THE FREIHEITSSATZ FOR GENERIC POISSON ALGEBRAS

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1. Introduction

One of the classical achievements of the combinatorial group theory is the decidability of the word problem in a finitely generated group with one defining relation [1]. This result was a corollary of a fundamental statement called Freiheitssatz: Every equation over a free group is solvable in some extension. For solvable and nilpotent groups, this complex of problems was studied in [2].

In the context of Lie algebras, similar statements were proved [3]. For associative algebras, the problem turns to be surprisingly difficult: Over a field of characteristic zero, the Freiheitssatz was proved in [5], but the question about decidability of the word problem for an associative algebra with one defining relation remains open.

In [6], the Freiheitssatz was proved for right-symmetric (pre-Lie) algebras, and in [7]—for Poisson algebras. In this paper, we consider a modified approach to the proof in [7] which allows to prove the Freiheitssatz also for generic Poisson algebras.

There is a plenty of varieties for which the Freiheitssatz is not true, e.g., so is the variety of Poisson algebras over a field of positive characteristic. One may find more examples of this kind in [8]. Also, for the variety of Leibniz algebras (as well as for every variety of di-algebras in the sense of [9]) the Freiheitssatz does not hold.

Throughout the paper $\Bbbk$ denotes a field of characteristic zero.

A generic Poisson algebra (GP-algebra) is a linear space with two operations and one constant:

(1) associative and commutative product $x \cdot y = xy$;
(2) anti-commutative bracket $\{x, y\}$;
(3) multiplicative identity 1, $x \cdot 1 = 1 \cdot x = x$,

satisfying the Leibniz identity

$$\{x, yz\} = \{x, y\}z + \{x, z\}y.$$

These algebras were introduced in [10] in the study of speciality and deformations of Malcev–Poisson algebras.

Let $AC(X)$ be the free anti-commutative algebra (AC-algebra) generated by a set X with respect to operation denoted by $\{\cdot, \cdot\}$, and let $GP(X)$ be the free GP-algebra with a set of generators X.

As a linear space, $GP(X)$ is isomorphic to the symmetric algebra $S(AC(X))$ [10].

2. Conditionally closed algebras and the Freiheitssatz

Suppose $\mathfrak{M}$ is a variety of algebras over a field $\Bbbk$. Denote by $\mathfrak{M}(X)$ the free algebra in $\mathfrak{M}$ generated by a set X. For $A, B \in \mathfrak{M}$, the notation $A *_{\mathfrak{M}} B$ stands for the free product of A and B in $\mathfrak{M}$.

If $A \in \mathfrak{M}$ then every $\Psi \in A *_{\mathfrak{M}} \mathfrak{M}(x)$ may be interpreted as an A-valued function on A. Moreover, for every extension $\bar{A}$ of A, $\bar{A} \in \mathfrak{M}$, $\Psi(x)$ is an $\bar{A}$-valued function on $\bar{A}$. An equation of the form $\Psi(x) = 0$ is *solvable over* A if there exists an extension $\bar{A}$ of A such that the equation has a solution in $\bar{A}$. If such a solution can be found in A itself then $\Psi(x) = 0$ is said to be *solvable in* A.

Recall the common definition (see, e.g., [11, 12]): An algebra A is *(existentially) algebraically closed* if every system of equations which is solvable over A is solvable in A. This definition is important for model theory, and it can be an efficient tool for studying algebras provided the principal question on the solvability of a particular equation is solved.

A stronger property (see [13]) can be stated as follows: An algebra $A \in \mathfrak{M}$ is called *algebraically closed in* $\mathfrak{M}$ if for every $\Psi \in A *_{\mathfrak{M}} \mathfrak{M}(x)$, $\Psi \notin A$, the equation $\Psi(x) = 0$ is solvable in A. We are going to propose an intermediate definition which is sufficient for our purpose.

Definition 1. *An algebra $A \in \mathfrak{M}$ is called* conditionally closed in $\mathfrak{M}$ *if for every $\Psi(x) \in A *_{\mathfrak{M}} F_{\mathfrak{M}}(x)$ which is not a constant function on A the equation $\Psi(x) = 0$ is solvable in A.*

Every algebraically closed in $\mathfrak{M}$ algebra is conditionally closed in $\mathfrak{M}$. However, there is a plenty of conditionally closed systems that are not algebraically closed in $\mathfrak{M}$.

For example, an algebraically closed field is conditionally closed but not algebraically closed in the variety of all associative algebras. Similarly, such a field may be considered as a Poisson algebra with respect to trivial bracket, and the Poisson algebra obtained is conditionally closed but not algebraically closed in the variety of all Poisson algebras.

Suppose $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two varieties of algebras over a field $\Bbbk$, and let $\omega : \mathfrak{M}_1 \to \mathfrak{M}_2$ be a functor which acts as follows: Given $A \in \mathfrak{M}_1$, $A^{(\omega)} \in \mathfrak{M}_2$ is the same linear space equipped with new operations expressed in terms of initial operations. For example, one may consider the classical functor from the variety of associative algebras into the variety of Lie algebras defined by $[x, y] = xy - yx$. In general, ω may be a functor induced by a morphism of the governing operads. Functors of this kind were closely studied in [8].

Proposition 1. *An algebra $A \in \mathfrak{M}_1$ is conditionally closed in $\mathfrak{M}_1$ if and only if $A^{(\omega)}$ is conditionally closed in $\mathfrak{M}_2$.*

Note that for algebraically closed algebras this statement does not hold.

