2^{-\tau} + T_2^\tau Q_x T_1^{-\sigma} + G^\tau Q_x G^{-\tau} - Q_{G^\tau(x)} - Q_{T_1^\tau(x), T_2^\tau(x)}$, where (2) again guarantees that $F^{-\sigma}(Q_s b) = Q_s F^\sigma(b) Q_s = 0$ from $Q_{t_i} Q_b Q_{t_1,t_2} + Q_{t_1,t_2} Q_b Q_{t_i} - Q_{Q_{t_i}(b), Q_{t_1,t_2}(b)} = 0$ [for $F^{1,\tau}$] and $Q_{t_1} Q_b Q_{t_2} + Q_{t_2} Q_b Q_{t_1} + Q_{t_1,t_2} Q_b Q_{t_1,t_2} - Q_{Q_{t_1,t_2}(b)} - Q_{Q_{t_1}(b), Q_{t_2}(b)} = 0$ [for $F^{2,\tau}$] by linearized (JP3). Thus for the third time $F^{-\sigma}(Q_s V^\sigma) = 0$ on V^σ and $F(V^\sigma) = 0$ on $Q_s V^\sigma$ as long as we can cancel Q_s. ■

We will spend the rest of the paper finding conditions (suitable for application to fractions) that guarantee (N, G, B) is a Bergmann triple on the entire pair $(V^\sigma, V^{-\sigma})$. Besides structionality (5.1.3-4) we still need glue (5.1.6). This modest proposal about glue translates, by (4.3.1-2), into 18 conditions[9] on Δ_0 and N, which we will group according to the parity $\pm\sigma$, the power α^k of the scalar α, and (in 4.3.1) the struction N or B. For (4.3.1) we first demand that the following **N-Gluing Conditions** relating $\mathcal{N}$ to the glue $\mathcal{G}$ hold for all $b, a \in V^\sigma, x \in V^{-\sigma}$:

$(1_N^{\sigma,0})\qquad N^\sigma Q_b\big(\Delta_0^{-\sigma} D_{s,a} - D_{n,a}\big) - \big(D_{a,s}\Delta_0^\sigma + D_{a,n}\big) Q_b N^{-\sigma} + Q_{N^\sigma(b), [D_{a,s}\Delta_0^\sigma + D_{a,n}](b)} = 0,$

$(1_N^{-\sigma,0})\qquad N^{-\sigma} Q_x\big(D_{a,s}\Delta_0^\sigma + D_{a,n}\big) - \big(\Delta_0^{-\sigma} D_{s,a} - D_{n,a}\big) Q_x N^\sigma + Q_{N^{-\sigma}(x), [\Delta_0^{-\sigma} D_{s,a} - D_{n,a}](x)} = 0,$

$(1_N^{\sigma,1})\qquad N^\sigma Q_b \Delta_0^{-\sigma} - \Delta_0^\sigma Q_b N^{-\sigma} + Q_{N^\sigma(b), \Delta_0^\sigma(b)} = 0,$

$(1_N^{-\sigma,1})\qquad N^{-\sigma} Q_x \Delta_0^\sigma - \Delta_0^{-\sigma} Q_x N^\sigma + Q_{N^{-\sigma}(x), \Delta_0^{-\sigma}(x)} = 0.$

Next we require that the **B-Gluing Conditions** relating $\mathcal{B}$ to the glue $\mathcal{G}$ hold for all $b, a \in V^\sigma, x \in V^{-\sigma}$:

[9] The reader may well be thinking of the scene in *Independence Day* when the monster is cut loose from its spacesuit.

$(1_B^{\sigma,0})$ $\quad Q_aQ_sQ_b\big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big) - \big(D_{a,s}\Delta_0^{-\sigma} + D_{a,n}\big)Q_bQ_sQ_a,$
$\qquad +Q_{Q_aQ_sb,[D_{a,s}\Delta_0^{\sigma}+D_{a,n}](b)} = 0,$

$(1_B^{-\sigma,0})$ $\quad Q_sQ_aQ_x\big(D_{a,s}\Delta_0^{\sigma}{+}D_{a,n}\big) - \big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big)Q_xQ_aQ_s$
$\qquad +Q_{Q_sQ_ax,[\Delta_0^{-\sigma}D_{s,a}-D_{n,a}](x)} = 0,$

$(1_B^{\sigma,1})$ $\quad D_{a,s}Q_b\big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big) - \big(D_{a,s}\Delta_0^{\sigma}{+}D_{a,n}\big)Q_bD_{s,a} + Q_aQ_sQ_b\Delta_0^{-\sigma} - \Delta_0^{\sigma}Q_bQ_sQ_a$
$\qquad +Q_{D_{a,s}b,D_{a,s}\Delta_0^{\sigma}(b)} + Q_{D_{a,s}b,D_{a,n}b} + Q_{Q_aQ_sb,\Delta_0^{\sigma}(b)} = 0,$

$(1_B^{-\sigma,1})$ $\quad D_{s,a}Q_x\big(D_{a,s}\Delta_0^{\sigma}{+}D_{a,n}\big) - \big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big)Q_xD_{a,s} + Q_sQ_aQ_x\Delta_0^{\sigma} - \Delta_0^{-\sigma}Q_xQ_aQ_s$
$\qquad +Q_{D_{s,a}x,\Delta_0^{-\sigma}D_{s,a}(x)} - Q_{D_{s,a}x,D_{n,a}x} + Q_{Q_sQ_ax,\Delta_0^{-\sigma}(x)} = 0,$

$(1_B^{\sigma,2})$ $\quad Q_b\big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big) - \big(D_{a,s}\Delta_0^{\sigma}{+}D_{a,n}\big)Q_b + D_{a,s}Q_b\Delta_0^{-\sigma} - \Delta_0^{\sigma}Q_bD_{s,a}$
$\qquad +Q_{b,[D_{a,s}\Delta_0^{\sigma}+D_{a,n}](b)} + Q_{D_{a,s}b,\Delta_0^{\sigma}(b)} = 0,$

$(1_B^{-\sigma,2})$ $\quad Q_x\big(D_{a,s}\Delta_0^{\sigma}{+}D_{a,n}\big) - \big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big)Q_x + D_{s,a}Q_x\Delta_0^{\sigma} - \Delta_0^{-\sigma}Q_xD_{a,s}$
$\qquad +Q_{x,[\Delta_0^{-\sigma}D_{s,a}-D_{n,a}](x)} + Q_{D_{s,a}x,\Delta_0^{-\sigma}(x)} = 0,$

$(1_B^{\sigma,3})$ $\quad Q_b\Delta_0^{-\sigma} - \Delta_0^{\sigma}Q_b + Q_{\Delta_0^{\sigma}(b),b} = 0,$

$(1_B^{-\sigma,3})$ $\quad Q_x\Delta_0^{\sigma} - \Delta_0^{-\sigma}Q_x + Q_{\Delta_0^{-\sigma}(x),x} = 0.$

Finally, for (4.3.2) we require that the **N-B-Gluing Conditions** relating $\mathcal{B}$ to the glue $\mathcal{G}$ hold for all $b, a \in V^{\sigma}, x \in V^{-\sigma}$:

$(2^{\sigma,0})$ $\quad N^{\sigma}Q_bQ_sQ_a + Q_aQ_sQ_bN^{-\sigma} - \big(D_{a,s}\Delta_0^{\sigma}{+}D_{a,n}\big)Q_b\big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big)$
$\qquad -Q_{[D_{a,s}\Delta_0^{\sigma}+D_{a,n}](b)} - Q_{N^{\sigma}(b),Q_aQ_s(b)} = 0,$

$(2^{-\sigma,0})$ $\quad N^{-\sigma}Q_xQ_aQ_s + Q_sQ_aQ_xN^{\sigma} - \big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big)Q_x\big(D_{a,s}\Delta_0^{\sigma}{+}D_{a,n}\big)$
$\qquad -Q_{[\Delta_0^{-\sigma}D_{s,a}-D_{n,a}](x)} - Q_{N^{-\sigma}(x),Q_sQ_s(x)} = 0,$

$(2^{\sigma,1})$ $\quad N^{\sigma}Q_bD_{s,a} + D_{a,s}Q_bN^{-\sigma} - \big(D_{a,s}\Delta_0^{\sigma}{+}D_{a,n}\big)Q_b\Delta_0^{-\sigma} - \Delta_0^{\sigma}Q_b\big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big)$
$\qquad -Q_{[D_{a,s}\Delta_0^{\sigma}+D_{a,n}](b),\Delta_0^{\sigma}(b)} - Q_{N^{\sigma}(b),D_{a,s}(b)} = 0,$

