GRADINGS ON $\mathfrak{g}_2$

CRISTINA DRAPER FONTANALS AND CÁNDIDO MARTÍN GONZÁLEZ

Abstract. We find all gradings of $\mathfrak{g}_2$ up to equivalence.

1. Introduction

In the last years, a renewed interest on gradings on simple Lie and Jordan algebras has arisen. The gradings of finite-dimensional simple Lie algebras ruling out $\mathfrak{a}_l$, $\mathfrak{d}_4$ and the exceptional cases ($\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$) by finite abelian groups are described in [4]. Also the gradings on simple Jordan algebras of type $\mathrm{H}(F_n)$ and $\mathrm{H}(Q_n)$ are given in the same reference, for an algebraically closed field F of characteristic zero and a quaternion algebra F-algebra Q. In this work, the authors use their previous results in [3] about gradings of associative algebras $\mathrm{M}_n(F)$. In [2] all gradings on the simple Jordan algebras of Clifford type have been described. The fine gradings in $\mathfrak{a}_l$ have been determined in [8] solving the related problem of finding maximal abelian groups of diagonalizable automorphisms of the algebras (not only in $\mathrm{gl}(n, \mathbb{C})$ but also in $\mathrm{o}(n, \mathbb{C})$ for $n \neq 8$ and $\mathrm{sp}(2n, \mathbb{C})$). The real case is treated in [9]. Notice that all the works about this topic make use of techniques related to the associative case. Our aim is to start the study of gradings of exceptional Lie algebras, solving completely the case $\mathfrak{g}_2$.

Our original feeling was that the description of the gradings on $\mathfrak{g}_2$ could be derived straightforwardly from the classification of gradings on octonion algebras C obtained in [5]. Recall that in the split case there is an algebraic groups isomorphism from the group of automorphisms of C to the group of automorphisms of $\mathfrak{g}_2 = \mathrm{Der}(C)$, mapping each $\varphi \in \mathrm{aut}(C)$ to the automorphism $d \mapsto \varphi d \varphi^{-1}$. Thus any grading of the exceptional simple Lie algebra $\mathfrak{g}_2$, which is given by a finitely generated abelian subgroup of its group of automorphisms, is induced by a grading on C, and it seems that we have all the information needed to compute explicitely all the possible gradings on $\mathfrak{g}_2$. But this mechanism for passing gradings from one algebra to another presents a serious handicap: it does not preserve equivalence. We will check that, although there are only 9 gradings up to equivalence on a split Cayley algebra, there are just 25 gradings (also up to equivalence) on $\mathfrak{g}_2$.

Our first steps in the search of gradings on $\mathfrak{g}_2$ led us to some general considerations, the first one being the definition of grading itself: must it be supposed the existence of a grading-group? Should this group be abelian? This is essential since the mechanism for translating gradings from C to $\mathfrak{g}_2$ works only for abelian groups. Although the work [18] contributes to the general theory with many results related to these questions, it does not provide a satisfactory answer to them, according to [6]. We shall deal with these topics in subsection 2.1, but only partially. Hence, all

Supported by the Spanish MCYT projects MTM2004-06580-C02-02, MTM2004-08115-C04-04 and by the Junta de Andalucía PAI project FQM-336.

through this paper we shall deal with gradings over groups and unless explicitely stated, the word grading will mean group grading.

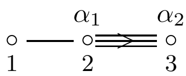
The problem of the study of fine gradings is posed in the just mentioned work [18]. The root space decomposition of a finite-dimensional semisimple complex Lie algebra $\mathcal{L}$ is a particular case of a fine grading on $\mathcal{L}$. So, given the great deal of applications (for instance to the study of representations) of such a decomposition, it is natural to consider the problem of the determination of all fine gradings. Indeed there could be problems admitting a simpler formulation by using basis formed no longer by root spaces but by homogeneous elements in some other fine grading. In fact this happens with the $\mathbb{Z}_2^2$-grading of gl(2) spanned by the Pauli's matrices, as pointed out in [8]. This is why this problem has also arisen in the Physics literature (see [7], [10] and [11]).

The study of fine gradings is obviously related to that of nontoral ones. A grading is toral if the homogeneous components are sums of root spaces. In section 2.4 we prove that the fact that a grading is toral is equivalent, from an algebraic viewpoint, to the existence of a Cartan subalgebra within the 0-homogeneous component, and from a geometric viewpoint, to the fact that the automorphisms producing the grading are in some torus of the automorphism group. Although the nontoral gradings of simple Lie algebras are not described, there are some works published in the early eighties on gradings with some extra conditions implying its nontoral nature. This is a source of examples of very nice gradings of exceptional Lie algebras. For instance the Jordan subgroups are studied in [1] and [17]. These are finite abelian subgroups of inner automorphisms of the algebra satisfying additional conditions such as the finiteness of its centralizer and the fact that they are minimal normal subgroups of their normalizers. Of course any grading obtained as the simultaneous eigenspaces of the automorphisms in one of these subgroups is nontoral (toral gradings have a torus contained in their centralizers). In case $\mathcal{L} = \mathfrak{g}_2$, there is one $\mathbb{Z}_2^3$-grading whose homogeneous components are Cartan subalgebras. This is analogous to the well-known $\mathbb{Z}_2^5$-grading on $\mathfrak{e}_8$ composed by 31 Cartan subalgebras, also induced by a Jordan subgroup.

The same gradings appear in a work by Hesselink [12] about special and pure gradings of Lie algebras. A grading of a complex semisimple Lie algebra $\mathcal{L}$ is said to be *special* if $\mathcal{L}_0 = 0$ (in particular it is nontoral). Recall that for any nonzero element $e \in \mathcal{L}_g$ there is a semisimple element $h \in \mathcal{L}_0$ and a nilpotent one $f \in \mathcal{L}_{-g}$ such that $\{h, e, f\}$ is a standard basis of sl(2) (see [20]). Therefore if a grading is special, all the homogeneous elements are semisimple and the homogeneous components are toral. If besides all the nonzero homogeneous components are Cartan subalgebras, the grading is called *very pure*. Hesselink classifies the very pure gradings, appearing again the gradings on $\mathfrak{g}_2$ and $\mathfrak{e}_8$ mentioned above.

A first purpose of this work has been to find out if there are nontoral gradings of $\mathfrak{g}_2$ apart from the previously mentioned one. In fact, in subsection 2.4 we develop a method for constructing nontoral gradings refining a given toral one when possible. Another aim of our work has been to give techniques which can be possibly applied in other contexts. Accordingly we have classified the gradings on the complex Cayley algebra C giving an alternative proof to that in [5], which does not make intensive use of the algebra itself but of the well-known conjugacy classes of elements in maximal tori of $\mathrm{aut}(C)$. Thus we get some tools which can be used in other nonassociative algebras like $\mathfrak{g}_2$ and hopefully in Albert's algebra and $\mathfrak{f}_4$, since their automorphism groups are related in a similar way as those of Cayley algebras and $\mathfrak{g}_2$.

In section 4 we give the classification of gradings on $\mathfrak{g}_2$. Although we do it for the complex field, the results are also valid over algebraically closed fields of characteristic zero. We obtain the toral gradings by taking advantage that the grading translating mechanism works better considering gradings of C by their universal grading groups, a notion introduced in subsection 2.2. Properties as the fine and toral character of gradings are well behaved under the translating mechanism. For the toral gradings we do not use directly the method in subsection 3.3, instead we use an algebraic tool basically equivalent but requiring less computations. This consists of the classification of group epimorphisms starting from $\mathbb{Z}^2$ under a suitable equivalence relation. We conclude that there are only 2 fine gradings, 1 nontoral, 24 toral ones, and of these, 20 are equivalent to gradings by cyclic groups. We highlight this fact because the gradings by cyclic groups have been essentially determined by Kac (see [16]). This author have found the automorphisms of finite order in a simple Lie algebra $\mathcal{L}$. His method uses the extended Dynkin diagrams and it is so easy that it deserves some explanation here. In the case of $\mathcal{L} = \mathfrak{g}_2$, if $\mathcal{L} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi} L_\alpha \right)$ is the root decomposition relative to a Cartan subalgebra $\mathfrak{h} = L_0$ and $\{\alpha_1, \alpha_2\}$ is a basis of the root system Φ, any $\mathbb{Z}_m$-grading comes from a triplet (p_0, p_1, p_2) of nonnegative integers associated to the edges of the affine diagram G_2^1:



such that $p_0 + 2p_1 + 3p_2 = m$, being the induced grading $\mathcal{L} = \oplus_{n\in\mathbb{Z}_m}\mathcal{L}_n$ where $\mathcal{L}_n$ is the sum of all root spaces L_α such that $\alpha = n_1\alpha_1 + n_2\alpha_2 \in \Phi \cup \{0\}$ and $n_1p_1 + n_2p_2 \equiv n (\mathrm{mod}\ m)$ for $n \in \{0, \ldots, m-1\}$. It is easy to obtain recursively the cyclic gradings but it is not so clear when to stop the process, and specially, it is not straightforward to deal with the case of mixed cyclic gradings. Subsection 3.2 and Lemma 3 show that gradings by noncyclic groups are not so easy to find in contrast with the simple algebraic method given for cyclic groups.

2. Generalities on gradings

2.1. **Gradings on Lie algebras.** Let A be an algebra (not necessarily associative, or Lie) over a field F. Also the dimension of A may be arbitrary. Let I be a nonempty set and

$$A = \bigoplus_{i\in I} A_i \tag{1}$$

a decomposition into nonzero subspaces such for any $i, j \in I$ we have $A_iA_j = 0$ or else there is a (necessarily unique) $k \in I$ such that $A_iA_j \subset A_k$. Then we shall say that the decomposition (1) is a *grading* of A. When A is a Lie algebra we shall say that (1) is a *Lie grading*. The main definitions on Lie gradings can be found in the reference [18]. Roughly speaking, a grading is a *coarsening* of a second one if the first arises collecting together some of the grading spaces of the second. In this case we say that the second grading is a *refinement* of the first one. Two gradings $A = \oplus_{i\in I}X_i = \oplus_{j\in J}Y_j$ of the same algebra are said to be *equivalent* when there is a bijection $\sigma\colon I \to J$ and an automorphism $f \in \mathrm{aut}(A)$ such that $f(X_i) = Y_{\sigma(i)}$ for all $i \in I$. When we have an algebra A and a semigroup G such that A decomposes in the way $A = \oplus_{i\in G} A_i$ and:

(1) $A_i A_j \subset A_{ij}$ for all $i, j \in G$,
(2) G is generated by $\{g \in G : A_g \neq 0\}$,

we shall say that the previous decomposition is a *G-grading on A*. A grading $A = \oplus_{i\in I} X_i$ is defined to be equivalent to a G-grading $A = \oplus_{g\in G} Y_g$ when taking $J := \{g \in G : Y_g \neq 0\}$, the grading $A = \oplus_{g\in J} Y_g$ is equivalent to the first one. It should be clear now when a G-grading is equivalent to a G'-grading of the same algebra. On the other hand, if we have a G-grading on the algebra $A = \oplus_{g\in G} A_g$ and a G'-grading on the same algebra $A = \oplus_{g'\in G'} A'_{g'}$ then we say that the given gradings are *isomorphic* if there is an isomorphism of semigroups $\sigma\colon G \to G'$ and an isomorphism of algebras $f\colon A \to A'$ such that $f(A_g) = A'_{\sigma(g)}$ for all $g \in G$.

The result in [6] shows that there are Lie gradings which are not gradings over any semigroup G. Therefore [18, Theorem 1 (d), p. 94] is not true, but as long as we know there is no example of a Lie grading on a simple finite-dimensional Lie algebra over an algebraically closed field of zero characteristic which is not equivalent to a group grading. For such an algebra, it follows from [18] that if we have a semigroup grading, then it is equivalent to a grading by an abelian group. This does not imply that gradings on simple Lie algebras by nonabelian groups are impossible. However we shall prove that this is the case. Before proving that, let us see an instance of a G-grading with G nonabelian which is equivalent to a G'-grading where G' is abelian. To do this, consider the triangle group $S_3 = \{1, g, g^2, \sigma, \sigma g, \sigma g^2\}$ with $g^3 = \sigma^2 = 1$ and $g\sigma = \sigma g^2$. Also consider in $\mathcal{L} := A_1 \times A_1 = \mathrm{sl}(V_1) \times \mathrm{sl}(V_2)$ (with $\dim V_i = 2$, $i = 1, 2$) the basis $\{h_1, h_2, x_1, x_2, y_1, y_2\}$ where $\{h_i, x_i, y_i\}$ is the standard basis of $\mathrm{sl}(V_i)$. Then, the following is a S_3-grading on $\mathcal{L}$:

$$\begin{array}{lll} \mathcal{L}_1 = \langle h_1, h_2\rangle, & \mathcal{L}_\sigma = \langle x_2, y_2\rangle, & \mathcal{L}_{\sigma g} = 0, \\ \mathcal{L}_{\sigma g^2} = 0, & \mathcal{L}_g = \langle x_1\rangle, & \mathcal{L}_{g^2} = \langle y_1\rangle. \end{array}$$

This S_3-grading is equivalent to the following $\mathbb{Z}_6$-grading on $\mathcal{L} = \mathcal{M}_0 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \oplus \mathcal{M}_4$ where $\mathcal{M}_0 = \langle h_1, h_2\rangle$, $\mathcal{M}_2 = \langle x_1\rangle$, $\mathcal{M}_3 = \langle x_2, y_2\rangle$ and $\mathcal{M}_4 = \langle y_1\rangle$.