Proof. Since ω is a functor, the universal property of the free product implies the existence of a homomorphism $\varphi : A^{(\omega)} *_{\mathfrak{M}_2} \mathfrak{M}_2(x) \to (A *_{\mathfrak{M}_1} \mathfrak{M}_1(x))^{(\omega)}$ such that $f(a) = \varphi(f)(a)$ for all $f = f(x) \in A^{(\omega)} *_{\mathfrak{M}_2} \mathfrak{M}_2(x)$, $a \in A$. Therefore, f is not a constant function on $A^{(\omega)}$ if and only if $\varphi(f)$ is not a constant function on A.

If A is conditionally closed then there exists $a \in A$ such that $\varphi(f)(a) = 0$ and thus $f(a) = 0$. The converse is similar. $\square$

Another important example comes from the following settings. Let $\mathfrak{M}_1 = Dif_{2n}$ be the variety of commutative associative algebras with $2n$ pairwise commuting derivations ∂_i, ∂_i', $i = 1, \dots, n$. Then, given $A \in Dif_{2n}$, the same space equipped with new binary operation

$$\{a, b\} = \sum_{i=1}^{n} \partial_i(a)\partial_i'(b) - \partial_i(b)\partial_i'(a), \quad a, b \in A, \tag{1}$$

is known to be a Poisson algebra denoted by $A^{(\partial)}$. If we allow the derivations ∂_i, ∂_i' to be non-commuting then (1) defines on a commutative algebra A a structure of a GP-algebra.

The *Freiheitssatz problem* for a variety $\mathfrak{M}$ is to determine whether every nontrivial equation over the free algebra $\mathfrak{M}(X)$, $X = \{x_1, x_2, \dots\}$, is solvable over $\mathfrak{M}(X)$.

It is obviously equivalent to the following question about free algebras: Is the intersection of the ideal generated by an element $f \in$

$\mathfrak{M}(X \cup \{x\})$ and the subalgebra $\mathfrak{M}(X) \subset \mathfrak{M}(X \cup \{x\})$ trivial if $f \notin \mathfrak{M}(X)$ (i.e., depends on x)? If the answer is positive for all such f then we say that the Freiheitssatz holds for $\mathfrak{M}$.

Lemma 1. *Suppose $\mathfrak{M}$ is a variety of algebras with at least one binary operation $\cdot$ in the language such that $\mathfrak{M}(X) = \mathfrak{M}(x_1, x_2, \dots)$ has no zero divisors with respect to $\cdot$. Then, if for every nonzero polynomial $h = h(x_1, \dots, x_n) \in \mathfrak{M}(X)$ there exists a conditionally closed algebra $A \in \mathfrak{M}$ which does not satisfy the polynomial identity $h(x_1, \dots, x_n) = 0$ then the Freiheitssatz holds for $\mathfrak{M}$.*

Proof. Suppose $X = \{x_1, x_2, \dots\}$, $x \notin X$, and let $f = f(x, x_1, \dots, x_n) \in \mathfrak{M}(X \cup \{x\}) \setminus \mathfrak{M}(X)$. Then $f = f_1 + f_0$, where f_1 belongs to the ideal generated by x, $f_0 \in \mathfrak{M}(X)$.

Assume $g \in (f) \cap \mathfrak{M}(X)$, $g \neq 0$. Then $h = f_1 \cdot g \neq 0$, hence, there exist a conditionally closed $A \in \mathfrak{M}$ such that $h(x, x_1, \dots, x_n)$ is not a polynomial identity on A. Therefore, there exist $a, a_1, \dots, a_n \in A$ such that $f_1(a, a_1, \dots, a_n) g(a_1, \dots, a_n) \neq 0$ in A, so $f_1(a_1, \dots, a_n) \neq 0$. On the other hand, $f_1(0, a_1, \dots, a_n) = 0$. Therefore, $\Psi(x) = f_1(x, a_1, \dots, a_n)$ is a non-constant function on A. Since A is conditionally closed, there exists $a \in A$ such that $\Psi(a) = f_1(a, a_1, \dots, a_n) = -f_0(a_1, \dots, a_n)$. Thus, $f(a, a_1, \dots, a_n) = 0$ but $g(a_1, \dots, a_n) \neq 0$ which is impossible if $g \in (f) \triangleleft \mathfrak{M}(X \cup \{x\})$. $\square$

Corollary 1. *The Freiheitssatz holds for the variety generated by special Jordan algebras.*

Proof. Consider the algebraically closed associative noncommutative algebra A from [5]. It is essential that A is a skew field and contains the first Weyl algebra W_1. Thus, A contains free associative algebra in any finite number of generators $x_1, \dots, x_n$ [4]. The special Jordan algebra $A^{(+)}$ is conditionally closed by Proposition 1 and contains free special Jordan algebra $SJ(x_1, \dots, x_n)$. Therefore, the variety SJ satisfies all conditions of Lemma 1. $\square$

3. Jacobian polynomials in free anti-commutative algebras

In order to prove the Freiheitssatz for a variety $\mathfrak{M}$ by means of Lemma 1, we have to construct an algebra in $\mathfrak{M}$ which is conditionally closed algebra and does not satisfy a given polynomial identity.

In this section, we discuss technical questions that are used in subsequent sections for the study of polynomial identities on generic Poisson algebras.

3.1. **Preliminaries on $AC(X)$.** Let X be a set of generators, and let X^* stand for the set of all (nonempty) associative words u in the alphabet X. Denote by X^{**} the set of all non-associative words in X. Given a word $u \in X^*$, denote by (u) a non-associative word obtained from u by some bracketing. We will also use $[X^*]$ to denote the set of all associative and commutative words in X. Given $u \in X^*$, $[u]$ stands for the commutative image of u.