$(2^{-\sigma,1})$ $\quad N^{-\sigma}Q_xD_{a,s} + D_{s,a}Q_xN^{\sigma} - \big(\Delta_0^{-\sigma}D_{s,a}{-}D_{n,a}\big)Q_x\Delta_0^{\sigma} - \Delta_0^{-\sigma}Q_x\big(D_{a,s}\Delta_0^{\sigma}{+}D_{a,n}\big)$
$\qquad -Q_{[\Delta_0^{-\sigma}D_{s,a}-D_{n,a}](x),\Delta_0^{-\sigma}(x)} - Q_{N^{-\sigma}(x),D_{s,a}(x)} = 0,$

$(2^{\sigma,2})$ $\quad N^{\sigma}Q_b + Q_bN^{-\sigma} - \Delta_0^{\sigma}Q_b\Delta_0^{-\sigma} - Q_{\Delta_0^{\sigma}(b)} - Q_{N^{\sigma}(b),b} = 0,$

$(2^{-\sigma,2})$ $\quad N^{-\sigma}Q_x + Q_xN^{\sigma} - \Delta_0^{-\sigma}Q_x\Delta_0^{\sigma} - Q_{\Delta_0^{-\sigma}(x)} - Q_{N^{-\sigma}(x),x} = 0.$

The major goal of our paper is to determine a small number of conditions besides (5.1.1-4) that will guarantee these 18 gluing conditions. The B-Gluing formulas (1_B) hold automatically for any derivation.[10]

Bergmann Glue Proposition 5.3 *The B-Gluing formulas* (1_B) *hold for any derivation* Δ_0 *of* $\mathcal{V}$ *as in* (5.1.4) *connected with* n *by the relation* (5.1.2).

[10]One suspects this follows immediately from properties of the Bergmann operator, but I could only prove it by breaking the operator into its constituent pieces.

PROOF: Formula $(1_B^{\sigma,0})$ follows (omitting superscripts, which are clear by context) from

$$
\begin{aligned}
&Q_aQ_s\big[Q_b\Delta_0-\Delta_0Q_b\big]D_{s,a}+D_{a,s}\big[-\Delta_0Q_b+Q_b\Delta_0\big]Q_sQ_a+Q_a\big[Q_s\Delta_0\big]Q_bD_{s,a}\\
&\quad -D_{a,s}Q_b\big[\Delta_0Q_s\big]Q_a+\big[-Q_aQ_sQ_bD_{n,a}-D_{a,n}Q_bQ_sQ_a+Q_{Q_aQ_s(b),D_{a,n}(b)}\big]\\
&\quad +\big[Q_{Q_aQ_s(b),D_{a,s}\Delta_0(b)}+Q_{Q_aQ_s\Delta_0(b),D_{a,s}(b)}\big]-Q_{Q_a(Q_s\Delta_0)(b),D_{a,s}(b)}\\
&=\big[-Q_aQ_sQ_{\Delta_0(b),b}D_{s,a}-D_{a,s}Q_{\Delta_0(b),b}Q_sQ_a+Q_{Q_aQ_s(b),D_{a,s}(\Delta_0(b))}+Q_{Q_aQ_s(\Delta_0(b)),D_{a,s}(b)}\big]\\
&\quad -\big[Q_aQ_sQ_bD_{n,a}+Q_aQ_{s,n}Q_bD_{s,a}+D_{a,n}Q_bQ_sQ_a+D_{a,s}Q_bQ_{s,n}Q_a-Q_{Q_aQ_s(b),D_{a,n}(b)}\\
&\qquad -Q_{Q_aQ_{s,n}(b),D_{a,s}(b)}\big], \qquad \text{[by (5.1.4), (5.1.2)]}
\end{aligned}
$$

which vanishes by linearizations $b\to\Delta_0(b),b$ and $s\to s,n$ of (0.1.5).

Formula $(1_B^{-\sigma,0})$ follows dually (though not by a dual proof, since the formulas (1_B) are all symmetric under reversal; because of asymmetry $(s,n\in V^{-\sigma})$ the "dual" proofs are really "inside-out"). We compute

$$
\begin{aligned}
&\big[Q_sQ_aQ_xD_{a,s}\big]\Delta_0-\Delta_0\big[D_{s,a}Q_xQ_aQ_s\big]+\big[Q_{Q_sQ_a(x),\Delta_0(D_{s,a}(x))}+Q_{\Delta_0(Q_sQ_a(x)),D_{s,a}(x)}\big]\\
&\quad -Q_{(\Delta_0Q_s)Q_a(x),D_{s,a}(x)}+\big[Q_sQ_aQ_xD_{a,n}+D_{n,a}Q_xQ_aQ_s-Q_{Q_sQ_a(x),D_{n,a}(x)}\big]\\
&=\big[Q_sQ_aQ_xD_{a,s}-Q_{Q_sQ_a(x),D_{s,a}(x)}\big]\Delta_0+\Delta_0\big[-D_{s,a}Q_xQ_aQ_s+Q_{Q_sQ_a(x),D_{s,a}(x)}\big]\\
&\quad -Q_{(Q_{s,n})Q_a(x),D_{s,a}(x)}-\big[Q_{s,n}Q_aQ_xD_{a,s}+D_{s,a}Q_xQ_aQ_{s,n}-Q_{Q_{s,n}Q_a(x),D_{s,a}(x)}\big]\\
&\qquad\qquad \text{[by (5.1.4), (0.1.3), and linearization } s\to s,n \text{ in (0.1.4)]}\\
&=-\big[D_{s,a}Q_xQ_a(Q_s\Delta_0)\big]+\big[(\Delta_0Q_s)Q_aQ_xD_{a,s}\big]-\big[D_{s,a}Q_xQ_aQ_{s,n}+Q_{s,n}Q_aQ_xD_{a,s}\big]
\end{aligned}
$$

[by (0.1.5)], which vanishes by (5.1.2).

The formula $(1_B^{\sigma,1})$ is

$$
\begin{aligned}
&D_{a,s}\big[Q_b\Delta_0-\Delta_0Q_b\big]D_{s,a}+Q_aQ_s\big[Q_b\Delta_0-\Delta_0Q_b\big]+\big[Q_b\Delta_0-\Delta_0Q_b\big]Q_sQ_a+Q_a\big[Q_s\Delta_0\big]Q_b\\
&\quad -Q_b\big[\Delta_0Q_s\big]Q_a+Q_{D_{a,s}(b),D_{a,s}(\Delta_0(b))}+\big[Q_{Q_aQ_s(b),\Delta_0(b)}+Q_{Q_aQ_s(\Delta_0(b)),b}\big]-Q_{Q_a(Q_s\Delta_0)(b),b}\\
&\quad +\big[-D_{a,s}Q_bD_{n,a}-D_{a,n}Q_bD_{s,a}+Q_{D_{a,s}(b),D_{a,n}(b)}\big]\\
&=\Big[D_{a,s}Q_{\Delta_0(b),b}D_{a,s}-Q_aQ_sQ_{\Delta_0(b),b}-Q_{\Delta_0(b),b}Q_sQ_a\\
&\quad +Q_{D_{a,s}(b),D_{a,s}(\Delta_0(b))}+Q_{Q_aQ_s(b),\Delta_0(b)}+Q_{Q_aQ_s(\Delta_0(b)),b}\Big]\\
&\quad +\Big[-Q_aQ_{n,s}Q_b-Q_bQ_{n,s}Q_a-D_{a,s}Q_bD_{n,a}-D_{a,n}Q_bD_{s,a}+Q_{Q_aQ_{n,s}(b),b}+Q_{D_{a,s}(b),D_{a,n}(b)}\Big]
\end{aligned}
$$

[by (5.1.4), (5.1.2), (0.1.1)], which vanishes by the linearizations $b \to \Delta_0(b), b$ and $s \to s, n$ of (0.1.4).