In particular, gradings by nonabelian groups are possible over semisimple Lie algebras. However they are not possible over simple Lie algebras:

Proposition 1. *If $\mathcal{L}$ is a simple (finite-dimensional) Lie algebra graded by a group G, then G is abelian.*

Proof. First of all we note that for any Lie algebra $\mathcal{K}$, if $\mathcal{K} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition (that is, $\mathfrak{h}$ is a subalgebra of $\mathcal{K}$ and $\mathfrak{m}$ is an $\mathfrak{h}$-module), then the annihilator of $\mathfrak{m}$ in $\mathfrak{h}$ defined as $\mathrm{ann}_{\mathfrak{h}}(\mathfrak{m}) := \{x \in \mathfrak{h} : [x, \mathfrak{m}] = 0\}$ is an ideal of $\mathcal{K}$ (hence if $\mathcal{K}$ is simple, it vanishes). Let now $\mathcal{L} = \oplus_{g\in G} \mathcal{L}_g$ be a G-grading on the simple Lie algebra $\mathcal{L}$. Each time we give a subgroup $G' \subset G$, we get a reductive decomposition by defining $\mathfrak{h} := \sum_{g\in G'} \mathcal{L}_g$ and $\mathfrak{m} := \sum_{g\in G-G'} \mathcal{L}_g$. We know that the set $\{g \in G : \mathcal{L}_g \neq 0\}$ is a set of generators of G. For each $g_1 \in G$ such that $\mathcal{L}_{g_1} \neq 0$ we can consider the set $S_{g_1} := \{g \in G : gg_1 = g_1 g\}$. If for all g_1 such that $\mathcal{L}_{g_1} \neq 0$ we have $S_{g_1} = G$, then there is a system of generators which are central and so G is abelian. Otherwise there is some g_1 such that $\mathcal{L}_{g_1} \neq 0$ and $G' := S_{g_1}$ is a proper subgroup of G. As before we have a reductive decomposition $\mathcal{L} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h} = \sum_{g\in G'} \mathcal{L}_g$ and $\mathfrak{m} = \sum_{g\in G-G'} \mathcal{L}_g$, hence $\mathrm{ann}_{\mathfrak{h}}(\mathfrak{m}) = 0$.

We must now realize that for two noncommuting elements $g_1, g_2 \in G$ we always have $[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}] = 0$ (otherwise: $0 \neq [\mathcal{L}_{g_1}, \mathcal{L}_{g_2}] = [\mathcal{L}_{g_2}, \mathcal{L}_{g_1}] \subset \mathcal{L}_{g_1 g_2} \cap \mathcal{L}_{g_2 g_1} = 0$). But then, as g_1 does not commute with any element in $G - G' \neq \emptyset$, we have $[\mathcal{L}_{g_1}, \mathfrak{m}] = 0$, that is, $\mathcal{L}_{g_1} \subset \mathrm{ann}_{\mathfrak{h}}(\mathfrak{m}) = 0$, a contradiction. $\square$

In this paper we shall be concerned mainly with simple finite-dimensional Lie algebras. Because of the results above, we must concentrate on gradings by finitely generated abelian groups. Another result implying the commutativity of grading groups is the one given in [5, Lemma 5, p. 348] following which, any group grading a Cayley-Dickson algebra is necessarily commutative.

2.2. **Universal grading group.** In this section and the remaining, we return to the convention that all the gradings will be equivalent to group gradings. But we know that different groups can produce equivalent gradings. Thus a universal procedure to select a unique group among all those producing the same grading (up to equivalence) would be interesting. So, let us start from a grading on a simple finite-dimensional Lie F-algebra

$$\mathcal{L} = \oplus_{i \in I} \mathcal{L}_i. \tag{2}$$

The set I must be finite given the finite-dimensional character of $\mathcal{L}$. Define now $\mathbb{Z}(I)$ as the free $\mathbb{Z}$-module generated by I, which has the following universal property: there is an injection $j \colon I \to \mathbb{Z}(I)$, $i \mapsto i$ such that for any map $j' \colon I \to G$ from I to a $\mathbb{Z}$-module G, there is a unique homomorphism of $\mathbb{Z}$-modules $f \colon \mathbb{Z}(I) \to G$ such that $f \circ j = j'$. Consider now the $\mathbb{Z}$-submodule M generated by the elements $i_1 + i_2 - i_3$ where (i_1, i_2, i_3) ranges over the triplets such that $0 \neq [\mathcal{L}_{i_1}, \mathcal{L}_{i_2}] \subset \mathcal{L}_{i_3}$. Finally define $G_I := \mathbb{Z}(I)/M$, which is a finitely generated abelian group. This is the abelian group constructed in [18, p. 93]. Suppose now that G is a finitely generated abelian group and $\mathcal{L} = \oplus_{g \in G} X_g$ is a coarsening of the grading (2). Then there is a surjective map $\sigma \colon I \to J := \{g \in G : X_g \neq 0\}$ such that $\mathcal{L}_i \subset X_{\sigma(i)}$, $(i \in I)$. Applying the universal property of $\mathbb{Z}(I)$ there is a unique homomorphism of abelian groups $f \colon \mathbb{Z}(I) \to G$ such that $f(i) = \sigma(i)$. But it is not difficult to prove that $M \subset \ker(f)$ hence there is a group homomorphism $\bar{f} \colon G_I \to G$ such that $\bar{f}(\bar{i}) = \sigma(i)$ for all $i \in I$. This is easily seen to be an epimorphism. The original grading (2) is in fact a G_I-grading writing $\mathcal{L} = \oplus_{k \in G_I} \mathcal{L}'_k$ where $\mathcal{L}'_k := \mathcal{L}_i$ if $k = \bar{i}$ and $\mathcal{L}'_k := 0$ otherwise. Summarizing this subsection we have:

Proposition 2. *Let $\mathcal{L} = \oplus_{i \in I} \mathcal{L}_i$ be a Lie grading (equivalent to a group grading) of a simple finite-dimensional Lie algebra $\mathcal{L}$. Then there is a finitely generated abelian group G_I containing I such that:*

1. *The previous grading can be rewritten as $\mathcal{L} = \oplus_{k \in G_I} \mathcal{L}'_k$ where $\mathcal{L}'_k = \mathcal{L}_k$ if $k \in I$ and $\mathcal{L}'_k = 0$ otherwise.*
2. *For any other finitely generated abelian group G and any coarsening $\mathcal{L} = \oplus_{g \in G} X_g$ of the previous grading, there is a unique group epimorphism $f \colon G_I \to G$ such that $X_g = \sum_{f(k)=g} \mathcal{L}'_k$ for any $g \in G$.*

The property above could be stated in terms of initial objects in a suitable category. So it is a universal property which justifies the name *universal grading group* for G_I.

To see how this universal property works in a concrete example consider the Lie algebra $\mathcal{L} = \mathrm{sl}(3, F)$ (the field F is still arbitrary). Let $\mathfrak{h}$ be the Cartan subalgebra $\mathfrak{h} = \langle h_1, h_2 \rangle$ where $h_1 = e_{11} - e_{33}$ and $h_2 = e_{22} - e_{33}$ being e_{ij} the elementary matrix whose entries are zero except for the (i,j) one which is 1. Then, the following decomposition of $\mathcal{L}$ is a Lie grading $\mathcal{L} = \mathcal{L}_i \oplus \mathcal{L}_j \oplus \mathcal{L}_k \oplus \mathcal{L}_m \oplus \mathcal{L}_n$, where:

$$\mathcal{L}_i = \mathfrak{h},\ \mathcal{L}_j = \langle e_{21}, e_{32}\rangle,\ \mathcal{L}_k = \langle e_{31}\rangle,\ \mathcal{L}_m = \langle e_{12}, e_{23}\rangle,\ \mathcal{L}_n = \langle e_{13}\rangle.$$

Indeed the multiplicative relations on the subspaces involved could be summarized in the table:

$\cdot$	$\mathcal{L}_i$	$\mathcal{L}_j$	$\mathcal{L}_k$	$\mathcal{L}_m$	$\mathcal{L}_n$
$\mathcal{L}_i$	0	$\mathcal{L}_j$	$\mathcal{L}_k$	$\mathcal{L}_m$	$\mathcal{L}_n$
$\mathcal{L}_j$		$\mathcal{L}_k$	0	$\mathcal{L}_i$	$\mathcal{L}_m$
$\mathcal{L}_k$			0	$\mathcal{L}_j$	$\mathcal{L}_i$
$\mathcal{L}_m$				$\mathcal{L}_n$	0
$\mathcal{L}_n$					0

where for instance the $(2,4)$ entry of the table means $0 \neq [\mathcal{L}_j, \mathcal{L}_m] \subset \mathcal{L}_i$. We want to compute the universal group G_I where $I = \{i,j,k,m,n\}$. So we must consider the free $\mathbb{Z}$-module $\mathbb{Z}(I)$ and the submodule M generated by all the elements $a+b-c$ where the triplet (a,b,c) satisfies $0 \neq [\mathcal{L}_a, \mathcal{L}_b] \subset \mathcal{L}_c$. The generators of M in our example are

$$i,\ 2j-k,\ j+m-i,\ j+n-m,\ k+m-j,\ k+n-i,\ 2m-n.$$

Thus we must compute $\mathbb{Z}(I)/M$ but identifying $\mathbb{Z}(I)$ with $\mathbb{Z}^5$ in such a way that i,j,k,m,n are identified with the canonical basis of $\mathbb{Z}^5$ respectively, we get an isomorphism $G_I \cong \mathbb{Z}^5/M'$ where M' is the submodule of $\mathbb{Z}^5$ generated by the rows of the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & -1 \end{pmatrix}.$$

Now, we can find the structure of $\mathbb{Z}^5/M'$ as direct sum of cyclic subgroups by computing the so called *normal form* of the matrix A (see [14, Sections 3.7 and 3.8, p. 181-188]). After some computations, we find by using elementary row and column transformations that the normal form of A is the matrix:

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As a consequence $G_I := \mathbb{Z}(I)/M \cong \mathbb{Z}^5/M' \cong \mathbb{Z}^5/M''$ where M'' is the $\mathbb{Z}$-submodule of $\mathbb{Z}^5$ generated by the rows of A'. So $M'' = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times 0$ and we get $\mathbb{Z}(I)/M \cong \mathbb{Z}$. This proves that the universal grading group is isomorphic to $\mathbb{Z}$. Indeed it is not difficult to realize that the given grading is a $\mathbb{Z}$-grading of the type $\mathcal{L} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$ for $\mathcal{L}_{-2} = \langle e_{13}\rangle = \mathcal{L}_n$, $\mathcal{L}_{-1} = \langle e_{12}, e_{23}\rangle = \mathcal{L}_m$, $\mathcal{L}_0 = \mathfrak{h} = \mathcal{L}_i$, $\mathcal{L}_1 = \langle e_{21}, e_{32}\rangle = \mathcal{L}_j$, $\mathcal{L}_2 = \langle e_{31}\rangle = \mathcal{L}_k$. This grading is also a $\mathbb{Z}_5$-grading if we define $\mathcal{L} = L_1 \oplus L_\omega \oplus L_{\omega^2} \oplus L_{\omega^3} \oplus L_{\omega^4}$ where $\omega = \exp(2\pi i/5)$ is a primitive fifth root of 1 and $L_1 = \mathfrak{h}$, $L_\omega = \langle e_{21}, e_{32}\rangle$, $L_{\omega^2} = \langle e_{31}\rangle$, $L_{\omega^3} = \langle e_{13}\rangle$ and $L_{\omega^4} = \langle e_{12}, e_{23}\rangle$ (identifying $\mathbb{Z}_5$ with $\{\omega^i : i \in \mathbb{Z}\}$). The universal property of G_I implies that there is an epimorphism $f\colon G_I = \mathbb{Z} \to \mathbb{Z}_5$ such that $\mathcal{L}_i = L_{f(i)}$ for $i \in \{-2, -1, 0, 1, 2\}$ and of course, this epimorphism is the induced by $1 \mapsto \omega$.

The computations needed for the determination of the universal grading group are easily converted into an algorithm. So the determination of G_I for gradings not so obvious as the given one, is an automatic task.

Another property of the universal grading group is that we can compute it when the starting Lie grading is not a group grading (even if the Lie algebra is not simple). In this case the map $I \to G_I$ mapping any $i \in I$ to its equivalence class in G_I will not be injective. So the computation of the universal group and the canonical map $I \to G_I$ will tell us if a given grading is certainly a group grading or not depending on the injectivity of the mentioned map. In the previous example we do not know a priori if the given grading is a group grading, but the study of its universal grading group provides an affirmative answer. On the contrary, for the grading $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ where $\mathcal{L}_i$ are simple ideals, the group G_I is trivial, so the map $I \to G_I$ is not injective and there does not exist a group G such that this grading is equivalent to a G-grading.