Suppose X^{**} is equipped with a linear order $\preceq$. A non-associative word $u \in X^{**}$ is *normal* if either $u = x \in X$ or $u = u_1u_2$, where u_1 and u_2 are normal and $u_1 \prec u_2$. Obviously, normal words in X^{**} form a linear basis of the free anti-commutative algebra $AC(X)$ generated by X (see [14, 15]).

Let us call the elements of $AC(X)$ *AC-polynomials*. Given $u \in X^{**}$, define $\deg u$ to be the length of u. Thus, we have a well-defined degree function on $AC(X)$.

Choose a generator $x_i \in X = \{x_1, \dots, x_n\}$ and denote by V_i the subspace of $AC(X)$ spanned by all nonassociative words linear in x_i. Fix a linear order $\preceq$ on X^{**} such that any nonassociative word which contains x_i is greater than any word without x_i (there exist many linear orders with this property). With respect to such an order, the unique normal form of a monomial $w \in V_i$ is

$$w = \{u_1, \{u_2, \dots \{u_k, x_i\} \dots \}\}, \tag{2}$$

where u_j, $j = 1, \dots, k$, are normal words in the alphabet $X \setminus \{x_i\}$. The number k is called *x_i-height* ([16]) of w, let us denote it by $ht(w, x_i)$.

Let $V_0 = \bigcap_{i=1}^{n} V_i$ be the space of polylinear AC-polynomials. It is easy to compute x_i-height of a nonassociative word $w \in V_0$ just by the number of brackets in w to the left of x_i, assuming $\{$ is counted as 1 and $\}$ as -1. For example, the x_4-height of $\{\{x_1, \{\{x_2, x_3\}, x_4\}\}, \{x_5, x_6\}\}$ is equal to 3.

Definition 2. *A linear transformation of V_i defined by the rule*

$$F_i : \{u_1, \{u_2, \dots \{u_k, x_i\} \dots \}\} \mapsto (-1)^{k-1}\{u_k, \{u_{k-1}, \dots \{u_1, x_i\} \dots \}\}$$

*is called an x_i-*flip.

Since the normal form (2) is unique, $F_i : V_i \to V_i$ is a well-defined map.

The set of all flips $\{F_1, \dots, F_n\}$ acts on the space V_0 and thus generates a group $\mathcal{F} \subseteq \mathrm{GL}(V_0)$. Given a normal word $u \in V_0$, the orbit $\mathcal{F}v$ consists of AC-monomials (polynomials of the form εv, v is a nonassociative word, $\varepsilon = \pm 1$).

Lemma 2. *Let $w = \{x_1, \{x_2, \dots \{x_{n-1}, x_n\} \dots \}\} \in V_0$. Then*

$$(-1)^{\sigma}\{x_{1\sigma}, \{x_{2\sigma}, \dots \{x_{(n-1)\sigma}, x_{n\sigma}\} \dots \}\} \in \mathcal{F}w$$

for every $\sigma \in S_n$ (here $(-1)^\sigma$ stands for the oddity of a permutation σ.)

Proof. It is straightforward to compute that

$$F_1(F_i w) = -\{x_i, \{x_2, \dots x_{i-1}, \{x_1, \{x_{i+1}, \dots \{x_{n-1}, x_n\} \dots \}\}\}\},$$

$i = 2, \dots, n$. Since transpositions of the form $(1i)$ generate the entire symmetric group S_n, the lemma is proved. □

3.2. **Jacobian AC-polynomials.** In this section, we describe polylinear AC-polynomials that have a specific property if considered as elements of the free GP-algebra.

Suppose $\Psi(x_1, \dots, x_n)$ is an element of the free GP-algebra $GP(X)$, $X = \{x_1, \dots, x_n\}$ which is linear with respect to x_n. Following [16], let us say that Ψ is a derivation with respect to x_n if

$$\Psi(x_1, \dots, x_{n-1}, yz) = y\Psi(x_1, \dots, x_{n-1}, z) + z\Psi(x_1, \dots, x_{n-1}, y)$$

in the free GP-algebra $GP(x_1, \dots, x_{n-1}, y, z)$.

Definition 3. *A polylinear AC-polynomial $\Psi = \Psi(x_1, \dots, x_n) \in V_0$ is said to be* jacobian *if Ψ is a derivation with respect to each variable x_i, $i = 1, \dots, n$. A polylinear element of $GP(X)$ with the same property is called a jacobian GP-polynomial.*

For free Lie algebra considered as a part of the free Poisson algebra, a similar notion was considered in [16]. Obviously, if $n = 2$ then $C_2 = \{x_1, x_2\}$ is a jacobian AC-polynomial. It was shown in [16] that there are no more jacobian Lie polynomials (up to a multiplicative constant). However, there exists a jacobian AC-polynomial of degree 3:

$$J_3 = \{\{x_1, x_2\}, x_3\} + \{\{x_2, x_3\}, x_1\} + \{\{x_3, x_1\}, x_2\}.$$

The main purpose of this section is to show that C_2 and J_3 exhaust all jacobian AC-polynomials.

For a generic Poisson algebra A, $a \in A$, consider the linear map $\operatorname{ad} a : x \mapsto \{a, x\}$, $x \in A$. The set of all such transformations $\{\operatorname{ad} a \mid a \in A\} \subset \operatorname{End}_{\Bbbk}(A)$ generates a Lie subalgebra $L(A) \subset \mathrm{gl}\,(A) = \operatorname{End}_{\Bbbk}(A)^{(-)}$.