Dually, the formula $(1_B^{-\sigma,1})$ becomes

$$\begin{aligned}
&\big[D_{s,a}Q_xD_{a,s}{+}Q_sQ_aQ_x\big]\Delta_0 - \Delta_0\big[D_{s,a}Q_xD_{a,s}{+}Q_xQ_aQ_s\big] + Q_{D_{s,a}x,\Delta_0(D_{s,a}x)}\\
&\qquad+\big[D_{s,a}Q_xD_{a,n}{+}D_{n,a}Q_xD_{a,s}{-}Q_{D_{s,a}(x),D_{n,a}(x)}\big] + Q_{D_{s,a}x,\Delta_0(D_{s,a}x)} + Q_{Q_sQ_ax,\Delta_0(x)}\\
&= [D_{s,a}Q_xD_{a,s}{+}Q_sQ_aQ_x{-}Q_{D_{s,a}x}{-}Q_{Q_sQ_ax,x}\big]\Delta_0\\
&\qquad-\Delta_0\big[D_{s,a}Q_xD_{a,s}{+}Q_xQ_aQ_s{-}Q_{D_{s,a}x}{-}Q_{Q_sQ_ax,x}\big]\\
&\qquad+\big[D_{s,a}Q_xD_{a,n}{+}D_{n,a}Q_xD_{a,s}{-}Q_{D_{s,a}(x),D_{n,a}(x)}\big] - Q_{\Delta_0(Q_sQ_ax),x} \qquad \text{[by (5.1.4) twice]}\\
&= \big[-Q_xQ_aQ_s\big]\Delta_0{-}\Delta_0\big[-Q_sQ_aQ_x\big]{+}\big[-Q_xQ_aQ_{s,n}{-}Q_{s,n}Q_aQ_x{+}Q_{Q_{s,n}Q_ax,x}\big]{-}Q_{(\Delta_0Q_s)Q_ax,x}
\end{aligned}$$

[by (0.1.4) twice and its linearization $s \to s, n$], which vanishes by (5.1.2).

The formula $(1_B^{\sigma,2})$ becomes $\big[Q_b\Delta_0{-}\Delta_0Q_b\big]D_{s,a}{+}D_{a,s}\big[Q_b\Delta_0{-}\Delta_0Q_b\big]{+}Q_{b,D_{a,s}(\Delta_0(b))}$ $+\big[-Q_bD_{n,a}{-}D_{a,n}Q_b{+}Q_{b,D_{a,n}b}\big]{+}Q_{\Delta_0(b),D_{a,s}(b)} = Q_{\Delta_0(b),b}D_{s,a}{+}D_{a,s}Q_{\Delta_0(b),b}{+}Q_{D_{a,s}(\Delta_0(b)),b}$ $+Q_{D_{a,s}(b),\Delta_0(b)}$ [by (5.1.4), (0.1.1)], which vanishes by the linearization $b \to b, \Delta_0(b)$ of (5.1.4).

Dually, $(1_B^{-\sigma,2})$ become $\big[Q_xD_{a,s}{+}D_{s,a}Q_x\big]\Delta_0{-}\Delta_0\big[D_{s,a}Q_x{+}Q_xD_{a,s}\big] + Q_{x,\Delta_0(D_{s,a}(x))}+$ $Q_{D_{s,a}(x),\Delta_0(x)} + \big[Q_xD_{a,n}{+}D_{n,a}Q_x{-}Q_{x,D_{n,a}x}\big] = Q_{D_{s,a}(x),x}\Delta_0{-}\Delta_0Q_{D_{s,a}(x),x}{+}Q_{\Delta_0(D_{s,a}x),x}-$ $Q_{D_{s,a}x,\Delta_0(x)}$ [by (0.1.1)], which vanishes by the linearization $x \to x, D_{s,a}x$ of (5.1.4).

Note that the final conditions $(1_B^{\pm\sigma,3})$ are just the conditions (5.1.4) that Δ_0 is a derivation of $\mathcal{V}$. ■

6 The Main Theorem

Our main result is that the 18 Gluing Conditions (5.1) which guarantee that $\mathcal{N}, \mathcal{X}$ are structions can be reduced to a small number of connections between N and Δ_0.

Dominance Theorem 6.1 *The Gluing Conditions* $(1_N), (1_B), (2)$ *of* (5.1) *will follow from the Dominance Conditions* $(1_N^{\pm\sigma,1}), (2^{\pm\sigma,2})$ *and* (5.1.4-4) *on* s, n *if we assume that the following additional conditions hold for elements* $w_1, w_2 \in V^\sigma$ *with* $z_i := Q_sw_i$ *and all* $a \in \widetilde{V}^\sigma$:

(6.1.1) $\quad \Delta_0^\tau\Delta_0^\tau = 2N^\tau + W_1^\tau \qquad (W_1^\sigma := D_{w_1,s}, W_1^{-\sigma} := D_{s,w_1})$,
(6.1.2) $\quad D_{a,s}N^\sigma + D_{a,n}\Delta_0^\sigma + D_{a,z_1} = 0.$
(6.1.2)* $\quad N^{-\sigma}D_{s,a} - \Delta_0^{-\sigma}D_{n,a} + D_{z_1,a} = 0.$
(6.1.3) $\quad D_{a,n}^\sigma N^\sigma + D_{a,z_1}^\sigma\Delta_0^\sigma + D_{a,z_2}^\sigma = 0,$
(6.1.3)* $\quad N^{-\sigma}D_{n,a}^{-\sigma} - \Delta_0^{-\sigma}D_{z_1,a}^{-\sigma} + D_{z_2,a}^{-\sigma} = 0,$

which implies the further condition

(6.1.4) $\quad \Delta_0 Q_{s,n} = -Q_{s,n}\Delta_0 = 2Q_n + Q_{z_1,s}$.

If the multiplications are regarded as universal in $\mathcal{UQE}(\mathcal{V})$, *rather than as maps, then the reversal* X^* *follows automatically from* X, *and we can omit* (6.1.2-3)*.

Proof: First note that (6.1.1) implies (6.1.4): $\Delta_0^{-\sigma} Q_{s,n} = \Delta_0^{-\sigma}(\Delta_0^{-\sigma} Q_s)$ [by (5.1.2)] $= (2N^{-\sigma} + D_{s,w_1})Q_s$ [by (1)] $= 2Q_n + Q_{Q_s w_1,s}$ [by (5.1.1), (JP1)] $= 2Q_n + Q_{z_1,s}$, and dually $-Q_{s,n}\Delta_0^{\sigma} = Q_s \Delta_0^{\sigma}\Delta_0^{\sigma} = Q_s(2N^{\sigma} + D_{w_1,s}) = 2Q_n + Q_{Q_s w_1,s} = 2Q_n + Q_{z_1,s}$.

To help the reader through the labyrinth of verifications of the Gluing Formulas, we indicate the migration of terms via superscripts; a superscript $\blacktriangle, \blacktriangledown, \bullet, \blacklozenge$ denotes a term which about to die, cancelled out by its evil twin. We also create terms and their anti-terms $*$ and $**$ at will.

We must show that the conditions $(1_N^{\pm\sigma,0})$ follow from the above Domnance Conditions. The relation $(1_N^{\sigma,0})$ follows from $(1_N^{\sigma,1}), (2^{\sigma,2}), (5.1.4), (6.1.2), (6.1.3)$ since it reduces to