2.3. **Automorphisms and gradings.** A useful way of seeing gradings is that of semisimple automorphisms (see the section with the same title in [17, §3, p. 104]). Following this reference, a commutative complex algebraic group whose identity component is an algebraic torus is called an algebraic *quasitorus.* Quasitori are direct products of tori and commutative finite groups though this is not essential for our study. Also an algebraic linear group is a quasitorus if and only if there is a basis relative to which the elements of the quasitorus are simultaneously diagonalizable. If S is a finitely generated abelian group, then its group of characters $\mathfrak{X}(S) = \hom(S, \mathbb{C}^\times)$ is a quasitorus and reciprocally, the group of characters of a quasitorus turns out to be a finitely generated abelian group.

The crucial point to have in mind is the following. Let S be a finitely generated abelian group. Of course $S = \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ where $r, n_j \in \mathbb{Z}$, $n_j > 1$, $r \geq 0$. The group of characters $\mathfrak{X}(S)$ is then $\mathfrak{X}(S) = (\mathbb{C}^*)^r \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ since $\mathfrak{X}(\mathbb{Z}) = \mathbb{C}^*$ and $\mathfrak{X}(\mathbb{Z}_n) = \mathbb{Z}_n$ for all $n > 1$. Moreover, any homomorphism $\varphi\colon \mathfrak{X}(S) \to \operatorname{aut}(A)$ to the group of automorphisms of the complex algebra A, provides a S-grading

$$A = \oplus_{i \in S} A_i$$

where $A_i := \{a \in A : \varphi(t)(a) = t(i)a, \forall t \in \mathfrak{X}(S)\}$. In [17, p. 106] it is proved that any grading on A by a finitely generated abelian group arises in this way.

In the special case of a $\mathbb{Z}$-grading on a complex algebra $\mathcal{L}$ (not necessarily a Lie algebra) we have a homomorphism $\varphi\colon \mathbb{C}^* \to \operatorname{aut}(\mathcal{L})$. Any character $t \in \mathfrak{X}(\mathbb{Z}) = \hom(\mathbb{Z}, \mathbb{C}^*)$ is of the form $t(n) = z^n$ for some $z \in \mathbb{C}^*$ and therefore the grading is $\mathcal{L} = \oplus_{n \in \mathbb{Z}} \mathcal{L}_n$ where $\mathcal{L}_n$ is the subspace of elements $x \in \mathcal{L}$ such that $\varphi(z)x = z^n x$ for all $z \in \mathbb{C}^*$. The map $\theta := d\varphi(1) : \mathbb{C} \to \operatorname{Der}(\mathcal{L})$ is linear and for $d_0 := \theta(1) \in \operatorname{Der}(\mathcal{L})$ we have $\theta(\lambda) = \lambda d_0$ for all $\lambda \in \mathbb{C}$. The derivation d_0 allows us to get a new description of the spaces $\mathcal{L}_n$ of the grading. Indeed, differentiating the equality $\varphi(z)x = z^n x$ at $z = 1$ we get that $x \in \mathcal{L}_n$ if and only if $d_0(x) = nx$. So the spaces $\mathcal{L}_n$ of the grading are the eigenspaces of d_0 and this is a diagonalizable derivation of $\mathcal{L}$ (with integer eigenvalues). We can go a step further by considering the diagonalizable automorphism $\psi := \exp(d_0)$ whose eigenvalues are of the form

$\exp(n)$ with $n \in \mathbb{Z}$, and which allows to describe $\mathcal{L}_n$ in the form $\mathcal{L}_n = \{x \in \mathcal{L} : \psi(x) = \exp(n)x\}$. If the grading is not trivial, $d_0 \neq 0$ and in case $\mathcal{L}$ is a complex semisimple finite-dimensional Lie algebra, the derivation d_0 is inner $d_0 = \mathrm{ad}(h_0)$ and h_0 can be taken in some Cartan subalgebra $\mathfrak{h}$ of $\mathcal{L}$. Thus $\mathfrak{h} \subset \ker(d_0) = \mathcal{L}_0$.

Suppose now that we have a $\mathbb{Z}_n$-grading of $\mathcal{L}$. Since $\mathfrak{X}(\mathbb{Z}_n) = \mathbb{Z}_n$, what we have now is a homomorphism $\varphi : \mathbb{Z}_n \to \mathrm{aut}(\mathcal{L})$ which is completely determined once we know $\varphi(1) =: f \in \mathrm{aut}(\mathcal{L})$. Moreover $f^n = 1_{\mathcal{L}}$ so that a $\mathbb{Z}_n$- grading is given by a finite order automorphism. Again, if the grading is not trivial $f \neq 1_{\mathcal{L}}$.

The information contained in the last two paragraphs allows us to conclude that a G-grading over a cyclic group G of a complex algebra $\mathcal{L}$ is just the decomposition of the algebra as direct sum of the eigenspaces relative to a diagonalizable automorphism of $\mathcal{L}$.

In the rest of the section all the groups considered are supposed to be finitely generated and abelian. Let G be a group which is the direct product of groups G_i $(i = 1, \ldots, n)$. The projection epimorphisms $\pi_i \colon G \to G_i$ induce monomorphisms $e_i \colon \mathfrak{X}(G_i) \to \mathfrak{X}(G)$, hence any G-grading of an algebra $\mathcal{L}$ given by $\varphi \colon \mathfrak{X}(G) \to \mathrm{aut}(\mathcal{L})$ produces G_i-gradings $\varphi_i \colon \mathfrak{X}(G_i) \to \mathrm{aut}(\mathcal{L})$ where $\varphi_i = \varphi \circ e_i$. Now, for any $\alpha_i \in \mathfrak{X}(G_i)$, $\alpha_j \in \mathfrak{X}(G_j)$ $(i, j = 1, \ldots, n)$ it is easy to prove the commutativity

$$\varphi_i(\alpha_i)\varphi_j(\alpha_j) = \varphi_j(\alpha_j)\varphi_i(\alpha_i). \tag{3}$$

Lemma 1. *Let $\{f_1, \ldots, f_k\}$ be a commutative family of diagonalizable automorphisms of a complex algebra $\mathcal{L}$ (not necessarily a Lie algebra). Then we get a grading of $\mathcal{L}$ in the form $\mathcal{L} = \oplus_{i \in I}\mathcal{L}_i$ where:*

1. *The $\mathcal{L}_i$ are f_j-invariant (for all i and j).*
2. *For any $i \in I$, the restriction of each of the maps f_j to $\mathcal{L}_i$ is a scalar multiple of the identity.*

Furthermore, each G-grading on $\mathcal{L}$ is of the previous form.

Proof. The direct part of the Lemma is a standard linear algebra result. Suppose now a G-grading of $\mathcal{L}$ given by a homomorphism $\varphi \colon \mathfrak{X}(G) \to \mathrm{aut}(\mathcal{L})$. We can argue by induction on the minimum number of cyclic factors of G, applying the commutativity condition (3). $\square$

Corollary 1. *Let $\mathcal{L}$ be a (finite-dimensional) semisimple complex Lie algebra with a G-grading $\varphi \colon \mathfrak{X}(G) \to \mathrm{aut}(\mathcal{L})$ where G is the torsion free group $G = \mathbb{Z}^n$. Then $d\varphi(1) \colon \mathbb{C}^n \to \mathrm{Der}(\mathcal{L})$ is a monomorphism.*

Proof. Let $\mathcal{L} = \oplus_{g \in G}\mathcal{L}_g$ be the grading and $\varphi \colon \mathfrak{X}(\mathbb{Z}^n) \to \mathrm{aut}(\mathcal{L})$ the homomorphism inducing the grading. Consider as before the gradings $\varphi_i \colon \mathfrak{X}(\mathbb{Z}) \to \mathrm{aut}(\mathcal{L})$ for $i = 1, \ldots, n$. Each of these comes from a diagonalizable derivation with integer eigenvalues defined by $d_i = d\varphi_i(1)(1)$. Since $d\varphi(1)(\alpha_1, \ldots, \alpha_n) = \sum_i \alpha_i d_i$, to prove the injectivity we only need to show that $\{d_1, \ldots, d_n\}$ is linearly independent.

Suppose $\sum_i \alpha_i d_i = 0$. For any $g = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ we have $x \in \mathcal{L}_g$ if and only if $d_i(x) = \lambda_i x$ for all $i = 1, \ldots, n$, hence $0 = \sum \alpha_i d_i(x) = (\sum \alpha_i \lambda_i)x$ and we conclude that $\sum \alpha_i \lambda_i = 0$ whenever $(\lambda_1, \ldots, \lambda_n) \in S := \{g \in \mathbb{Z}^n : \mathcal{L}_g \neq 0\}$. Taking into account that S is a system of generators of $\mathbb{Z}^n$ we can prove now that $\alpha_i = 0$. For simplicity we are proving the equality $\alpha_1 = 0$. There exist integers n_i and l such that

$$(1, 0, \ldots, 0) = \sum_1^l n_i s_i \tag{4}$$

where $s_i = (\lambda_{1i}, \dots, \lambda_{ni}) \in S$ and therefore $\sum_k \alpha_i \lambda_{ki} = 0$ for each i. Moreover by (4) one has $1 = \sum_1^l n_i \lambda_{1i}$ and $0 = \sum_1^l n_i \lambda_{ki}$ for $k \neq 1$. So $\alpha_1 = \alpha_1 \sum_1^l n_i \lambda_{1i} + \sum_{k=2}^l \alpha_k \sum_{i=1}^l n_i \lambda_{ki} = \sum_{i,k=1}^l \alpha_k n_i \lambda_{ki} = \sum_i (\sum_k \alpha_k \lambda_{ki}) = 0$. $\square$

As a corollary to the previous result, if $\mathcal{L}$ is graded by a finitely generated abelian group, its torsion-free component $\mathbb{Z}^n$ satisfies $n \leq \mathrm{rank}(\mathcal{L})$ (see [18, Theorem 4, p. 149] for an alternative proof when $\mathcal{L}$ is simple).

2.4. **Toral gradings.** In this section we shall work with complex semisimple and finite-dimensional Lie algebras. A particular grading in any such Lie algebra $\mathcal{L}$ is the so called *Cartan grading* $\mathcal{L} = \mathfrak{h} \oplus \left(\oplus_{\alpha \in \Phi} L_\alpha \right)$ where $\mathfrak{h}$ is a Cartan subalgebra, Φ a root system and L_α the various root spaces. When $\mathcal{L}$ is simple, this grading is *fine* in the sense that it can not be further refined. The Cartan grading is a $\mathbb{Z}^n$-grading for $n = \mathrm{rank}(\mathcal{L})$ since the root system can be considered as a generating system of $\mathbb{Z}^n$. We shall say that a grading on $\mathcal{L}$ is *toral* if it is a coarsening of a Cartan grading.

We shall use some standard results of the theory of algebraic groups. For a connected algebraic group G, each semisimple element lies in a maximal torus of G (see for instance [13, Theorem 22.2, p. 139]). As maximal tori are conjugated ([13, Corollary A, p. 135]), another way to state the above result is that fixed a maximal torus T in G, any semisimple element of G is conjugated to some element in T.

Let now G be the group $\mathrm{aut}(\mathcal{L})$, and denote by G_0 the subgroup of inner automorphisms of $\mathcal{L}$. As mentioned in [15, Remark, p. 281], the group G_0 is the algebraic component of the identity element of the linear algebraic group G. Moreover in [15, Theorem 4, p. 281] it is proved that for $\mathcal{L}$ simple, $G = G_0$ unless $\mathcal{L}$ is of one of the following types: $\mathfrak{a}_l$, $l > 1$, $\mathfrak{d}_l$ or $\mathfrak{e}_6$. If $\mathcal{L}$ is semisimple, it is proved in [15, Proposition 3, p. 278] that any $f \in G$ fixing pointwise a Cartan subalgebra of $\mathcal{L}$ is necessarily an element of G_0.

For a simple algebra $\mathcal{L}$ fix a Cartan subalgebra $\mathfrak{h}$ and consider the Cartan decomposition

$$\mathcal{L} = \mathfrak{h} \oplus \left(\oplus L_{\alpha_i} \right) \tag{5}$$

relative to $\mathfrak{h}$. Fix any basis $\{h_1, \dots, h_r\}$ in $\mathfrak{h}$ and enlarge this to a basis $B = \{h_1, \dots, h_r, v_{\alpha_1}, \dots v_{\alpha_l}\}$ of $\mathcal{L}$ with $v_{\alpha_i} \in L_{\alpha_i}$. Suppose now that $f \in \mathrm{aut}(\mathcal{L})$ is diagonal relative to B. Then it is easy to see that f is the identity on $\mathfrak{h}$. We shall call these elements *diagonal relative to the given Cartan decomposition*. As we mentioned before, these automorphisms are in the identity component G_0 of the group $G = \mathrm{aut}(\mathcal{L})$. Another known result ensures that the subgroup of G_0 given by the diagonal elements relative to this Cartan decomposition is a maximal torus T_0 of G_0.

Let $\mathcal{L} = \oplus_{i \in I} \mathcal{L}_i$ be a toral grading of $\mathcal{L}$ (a coarsening of (5), up to conjugation) induced by the diagonalizable automorphisms $\{f_1, \dots, f_n\}$. So the restriction of f_j to $\mathcal{L}_i$ is a scalar multiple of the identity and any f_j is diagonal relative to the Cartan decomposition (5). Therefore $\{f_1, \dots, f_n\}$ is contained in the maximal torus T_0.