Given $L \in L(GP(x_1, \dots, x_{n-1}))$, one may easily note that $L(x_n) \in GP(x_1, \dots, x_n)$ is a derivation with respect to x_n. Indeed, the Leibniz identity implies that $\operatorname{ad} u$, $u \in GP(x_1, \dots, x_{n-1})$, is a derivation with respect to x_n, and the commutator of derivations is a derivation itself.

Lemma 3. *Let $\Psi(x_1, \dots, x_n) \in AC(X) \subset GP(X)$ be a polylinear element such that Ψ is a derivation with respect to x_n. Then there exists $L \in L(AC(X))$ such that $\Psi = L(x_n)$.*

Proof. Consider a polylinear word from $AC(X)$:

$$w(x_1, \dots, x_n) = \{u_1, \{u_2, \dots \{u_k, x_n\} \dots\}\}.$$

Denote

$$D(w, x_n; x, y) = w(x_1, \dots, xy) - xw(x_1, \dots, y) - yw(x_1, \dots, x)$$

and extend $D(\cdot, x_n; x, y)$ to the entire space V_0 by linearity.

By the condition of the statement, $D(\Psi, x_n; x, y) = 0$.

It is easy to compute $D(w, x_n; x, y)$:

$$D(w, x_n; x, y) = \sum_{l=1}^{k-1} \sum_{s \in \mathrm{Sh}\,(k,l)} \{u_{1s}, \dots \{u_{ls}, x\} \dots\} \cdot \{u_{(l+1)s}, \dots \{u_{ks}, y\} \dots\}$$

where $\mathrm{Sh}\,(k, l)$ is the set of all shuffles, i.e., permutations $s \in S_k$ such that $1s < 2s < \dots < ls$ and $(l + 1)s < \dots < ks$.

Comparing the x-heights and y-heights of factors conclude that all summands of Ψ split into groups of equal x_n-height; each group is a derivation in x_n. Hence, we may suppose Ψ is homogeneous with respect to x_n-height.

Moreover, the formula for $D(\cdot)$ shows that it is enough to consider

$$\Psi = \sum_{s\in S_k} \alpha_s w_s, \quad w_s = \{u_{1s}, \dots \{u_{ks}, x_n\} \dots\}$$

(computation of $D(\cdot)$ does not affect the inner structure of all u_i). Now we may proceed with description of all jacobian AC-polynomials. Suppose u_1 is maximal among all u_i with respect to $\prec$. Let us show that there exists $s \in S_k$ such that $1s = 1$ and $\alpha_s \neq 0$, i.e., the maximal word appears as the first entry.

Assume the converse: u_1 never appears in the beginning. Choose the minimal $j > 1$ such that $js = 1$ for some $s \in S_k$, $\alpha_s \neq 0$. Then $D(w_s, x_n; x, y)$ contains a summand like

$$\alpha_s\{u_{js}, \dots \{u_{ks}, x\} \dots\} \cdot \{u_{1s}, \dots \{u_{(j-1)s}, y\} \dots\}.$$

This summand must coincide with at least one similar term coming from $D(w_t, x_n; x, y)$, for some $t \in S_k$, $\alpha_t \neq 0$ (since in $D(\Psi, x_n; x, y)$ all terms vanish). Since j is minimal, we have $jt = 1$, and the shuffle corresponding to the desired term of $D(w_t, x_n; x, y)$ is the same as we have for $D(w_s, x_n; x, y)$. Hence, $t = s$, which proves the claim.

Note that for

$$L = [\operatorname{ad} u_k[\operatorname{ad} u_{k-1} \dots [\operatorname{ad} u_2, \operatorname{ad} u_1] \dots]] \in \operatorname{gl}(AC(X))$$

the element $L(x_n)$ contains only one summand of the form $\{u_1, \dots\}$ (i.e., starting with u_1) which is equal to $\pm\{u_1, \{u_2, \dots \{u_k, x_n\} \dots\}\}$.

Therefore, subtracting $L(x_n)$ with an appropriate coefficient we may cancel one term of Ψ starting with u_1, but the result is still a derivation with respect to x_n. When we cancel all such terms, the polynomial obtained must be zero since it does not contain a summand starting with u_1. $\square$

Remark 1. *In the same way as the claim above was proved, one may show that there exists $s \in S_k$ such that $ks = 1$ and $\alpha_s \neq 0$, i.e., the maximal word u_1 appears as the last entry in*

$$w_s = \{u_{1s}, \dots \{u_{ks}, x_n\} \dots\}, \quad u_{ks} = u_1.$$

Corollary 2. *Let $\Psi(x_1, \dots, x_n) \in AC(X)$ be a poly-linear AC-polynomial of degree n such that Ψ is a derivation with respect to x_n. Suppose $\Psi = \sum\limits_w \alpha_w w$, $w \in X^{**}$ are normal words, and*

$$\max_{\alpha_w \neq 0} ht(w, x_n) \geq \max_{\alpha_w \neq 0} ht(w, x_i), \quad i = 1, \dots, n.$$

(x_n has maximal height in Ψ). Then

$$\max_{\alpha_w \neq 0} ht(w, x_n) = n - 1$$

and thus Ψ contains a monomial of the form

$$w = \{x_{1s}, \dots \{x_{(n-1)s}, x_n\} \dots\}$$

for some $s \in S_{n-1}$.