$$
\begin{aligned}
&NQ_b\big[\Delta_0 D_{s,a} - D_{n,a}\big] - \big[D_{a,s}\Delta_0 + D_{a,n}\big]Q_b N + Q_{N(b),D_{a,s}\Delta_0(b)} - Q_{N(b),D_{a,n}(b)} \\
&= \big(NQ_b\Delta_0\big)^{(1)} D_{s,a} - \big(NQ_b\big)^{(2)} D_{n,a} - D_{a,s}\big(\Delta_0 Q_b N\big)^{(3)} - D_{a,n}\big(Q_b N\big)^{(4)} \\
&\qquad + Q^{(5)}_{N(b),D_{a,s}(\Delta_0(b))} + Q^{(6)}_{N(b),D_{a,n}(b)} \\
&= \big[\Delta_0 Q_b^{(1a)} N - Q^{(1b)\blacktriangle}_{N(b),\Delta_0(b)}\big] D_{s,a} - \big[-Q_b N^{(2a)} + \Delta_0 Q_b^{(2b)}\Delta_0 + Q^{(2c)}_{\Delta_0(b)} + Q^{(2d)\blacktriangledown}_{N(b),b}\big] \\
&\qquad - D_{a,s}\big[NQ_b^{(3a)} + Q^{(3b)\blacklozenge}_{N(b),\Delta_0(b)}\big] - D_{a,n}\big[-NQ_b^{(4a)} + \Delta_0 Q_b^{(4b)}\Delta_0 + Q^{(4c)}_{\Delta_0(b)} + Q^{(4d)\bullet}_{N(b),b}\big] \\
&\qquad + \big[D_{a,s} Q^{(5a)\blacklozenge)}_{N(b)\Delta_0(b)} + Q^{(5b)\blacktriangle}_{N(b),\Delta_0(b)} D_{s,a} - Q^{(5c)}_{[D_{a,s}N](b),\Delta_0(b)}\big] \\
&\qquad + \big[-Q^{(6a)}_{[D_{a,n}N](b),b} + D_{a,n} Q^{(6b)\bullet}_{N(b),b} + Q^{(6c)\blacktriangledown}_{N(b),b} D_{n,a}\big] \\
&\qquad\qquad\qquad \text{[by } (1_N^{\sigma,1}) \text{ for (1),(3), } (2^{\sigma,2}) \text{ for (2),(4), and linearized (0.1.1) for (5),(6)]} \\
&= \Delta_0 Q_b\big[\Delta_0 D^{(1a1)\blacktriangle}_{n,a} - D^{(1a2)}_{z_1,a}\big] + Q_b\big[\Delta_0 D^{(2a1)}_{z_1,a} - D^{(2a2)}_{z_2,a}\big] - \Delta_0 Q_b^{(2b)\blacktriangle}\Delta_0 D_{n,a} \\
&\qquad + \big[D^{(3a1)\blacktriangledown)}_{a,n}\Delta_0 + D^{(3a2)}_{a,z_1}\big] Q_b\Delta_0 - \big[Q_{\Delta_0(b)} D^{(2c)\blacklozenge}_{n,a} + D_{a,n} Q^{(4c)\bullet}_{\Delta_0(b)}\big] \\
&\qquad - \big[D_{a,z_1}\Delta_0 + D_{a,z_2}\big] Q_b - D_{a,n}\Delta_0 Q_b^{(4b)\blacktriangledown}\Delta_0 \\
&\qquad + \big[Q^{(5c1)\blacklozenge}_{D_{a,n}\Delta_0(b),\Delta_0(b)} + Q^{(5c2)}_{D_{a,z_1}(b),\Delta_0(b)}\big] = \big[Q^{(6a1)}_{D_{a,z_1}\Delta_0(b),b} + Q^{(6a2)}_{D_{a,z_2}(b),b}\big]
\end{aligned}
$$

[by (6.1.2)* for (1a), (6.1.2) for (3a),(5c), (6.1.3)* for (2a), (6.1.3) for (4a),(6a)]

$$
\begin{aligned}
&= \big[-\Delta_0 Q_b^{(1a2)} + Q_b^{(2a1)}\Delta_0\big] D_{z_1,a} + \big[-Q_b D^{(2a2)\blacklozenge)}_{z_2,a} - D^{(4a2)\blacklozenge}_{a,z_2} Q_b + Q^{(6a2)\blacklozenge}_{D_{a,z_2}(b),b}\big] \\
&\qquad + D_{a,z_1}\big[-\Delta_0 Q_b^{(4a1)} + Q_b^{(3a2)}\Delta_0\big] + Q^{(6a1)}_{D_{a,z_1}(\Delta_0(b)),b} + Q^{(5b2)}_{D_{a,z_1}(b),\Delta_0(b)} \\
&= -Q_{\Delta_0(b),b} D_{z_1,a} - D_{a,z_1} Q_{\Delta_0(b),b} + Q_{D_{a,z_1}(\Delta_0(b)),b} + Q_{D_{a,z_1}(b),\Delta_0(b)}
\end{aligned}
$$

[by (5.1.4)], which vanishes by linearized (0.1.1)].

The formula $(1_N^{-\sigma,0})$ follows dually from $(1_N^{-\sigma,1}), (2^{-\sigma,2})$, (5.1.4), (6.1.2-3), (6.1.2-3)* :

$$NQ_xD^{(1)}_{a,s}\Delta_0+NQ_xDa,n^{(2)}-\Delta_0D_{s,a}Q^{(3)}_xN+D_{n,a}Q^{(4)}_xN+Q^{(5)}_{N(x),\Delta_0D_{s,a}(x)}-Q^{(6)}_{N(x),D^{(1)}_{n,a}(x)}$$
$$=\big[NQ^{(1a)}_{\{s,a,x\},x}\Delta_0-\Delta_0Q^{(3a)}_{\{s,a,x\},x}N\big]^{(7)}+\big[NQ^{(2a)}_{\{s,a,x\},x}+Q^{(4a)}_{\{s,a,x\},x}N\big]^{(8)}$$
$$-\big(ND_{s,a}\big)^{(1b)}Q_x\Delta_0+\Delta_0Q_x\big(D_{a,s}N\big)^{(3b)}-\big(ND_{n,a}\big)^{(2b)}Q_x-Q_x\big(D_{a,n}N\big)^{(4b)}$$
$$+Q^{(5)}_{N(x),\Delta_0(\{s,a,x\})}-Q^{(6)}_{N(x),D_{n,a}(x)}\qquad\text{[by (0.1.1) on (1),(2),(3),(4)]}$$
$$=\big[-Q^{(7a)}_{N(\{s,a,x\}),\Delta_0(x)}-Q^{(7b)\blacktriangle}_{N(x),\Delta_0(\{s,a,x\})}\big]$$
$$+\big[\Delta_0Q^{(8a)}_{\{n,a,x\},x}\Delta_0+Q^{(8b)}_{\Delta_0\{n,a,x\},\Delta_0(x)}+Q^{(8c)}_{N(\{n,a,x\}),x}+Q^{(8d)\blacktriangle}_{N(x),\{n,a,x\}}\big]^{(8)}$$
$$-\big[\Delta_0D^{(1b1)}_{n,a}-D^{(1b2)}_{z_1,a}\big]Q_x\Delta_0-\Delta_0Q_x\big[D_{a,n}\Delta^{(3b1)}_0+D^{(3b2)}_{a,z_1}\big]-\big[\Delta_0D^{(2b1)}_{z_1,a}-D^{(2b2)}_{z_2,a}\big]Q_x$$
$$+Q_x\big[D^{(4b1)}_{a,z_1}+D^{(4b2)}_{a,z_2}\big]+Q^{(5)\blacktriangle}_{N(x),\Delta_0(\{s,a,x\})}-Q^{(6)\blacktriangledown}_{N)x),\{n,a,x\}}$$

[by $(1_N^{-\sigma,1})$ on (7), $(2^{-\sigma,2})$ on (8), (6.1.2)* on (1b), (6.1.3)* on (2b), (6.1.2) on (3b), (6.1.3) on (4b)]

$$=Q^{(7a8b)}_{\big[-ND_{s,a}+\Delta_0D_{n,a}\big](x),\Delta_0(x)}+Q^{(8c)}_{ND_{n,a}(x),x}+\Delta_0\big[-D_{na,}Q^{(1b1)}_x-Q_xD^{(3b1)}_{a,n}+Q^{(8a)}_{\{n,a,x\},x}\big]^{(9)}\Delta_0$$
$$-\Delta_0\big[D_{z_1,a}Q^{(2b1)}_x+Q_xD^{(3b2)}_{z_1,a}\big]^{(10)}+\big[Q_xD^{(4b1)}_{a,z_1}+D_{z_1,a}Q^{(1b2)}_x\big]^{(11)}$$
$$\Delta_0\big[D_{z_2,a}Q^{(2b2)}_x+Q_xD^{(4b2)}_{a,z_2}\big]^{(12)}$$
$$=Q^{(7a8b)\blacktriangle}_{D_{z_1,a}(x),\Delta_0(x)}+\big[Q^{(8c1)\blacktriangle}_{\Delta_0D_{z_1,a}(x),x}-Q^{(8c2)\blacktriangledown}_{D_{a,z_2}(x),x}$$
$$-\Delta_0\big[Q_{\{z_1,a,x\},x}\big]^{(10)\blacktriangle}+\big[Q_{\{z_1,a,x\},x}\big]^{(11)\blacktriangle}\Delta_0+Q^{(12)\blacktriangledown}_{\{z_2,,x\},x}$$

[by (6.1.2)* on (7a8b), (6.1.3)* on (8c), and (0.1.1) on (9),(10),(11),(12)], which vanishes by linearized (5.1.4).