Reciprocally take $\{f_1, \dots, f_n\}$ a set of commuting semisimple elements contained in a maximal torus T of G. Note that $T \subset G_0$ since T is connected and $1 \in T$. As any two maximal tori of G_0 are conjugated then there is $p \in G_0$ such that defining $f_i' := p f_i p^{-1}$, we have $\{f_1', \dots, f_n'\} \subset T_0$. The grading induced by the f_i''s is toral since these automorphisms are diagonal relative to the Cartan decomposition (5), therefore the original grading coming from the f_i's is also toral since the f_i's are diagonal relative to the Cartan decomposition

$$\mathcal{L} = p^{-1}(\mathfrak{h}) \oplus \big(\oplus p^{-1}(L_{\alpha_i}) \big).$$

Summarizing, the grading induced by a set of automorphisms $\{f_1, \dots, f_n\}$ is toral if and only if there is some maximal torus in $\mathrm{aut}(\mathcal{L})$ containing the whole set $\{f_1, \dots, f_n\}$.

Let us remark another feature of the toral gradings. Note that, according to subsection 2.3, if $\mathcal{L} = \oplus_{g \in A} \mathcal{L}_g$ is a grading produced by $\{f_1, \dots, f_n\}$, the identity component is $\mathcal{L}_0 = \{x \in \mathcal{L} : f_i(x) = x \, \forall i = 1, \dots n\}$. If this grading is toral (relative to (5)), as before f_i is the identity in $\mathfrak{h}$ and hence $\mathfrak{h} \subset \mathcal{L}_0$. Reciprocally if $\mathfrak{h} \subset \mathcal{L}_0$, $f_i|_{\mathfrak{h}} = \mathrm{id}$ and an easy computation shows that $f_i(L_\alpha) \subset L_\alpha$ for any root space L_α so that the grading is toral. This characterizes toral gradings as those whose zero component contains a Cartan subalgebra of $\mathcal{L}$. It is well-known that $\mathcal{L}_0$ is a reductive Lie algebra (see [12, Remark 3.5.]). Hence the number $\mathrm{rank}(\mathcal{L}_0)$ takes sense and therefore toral gradings are characterized by the formula: $\mathrm{rank}(\mathcal{L}_0) = \mathrm{rank}(\mathcal{L})$ (a useful criterion for testing the toral nature of a given grading).

Now we will design a mechanism for refining toral gradings to nontoral ones (if possible). Given a toral grading by a set of commuting diagonalizable automorphisms $f_1, \dots, f_n \in \mathrm{aut}(\mathcal{L})$, we can define the subgroup $Z := C_G(f_1, \dots, f_n)$ of those $g \in G = \mathrm{aut}(\mathcal{L})$ such that $gf_i = f_i g$ for all i (that is, the centralizer of the f_i's in G). This is a closed subgroup of the algebraic group G and we can consider the decomposition of Z into connected components (as usual Z_0 will denote the unit connected component). Let us see that $\{f_1, \dots, f_n\} \subset Z_0$. Since the grading is toral there is a (maximal) torus T such that $\{f_1, \dots, f_n\} \subset T$. But T being abelian, it is necessarily contained in Z and thus in Z_0. In particular if Z is not connected and there exists some diagonalizable element out of the connected component of the unit $f_{n+1} \in Z - Z_0$, the grading produced by $\{f_1, \dots, f_{n+1}\}$ is nontoral.

On the other hand, it is easy to see that taking a diagonalizable element $f_{n+1} \in Z_0$, the grading produced by $\{f_1, \dots, f_{n+1}\}$ is toral. Indeed, as $T \subset Z_0$ and it is a maximal torus in Z_0, there is some $p \in Z_0$ such that $pf_{n+1}p^{-1} \in T$. Hence the set $\{f_1, \dots, f_{n+1}\} \subset p^{-1}Tp$. We can summarize this in the following theorem.

Theorem 1. *Let $\mathcal{L}$ be a complex semisimple Lie algebra with a toral grading $\mathcal{L} = \oplus_{g \in A} \mathcal{L}_g$ induced by the automorphisms $\{f_1, \dots, f_n\}$ in $G := \mathrm{aut}(\mathcal{L})$. Let $Z \subset G$ be the centralizer in G of the f_i's and Z_0 its unit connected component. Then the grading can be refined to a nontoral one if and only if there exists some diagonalizable $f_{n+1} \in Z - Z_0$ and considering the grading induced by $\{f_1, \dots, f_{n+1}\}$.*

An example may be convenient at this point. So consider the Lie algebra $\mathcal{L} = \mathrm{sl}(n+1, \mathbb{C})$, $(n \geq 1)$ and the $\mathbb{Z}_2$-grading $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ where $\mathcal{L}_i$ is the eigenspace of eigenvalue $(-1)^i$ relative to the involutive automorphism $f_1 \colon \mathcal{L} \to \mathcal{L}$ given by $f_1(x) := pxp^{-1}$ where p is the block-diagonal matrix

$$p = \begin{pmatrix} 1_n & 0 \\ 0 & -1 \end{pmatrix},$$

being 1_n the $n \times n$ identity matrix. Thus the matrices in $\mathcal{L}_0$ are of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with the same block structure, while $\mathcal{L}_1$ consists of the matrices $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. We want to know if this grading, which is toral of course, can be refined to a nontoral one. Thus we compute the centralizer Z of f_1 in $\mathrm{aut}(\mathcal{L})$. This turns out to be the group of automorphisms $x \mapsto qx^{\square}q^{-1}$ where $x \mapsto x^{\square}$ is the identity or the opposite of the transposition, and q is of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, for $a \in \mathrm{SL}(n, \mathbb{C})$. Thus $Z = Z_0 \cup Z_1$ has two connected components which are precisely the intersection of Z with the two components of $\mathrm{aut}(\mathcal{L})$ (its unit component, isomorphic to $\mathrm{PSL}(n+1, \mathbb{C})$, corresponds to $\square = \mathrm{id}$). Therefore taking a diagonalizable element $f_2 \in Z_1$ in the complementary of the unit component Z_0 we shall get a nontoral grading induced by $\{f_1, f_2\}$. For instance we can take $f_2(x) := -x^t$ (minus transposition operator). We get a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading where $\mathcal{L}_{00}$ is the set of matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, being a antisymmetric; $\mathcal{L}_{01}$ is the set of matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, being $b \in \mathbb{C}$ and a a symmetric matrix; $\mathcal{L}_{10}$ is the set of matrices $\left(\begin{smallmatrix} 0 & b \\ -b^t & 0 \end{smallmatrix}\right)$, and $\mathcal{L}_{11}$ is the set of matrices $\left(\begin{smallmatrix} 0 & b \\ b^t & 0 \end{smallmatrix}\right)$. We see that $\mathrm{rank}(\mathcal{L}_{00}) \leq \frac{n}{2} < n = \mathrm{rank}(\mathcal{L})$, in agreement with the fact that the grading is not toral.

3. GRADINGS ON OCTONIONS

In this section we are dealing with the simple alternative algebra of complex octonions $\mathbb{O}_{\mathbb{C}}$, isomorphic to the Zorn matrices algebra (see [21]). In this algebra we shall use intensively the so called *standard basis* $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ whose multiplication table is

$\cdot$	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	e_1	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	e_1	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	e_1
v_1	v_1	0	e_2	0	0	0	$-u_3$	u_2
v_2	v_2	0	0	e_2	0	u_3	0	$-u_1$
v_3	v_3	0	0	0	e_2	$-u_2$	u_1	0

.

As we are concerned with group gradings of Lie algebras we also must focus on gradings over groups (necessarily abelian as proved in [5, Lemma 5, p. 348]) of $\mathbb{O}_{\mathbb{C}}$. Thus, when speaking of gradings over $\mathbb{O}_{\mathbb{C}}$ we shall mean gradings over abelian groups.

The algebra $\mathcal{L} := \mathfrak{g}_2 = \mathrm{Der}(\mathbb{O}_{\mathbb{C}})$ is the well-known exceptional 14-dimensional Lie algebra. As one learns for instance from [15], the automorphism group $G_2 = \mathrm{aut}(\mathbb{O}_{\mathbb{C}})$ and the automorphism group $\mathrm{aut}(\mathfrak{g}_2)$ are isomorphic via the map $\mathrm{Ad}\colon G_2 \to \mathrm{aut}(\mathfrak{g}_2)$ such that $\mathrm{Ad}(f)d := fdf^{-1}$ for any $f \in G_2$ and $d \in \mathfrak{g}_2$. This is an isomorphism of algebraic groups so it maps semisimple elements of one group to semisimple elements in the other one. Thus any group grading on $\mathbb{O}_{\mathbb{C}}$, which is given by a set of diagonalizable automorphisms $\{f_1, \ldots, f_n\}$, induces a grading on $\mathfrak{g}_2$ by means of $\{\mathrm{Ad}(f_1), \ldots, \mathrm{Ad}(f_n)\}$ and conversely. Therefore we have a device for passing gradings from $\mathbb{O}_{\mathbb{C}}$ to $\mathfrak{g}_2$ and reciprocally. Since the gradings of $\mathbb{O}_{\mathbb{C}}$ has been fully described by A. Elduque in a more general context (see [5]), it could be thought that, passing gradings from $\mathbb{O}_{\mathbb{C}}$ to $\mathfrak{g}_2$, we have a complete description of gradings on $\mathfrak{g}_2$. However the device for translating gradings from one algebra to the other one does not preserve equivalences though it does preserve isomorphisms. In other words we can have two equivalent gradings on $\mathbb{O}_{\mathbb{C}}$ whose induced gradings on $\mathfrak{g}_2$ are not equivalent. Since our aim is the classification up to equivalence of gradings on $\mathfrak{g}_2$, it is clear that we must introduce some other tool to complete this task. In spite of this bad behavior, some properties of gradings on $\mathbb{O}_{\mathbb{C}}$ pass to their induced gradings on $\mathfrak{g}_2$ and conversely. For instance, define a grading $\rho\colon \mathfrak{X}(G) \to \mathrm{aut}(\mathbb{O}_{\mathbb{C}})$ on $\mathbb{O}_{\mathbb{C}}$ to be *toral* if $\rho(\mathfrak{X}(G))$ is contained in a maximal torus of $\mathrm{aut}(\mathbb{O}_{\mathbb{C}})$. Then it is easy to prove that a grading on $\mathbb{O}_{\mathbb{C}}$ is toral if and only if the grading induced on $\mathfrak{g}_2$ is toral. One can check that a grading on $\mathbb{O}_{\mathbb{C}}$ is toral if and only if it is equivalent to a coarsening of the fine grading whose homogeneous spaces are $\langle e_1, e_2\rangle$, $\langle u_i\rangle$, $\langle v_i\rangle$, $(i = 1, 2, 3)$ for a standard basis $\{e_1, e_2, u_i, v_i\}_{i=1}^3$.

3.1. **Some subgroups of G_2.** Some interesting subgroups of G_2 will be necessary for our study of gradings. Consider first the group $G := \{f \in G_2 : f(\langle e_1, e_2\rangle) = \langle e_1, e_2\rangle\}$. Of course this is a linear algebraic group and its elements either fix the idempotents e_1 and e_2 or permute them. Thus G has two components, the identity component being the subgroup G_0 of automorphisms of $\mathbb{O}_{\mathbb{C}}$ fixing e_i $(i = 1, 2)$. It is immediate to see that G_0 is the subgroup of automorphisms whose matrix relative to the standard basis is of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & (M^t)^{-1} \end{pmatrix} \tag{6}$$

where $M \in \mathrm{SL}(3)$ is a 3×3 matrix with determinant 1 and $M \mapsto M^t$ is the matrix transposition. Thus in fact we have an isomorphism $G_0 \cong \mathrm{SL}(3)$. Another interesting subgroup is the maximal torus T of G_2 of all automorphisms $t_{\alpha,\beta}$ whose matrix in the standard basis is the diagonal matrix

$$\mathrm{diag}(1, 1, \alpha, \beta, (\alpha\beta)^{-1}, \alpha^{-1}, \beta^{-1}, \alpha\beta),$$

where $\alpha, \beta \in \mathbb{C}^{\times}$. We now compute the centralizers of some specific elements in the torus T. Let us define the set $S = \{1, \alpha, \beta, (\alpha\beta)^{-1}, \alpha^{-1}, \beta^{-1}, \alpha\beta\}$ of eigenvalues of $t_{\alpha,\beta}$. For any subset $V \subset G_2$ let $C_{G_2}(V)$ denote the centralizer of V in G_2. The diagonalizable elements of $C_{G_2}(V)$ not contained in its unit connected component will be called its *outer diagonalizable elements.* If $C_{G_2}(V)$ is connected then it obviously has no outer diagonalizable automorphisms. On the other hand, consider now a subset $V \subset T$ (the maximal torus of G_2 defined above). Then $T \subset C_{G_2}(V)$ and in fact T is a maximal torus in $C_{G_2}(V)$. Suppose now that any diagonalizable element in $C_{G_2}(V)$ is conjugated (in $C_{G_2}(V)$) to some element in its unit component $C_{G_2}(V)_0$. Then we can ensure that $C_{G_2}(V)$ has no outer diagonalizable automorphism. Indeed: let $f \in C_{G_2}(V)$ be a diagonalizable element, then for some $p \in C_{G_2}(V)$ we have $pfp^{-1} \in C_{G_2}(V)_0$. Since $C_{G_2}(V)_0$ is a normal subgroup of $C_{G_2}(V)$ then f itself is an element in $C_{G_2}(V)_0$ and so the diagonalizable elements can not be outer.