Proof. Assume $k < n - 1$ is the maximal height of x_n in Ψ, i.e., Ψ contains a summand of the form $\{u_1, \dots \{u_k, x_n\} \dots\}$, $k < n-1$. Then at least one of u_i has the degree greater than 1. Choose the maximal of these u_i with respect to $\prec$. Remark 1 implies Ψ to contain a summand $\alpha_w w$, where $w = \{u_{j_1}, \dots \{u_{j_k}, x_n\} \dots\}$, $j_k = i$, $\alpha_w \neq 0$. There exists x_j such that $ht(u_i, x_j) > 1$, so $ht(w, x_j) > k$, which contradicts to the condition $ht(w, x_j) \leq k$. Hence, $k = n - 1$. □

Lemma 4. *Suppose $\Psi = \Psi(x_1, \dots, x_n)$ is a jacobian AC-polynomial. Then Ψ is invariant with respect to the action of the group $\mathcal{F}$ generated by all x_i-flips, $i = 1, \dots, n$.*

Proof. Let us fix $i \in \{1, \ldots, n\}$. Without loss of generality we may assume $i = n$. Then by Lemma 3 $\Psi = L(x_n)$, where L is a linear operator constructed by commutators of operators $\operatorname{ad} u$, $u \in (X \setminus \{x_n\})^{**}$ are normal words. The set U of all such $\operatorname{ad} u$ generates an associative subalgebra $\mathcal{U} \subset \operatorname{End}_{\Bbbk} V_n$. It follows from the description of the basis in $AC(X)$ that $\mathcal{U}$ is isomorphic to the free associative algebra generated by U. Note that since Ψ is polylinear, L naturally splits into a sum of operators presented by polylinear elements of $\mathcal{U}$.

Since L belongs to the Lie subalgebra of $\mathcal{U}^{(-)}$ generated by U (which is well-known to be free), it is invariant with respect to the natural involution τ of $\mathcal{U}$ given by

$$\tau : u_1 \ldots u_k \mapsto (-1)^{k-1} u_k \ldots u_1, \quad u_i \in U$$

(it follows from the obvious observation $\tau([u, v]) = [\tau(u), \tau(v)]$ for $u, v \in \mathcal{U}$).

By Definition 2,

$$F_n(\Psi) = F_n(L(x_n)) = \tau(L)(x_n) = L(x_n) = \Psi.$$

As Ψ is invariant with respect to all flips, we have $\mathcal{F}(\Psi) = \{\Psi\}$. $\square$

Lemma 5. *Suppose $U = \{u_1, \ldots, u_m\}$ is a set, $\mathcal{U} = As(U)$ is the free associative algebra generated by U. Denote by $Lie(U)$ the free Lie algebra generated by U, $Lie(U) \subset As(U)^{(-)}$ Consider*

$$A_m = \sum_{s \in S_m} (-1)^s u_{1s} \ldots u_{ms}.$$

Then $A_m \in Lie(U)$ if and only if $m = 1$ or $m = 2$.

Proof. For $m = 1, 2$ it is obvious that $A_m \in Lie(U)$.

Assume $m \geq 3$ and $A_m \in Lie(U)$. Consider the homomorphism $\Phi : As(U) \to \wedge(\Bbbk U)$ given by $u \mapsto u$, where $\wedge(\Bbbk U)$ is the exterior algebra of the linear space spanned by U. Note that $\Phi(A_m) = m!u_1 \dots u_m \neq 0$ in $\wedge(\Bbbk U)$. However, $\Phi(Lie(U)) \subset \wedge(\Bbbk U)^{(-)}$ is a Lie subalgebra generated by U. It is easy to see that $\wedge(\Bbbk U)^{(-)}$ is a 3-nilpotent Lie algebra, so $\Phi(Lie(U))$ does not contain elements of degree $m \geq 3$. □

Theorem 1. *Let $X = \{x_1, \dots, x_n\}$, and let $\Psi = \Psi(x_1, \dots, x_n) \in AC(X)$ be a jacobian AC-polynomial. Then either $n = 2$ and $\Psi = \alpha C_2$, or $n = 3$ and $\Psi = \alpha J_3$, where $\alpha \in \Bbbk^*$.*

Proof. By Lemma 4 $F\Psi = \Psi$ for every $F \in \mathcal{F}$. Corollary 2 implies that Ψ contains a summand of the form αw, where $\alpha \in \Bbbk^*$, $w = \{x_{1s}, \dots \{x_{(n-1)s}, x_n\} \dots\}$ for some $s \in S_{n-1}$. Without loss of generality, $\alpha = 1$ and $s = \mathrm{id}$. By Lemma 2, Ψ contains all monomials obtained from w by all permutations of variables, i.e.,

$$\Psi = \sum_{s \in S_{n-1}} (-1)^s \{x_{1s}, \dots \{x_{(n-1)s}, x_n\} \dots\} + \Phi(x_1, \dots, x_n),$$

where the x_n-height of all monomials in Ψ is smaller than $n-1$. Since all summands of Ψ with the same x_n-height form a derivation with respect to x_n, the AC-polynomial

$$\Psi_1 = \sum_{s \in S_{n-1}} (-1)^s \{x_{1s}, \dots \{x_{(n-1)s}, x_n\} \dots\}$$

must be a derivation with respect to x_n. But

$$\Psi_1 = A_{n-1}(u_1, \dots, u_{n-1})(x_n), \quad u_i = \operatorname{ad} x_i,$$

so $A_{n-1}(u_1, \dots, u_{n-1}) \in L(AC(X))$. By Lemma 5, $n-1 \leq 2$, so $n \leq 3$. Obviously, C_2 and J_3 are the only jacobian AC-polynomials for $n = 2$ and $n = 3$, respectively. □

4. Identities of generic Poisson algebras

Let A be a GP-algebra, and let $f \in GP(x_1, \dots, x_n)$, $f \neq 0$. As usual, we say that f is a polynomial identity on A if for every homomorphism

$\varphi : GP(x_1, \dots, x_n) \to A$ we have $\varphi(f) = 0$. In this case we also say that A satisfies the polynomial identity f.