The formula $(2^{\sigma,0})$ follows from $(2^{\sigma,2})$, (5.1.1), (5.1.2), (5.1.4), (6.1.2)(6.1.4) via

$$NQ_bQ_sQ^{(1)}_a+Q_aQ_sQ_bN^{(2)}-D_{a,s}\big[\Delta_0Q_b\Delta_0\big]^{(3)}D_{s,a}-Q^{(4)}_{D_{a,s}(\Delta_0(b))}$$
$$+D_{a,s}\big[\Delta_0Q_b\big]^{(5)}D_{n,a}-D_{a,n}\big[Q_b\Delta_0\big]^{(6)}D_{s,a}-Q^{(7)}_{D_{a,s}\Delta_0(b),D_{a,n}(b)}$$
$$+\big[D_{a,n}Q_bD_{n,a}-Q_{D_{a,n}(b)}\big]^{(8)}-Q^{(9)}_{N(b),Q_aQ_s(b)}$$
$$=NQ_bQ_sQ^{(1)}_a+Q_aQ_sQ_bN^{(2)}+D_{a,s}\big[-NQ^{(3a)}_b-Q_bN^{(3b)}+Q^{(3c)\blacktriangle}_{\Delta_0(b)}+Q^{(3d)}_{N(b),b}\big]D_{s,a}$$
$$+\big[Q^{(4a)}_{Q_aQ_s\Delta_0(b),\Delta_0(b)}-Q_aQ_sQ^{(4b)}_{\Delta_0(b)}-Q_{\Delta_0(b)}Q_sQ^{(4c)}_a-D_{a,s}Q^{(4d)\blacktriangle}_{\Delta_0(b)}D_{s,a}\big]$$
$$+D_{a,s}\big[Q^{(5a)}_{\Delta_0(b),b}+Q_b\Delta^{(5b)}_0\big]D_{n,a}-D_{a,n}\big[-Q^{(6a)}_{\Delta_0(b),b}+\Delta_0Q^{(6b)}_b\big]D_{s,a}$$
$$-\big[Q^{(7)}_{D_{a,s}\Delta_0(b),D_{a,n}(b)}+Q^{(7*)}_{D_{a,n}\Delta_0(b),D_{a,s}(b)}-Q^{(7**)}_{(D_{a,n}\Delta_0)(b),D_{a,s}(b)}\big]$$
$$+\big[-Q_aQ_nQ^{(8a)}_b-Q_bQ_nQ^{(8b)}_a+Q^{(8c)}_{Q_aQ_n(b),b}\big]+\big[-Q^{(9)}_{N(b),Q_aQ_s(b)}-Q^{(9*)}_{b,Q_aQ_sN(b)}+Q^{(9**)}_{b,Q_aQ_n(b)}\big]$$

[by $(2^{\sigma,2})$ for (3); (0.1.4) in (4),(8); (5.1.4) in (5),(6); (5.1.1) in (9*)]

$$
\begin{aligned}
&= \big[NQ_b^{(1)} - Q^{(4c)}_{\Delta_0(b)}\big]Q_sQ_a + Q_aQ_s\big[Q_bN^{(2)} - Q^{(4b)}_{\Delta_0(b)}\big] - \big[\big(D_{a,s}N\big)Q_bD^{(3a)}_{s,a} + D_{a,s}Q_b\big(ND_{s,a}\big)^{(3b)}\big] \\
&\quad + \big[D_{a,s}Q_{\Delta_0(b),b}D^{(5a)}_{n,a} + D_{a,n}Q_{\Delta_0(b),b}D^{(6a)}_{s,a} - Q^{(7)}_{D_{a,s}(\Delta_0(b)),D_{a,n}(b)} - Q^{(10)}_{D_{a,s}(b),D_{a,n}(\Delta_0(b))}\big]^{(11)} \\
&\quad - Q^{(10*)}_{[D_{a,n}\Delta_0](b),D_{a,s}(b)} + \big[-Q_aQ_nQ^{(8a)}_b - Q_bQ_nQ^{(8b)}_a + 2Q^{(8c9*)}_{Q_aQ_n(b),b}\big] \\
&\quad + \big[D_{a,s}Q_{N(b),b}D^{(3d)}_{s,a} - Q^{(9b)}_{Q_aQ_sN(b),b} - Q^{(9a)}_{Q_aQ_s(b),N(b)}\big]^{(12)} + D_{a,s}Q_b\big[\Delta_0D_{n,a}\big]^{(5b)} \\
&\quad + \big[-D_{a,n}\Delta_0\big]^{(6b)}Q_bD_{a,s} - Q^{(7*)}_{D_{a,s}(b),D_{a,n}\Delta_0(b)} + Q^{(7**)}_{D_{a,s}(b),(D_{a,n}\Delta_0)(b)} \\
&= \big[-Q_bN^{(1a)} + \Delta_0Q^{(1b)}_b\Delta_0 + Q^{(1c)\blacktriangle}_{N(b),b}\big]Q_sQ_a + Q_aQ_s\big[-NQ^{(2a)}_b + \Delta_0Q^{(2b)}_b\Delta_0 + Q^{(2c)\blacktriangledown}_{N(b),b}\big] \\
&\quad + \big[\big(D_{a,n}\Delta_0^{(3a1)\blacklozenge} + D^{(3a2)}_{a,z_1}\big)Q_bD_{s,a} - D_{a,s}Q_b\big(\Delta_0D^{(3b1)\bullet}_{n,a} - D^{(3b2)}_{z_1,a}\big)\big] + \big[-Q_aQ_{n,s}Q^{(11a)}_{\Delta_0(b),b} \\
&\quad - Q_{\Delta_0(b),b}Q_{n,s}Q^{(11b)}_a + Q^{(11c)}_{Q_aQ_{s,n}\Delta_0(b),b}\big] + \big[-Q_aQ_nQ^{(8a)}_b - Q_bQ_nQ^{(8b)}_a + 2Q^{(8c9*)}_{Q_aQ_nb,b}\big] \\
&\quad + \big[-Q_aQ_sQ^{(12a)\blacktriangledown}_{N(b),b} - Q_{N(b),b}Q_sQ^{(12b)\blacktriangle}_a + Q^{(12c)\blacktriangleleft}_{D_{a,s}(N(b)),D_{a,s}(b)}\big] + D_{a,s}Q_b\big[\Delta_0D^{(5b)\bullet}_{n,a}\big] \\
&\quad - \big[D_{a,n}\Delta_0^{(6b)\blacklozenge}\big]Q_bD_{s,a} - \big[Q^{(10*)\blacktriangleleft}_{D_{a,s}(b),D_{a,s}N(b)} + Q^{(10*a)}_{D_{a,s}(b),D_{a,z_1}(b)}\big]
\end{aligned}
$$

[by $(2^{\sigma,2})$ for (1),(2) and its linearization for (11), (12); (6.1.2) for (3a), (10*); $(6.1.2)^*$ for (3b)]

$$
\begin{aligned}
&= \big[D_{a,z_1}Q_bD^{(3a2)}_{s,a} + D_{a,s}Q_bD^{(3b2)}_{z_1,a} - Q^{(10*a)}_{D_{a,s}(b),D_{a,z_1}(b)}\big]^{(13)} \\
&\quad + \big[\big(\Delta_0Q^{(1b)}_b - Q^{(10b)}_{\Delta_0(b),b}\big)Q_{s,n}Q_a - Q_aQ_{s,n}\big(Q_b\Delta_0^{(2b)} - Q^{(10a)}_{\Delta_0(b),b}\big) + Q^{(10c)}_{Q_aQ_{s,n}\Delta_0(b),b}\big]^{(14)} \\
&\qquad + 2\big[-Q_bQ_nQ^{(1a8b)}_a - Q_aQ_nQ^{(2a8a)}_b + Q^{(8c)}_{Q_aQ_n(b),b}\big]
\end{aligned}
$$

[by (5.1.1) for (1a), (2a) and (5.1.2) for (1b), (2b)]

$$
\begin{aligned}
&= \big[-Q_aQ_{z_1,s}Q_b - Q_bQ_{z_1,s}Q_a + Q_{Q_aQ_{z_1,s}(b),b}\big]^{(13')} + \big[Q_b\big(\Delta_0Q^{(14a)}_{s,n}\big)Q_a - Q_a\big(Q^{(14b)}_{s,n}\Delta_0\big)Q_b \\
&\quad + Q^{(14c)}_{Q_aQ_{s,n}\Delta_0(b),b}\big] + 2\big[-Q_bQ_nQ^{(1a8b)}_a - Q_aQ_nQ^{(2a8a)}_b + Q^{(8c)}_{Q_aQ_n(b),b}\big]
\end{aligned}
$$

[by (5.1.4) for (14), linearized (0.1.4) $s \to s, z_1$ for (13′)]

$$
= Q_a\big[-Q_{z_1,s} - Q_{s,n}\Delta_0 - 2Q_n\big]Q_b + Q_b\big[-Q_{z_1,s} + \Delta_0Q_{s,n} - 2Q_n\big]Q_a + Q_{Q_a[Q_{z_1,s}+Q_{s,n}\Delta_0+2Q_n](b),b}
$$

which vanishes by assumption (6.1.4).