Lemma 2. *The group $C_{G_2}(t_{\alpha,\beta})$ has no outer diagonalizable automorphisms in the following cases:*

- $|S| = 7$.
- $\alpha = -1$ *and* $|S| = 6$.
- $\alpha = \beta$ *with* $|S| = 5$.
- $\alpha = \beta = \omega$ *a primitive fourth root of unit.*
- $\alpha = 1$, $\beta = -1$.
- $\alpha = 1$ *and* $|S| = 3$.
- $\alpha = \beta = \rho$ *a primitive cubic root of unit.*

Proof. If S has cardinal 7 then it is immediate that the centralizer $C_{G_2}(t_{\alpha,\beta})$ of $t_{\alpha,\beta}$ in G_2 is just the maximal torus T. If we take now the element $t_{-1,\beta}$ with $|S| = 6$, after some easy considerations also $C_{G_2}(t_{-1,\beta}) = T$. Consider now the element $t_{\alpha,\alpha}$ such that $S = \{1, \alpha, \alpha^{-1}, \alpha^2, \alpha^{-2}\}$ has cardinal 5. Then any $f \in C_{G_2}(t_{\alpha,\alpha})$ is an element in G and $f(u_3) \in \langle u_3\rangle$, $f(v_3) \in \langle v_3\rangle$. Thus necessarily $f \in G_0$ and f is of the form (6) with $M = \operatorname{diag}(N, |N|^{-1})$ for a 2×2 invertible matrix N. Thus, the map $f \mapsto N$ is an isomorphism from $C_{G_2}(t_{\alpha,\alpha})$ to GL(2). Next we consider the automorphism $t_{\omega,\omega}$ where ω is a primitive fourth root of the unit. Any $f \in C_{G_2}(t_{\omega,\omega})$ is again an element in G_0 and the subspaces $\langle u_1, u_2\rangle$, $\langle v_1, v_2\rangle$ and $\langle u_3, v_3\rangle$ are f-invariant. Hence f is again of the form (6), which implies that $\langle u_3\rangle$ and $\langle v_3\rangle$ are f-invariant. Thus $C_{G_2}(t_{\omega,\omega})$ is as before isomorphic to GL(2). Consider now the automorphism $t_{1,-1}$ and $f \in C_{G_2}(t_{1,-1})$ diagonalizable. Then f fixes the subspaces $Q := \langle e_1, e_2, u_1, v_1\rangle$ and $Q^\perp := \langle u_2, u_3, v_2, v_3\rangle$. The first one, Q, is a split quaternion algebra which we identify with $\mathcal{M}_2(\mathbb{C})$. We are using the terminology *standard basis of* Q to mean the basis $\{e_{11}, e_{22}, e_{12}, e_{21}\}$ where e_{ij} are the standard elementary matrices. There is an epimorphism $\mathrm{Ad}\colon \mathrm{SL}(2) \cong \mathrm{aut}(Q)$ which maps maximal tori of SL(2) to maximal tori of aut(Q). Since a maximal torus of SL(2) is the group of all matrices $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right)$ with $\lambda \in \mathbb{C}^\times$, passing through Ad we get a maximal torus in aut(Q) to be the group of automorphisms whose matrices relative to the standard basis of Q are of the form $\operatorname{diag}(1, 1, \lambda, \lambda^{-1})$, with $\lambda \in \mathbb{C}^\times$. As the restriction of f to Q is diagonalizable, the element f must be conjugated to some in the maximal torus of Q just described. Therefore there exists a basis $\{e_1', e_2', u_1', v_1'\}$ of Q relative to which the matrix of $f|_Q$ is $\operatorname{diag}(1, 1, \lambda, \lambda^{-1})$ for some nonzero $\lambda \in \mathbb{C}$. This basis can be extended to a standard basis $\{e_1', e_2', u_1', u_2', u_3', v_1', v_2', v_3'\}$ of $\mathbb{O}_\mathbb{C}$ formed by eigenvectors of f such that $Q = \langle e_1, e_2, u_1, v_1\rangle = \langle e_1', e_2', u_1', v_1'\rangle$ and $Q^\perp = \langle u_2, u_3, v_2, v_3\rangle = \langle u_2', u_3', v_2', v_3'\rangle$. In other words f is conjugated in $C_{G_2}(t_{1,-1})$ to an element in the maximal torus T. This implies, according to the paragraph before Lemma 2, that $C_{G_2}(t_{1,-1})$ has no outer diagonalizable elements. Let us consider now $t_{1,\beta}$ with $\beta \neq \pm 1$ so that $S = \{1, \beta, \beta^{-1}\}$ has cardinal 3. If $f \in C_{G_2}(t_{1,\beta})$ is diagonalizable, it fixes again Q as well as the subspaces $\langle u_2, v_3\rangle$ and $\langle u_3, v_2\rangle$. Arguing as before there is a new standard basis of $\mathbb{O}_\mathbb{C}$ given by $\{e_1', e_2', u_1', u_2', u_3', v_1', v_2', v_3'\}$ such that each element in the new basis is an eigenvector of f and $Q = \langle e_1, e_2, u_1, v_1\rangle = \langle e_1', e_2', u_1', v_1'\rangle$, $\langle u_2, v_3\rangle = \langle u_2', v_3'\rangle$, $\langle u_3, v_2\rangle = \langle u_3', v_2'\rangle$. This implies again that f is conjugated in $C_{G_2}(t_{1,\beta})$ to some element in T. So outer diagonalizable elements do not exist in $C_{G_2}(t_{1,\beta})$. The last case to study is $C_{G_2}(t_{\rho,\rho})$ where ρ is a primitive cubic root of the unit. Then any $f \in C_{G_2}(t_{\rho,\rho})$ fixes the subspaces $\langle e_1, e_2\rangle$, $\langle u_1, u_2, u_3\rangle$ and $\langle v_1, v_2, v_3\rangle$. Then necessarily f fixes the idempotents e_i and if moreover f is diagonalizable, then we can obtain a new standard basis $\{e_1, e_2, u_1', u_2', u_3', v_1', v_2', v_3'\}$ of eigenvectors of f such that $\langle u_1, u_2, u_3\rangle = \langle u_1', u_2', u_3'\rangle$ and $\langle v_1, v_2, v_3\rangle = \langle v_1', v_2', v_3'\rangle$. So f is conjugated within $C_{G_2}(t_{\rho,\rho})$ to an element in the torus T. Again $C_{G_2}(t_{\rho,\rho})$ has no outer diagonalizable elements. □

3.2. **Description of the gradings.** In this section we are summarizing the equivalence classes of possible gradings in $\mathbb{O}_\mathbb{C}$ found in [5]. We also add some information on the universal grading group, obtained using the methods in subsection 2.2. This added information is useful for our purposes by different reasons: (1) the set of toral gradings on $\mathfrak{g}_2$ are obtained by epimorphisms of the universal grading group of its Cartan grading, (2) the set of gradings on $\mathfrak{g}_2$ induced by the different representatives of an equivalence class of a fixed grading on $\mathbb{O}_\mathbb{C}$ is obtained also by epimorphisms from the universal grading group, and (3) the universal grading group plays an essential role in the determination of the (unique up to equivalence) nontoral grading on $\mathfrak{g}_2$.

Let us use in this paragraph the more convenient notation C for the complex octonion algebra $\mathbb{O}_{\mathbb{C}}$. Denoting by G the grading group in [5] and by $G_{\mathcal{U}}$ the universal grading group, the equivalence classes of gradings in C are those whose representatives are the following:

(1) $G = \mathbb{Z} = G_{\mathcal{U}}$, $C = C_{-1} \oplus C_0 \oplus C_1$ with $C_0 = \langle e_1, e_2, u_1, v_1\rangle$, $C_1 = \langle u_2, v_3\rangle$, $C_{-1} = \langle u_3, v_2\rangle$.

(2) $G = \mathbb{Z} = G_{\mathcal{U}}$, $C = C_{-2} \oplus C_{-1} \oplus C_0 \oplus C_1 \oplus C_2$ with $C_0 = \langle e_1, e_2\rangle$, $C_1 = \langle u_1, u_2\rangle$, $C_2 = \langle v_3\rangle$, $C_{-1} = \langle v_1, v_2\rangle$, $C_{-2} = \langle u_3\rangle$.

(3) $G = \mathbb{Z}$, $C = C_{-3} \oplus C_{-2} \oplus C_{-1} \oplus C_0 \oplus C_1 \oplus C_2 \oplus C_3$ with $C_0 = \langle e_1, e_2\rangle$, $C_1 = \langle u_1\rangle$, $C_2 = \langle u_2\rangle$, $C_3 = \langle v_3\rangle$, $C_{-1} = \langle v_1\rangle$, $C_{-2} = \langle v_2\rangle$, $C_{-3} = \langle u_3\rangle$. Here, the universal grading group is $G_{\mathcal{U}} = \mathbb{Z} \times \mathbb{Z}$ and the corresponding grading is: $C_{0,0} = \langle e_1, e_2\rangle$, $C_{1,0} = \langle u_1\rangle$, $C_{0,1} = \langle u_2\rangle$, $C_{1,1} = \langle v_3\rangle$, $C_{-1,0} = \langle v_1\rangle$, $C_{0,-1} = \langle v_2\rangle$, $C_{-1,-1} = \langle u_3\rangle$.

(4) $G = \mathbb{Z}_2 = G_{\mathcal{U}}$, with $C_0 = \langle e_1, e_2, u_1, v_1\rangle$, $C_1 = \langle u_2, u_3, v_2, v_3\rangle$.

(5) $G = \mathbb{Z}_3 = G_{\mathcal{U}}$, with $C_0 = \langle e_1, e_2\rangle$, $C_1 = \langle u_1, u_2, u_3\rangle$, $C_2 = \langle v_1, v_2, v_3\rangle$.

(6) $G = \mathbb{Z}_4 = G_{\mathcal{U}}$, with $C_0 = \langle e_1, e_2\rangle$, $C_1 = \langle u_1, u_2\rangle$, $C_2 = \langle u_3, v_3\rangle$, $C_3 = \langle v_1, v_2\rangle$.

(7) $G = \mathbb{Z}_6$, with $C_0 = \langle e_1, e_2\rangle$, $C_1 = \langle u_1\rangle$, $C_2 = \langle u_2\rangle$, $C_3 = \langle u_3, v_3\rangle$, $C_4 = \langle v_2\rangle$, $C_5 = \langle v_1\rangle$. In this case the universal grading group is $G_{\mathcal{U}} = \mathbb{Z} \times \mathbb{Z}_2$, and the grading is $C_{0,0} = \langle e_1, e_2\rangle$, $C_{0,1} = \langle u_3, v_3\rangle$, $C_{1,0} = \langle u_1\rangle$, $C_{-1,0} = \langle v_1\rangle$, $C_{1,1} = \langle v_2\rangle$, $C_{-1,1} = \langle u_2\rangle$.

(8) $G = \mathbb{Z}_2^2 = G_{\mathcal{U}}$, with $C_{00} = \langle e_1, e_2\rangle$, $C_{10} = \langle u_1, v_1\rangle$, $C_{01} = \langle u_2, v_2\rangle$, $C_{11} = \langle u_3, v_3\rangle$.

(9) $G = \mathbb{Z}_2^3 = G_{\mathcal{U}}$, with $C_{000} = \langle e_1 + e_2\rangle$, $C_{001} = \langle e_1 - e_2\rangle$, $C_{100} = \langle u_1 + v_1\rangle$, $C_{010} = \langle u_2 + v_2\rangle$, $C_{101} = \langle u_1 - v_1\rangle$, $C_{011} = \langle u_2 - v_2\rangle$, $C_{110} = \langle u_3 + v_3\rangle$, $C_{111} = \langle u_3 - v_3\rangle$.

3.3. **Algebraic groups approach.** Though the description of equivalence classes of gradings on $\mathbb{O}_{\mathbb{C}}$ is fully achieved in [5], we give here an alternative proof which is of independent interest since the used methods are easily exported to other nonassociative algebras. We shall use some elements of the theory of algebraic groups. First, we shall consider gradings on $C = \mathbb{O}_{\mathbb{C}}$ by cyclic groups. Let φ denote the semisimple automorphism of C producing the grading. Then φ is conjugated to some element $t_{\alpha,\beta}$ in the maximal torus T of G_2. Thus the initial grading is isomorphic to the one given by $t_{\alpha,\beta}$. Denote by S the set of eigenvalues $S = \{1, \alpha, \beta, \alpha^{-1}\beta^{-1}, \alpha^{-1}, \beta^{-1}, \alpha\beta\}$. We now make a discussion of the different possibilities arising from the cardinal of S.

· (a) If $|S| = 7$ then any of the automorphisms $t_{\alpha,\beta}$ produces the same grading: the fine grading given in (3).
· (b) Suppose $|S| = 6$. The only way to get this possibility (up to automorphism) is to write $\alpha = -1$, $\beta \neq \pm 1, \pm i$. So $t_{-1,\beta}$ induces the decomposition

$C_1 = \langle e_1, e_2\rangle$, $C_{-1} = \langle u_1, v_1\rangle$, $C_\beta = \langle u_2\rangle$, $C_{\beta^{-1}} = \langle v_2\rangle$, $C_{-\beta} = \langle v_3\rangle$ and $C_{-\beta^{-1}} = \langle u_3\rangle$. This grading is isomorphic to the one in (7).