Proposition 2. *Suppose a GP-algebra A satisfies a polynomial identity. Then there exists a polynomial identity Ψ on A which is a jacobian GP-polynomial.*

This statement, as well as its proof, is similar to the result by Farkas [16] on polynomial identities of Poisson algebras.

Proof. The standard linearization procedure (see, e.g., [17, Chapter 1]) allows to assume that A satisfies a polylinear polynomial identity $f \in GP(X)$, $X = \{x_1, \dots, x_n\}$.

As an element of $GP(X)$, f may be uniquely presented as a linear combination of GP-monomials $w = u_1 \dots u_k$, $u_j \in U$, where $U \subset AC(X)$ is the set of normal words. Note that u_j are of degree two or more (if an AC-monomial of degree one appears, e.g., $u_j = x_i$, then one may plug in $x_i = 1$ and obtain a polylinear polynomial identity without x_i). Denote by $FH_i(w)$ (the Farkas height) the degree of u_j in which the variable x_i occurs, and let $FH_i(f)$ be the maximal of $FH_i(w)$ among all GP-monomials w that appear in f with a nonzero coefficient. Finally, set

$$FH(f) = \sum_{i=1}^{n} 3^{FH_i(f)}$$

Observe that if f is not a derivation in x_i then the derivation difference $D(f, x_i; x_i, x_{n+1})$ is a nonzero polylinear element of $GP(X \cup \{x_{n+1}\})$ which has a smaller Farkas height. Indeed, for a GP-monomial w from

f we have

$$FH_j(D(w, x_i; x_i, x_{n+1})) \leq FH_j(w),$$

$$FH_i(D(w, x_i; x_i, x_{n+1})) \leq FH_i(w) - 1,$$

$$FH_{n+1}(D(w, x_i; x_i, x_{n+1})) \leq FH_i(w) - 1,$$

which implies

$$FH(w) - FH(D(w, x_i; x_i, x_{n+1})) \leq 3^{FH_i(w)} - 2 \cdot 3^{FH_i(w)-1} > 0.$$

Obviously, $D(f, x_i; x_i, x_{n+1})$ is a polynomial identity on A.

Therefore, after a finite number of steps we obtain a nonzero polynomial identity on A which is a jacobian GP-polynomial in a larger set of variables $\widetilde{X} \supseteq X$. $\square$

Let us recall the notion of fine grading [16]. First, given a set X, the free anti-commutative algebra $AC(X)$ carries $[X^*]$-grading such that $u \in X^{**}$ has weight $[u]$. Next, if $w = (u_1) \dots (u_n) \in GP(X)$, $u_i \in X^*$, then the weight of w is $[u_1] + \cdots + [u_n] \in \Bbbk[X^*]$. As a result,

$$GP(X) = \bigoplus_{p \in \Bbbk[X^*] \setminus \{0\}} GP_p(X),$$

where $GP_p(X)$ is the space spanned by all generic Poisson monomials of degree p. An element $f \in GP_p(X)$ is said to be *finely homogeneous.*

Proposition 3. *A jacobian GP-polynomial Ψ can be presented as a linear combination of products of jacobian AC-polynomials (on the appropriate set of variables).*

Proof. Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of variables, and let $U = \{u_1, u_2, \dots\}$ be the set of normal nonassociative words in X (with respect to some ordering), then $GP(X) = \Bbbk[U]$. For $\Psi \in GP(X)$, denote by $\mathrm{supp}(\Psi)$ all variables from X that appear in Ψ and by $\mathrm{psupp}(\Psi)$ all variables from U that appear in Ψ.

Suppose $f \in GP(x_1, \dots, x_n) \subset GP(X)$ is a jacobian GP-polynomial. Without loss of generality we may assume f to be finely homogeneous and $f \notin AC(X)$. Proceed by induction on $|\operatorname{psupp}(f)|$.

Consider a GP-monomial w in f. If $w = u_i w'$ where $w' \neq 1$ then write $f = u_i g + h$, $g, h \in GP(X)$, $g \neq 1$, where all GP-monomials of h are not divisible by u_i (in $\Bbbk[U]$). Since f is polylinear, $\operatorname{supp}(g) \cap \operatorname{supp}(u_i) = \emptyset$.

Denote by D_i a map $GP(X) \to GP(X \cup \{y, z\})$ defined as follows:

$$D_i(\Psi) = \begin{cases} D(\Psi, x_i; y, z), & x_i \in \operatorname{supp}(\Psi), \\ \Psi, & x_i \notin \operatorname{supp}(\Psi). \end{cases}$$

Then $D_j(f) = u_i D_j(g) + D_j(h) = 0$ if $x_j \in \operatorname{supp}(g)$.

Consider $GP(X \cup \{y, z\})$ as a polynomial algebra with a set $\widetilde{U}$ of generators including U. Then $u_i \notin \operatorname{psupp}(h)$ and $u_i \notin \operatorname{psupp}(D_j(h))$. Hence, $D_j(g) = 0$ and g is a jacobian GP-polynomial.