Formula $(2^{-\sigma,0})$ follows dually by an equally tortuous computation: it follows from $(2^{-\sigma,2})$, (5.1.1), (5.1.2), (5.1.4), (6.1.2), (6.1.4) since it reduces to

$$
\begin{aligned}
&\big[NQ_xQ_aQ_s + Q_sQ_aQ_xN\big]^{(1)} - \Delta_0\big[D_{s,a}Q_xD_{a,s}\big]^{(2)}\Delta_0 - Q^{(3)}_{\Delta_0D_{a,s}(x)} \\
&\quad + \big[D_{n,a}Q^{(4)}_xD_{a,s}\Delta_0 - \Delta_0D_{s,a}Q^{(5)}_xD_{a,n} + Q^{(6)}_{\Delta_0D_{s,a}(x),D_{n,a}(x)}\big] + \big[D_{n,a}Q_xD_{a,n} - Q_{D_{n,a}(x)}\big]^{(7)} \\
&\qquad - Q^{(8)}_{N(x),Q_sQ_a(x)} \\
&= \big[NQ_xQ^{(1a)}_aQ_s + Q_sQ^{(1b)}_aQ_xN\big] - \Delta_0\big[-Q_sQ^{(2a)}_aQ_x - Q_xQ^{(2b)}_aQ_s + Q^{(2c)\blacktriangle}_{D_{s,a}(x)} + Q^{(2d)}_{Q_sQ_a(x),x}\big]\Delta_0
\end{aligned}
$$

$$+\big[\Delta_0 Q^{(3a)\blacktriangle}_{D_{s,a}(x)}\Delta_0 + Q^{(3b)}_{ND_{s,a}(x),D_{s,a}(x)} - NQ^{(3c)}_{D_{s,a}(x)} - Q^{(3d)}_{D_{s,a}(x)}N\big]$$
$$+\big[D_{n,a}Q^{(4)}_x D_{a,s} - Q^{(6a)}_{D_{s,a}(x),D_{n,a}(x)}\big]\Delta_0 + \Delta_0\big[-D_{s,a}Q^{(5)}_x D_{a,n} + Q^{(6b)}_{D_{s,a}(x),D_{n,a}(x)}\big]$$
$$-Q^{(6c)}_{D_{s,a}(x),(\Delta_0 D_{n,a})(x)} + \big[-Q_n Q^{(7a)}_a Q_x - Q_x Q^{(7b)}_a Q_n + Q^{(7c)}_{Q_n Q_a(x),x}\big]$$
$$-\big[Q^{(8)}_{N(x),Q_sQ_a(x)} + Q^{(8*)}_{N(Q_sQ_ax),x}\big] + Q^{(8**)}_{Q_nQ_a(x),x}$$

[by $(2^{-\sigma,2})$ on (3); (0.1.4) on (2), (7); (5.1.4) on (6); (5.1.1) on (8**)]

$$= N\big[Q_xQ_aQ^{(1a)}_s - Q^{(3c)}_{D_{s,a}(x)}\big] + \big[Q_sQ^{(1b)}_a Q_x - Q^{(3d)}_{D_{s,a}(x)}\big]N + \big[Q_{s,n}Q^{(2a)}_a Q_x$$
$$+D_{n,a}Q^{(4)}_x D_{a,s} - Q^{(6a)}_{D_{s,a}(x),D_{n,a}(x)}\big]^{(9)}\Delta_0 - \Delta_0\big[Q_xQ^{(2b)}_a Q_{s,n} + D_{s,a}Q^{(5)}_x D_{a,n} - Q^{(6b)}_{D_{s,a}(x),D_{n,a}(x)}\big]^{(10)}$$
$$+\big[-NQ^{(2d1)}_{Q_sQ_ax,x} - Q^{(2d2)}_{Q_sQ_ax,x}N + Q^{(2d3)}_{\Delta_0(Q_sQ_ax),\Delta_0(x)} + Q^{(2d4)\blacktriangle}_{N(Q_sQ_ax),x} + Q^{(2d5)\blacktriangledown}_{N(x),Q_sQ_ax}\big]$$
$$+\big[-Q_nQ^{(7a)}_a Q_x - Q_xQ^{(7b)}_a Q_n + 2Q^{(7c8**)}_{Q_nQ_ax,x}\big] - \big[Q^{(8a)\blacktriangledown}_{N(x),Q_sQ_ax} + Q^{(8*)\blacktriangle}_{N(Q_sQ_ax),x}\big]$$
$$+Q^{(3b6c)}_{[ND_{s,a}-\Delta_0 D_{n,a}](x),D_{s,a}(x)}$$ [by linearized $(2^{-\sigma,2})$ on (2d); (5.1.2) on (2a), (2b)]

$$= N\big[Q_xQ^{(1a)}_a Q_s - Q^{(3c)}_{D_{s,a}(x)} - Q^{(2d1)}_{Q_sQ_ax,x}\big]^{(11)} + \big[Q_sQ^{(1b)}_a Q_x - Q^{(3d)}_{D_{s,a}(x)} - Q^{(2d2)}_{Q_sQ_ax,x}\big]^{(12)}N$$
$$+\big[-Q_xQ^{(9a)}_a Q_{s,n} - D_{s,a}Q^{(9b)}_x D_{a,n} + Q^{(9c)}_{Q_{s,n}Q_ax,x}\big]\Delta_0 - \Delta_0\big[-Q_{s,n}Q^{(10a)}_a Q_x - D_{n,a}Q^{(10b)}_x D_{a,s}$$
$$+Q^{(10c)}_{Q_{s,n}Q_ax,x}\big] + \big[-Q_nQ^{(7a)}_a Q_x - Q_xQ^{(7b)}_a Q_n + 2Q^{(7c8**)}_{Q_nQ_ax,x}\big] + Q^{(2d3)}_{Q_{s,n}Q_ax,\Delta_0(x)} - Q^{(3b6c)}_{D_{z_1,a}(x),D_{s,a}(x)}$$

[by linearized (0.1.4) $s \to s,n$ on (9),(10); (5.1.2) on (2d3); and $(6.1.2)^*$ on (3b6c)]

$$= -N\big[D_{s,a}Q^{(11a)\blacktriangle}_x D_{a,s} + Q_sQ^{(11b)}_a Q_x\big] - \big[D_{s,a}Q^{(12a)\blacktriangledown}_x D_{a,s} + Q_xQ^{(12b)}_a Q_s\big]N + Q_xQ_a\big[2Q^{(9a1)}_n + Q^{(9a2)}_{z_1,s}\big]$$
$$+\big[2Q^{(10a1)}_n + Q^{(10a2)}_{z_1,s}\big]Q_aQ_x + D_{s,a}Q_x\big[D_{a,s}N^{(9b1)\blacktriangledown} + D^{(9b2)}_{a,z_1}\big] + \big[ND^{(10b1)\blacktriangle}_{s,a} + D^{(10b2)}_{z_1,a}\big]Q_xD_{a,s}$$
$$-\big[Q^{(9c10c1)}_{\Delta_0(Q_{s,n}Q_ax),x} + Q^{(9c10c2)\blacklozenge}_{Q_{s,n}Q_ax,\Delta_0(x)}\big] + \big[-Q_nQ^{(7a)}_a Q_x - Q_xQ^{(7b)}_a Q_n + 2Q^{(7c8**)}_{Q_nQ_ax,x}\big]$$
$$+Q^{(2d3)\blacklozenge}_{Q_{s,n}Q_ax,\Delta_0(x)} - Q^{(3a6c)}_{D_{z_1,a}(x),D_{s,a}(x)}$$

[by (0.1.4) on (11),(12); (6.1.4) on (9a),(10a); (5.1.4) on (9c),(10c); (6.1.2) on (9b) and $(6.1.2)^*$ on (10b)]

$$= \big[-NQ^{(11b)}_s + 2Q^{(10a1)}_n - Q^{(7a)}_n\big]^{(13)\blacktriangle}Q_aQ_x + Q_xQ_a\big[-Q_sN^{(12b)} + 2Q^{(9a1)}_n - Q^{(7b)}_n\big]^{(14)\blacktriangledown}$$
$$-Q^{(9c10c1)}_{[2Q_n+Q_{z_1,s}]Q_ax,x} + \big[Q_xQ^{(9a2)}_a Q_{z_1,s} + Q_{z_1,s}Q^{(10a2)}_a Q_x + D_{s,a}Q^{(9b2)}_x D_{a,z_1} + D_{z_1,a}Q^{(10b2)}_x D_{a,s}$$
$$-Q^{(3a6c)}_{D_{z_1,a}(x),D_{s,a}(x)}\big]^{(15)} + 2Q^{(7c8c*)}_{Q_nQ_ax,x}$$ [by (6.1.4) on (9c10c1)]

$$= -Q_{[2Q_n+Q_{z_1,s}]Q_ax,x} + \big[Q_{Q_{z_1,s}Q_ax,x}\big] + 2Q_{Q_nQ_ax,x} = 0$$

[by (5.1.1) on (13), (14), and by linearized (0.1.4) $s \to s, z_1$ on (15)].