- (c) Consider next the case $|S| = 5$. Up to automorphism, this is achieved only by making $\alpha = \beta$, $\alpha^n \neq 1$ for $n \leq 4$. So the grading automorphism is $t_{\alpha,\alpha}$, giving the grading $C = C_1 \oplus C_\alpha \oplus C_{\alpha^{-1}} \oplus C_{\alpha^2} \oplus C_{\alpha^{-2}}$ where $C_1 = \langle e_1, e_2\rangle$, $C_\alpha = \langle u_1, u_2\rangle$, $C_{\alpha^{-1}} = \langle v_1, v_2\rangle$, $C_{\alpha^2} = \langle v_3\rangle$ and $C_{\alpha^{-2}} = \langle u_3\rangle$. This is the grading in (2).
- (d) Now we suppose $|S| = 4$. Up to automorphism the only way to get this is given by $\alpha = \beta = \omega$ a primitive fourth root of unit. The grading automorphism is $t_{\omega,\omega}$ and the grading is $C = C_1 \oplus C_\omega \oplus C_{\omega^2} \oplus C_{\omega^3}$ where $C_1 = \langle e_1, e_2\rangle$, $C_\omega = \langle u_1, u_2\rangle$, $C_{\omega^2} = \langle u_3, v_3\rangle$ and $C_{\omega^3} = \langle v_1, v_2\rangle$. This is the grading in (6).
- (e) The possibility $|S| = 3$ can be accomplished in two different ways (always up to isomorphism). The first one is given by $\alpha = 1$, $\beta \neq \pm 1$. Thus the grading automorphism $t_{1,\beta}$ gives the grading $C = C_1 \oplus C_\beta \oplus C_{\beta^{-1}}$ where $C_1 = \langle e_1, e_2, u_1, v_1\rangle$, $C_\beta = \langle u_2, v_3\rangle$ and $C_{\beta^{-1}} = \langle u_3, v_2\rangle$. This is the grading in (1). The second possibility is $\alpha = \beta = \omega$ a primitive cubic root of unit. In this case the grading automorphism $t_{\omega,\omega}$ gives the grading $C = C_1 \oplus C_\omega \oplus C_{\omega^2}$ where $C_1 = \langle e_1, e_2\rangle$, $C_\omega = \langle u_1, u_2, u_3\rangle$ and $C_{\omega^2} = \langle v_1, v_2, v_3\rangle$. This is the grading given in (5).
- (f) It remains to study the case $|S| = 2$. Up to automorphism, the only way to get this possibility is to write $\alpha = 1$, $\beta = -1$ so that the grading automorphism is $t_{1,-1}$ and the grading is $C = C_1 \oplus C_{-1}$ with $C_1 = \langle e_1, e_2, u_1, v_1\rangle$ and $C_{-1} = \langle u_2, u_3, v_2, v_3\rangle$. This is the grading in (4).

So far, we have described the equivalence classes of gradings on $\mathbb{O}_{\mathbb{C}}$ by cyclic groups. Now we must complete this task with the determination of equivalence classes of gradings by noncyclic groups. Since noncyclic groups (abelian and finitely-generated) are products of cyclic ones, any grading by a noncyclic group is a refinement of a grading by a cyclic group. So we continue by studying the possible refinements of the gradings (a)-(f) above. To do that, we consider the grading automorphism $t_{\alpha,\beta}$ in each case and analyze a refinement to a grading induced by a commuting diagonalizable set of automorphisms $\{t_{\alpha,\beta}, f\}$. So $f \in C_{G_2}(t_{\alpha,\beta})$ is a diagonal element. According to Lemma 2 the automorphism f can not be taken outer. Therefore, applying Theorem 1, f can also be taken in the maximal torus T and we can write $f = t_{\lambda,\mu}$. In the following, we list in upper case the possible refinements of the cyclic grading with the same letter in lower case:

- (A) This grading is fine so it can not be further refined.
- (B) The grading is produced by $t_{-1,\beta}$. Now, if $\lambda \neq \pm 1$ the grading $\{t_{-1,\beta}, t_{\lambda,\mu}\}$ is the fine grading in (a). For $\lambda = \pm 1$ the grading given by $\{t_{-1,\beta}, t_{\lambda,\mu}\}$ coincides with the induced by $t_{-1,\beta}$ alone.
- (C) The grading to refine comes from $t_{\alpha,\alpha}$. If $\lambda \neq \mu$ the grading $\{t_{\alpha,\alpha}, t_{\lambda,\mu}\}$ is again the fine grading in (a). If $\lambda = \mu$ then $\{t_{\alpha,\alpha}, t_{\lambda,\mu}\}$ induces the same grading as $t_{\alpha,\alpha}$ alone.
- (D) We have the grading $\{t_{\omega,\omega}, t_{\lambda,\mu}\}$. Then:
  - (i) $\lambda \neq \mu$, $\lambda\mu = \pm 1$, the resulting grading is isomorphic to the one in (b).
  - (ii) $\lambda \neq \mu$, $\lambda\mu \neq \pm 1$, the resulting grading is isomorphic to the one in (a).

- (iii) $\lambda = \mu$, $\lambda^2 = \pm 1$, the resulting grading is isomorphic to the one in (d).
- (iv) $\lambda = \mu$, $\lambda^2 \neq \pm 1$, the resulting grading is isomorphic to the one in (c).

- (E) There are two cases. In the first one, the grading to refine is the one coming from $t_{1,\beta}$ with $\beta \neq \pm 1$ so that $S = \{1, \beta, \beta^{-1}\}$. We have the grading induced by $\{t_{1,\beta}, t_{\lambda,\mu}\}$. There are several cases to take into account:
  - (i) $\lambda \neq \pm 1$. Then it is easily seen that $\{t_{1,\beta}, t_{\lambda,\mu}\}$ induces the fine grading (a).
  - (ii) $\lambda = 1$. In this case the grading induced by $\{t_{1,\beta}, t_{\lambda,\mu}\}$ is the same as the one induced by $t_{1,\beta}$ alone.
  - (iii) $\lambda = -1$. This produces the grading in (b).

  Next we must consider the second possibility in (e), that is, the grading automorphism is $t_{\omega,\omega}$ for a primitive cubic root of the unit ω. Thus the refinement to consider is now $\{t_{\omega,\omega}, t_{\lambda,\mu}\}$. Define $S' = \{\lambda, \mu, (\lambda\mu)^{-1}\}$. We consider the following cases:
  - (i) $|S'| = 1$. This possibility does not give any proper refinement.
  - (ii) $|S'| = 2$. We can suppose without loss of generality that $\lambda = \mu \neq (\lambda\mu)^{-1}$. In this case the grading induced is the one in (c).
  - (iii) $|S'| = 3$. We obtain the fine grading in (a).
- (F) Finally, the grading automorphism is $t_{1,-1}$ and the refinement $\{t_{1,-1}, t_{\lambda,\mu}\}$. We distinguish again different cases:
  - (i) $\lambda \neq \lambda^{-1}$, $\mu = \mu^{-1}$. We obtain a grading equivalent to the one in (b).
  - (ii) $\lambda \neq \lambda^{-1}$, $\mu \neq \mu^{-1}$. If $\mu = \lambda^{-1}\mu^{-1}$ we get the grading in (c). Otherwise we get the grading (b) if $\lambda\mu = \pm 1$ and the one in (a) if $\lambda\mu \neq \pm 1$.
  - (iii) $\lambda = \lambda^{-1}$. We consider the following cases:
    1. $\lambda = 1$, $\mu = \mu^{-1}$. The grading $\{t_{1,-1}, t_{\lambda,\mu}\}$ is not proper (it agrees with the one in (f)).
    2. $\lambda = 1$, $\mu \neq \mu^{-1}$. The refinement is the first grading in item (e).
    3. $\lambda = -1$, $\mu = \mu^{-1}$. We get here a new grading coming from $\{t_{1,-1}, t_{-1,\mu}\}$ ($\mu = \pm 1$). The sign of μ is irrelevant. This is a $\mathbb{Z}_2^2$-grading given by $C_{1,1} = \langle e_1, e_2\rangle$, $C_{1,-1} = \langle u_1, v_1\rangle$, $C_{-1,\mu} = \langle u_2, v_2\rangle$, $C_{-1,-\mu} = \langle u_3, v_3\rangle$. This is the first grading which is new, that is, it is not in the list (a)-(f) above.
    4. $\lambda = -1$, $\mu \neq \mu^{-1}$. Here the refinement $\{t_{1,-1}, t_{\lambda,\mu}\}$ is the one in (d) if $\mu = -\mu^{-1}$ and the one in (b) in case $\mu \neq -\mu^{-1}$.

Thus the unique proper refinement (up to isomorphism) of gradings by cyclic groups is the $\mathbb{Z}_2^2$-grading in (F)(iii)(3). Hence to complete our study of refinements of gradings by cyclic groups, we must describe now the possible refinements $\{t_{1,-1}, t_{-1,1}, f\}$ where $f \in Z := C_{G_2}(\{t_{1,-1}, t_{-1,1}\})$ is a diagonalizable automorphism. It is very easy to describe the subgroup Z. Its elements either fix the idempotents e_i or permute them. In the first case it is straightforward to see that $f \in T$ while in the second one the matrix of f is of the form:

$$f = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\alpha\beta)^{-1} \\ 0 & 0 & \alpha^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha\beta & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

for $\alpha, \beta \in \mathbb{C}^\times$. Therefore $Z = Z_0 \cup Z_1$ has two connected components, $Z_0 = T$ is the maximal torus of G_2 and the other component $Z_1 = T \cdot f_0$ where f_0 is f with $\alpha = \beta = 1$. All the elements in Z are diagonalizable and if $f \in Z_0$, clearly the induced grading is not new. On the contrary for any $f \in Z_1$, Theorem 1 implies that the grading induced by $\{t_{1,-1}, t_{-1,1}, f\}$ is nontoral. Indeed taking different f's in Z_1 we get equivalent gradings so we can take f_0 (which is of order two) and obtain the following grading:

$$C_{1,1,1} = \langle e_1 + e_2 \rangle, \quad C_{1,1,-1} = \langle e_1 - e_2 \rangle, \quad C_{1,-1,1} = \langle u_1 + v_1 \rangle,$$

$$C_{1,-1,-1} = \langle u_1 - v_1 \rangle, \quad C_{-1,1,1} = \langle u_2 + v_2 \rangle, \quad C_{-1,1,-1} = \langle u_2 - v_2 \rangle,$$

$$C_{-1,-1,1} = \langle u_3 + v_3 \rangle, \quad C_{-1,-1,-1} = \langle u_3 - v_3 \rangle,$$

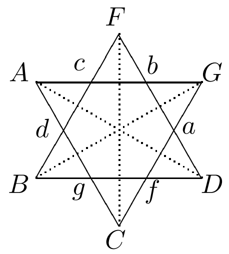
in which $C_{i,j,k}$ denotes the intersection of the kernels of $t_{1,-1} - i \cdot 1_C$, $t_{-1,1} - j \cdot 1_C$ and $f - k \cdot 1_C$. This grading is fine so we have finished our study of refinements of gradings by cyclic groups. Summarizing: up to equivalence the gradings on $C = \mathbb{O}_\mathbb{C}$ are the given in (1)-(9) of subsection 3.2.