Let us now fix the deg-lex order on the set $[U^*]$, i.e., commutative monomials in U are first compared by their length and then lexicographically, assuming $u_1 < u_2 < \dots$. Recall that $f = u_i g + h$, where $\operatorname{psupp}(h) \not\ni u_i$, and presented h as $h = gp + r$, where all GP-monomials of r are not divisible (in $\Bbbk[U]$) by the leading GP-monomial $\bar{g}$ of g. Then $f = gq + r$, $q = u_i + p$, and $\operatorname{psupp}(r) \not\ni u_i$. In particular, $\operatorname{psupp}(r) \subset \operatorname{psupp}(f)$.

By definition, $D_j(f) = gD_j(q) + D_j(r) = 0$ if $x_j \in \operatorname{supp}(q)$. If $D_j(q) \neq 0$ then some of the monomials in $D_j(r)$ are divisible by $\bar{g}$. Consider a GP-monomial M of r. Since it is not divisible by $\bar{g}$ there is at least one variable u_a which appears in $\bar{g}$ and does not appear in M. Note that if $\operatorname{supp}(u_b) \not\ni x_i$ then $D_i(u_b) = u_b$, and if $\operatorname{supp} u_b \ni x_i$ then $D_i(u_b)$ is a GP-polynomial of degree two (in $\Bbbk[\widetilde{U}]$) in which neither of variables belongs to U. Hence, $D_j(M)$ is not divisible by u_a and neither of the GP-monomials of $D_j(r)$ is divisible by $\bar{g}$. Therefore, $D_j(q) = 0$ and q is a jacobian GP-polynomial.

Since a product of two jacobian GP-polynomials is also jacobian (with respect to the corresponding sets of variables), $r = f - gq$ is a jacobian GP-polynomial. By induction, the statement holds for r, as well as for g and q. □

Corollary 3. *Let $F(t_1, \dots, t_n) \in GP(t_1, t_2, \dots)$ be a finely homogeneous jacobian GP-polynomial. Then F contains a summand $\alpha u_1 \dots u_k$, where $\alpha \in \Bbbk^*$, $u_i \in AC(t_1, t_2, \dots)$ are of the form*

$$\{t_{i_1}, t_{i_2}\} \quad or \quad \{t_{i_1}, \{t_{i_2}, t_{i_3}\}\}$$

5. The Freiheitssatz for (generic) Poisson algebras

The following statement is well-known in the theory of differential fields [18, 19]. We will sketch a proof below in order to make the exposition more convenient for a reader. Recall that the characteristic of the base field $\Bbbk$ is assumed to be zero.

Theorem 2. *Every algebra from Dif_n which is a field can be embedded into an algebraically closed algebra in Dif_n.*

Proof. Let F be a differential field of characteristic zero with a set $\Delta = \{\partial_i \mid i = 1, \dots, n\}$ of pairwise commuting derivations. Denote by $F[x; \Delta] = F *_{Dif_n} Dif_n(x)$ the set of all differential polynomials in one variable x over F. Suppose $f(x) \in F[x; \Delta] \setminus F$. Then there exists a differential field K which is an extension of the differential field F such that the equation $f(x) = 0$ has a solution in K.

Indeed, differential polynomials $F[x; \Delta]$ may be considered as ordinary polynomials in infinitely many variables

$$X = \{x^{(i_1, \dots, i_n)} \mid (i_1, \dots, i_n) \in \mathbb{Z}_+^n\},$$

where $x^{(i_1,\dots,i_n)}$ is identified with $\partial_1^{i_1}\dots\partial_n^{i_n}(x)$. Then the differential ideal $I(f;\Delta)$ generated by $f(x)$ in $F[x;\Delta]$ coincides with the ordinary ideal in $F[X]$ generated by f and all its derivatives $\partial_1^{i_1}\dots\partial_n^{i_n}(f)$.

Note that if $f\notin F$ then $I(f;\Delta)$ is proper: One may apply the notion of a characteristic set (see, e.g., [19, Ch. I.10]) or simply note that the set of all derivatives of f is a Gröbner basis provided that we choose an ordering of monomials in such a way that highest derivative (leader) is contained in the leading monomial (e.g., rank ordering in [19, Ch. I.8]). Indeed, if uy is the leading monomial of f ($y\in X$ is the leader of f, u is an ordered monomial in X) then $uy^{(i_1,\dots,i_n)}$ is the leading monomial of $\partial_1^{i_1}\dots\partial_n^{i_n}(f)$. It is easy to see that there are no compositions (we follow the terminology of Shirshov [3], see [20] for details) among f and its derivatives except for the case when $uy=y^k$, but in the latter case the only series of compositions of intersection of f with itself is obviously trivial.

Hence, if $f\notin F$ then $I=I(f;\Delta)$ is proper, and so is its radical $\sqrt{I}$. By the differential prime decomposition theorem (see, e.g., [18, Ch. 1]), $I=p_1\cap\dots\cap p_k$, where p_i are prime differential ideals in $F[x;\Delta]$. In particular, $f\in p_1$, and $F[x;\Delta]/p_1$ is a differential domain containing a root $x+p_1$ of f. Finally, the quotient field of that domain $Q(F[x;\Delta]/p_1)$ is the desired differential field.