Formulas $(2^{\pm\sigma,1})$ are much easier. $(2^{\sigma,1})$ follows from $(2^{\sigma,2})$, (5.1.4), (6.1.2) since it reduces to

$$NQ_bD^{(1)}_{s,a} + D^{(2)}_{a,s}Q_bN - D_{a,s}\Delta_0Q^{(3)}_b\Delta_0 - \Delta_0Q^{(4)}_b\Delta_0D_{s,a} - D_{a,n}Q^{(5)}_b\Delta_0 + \Delta_0Q^{(6)}_bD_{n,a}$$
$$-Q^{(7)}_{D_{a,s}(\Delta_0(b)),\Delta_0(b)} - Q^{(8)}_{D_{a,n}(b),\Delta_0(b)} - Q^{(9)}_{N(b),D_{a,s}(b)}$$

$$
\begin{aligned}
&= \big[NQ_b^{(1)} - \Delta_0 Q_b^{(4)} \Delta_0 - Q_{\Delta_0(b)}^{(7a)}\big]^{(10)} D_{s,a} + D_{a,s}\big[Q_b^{(2)} N - \Delta_0 Q_b^{(3)} \Delta_0 - Q_{\Delta_0(b)}^{(7b)}\big]^{(11)} \\
&\quad -\big[Q_{D_{a,s}(b),N(b)}^{(9)} + Q_{D_{a,s}(N(b)),b}^{(9*)}\big]^{(12)} + Q_{(D_{a,s}N)(b),b}^{(9**)} - D_{a,n}\big[\Delta_0 Q_b^{(5a)} - Q_{\Delta_0(b),b}^{(5b)}\big] \\
&\quad +\big[Q_{\Delta_0 b,b}^{(6a)} + Q_b^{(6b)}\Delta_0\big] D_{n,a} - \big[Q_{D_{a,n}(b),\Delta_0(b)}^{(8)} + Q_{D_{a,n}(\Delta_0(b)),b}^{(8*}\big] + Q_{(D_{a,n}\Delta_0)(b),b}^{(8**)}
\end{aligned}
$$

[by (5.1.4) for (5),(6), (0.1.1) for (7)]

$$
\begin{aligned}
&= \big[-Q_b N^{\blacktriangleright} + Q_{N(b),b}^{\blacktriangle}\big]^{(10')} D_{s,a} + D_{a,s}\big[-NQ_b^{\blacktriangledown} + Q_{N(b),b}^{\blacktriangleleft}\big]^{(11)'} \\
&\quad -\big[D_{a,s}Q_{b,N(b)}^{\blacktriangleleft} + Q_{N(b),b}^{\blacktriangle} D_{s,a}\big]^{(12')} - Q_{D_{a,z_1}(b),b}^{(8**9**)\bullet} + \big[D_{a,s}^{(5a1)\blacktriangledown} N + D_{a,z_1}^{(5a2)\bullet}\big] Q_b + Q_b\big[ND_{s,a}^{(6b1)\blacktriangleright} + D_{z_1,a}^{(6b2)\bullet}\big] \\
&\qquad +\big[D_{a,n}Q_{\Delta_0(b),b}^{(5b)} + Q_{\Delta_0(b),b}^{(6a)} D_{n,a} - Q_{D_{a,n}(b),\Delta_0(b)}^{(8)} - Q_{D_{a,n}(\Delta_0(b)),b}^{(8*)}\big]^{\blacklozenge}
\end{aligned}
$$

[by $(2^{\sigma,2})$ for (10),(11), (0.1.1) for (12), (6.1.2) for (5a), $(6.1.2)^*$ for (6b)], which vanishes by (0.1.1) on $\bullet, \blacklozenge$.

Dually, formula $(2^{-\sigma,1})$ follows from $(2^{-\sigma,2})$, (5.1.4), (6.1.2) since

$$
\begin{aligned}
&NQ_x^{(1)} D_{a,s} + D_{s,a}^{(2)} Q_x N - \Delta_0\big[D_{s,a}Q_x + Q_x D_{a,s}\big]^{(3)}\Delta_0 \\
&\qquad +\big[D_{n,a}Q_x^{(4)}\Delta_0 - \Delta_0 Q_x^{(5)} D_{a,n} + Q_{D_{n,a}(x),\Delta_0(x)}^{(6)}\big] - Q_{\Delta_0 D_{s,a}(x),\Delta_0(x)}^{(7)} - Q_{N(x),D_{s,a}(x)}^{(8)} \\
&= N\big(Q_{D_{s,a}(x),x}^{(1a)\blacktriangledown} - D_{s,a}Q_x^{(1b)}\big) + \big(Q_{D_{s,a}(x),x}^{(2a)\blacktriangledown} - Q_x D_{s,a}^{(2b)}\big)N - \Delta_0\big[Q_{D_{s,a}(x),x}^{(3)\blacktriangledown}\big]\Delta_0 \\
&\quad +\big(Q_{D_{n,a}(x),x}^{(4a)\blacktriangle} - Q_x D_{a,n}^{(4b)}\big)\Delta_0 + \Delta_0\big(D_{n,a}Q_x^{(5a)} - Q_{D_{n,a}(x),x}^{(5b)\blacktriangle}\big) + Q_{D_{n,a}(x),\Delta_0(x)}^{(6)\blacktriangle} \\
&\qquad -\big[Q_{\Delta_0 D_{s,a}(x),\Delta_0(x)}^{(7)\blacktriangledown} + Q_{N(x),D_{s,a}(x)}^{(8)\blacktriangledown} + Q_{ND_{s,a}(x),x}^{(9*)\blacktriangledown}\big] \\
&\qquad + Q_{\Delta_0(D_{n,a}x),x}^{(10*)\blacktriangle} + \big[Q_{-\Delta_0 D_{n,a}(x),x}^{(10**)} + Q_{ND_{s,a}(x),x}^{(9**)}\big]^{(11)} \qquad \text{[by (0.1.1) for (1)-(5)]} \\
&= -\big[ND_{s,a}^{(1b)\bullet}Q_x + Q_x^{(2b)\blacklozenge} D_{s,a}N\big] + Q_x\big[D_{a,s}^{(4b1)\blacklozenge} N + D_{a,z_1}^{(4b2)}\big] + \big[ND_{s,a}^{(5a1)\bullet} + D_{z_1,a}^{(5a2)}\big]Q_x - Q_{D_{z_1,a}(x),x}^{(11')}
\end{aligned}
$$

[by (5.1.4) for $\blacktriangle$, linearized $(2^{-\sigma,2})$ for $\blacktriangledown$, $(6.1.2)^*$ for (5a),(11)], which vanishes by (0.1.1) for $D_{z_1,a}$. This completes the verification of the gluing axioms under the given hypotheses. ■

Remark 6.2 *The formula* (3.3.5) *shows* Δ_0' *is inner, but the formula* (3.3.4) *does not make clear that* N' *is inner* (*though perhaps Innner Multiplication from the Black Lagoon*!) *even if* N *isn't: whenever* $(2^{\sigma,2})$ *holds* (*but not necessarily any of the other structure conditions*)

$$
N'^{\tau} = Q_n Q_c + Q_{z_1,s}Q_c - Q_s Q_{\Delta_0(c),c} + Q_s Q_{\Delta_0(c)} + Q_s Q_{N^{-\tau}(c),c} \in Q_{V^\tau} Q_{V^{-\tau}}
$$

and dually for $N'^{-\tau}$.