4. Gradings on $\mathfrak{g}_2$

We mentioned in section 3 the fact that the automorphism group $G_2 = \mathrm{aut}(\mathbb{O}_\mathbb{C})$ and the automorphism group $\mathrm{aut}(\mathfrak{g}_2)$ are isomorphic via the map $\mathrm{Ad}\colon G_2 \to \mathrm{aut}(\mathfrak{g}_2)$. Thus, gradings on $\mathbb{O}_\mathbb{C}$ induce gradings on $\mathfrak{g}_2$ and conversely. In this correspondence, toral gradings on $\mathbb{O}_\mathbb{C}$ induce toral gradings on $\mathfrak{g}_2$ and reciprocally. Let us illustrate this grading-inducing procedure in the toral case. Fix a standard basis $B = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of the split Cayley algebra $C = \mathbb{O}_\mathbb{C}$. Recall that any derivation of C is in the linear span of the set of derivations $D_{x,y} = [l_x, l_y] + [l_x, r_y] + [r_x, r_y]$, $(x, y \in C)$, where r_x and l_x denote the right and left multiplication operators respectively (see [19, Corollary 3.29, p. 87]). Define $\mathfrak{h} = \sum_1^3 \mathbb{C} D_{u_i,v_i}$ which is obviously a Cartan subalgebra of $\mathcal{L} = \mathfrak{g}_2$ since any element $D_{x,y}$ with $x, y \in B$ is an eigenvector of $\mathrm{ad}\, D_{u_i,v_i}$ for any $i = 1, 2, 3$. Fix a basis $\{h_1, h_2\}$ of $\mathfrak{h}$ defined by $h_1 = \frac{1}{3}(D_{u_1,v_1} + 2D_{u_2,v_2})$ and $h_2 = \frac{1}{3}(2D_{u_1,v_1} + D_{u_2,v_2})$. For $i = 1, 2$ take $\alpha_i\colon \mathfrak{h} \to \mathbb{C}$ given by $\alpha_1(w_1 h_1 + w_2 h_2) = w_1 - w_2$ and $\alpha_2(w_1 h_1 + w_2 h_2) = w_2$. Thus $\{\alpha_1, \alpha_2\}$ is a basis of the root system $\Phi = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$ relative to $\mathfrak{h}$ and the root spaces are generated by

$$\begin{array}{lll} A := D_{v_1,u_2} \in L_{\alpha_1}, & a := D_{v_2,v_3} \in L_{\alpha_2}, & c := D_{v_1,v_3} \in L_{\alpha_1+\alpha_2}, \\ b := D_{u_1,u_2} \in L_{\alpha_1+2\alpha_2}, & G := D_{u_1,v_3} \in L_{\alpha_1+3\alpha_2}, & F := D_{u_2,v_3} \in L_{2\alpha_1+3\alpha_2}, \\ D := D_{u_1,v_2} \in L_{-\alpha_1}, & d := D_{u_2,u_3} \in L_{-\alpha_2}, & f := D_{u_1,u_3} \in L_{-\alpha_1-\alpha_2}, \\ g := D_{v_1,v_2} \in L_{-\alpha_1-2\alpha_2}, & B := D_{v_1,u_3} \in L_{-\alpha_1-3\alpha_2}, & C := D_{v_2,u_3} \in L_{-2\alpha_1-3\alpha_2}. \end{array}$$

Thus, the root system can be represented as



identifying (only pictorially) the root vectors with the roots themselves. For any $f \in G_2 = \mathrm{aut}(\mathbb{O}_{\mathbb{C}})$, notice that $\mathrm{Ad}\, f(D_{x,y}) = D_{f(x),f(y)}$, hence the matrix of $s_{\alpha,\beta} := \mathrm{Ad}(t_{\alpha,\beta})$ relative to the basis of $\mathfrak{g}_2$ given by $\{h_1, h_2, A, a, c, b, G, F, D, d, f, g, B, C\}$ is

$$\mathrm{diag}(1, 1, \alpha^{-1}\beta, \alpha, \beta, \alpha\beta, \alpha^2\beta, \alpha\beta^2, \alpha\beta^{-1}, \alpha^{-1}, \beta^{-1}, \alpha^{-1}\beta^{-1}, \alpha^{-2}\beta^{-1}, \alpha^{-1}\beta^{-2}).$$

In particular, the $\mathbb{Z}\times\mathbb{Z}$-grading of $\mathfrak{g}_2$ induced by the $\mathbb{Z}\times\mathbb{Z}$-grading of $\mathbb{O}_{\mathbb{C}}$ specified in (3) of 3.2, is just the one such that

$$s_{\alpha,\beta}(v_{i,j}) = \alpha^i\beta^j v_{i,j} \quad \text{for all } v_{i,j} \in \mathcal{L}_{i,j}. \tag{8}$$

Now it is easy to give examples of equivalent gradings on $\mathbb{O}_{\mathbb{C}}$ which do not give equivalent gradings on $\mathfrak{g}_2$. The grading (3) of 3.2 is produced, for instance, by any of the automorphisms t_{e,e^2}, t_{e^4,e^5} or t_{ω,ω^2}, where $e := \exp(1)$ and ω is a primitive 7th-root of the unit (we only need the set $\{1, \alpha, \beta, \alpha^{-1}\beta^{-1}, \alpha^{-1}, \beta^{-1}, \alpha\beta\}$ to have seven different elements). The automorphisms $\mathrm{Ad}(t_{e,e^2})$, $\mathrm{Ad}(t_{e^4,e^5})$ and $\mathrm{Ad}(t_{\omega,\omega^2})$ produce however nonequivalent gradings of $\mathcal{L} = \mathfrak{g}_2$. The first one produces the $\mathbb{Z}$-grading $\mathcal{L} = \oplus_{i=-5}^{5}\mathcal{L}_i$ where $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, a\rangle$, $\mathcal{L}_2 = \langle c\rangle$, $\mathcal{L}_3 = \langle b\rangle$, $\mathcal{L}_4 = \langle G\rangle$, $\mathcal{L}_5 = \langle F\rangle$, and $\mathcal{L}_{-i} = \mathcal{L}_i^*$ for $i = 1, \ldots, 5$, where $* : \mathcal{L} \to \mathcal{L}$ is the involution acting as the identity on $\mathfrak{h}$ and such that $L_\alpha^* = L_{-\alpha}$ for each root $\alpha \in \Phi$ (if we represent the elements in $\mathfrak{g}_2$ as matrices relative to B, this involution is just the matrix transposition). This grading is equivalent to the Kostant grading described in [18, p. 91]. The automorphism s_{e^4,e^5} induces the Cartan grading of $\mathfrak{g}_2$ and s_{ω,ω^2} yields the $\mathbb{Z}_7$-grading $\mathcal{L} = \oplus_{i\in\mathbb{Z}_7}\mathcal{L}_i$ such that $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, a\rangle$, $\mathcal{L}_2 = \langle c, C\rangle$, $\mathcal{L}_3 = \langle b, B\rangle$, $\mathcal{L}_4 = \langle g, G\rangle$, $\mathcal{L}_5 = \langle f, F\rangle$ and $\mathcal{L}_6 = \langle d, D\rangle$. As a consequence of the non-preserving equivalence phenomenon, we need some other techniques to describe all the gradings of $\mathfrak{g}_2$ up to equivalence. Paradoxically, the hardest problem in the classification of the gradings on $\mathfrak{g}_2$ is the description of the toral ones.

4.1. **Toral gradings: translation to an algebraic problem.** We recall that all the grading groups G under consideration are abelian and finitely generated, and the Lie algebras (unless specified) are semisimple. As mentioned in previous sections, a G-grading on a Lie algebra $\mathcal{L}$ comes from a group homomorphism $\rho : \mathfrak{X}(G) \to \mathrm{aut}(\mathcal{L})$ so that $\mathcal{L}_g$ is the set of all x such that $\rho(\varphi)(x) = \varphi(g)x$ for all $\varphi \in \mathfrak{X}(G)$. If the grading is toral then the image of ρ is contained in a maximal torus $T \cong (\mathbb{C}^\times)^r$ of $\mathcal{L}$ (r being the rank of $\mathcal{L}$). Thus we can consider from the beginning $\rho\colon \mathfrak{X}(G) \to T$. Taking characters we have a group homomorphism $\rho^*\colon \mathfrak{X}(T) \to \mathfrak{X}^2(G)$ where $\mathfrak{X}^2 = \mathfrak{X}\circ\mathfrak{X}$. Of course $\mathfrak{X}^2(G) \cong G$ and $\mathfrak{X}(T) \cong \mathbb{Z}^r$ since $\mathfrak{X}(\mathbb{C}^\times) = \mathbb{Z}$. Therefore ρ^* can be identified with a map $\mathbb{Z}^r \to G$ and ρ itself is completely determined by ρ^* given the dual nature of $\mathfrak{X}$. In our case $\mathcal{L} = \mathfrak{g}_2$, $r = 2$ and so $\rho^*\colon \mathbb{Z}^2 \to G$. Thus we can conclude that a toral grading on $\mathfrak{g}_2$ comes from a group homomorphism $f\colon \mathbb{Z}^2 \to G$ which induces $f^*\colon \mathfrak{X}(G) \to T$ and the grading is $\mathcal{L} = \oplus_g \mathcal{L}_g$ where $\mathcal{L}_g$ is as before. It can be proved that $\mathcal{L}_g$ agrees with the sum of all subspaces $\mathcal{L}_{i,j}$ (see formula (8)) such that $f(i,j) = g$ (this can be proved first for a cyclic group G with the help of the mentioned formula, and then for a product of such groups). This easily implies that f is an epimorphism since G is generated by the set of all g such that $\mathcal{L}_g \neq 0$. In this way the group homomorphism f is precisely the given by Proposition 2 by the fact that the grading $\mathcal{L} = \oplus_g \mathcal{L}_g$ (being toral) is a coarsening of the Cartan grading.

Now we will introduce the action of the Weyl group. Let us denote by $T' = \mathrm{Ad}(T)$, the maximal torus in $\mathrm{aut}(\mathfrak{g}_2)$. The Weyl group is the quotient $W := N_{\mathrm{aut}(\mathfrak{g}_2)}(T')/T'$. It is a well-known fact that it is a semidirect product of S_3 and $\mathbb{Z}_2$. Denote by $\bar{w}$ the class of $w \in N := N_{\mathrm{aut}(\mathfrak{g}_2)}(T')$ in W. The map $\mathrm{Ad}(w)\colon\ \mathrm{aut}(\mathfrak{g}_2) \to \mathrm{aut}(\mathfrak{g}_2)$ can be restricted to $\mathrm{Ad}(w)\colon T' \to T'$. Moreover for any $w' \in N$ such that $\bar{w} = \bar{w}'$, we have $\mathrm{Ad}(w) = \mathrm{Ad}(w')$ (restricted to T'). Thus we have a natural action of W on T' given by $\bar{w} \cdot t = \mathrm{Ad}(w)(t)$ for all $\bar{w} \in W$ and $t \in T'$. Given two gradings by their homomorphisms $\rho, \rho'\colon \mathfrak{X}(G) \to \mathrm{aut}(\mathfrak{g}_2)$, we shall say that they are *related* when there is an element $\bar{w} \in W$ and an automorphism $\xi \in \mathrm{aut}(G)$ such that $\rho' = \mathrm{Ad}(w)\rho\xi^*$, where $\xi^*\colon \mathfrak{X}(G) \to \mathfrak{X}(G)$ is given by $\xi^*(h) = h\xi$ for all $h \in \mathfrak{X}(G)$. It is immediate to see that if ρ and ρ' are related, then the induced gradings are equivalent. Thus, we must study equivalence classes of homomorphisms $\rho\colon \mathfrak{X}(G) \to \mathrm{aut}(\mathfrak{g}_2)$.

Since any toral grading $\rho\colon \mathfrak{X}(G) \to T'$ comes from a group epimorphism $\rho^*\colon \mathbb{Z}^2 \to G$, we want to describe the induced equivalence relation on group epimorphisms $\mathbb{Z}^2 \to G$. If $\rho' = \mathrm{Ad}(w)\rho\xi^*$, then taking characters we have $\rho'^* = \xi\rho^*\,\mathrm{Ad}(w)^*$ where $\mathrm{Ad}(w)^*\colon \mathbb{Z}^2 \to \mathbb{Z}^2$. Thus, to describe the induced action of W on group epimorphisms $\mathbb{Z}^2 \to G$, it suffices to describe W as a group of automorphisms of $\mathbb{Z}^2$. This action is natural because W is isomorphic to a group of isometries of the euclidean two-dimensional space E and it acts on the root lattice $\Phi \subset E$, which is identified with $\mathbb{Z}^2$ by means of the coordinates relative to the basis $\{\alpha_1, \alpha_2\}$. According to these identifications, the set $\mathfrak{M} = \{\pm(2,3), \pm(1,3), \pm(1,0)\}$ contains the long roots of Φ, $\mathfrak{m} = \{\pm(1,2), \pm(1,1), \pm(0,1)\}$ the short ones and $\mathfrak{S} = \mathfrak{M} \cup \mathfrak{m}$ all of them. It is easy to check that $\sigma_1(x,y) := (-2x+y, -3x+y)$ acts as the clockwise rotation of angle $\frac{4\pi}{3}$, $\sigma_2(x,y) := (y-x, y)$ is the symmetry fixing $b = (1,2)$ and $\sigma_3(x,y) := (-x,-y)$ is the rotation of angle π; therefore W is identified with the subgroup of $\mathrm{aut}(\mathbb{Z}^2)$ generated by $\{\sigma_i : i = 1,2,3\}$. These generators of W can be related to certain automorphisms of $\mathfrak{g}_2$. Thus σ_1 comes from an automorphism permuting cyclically u_1, u_2 and u_3 (and similarly the v_i's), while σ_2 would permute the indexes u_1 and u_2 fixing u_3 (and similarly with the v_i's). Finally σ_3 would exchange u_i's with v_i's. Recall that the Weyl group acts transitively on $\mathfrak{M}$ and $\mathfrak{m}$.

Summarizing the ideas in the previous paragraphs, the Weyl group and $\mathrm{aut}(G)$ act on the set of epimorphisms $\mathbb{Z}^2 \to G$ so that two epimorphisms $f, f'\colon \mathbb{Z}^2 \to G$ induce equivalent gradings if there are $\omega \in W \subset \mathrm{aut}(\mathbb{Z}^2)$ and $\xi \in \mathrm{aut}(G)$ such that $f' = \xi f \omega$. Thus we will classify group epimorphisms $\mathbb{Z}^2 \to G$ via this relation, describing the induced G-gradings from the $\mathbb{Z}^2$-grading $\mathcal{L} = \oplus \mathcal{L}_{(n,m)}$ whose homogeneous spaces are $\mathcal{L}_{(n,m)} = L_{n\alpha_1 + m\alpha_2}$ (of course equivalent to the one characterized by formula (8)).

4.2. **Cyclic gradings.** In this section we classify gradings by cyclic groups G. These gradings are necessarily toral and there are many possible ways to achieve the mentioned classification. One should see the reference [16] or [17] for a classification mechanism based on maximal tori and Dynkin diagrams (eventually extended). Also a direct inspection of the conjugacy classes in the maximal torus T' leads to the classification we are searching (similarly as we made in the case of cyclic gradings on octonions).