Therefore, every nontrivial equation over an arbitrary differential field F has a solution in an extension K of F. If F is infinite then K has the same cardinality as F, so the standard transfinite induction arguments similar to those applied to ordinary fields show that F can be embedded into a differential field $\bar{F}\in Dif_n$ in which every nontrivial differential polynomial has a root. $\square$

Corollary 4 ([7])**.** *The Freiheitssatz holds for the variety of Poisson algebras.*

Proof. Let $A_{2n} = \Bbbk(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ be the algebra of (commutative) rational functions over $\Bbbk$, $\partial_i = \partial_{x_i}$, $\partial_i' = \partial_{y_i}$ be ordinary partial derivatives with respect to x_i, y_i, respectively. As $A_{2n} \in Dif_{2n}$, there exists its algebraically closed extension $\bar{A}_{2n} \in Dif_{2n}$. Let $PS_n = A_{2n}^{(\partial)}$ be the Poisson algebra defined by (1). Then $PS_n \subseteq \bar{A}_{2n}^{(\partial)}$, where the latter is a conditionally closed Poisson algebra by Proposition 1.

It was shown in [16] that for every nonzero Poisson polynomial $h = h(x_1, \dots, x_m)$, $m \geq 1$, there exists a sufficiently large N such that PS_N (and thus $\bar{A}_{2N}^{(\partial)}$) does not satisfy the identity $h(x_1, \dots, x_m) = 0$. Lemma 1 implies the claim. $\square$

Let us twist the functor $\partial : Dif_{2n} \to Pois$ in order to obtain a conditionally closed generic Poisson algebra that does not satisfy a fixed polynomial identity.

Consider the variety $CDif_n$ of commutative differential algebras with pairwise commuting derivations ∂_i and constants c_i, $i = 1, \dots, n$, such that $\partial_i(c_j) = \delta_{ij}$. Then there exists a natural forgetting functor $\omega : CDif_n \to Dif_n$ erasing the information about constants.

In particular, A_{2n} may be considered as an algebra from $CDif_{2n}$ with derivations $\partial_i = \partial_{x_i}$, $\partial_i' = \partial_{y_i}$, and constants $c_i = x_i$, $c_i' = y_i$, $i = 1, \dots, n$. Moreover, if $A_{2n} \subseteq A \in Dif_{2n}$ then $A = B^{(\omega)}$ for an appropriate $B \in CDif_{2n}$. Hence, by Proposition 1, for every $n \geq 1$ there exists a conditionally closed algebra $\bar{B}_{2n}$ in $CDif_{2n}$, $\bar{B}_{2n}^{(\omega)} = \bar{A}_{2n}$.

Suppose $B \in CDif_{2n}$ with derivations ∂_i, ∂_i' and constants c_i, c_i', $i = 1, \dots, n$. Let us consider the following functor τ from $CDif_{2n}$

to the variety $NDif_{2n}$ of commutative differential algebras with non-commuting derivations ξ_i, ξ_i', $i=1,\dots,n$. On the same space B, define new derivations by

$$\begin{gathered}\xi_i(a) = c'_{i+1}\partial_i, \quad i=1,\dots,n-1,\\ \xi_n(a) = c'_1\partial_n, \\ \xi_i'(a) = \partial_i'(a), \quad i=1,\dots,n,\end{gathered} \tag{3}$$

for $a\in B$. If B is conditionally closed in $CDif_{2n}$ then $B^{(\tau)}$ is conditionally closed in $NDif_{2n}$.

Finally, define a functor ξ from $NDif_{2n}$ to the variety GP of generic Poisson algebras by means of

$$\{a,b\} = \sum_{i\ge 1} \xi_i(a)\xi_i'(b) - \xi_i(b)\xi_i'(a). \tag{4}$$

Denote by GPS_N the GP-algebra $\big(A_{2n}^{(\tau)}\big)^{(\xi)}$.

Proposition 4. *For every $n\ge 1$ there exists $N\ge 1$ such that the GP-algebra GPS_m does not satisfy a polynomial identity of degree n for all $m\ge N$.*

Proof. Suppose $f\in GP(t_1,t_2,\dots)$ is a GP-polynomial of degree n which is an identity on GPS_m. By Proposition 2 there also exists a polylinear identity g on GPS_m which is a jacobian GP-polynomial.

Let us split g into finely homogeneous components:

$$g = g_1 + \dots + g_k,$$

each g_i is a jacobian GP-polynomial (but not an identity on GPS_m).

According to Corollary 3, g_1 contains a summand $\alpha u_1\dots u_l$, $\alpha\in\Bbbk^*$,

$$u_i = \{t_{i_1},\dots\{t_{i_{m_i}}, t_{i_{m_i+1}}\}\dots\}, \quad m_i = 1,2.$$

Assume m is large enough (e.g., $m > 2l$), and evaluate the variables in such a way that

$$t_{i_{m_i+1}} = y_{k_i},$$

$$t_{i_{m_i}} = x_{k_i},\ t_{i_{m_i-1}} = x_{k_i+1}, \dots, t_{i_1} = x_{k_i+m_i-1},$$

$$k_{i+1} \geq k_i + m_i, \quad k_l + m_l < m.$$

Then the only summand in $g_1(t_1, \dots, t_n)$ is nonzero, namely, the summand mentioned by Corollary 3: It turns into $\alpha y_{k_1+m_1} \dots y_{k_l+m_l} \neq 0$. Other g_is turn into zero.

Hence, g can not be a polynomial identity on GPS_m. □

Theorem 3. *The Freiheitssatz holds for the variety of generic Poisson algebras.*

Proof. Given $N \geq 1$, $G_N = \left(\bar{B}_{2N}^{(\tau)}\right)^{(\xi)}$ is a conditionally closed algebra in GP by Proposition 1, and $GPS_N \subseteq G_N$. The claim now follows from Proposition 4 and Lemma 1. □

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