Proof: From $(2^{\pm,2})$ we have $Q_s Q_c N^\tau = Q_s\big[Q_c N^\tau + N^{-\tau}Q_c\big] - Q_n Q_c$ [by (5.1.1)] $= Q_s\big[\Delta_0 Q_c \Delta_0 + Q_{\Delta_0(c)} + Q_{N^{-\tau}(c),c}\big] - Q_n Q_c = Q_s\Delta_0\big[\Delta_0 Q_c - Q_{\Delta_0(c),c}\big] + Q_s Q_{\Delta_0(c)} + Q_s Q_{N(c),c} - Q_n Q_c$ [by (5.1.4)] $= \big[2Q_n + Q_{z_1,s}\big]Q_c - Q_s Q_{\Delta_0(c),c} + Q_s Q_{\Delta_0(c)} + Q_s Q_{N^{-\tau}(c),c} - Q_n Q_c$ [by (6.1.4)] $= Q_n Q_c + Q_{z_1,s}Q_c - Q_s Q_{\Delta_0(c),c} + Q_s Q_{\Delta_0(c)} + Q_s Q_{N^{-\tau}(c),c}$.[11] ■

[11] It is not hard to check that if we take $c = w_1$ then with relations such as those in (6.1) (c.f. [[**?**],

In the presence of scalars $\frac{1}{2}$ or $\frac{1}{3}$ certain conditions become redundant.

Proposition 6.3 *If* $\mathcal{N}, \Delta_0$ *satisfy condition* (4.7.9),

$$\text{(M):}\qquad N^\tau \Delta_0^\tau = \Delta_0^\tau N^\tau = M^\tau \quad \big(M^\sigma := -D_{w_1}^\sigma,\ M^{-\sigma} := D_{w_1}^{-\sigma}\big),$$

then $(2^{\sigma,2})$ *implies* $3(1_N^{\sigma,1})$*, so that if* $\frac{1}{3} \in \Phi$ *we can replace condition* $(1_N^{\sigma,1})$ *by condition* (M).

Proof: For generic $\tau = \pm\sigma$ we omit superscripts and compute $3(1_N^{\tau,1})$ as

$$\big[NQ_w\Delta_0 - \Delta_0 Q_w N + Q_{N(w),w}\big] + \big[(2N)Q_w\Delta_0 - \Delta_0 Q_w(2N) + Q_{[2N](w),w}\big]$$

$$= \big[(NQ_w + Q_w N)^{(1)}\Delta_0 - Q_w^{(2)}M\big] - \big[\Delta_0(NQ_w + Q_w N)^{(3)} - MQ_w^{(4)}\big] + Q_{N(w),w}^{(5)}$$
$$+ \big[(\Delta_0\Delta_0^{(6)} - W_1^{(7)})Q_w\Delta_0 - \Delta_0 Q_w(\Delta_0\Delta_0^{(8)} - W_1^{(9)})\big] + \big[Q_{\Delta_0\Delta_0(w),w}^{(10)} - Q_{W_1(w),\Delta_0(w)}^{(11)}\big]$$

[by (M) in (2),(4) and (6.1.1) for $2N$]

$$= \big[\Delta_0 Q_w^{(1a)\blacktriangle}\Delta_0 + Q_{\Delta_0(w)}^{(1b)\bullet} + Q_{N(w),w}^{(1c)\blacklozenge}\big]\Delta_0 - \Delta_0\big[\Delta_0 Q_w^{(3a)\blacktriangledown}\Delta_0 + Q_{\Delta_0(w)}^{(3b)\bullet} + Q_{N(w),w}^{(3c)\blacklozenge}\big]$$
$$+ MQ_w^{(2)} - Q_w M^{(4)} + Q_{N(w),w}^{(5)\blacklozenge} + \big[\Delta_0\Delta_0 Q_w^{(6)\blacktriangledown}\Delta_0 - \Delta_0 Q_w^{(8)\blacktriangle}\Delta_0\Delta_0$$
$$- W_1 Q_w^{(7)}\Delta_0 + \Delta_0 Q_w^{(9)} W_1\big] + \big[Q_{\Delta_0\Delta_0(w),\Delta_0(w)}^{(10)\bullet} - Q_{W_1(w),\Delta_0(w)}^{(11)}\big]$$

[by $(2^{\tau,2})$ on (1), (2)]

$$= -Q_{\Delta_0 N(w),\Delta_0(w)}^{(1c3c)} + MQ_w^{(2)} - Q_w^{(4)}M - W_1\big[\Delta_0 Q_w^{(7a)} - Q_{\Delta_0(w),w}^{(7b)\blacktriangleright}\big] + \big[Q_{\Delta_0(w),w}^{(9a)\blacktriangleright} + Q_w^{(9b)}\Delta_0\big]W_1$$
$$- Q_{W_1(w),\Delta_0(w)}^{(11)\blacktriangleright}$$

[Δ_0 is a derivation on $\bullet$, $\blacklozenge$]

$$= \big[M^\tau - W_1^\tau\Delta_0^\tau\big]Q_w + Q_w\big[\Delta_0^{-\tau}W_1^{-\tau} - M^{-\tau}\big] - Q_{[M - W_1\Delta_0](w),w}.$$

[by (M), W_1 a derivation on $\blacktriangleright$]

At this point we take a break to establish

$$M^\tau - W_1^\tau\Delta_0^\tau = \Delta_{w_1}^\tau, \quad (\Delta_{w_1}^\tau)^* = -\Delta_{w_1}^{-\tau} \qquad (\Delta_{w_1}^\sigma := D_{w_1,n},\ \Delta_{w_1}^{-\sigma} := -D_{n,w_1}).$$

Indeed, for $\tau = \sigma$ we have $\big(D_{w_1,s}\Delta_0^\sigma + D_{w_1,n}\big) - D_{w_1,s}\Delta_0 = D_{w_1,n}$, and for $\tau = -\sigma$ we have $\big(\Delta_0^{-\sigma}D_{s,w_1} - D_{n,w_1}\big) - D_{s,w_1}\Delta_0^{-\sigma} = -D_{n,w_1}$ since $\Delta_0^{-\sigma}$ commutes with $D_{s,w_1} = \Delta_0^{-\sigma}\Delta_0^{-\sigma} - 2N^{-\sigma}$ [by (6.1.1) by the commutativity in condition (M)]. Returning where we left off in the proof, we get

$$= \big[\Delta_{w_1}^\tau\big]Q_w + Q_w\big[-\Delta_{w_1}^{-\tau}\big] - Q_{[\Delta_{w_1}^\tau](w),w}$$
$$= \big[\Delta_{w_1}^\tau\big]Q_w + Q_w\big[\Delta_{w_1}^\tau\big]^* - Q_{[\Delta_{w_1}^\tau](w),w} = 0$$

by (0.1.1) since $\Delta_{w_1}^\tau = D_{p^\tau,q^{-\tau}}$. This shows $3(1_N^{\sigma,1})$ does indeed vanish. ■

We make a final remark about the notation for $w_i \in V^\sigma$ and $z_i \in V^{-\sigma}$: from the Pseudo-Principal Example 4.7 we see that $w_1 = Q_{\tilde{q}}s$, $w_2 = Q_{\tilde{q}}(n) = Q_{\tilde{q}}Q_s\tilde{q}$, $z_1 =$

[?]]) the resulting subdominion has $N'^{\tau} = Q_{w_1}Q_n, N'^{-\tau} = Q_nQ_{w_1}$ a principal struction. However, in the theory of fractions we want only injective denominators, and $s' = Q_s w_1$ is usually not injective.

$N(s)$, $z_2 = N(n)$. A more consistent notation would set $s_1 = s$, $s_2 = s^{(2,\tilde{q})} = Q_s\tilde{q} = n$, $s_3 = s^{(3,\tilde{q})} = Q_sQ_{\tilde{q}}s = Q_sw_1 = z_1$, $s_4 = s^{(4,\tilde{q})} = Q_sQ_{\tilde{q}}n = z_2$, and in general $s_k = s^{(k,\tilde{q})} = z_{k-2}$ with $s = z_{-1}$ and $n = z_0$. Similarly $\tilde{q}_1 = \tilde{q} = w_0$, $\tilde{q}_2 = \tilde{q}^{(2,s)} = Q_{\tilde{q}}s = w_1$, $\tilde{q}_3 = Q_{\tilde{q}}Q_s\tilde{q} = w_2$, and in general $\tilde{q}_k = \tilde{q}^{(k,s)} = w_{k-1}$. Only w_i, z_i for $i = 1, 2$ play much of a role in the struction conditions or Jordan derivations [**?**].

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