However we are using the above translation to an algebraic problem: classify epimorphisms from $\mathbb{Z}^2$ to a cyclic group G module the actions of the Weyl group and $\mathrm{aut}(G)$. Consider first the case $G = \mathbb{Z}$, then any epimorphism $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ is of the form $f(x, y) := mx + ny$ for some $m, n \in \mathbb{Z}$.

- If $f(P) = 0$ for some $P \in \mathfrak{S}$ (see the previous subsection for notations), letting the Weyl group act, we can suppose that $f(1, 0) = 0$ if $P \in \mathfrak{M}$, and $f(0, 1) = 0$ in case $P \in \mathfrak{m}$. So f may be supposed to be of the form either $f(x, y) = mx$ or $f(x, y) = ny$. But f is an epimorphism so that $n, m = \pm 1$. By composing with the automorphism of the group $\mathbb{Z}$ such that $x \mapsto -x$ we can suppose $n = 1$ and $m = 1$. So f can be taken to be one of the projections $\mathbb{Z}^2 \to \mathbb{Z}$, either $(x, y) \mapsto x$ or $(x, y) \mapsto y$. In the first case the grading on $\mathcal{L} = \mathfrak{g}_2$ is $\mathcal{L} = \oplus_{-2}^{2}\mathcal{L}_i$ where $\mathcal{L}_{-2} = \langle C\rangle$, $\mathcal{L}_{-1} = \langle D, f, g, B\rangle$, $\mathcal{L}_0 = \mathfrak{h} + \langle a, d\rangle$, $\mathcal{L}_1 = \langle A, c, b, G\rangle$ and $\mathcal{L}_2 = \langle F\rangle$. In the second case the grading is $\mathcal{L} = \oplus_{-3}^{3}\mathcal{L}_i$ where $\mathcal{L}_{-3} = \langle B, C\rangle$, $\mathcal{L}_{-2} = \langle g\rangle$, $\mathcal{L}_{-1} = \langle d, f\rangle$, $\mathcal{L}_0 = \mathfrak{h} + \langle A, D\rangle$, $\mathcal{L}_1 = \langle a, c\rangle$, $\mathcal{L}_2 = \langle b\rangle$ and $\mathcal{L}_3 = \langle G, F\rangle$.
- Suppose then that $f(P) \neq 0$ for every $P \in \mathfrak{S}$. Now if $f(P) \neq f(Q)$ for all different $P, Q \in \mathfrak{S}$, obviously the grading is the Cartan grading. The epimorphism $f(x, y) = mx+ny$ induces then the grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_m = \langle A\rangle$, $\mathcal{L}_n = \langle a\rangle$, $\mathcal{L}_{m+n} = \langle c\rangle$, $\mathcal{L}_{m+2n} = \langle b\rangle$, $\mathcal{L}_{m+3n} = \langle G\rangle$, $\mathcal{L}_{2m+3n} = \langle F\rangle$, $\mathcal{L}_{-m} = \langle D\rangle$, $\mathcal{L}_{-n} = \langle d\rangle$, $\mathcal{L}_{-m-n} = \langle f\rangle$, $\mathcal{L}_{-m-2n} = \langle g\rangle$, $\mathcal{L}_{-m-3n} = \langle B\rangle$ and $\mathcal{L}_{-2m-3n} = \langle C\rangle$. Taking for instance $m = 1$, $n = 2$ we get the Cartan grading as a $\mathbb{Z}$-grading $\mathcal{L} = \oplus_{-8}^{8}\mathcal{L}_i$ such as it appears in the forthcoming Theorem 2. So suppose in the sequel that there are different $P, Q \in \mathfrak{S}$ such that $f(P) = f(Q)$. If P or Q is in $\mathfrak{M}$ we can suppose $P = (1, 0)$. So equating $f(1, 0) = f(Q)$ for Q ranging in $\mathfrak{S}$ we get the following possibilities:
  - $f(1, 0) = f(0, 1)$, then $f(x, y) = m(x+y)$ and we can take $m = 1$. The induced grading is then $\mathcal{L} = \oplus_{-5}^{5}\mathcal{L}_i$ where $\mathcal{L}_{-5} = \langle C\rangle$, $\mathcal{L}_{-4} = \langle B\rangle$, $\mathcal{L}_{-3} = \langle g\rangle$, $\mathcal{L}_{-2} = \langle f\rangle$, $\mathcal{L}_{-1} = \langle d, D\rangle$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle a, A\rangle$, $\mathcal{L}_2 = \langle c\rangle$, $\mathcal{L}_3 = \langle b\rangle$, $\mathcal{L}_4 = \langle G\rangle$, $\mathcal{L}_5 = \langle F\rangle$.
  - $f(1, 0) = f(-1, -1)$, implying $n = -2m$ and $f(x, y) = m(x - 2y)$. Again we can suppose $m = 1$ and the induced grading is equivalent to the previous one by applying the symmetry fixing A, that is, $\sigma_3\sigma_2(x, y) = (x - y, -y)$.

  The other equalities $f(1, 0) = f(Q)$ lead us to $0 \in f(\mathfrak{S})$. Finally we must analyze the possibility $f(P) = f(Q)$ for $P, Q \in \mathfrak{m}$. We may suppose $P = (0, 1)$ but the equations $f(0, 1) = f(Q)$ with $Q \in \mathfrak{m}$ are not compatible with the hypothesis that f does not vanish on $\mathfrak{S}$.

Now we consider the case $G = \mathbb{Z}_k$ for some $k > 1$. Any epimorphism $f : \mathbb{Z}^2 \to \mathbb{Z}_k$ decomposes as $f = \pi\bar{f}$ where $\pi : \mathbb{Z} \to \mathbb{Z}_k$ is the natural projection and $\bar{f} : \mathbb{Z}^2 \to \mathbb{Z}$ is a homomorphism, that is, there are $m, n \in \mathbb{Z}$ such that $\bar{f}(x, y) = mx+ny$. Thus, the grading induced by f is obtained by 'folding' module k the $\mathbb{Z}$-grading induced by $\bar{f}$. However it must be noted that there are equivalent $\mathbb{Z}$-gradings such that the induced $\mathbb{Z}_k$-gradings by the epimorphism π are not equivalent. This makes more convenient an analysis based on the epimorphism $f\colon \mathbb{Z}^2 \to \mathbb{Z}_k$ directly. This is given by $f(x,y) := m\bar{x} + n\bar{y}$, where $\bar{x}$ denotes the class of the integer x in $\mathbb{Z}_k$. As before we can distinguish different cases:

- f vanishes on $\mathfrak{S}$. We may assume $f(1,0) = 0$ or $f(0,1) = 0$. Thus f is of the form $f(x,y) = \bar{m}\bar{x}$ or $f(x,y) = \bar{m}\bar{y}$ for some integer m which is prime to k. But considering the group automorphism $\beta\colon \mathbb{Z}_k \to \mathbb{Z}_k$ such that $\beta(\bar{x}) := \bar{m}^{-1}\bar{x}$, the epimorphism βf induces a grading isomorphic to the one given by f alone. Replacing f by βf, we can suppose that $f(x,y) = \bar{x}$ or $f(x,y) = \bar{y}$. In the first case the grading is $\mathcal{L} = \sum_{-2}^{2} \mathcal{L}_i$ where $\mathcal{L}_0 = \mathfrak{h} + \langle d, a\rangle$, $\mathcal{L}_{-1} = \langle D, B, f, g\rangle$, $\mathcal{L}_1 = \langle A, G, b, c\rangle$, $\mathcal{L}_{-2} = \langle C\rangle$ and $\mathcal{L}_2 = \langle F\rangle$, provided that the set $\{0, \pm\bar{1}, \pm\bar{2}\}$ has cardinal 5 in $\mathbb{Z}_k$. But this grading has already been found as a $\mathbb{Z}$-grading. If the set has not cardinal 5, we have the possibilities:
  - $\bar{2} = \bar{0}$, then the grading is a $\mathbb{Z}_2$-grading with $\mathcal{L}_0 = \mathfrak{h} + \langle a, d, C, F\rangle$, $\mathcal{L}_1 = \langle D, B, f, g, A, G, b, c\rangle$.
  - $\bar{3} = \bar{0}$, then the grading is the $\mathbb{Z}_3$-grading $\mathcal{L}_0 = \mathfrak{h} + \langle d, a\rangle$, $\mathcal{L}_1 = \langle A, G, C, b, c\rangle$ and $\mathcal{L}_2 = \langle D, B, F, f, g\rangle$.
  - $\bar{4} = 0$, $\bar{2} \neq 0$, in which case the grading is the $\mathbb{Z}_4$-grading $\mathcal{L}_0 = \mathfrak{h} + \langle d, a\rangle$, $\mathcal{L}_1 = \langle A, G, b, c\rangle$, $\mathcal{L}_2 = \langle F, C\rangle$ and $\mathcal{L}_3 = \langle D, B, f, g\rangle$.

  If $f(x,y) = \bar{y}$ then the grading is $\mathcal{L} = \sum_{-3}^{3} \mathcal{L}_i$ where $\mathcal{L}_0 = \mathfrak{h} + \langle A, D\rangle$, $\mathcal{L}_1 = \langle a, c\rangle$, $\mathcal{L}_{-1} = \langle d, f\rangle$, $\mathcal{L}_2 = \langle b\rangle$, $\mathcal{L}_{-2} = \langle g\rangle$, $\mathcal{L}_3 = \langle G, F\rangle$, $\mathcal{L}_{-3} = \langle B, C\rangle$, provided that the set $\{0, \pm\bar{1}, \pm\bar{2}, \pm\bar{3}\}$ has cardinal 7. This grading has already been found as a $\mathbb{Z}$-grading. If the set has not cardinal 7, then the possibilities are:
  - $\bar{2} = \bar{0}$, which gives (up to isomorphism) the same $\mathbb{Z}_2$-grading previously found.
  - $\bar{3} = \bar{0}$, which gives the $\mathbb{Z}_3$-grading $\mathcal{L}_0 = \mathfrak{h} + \langle A, D, G, F, B, C\rangle$, $\mathcal{L}_1 = \langle a, c, g\rangle$ and $\mathcal{L}_2 = \langle d, f, b\rangle$.
  - $\bar{4} = \bar{0}$, $\bar{2} \neq 0$, which gives the $\mathbb{Z}_4$-grading $\mathcal{L}_0 = \mathfrak{h} + \langle A, D\rangle$, $\mathcal{L}_1 = \langle a, c, B, C\rangle$, $\mathcal{L}_2 = \langle b, g\rangle$ and $\mathcal{L}_3 = \langle d, f, G, F\rangle$.
  - $\bar{5} = \bar{0}$, which gives the $\mathbb{Z}_5$-grading $\mathcal{L}_0 = \mathfrak{h} + \langle A, D\rangle$, $\mathcal{L}_1 = \langle a, c\rangle$, $\mathcal{L}_2 = \langle b, B, C\rangle$, $\mathcal{L}_3 = \langle g, G, F\rangle$ and $\mathcal{L}_4 = \langle d, f\rangle$.
  - $\bar{6} = \bar{0}$, $\bar{2} \neq 0$, $\bar{3} \neq 0$, which gives the $\mathbb{Z}_6$-grading $\mathcal{L}_0 = \mathfrak{h} + \langle A, D\rangle$, $\mathcal{L}_1 = \langle a, c\rangle$, $\mathcal{L}_2 = \langle b\rangle$, $\mathcal{L}_3 = \langle G, F, B, C\rangle$, $\mathcal{L}_4 = \langle g\rangle$ and $\mathcal{L}_5 = \langle d, f\rangle$.
- f does not vanish on $\mathfrak{S}$. Ruling out the Cartan grading, which has already appeared, there must be some different $P, Q \in \mathfrak{S}$ such that $f(P) = f(Q)$. If P or Q is in $\mathfrak{M}$ we can suppose $P = (1,0)$, so that equating $f(1,0) = f(Q)$ for Q ranging in $\mathfrak{S}$ we have the following possibilities:
  - $f(1,0) = f(1,3)$, hence $3f(0,1) = 0$. Define $g_1 = f(1,0)$ and $g_2 = f(0,1)$. Then $\{g_1, g_2\}$ generates $\mathbb{Z}_k$ being g_2 of order 3. Thus $(g_1) + (g_2) = \mathbb{Z}_k$. If $(g_1) \cap (g_2) = 0$ then g_1 has order n prime to 3 and such that $k = 3n$. Thus the epimorphism f can be identified with $f\colon \mathbb{Z}^2 \to \mathbb{Z}_n \times \mathbb{Z}_3$ such that $f(1,0) = (\bar{1}, 0)$ and $f(0,1) = (0, \bar{1})$. Now it is easy to see that for $n \geq 5$ the grading induced by f is equivalent to the following $\mathbb{Z}_{12}$-grading: $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, G\rangle$, $\mathcal{L}_2 = \langle F\rangle$, $\mathcal{L}_3 = \langle g\rangle$, $\mathcal{L}_4 = \langle a\rangle$, $\mathcal{L}_5 = \langle c\rangle$, $\mathcal{L}_7 = \langle f\rangle$, $\mathcal{L}_8 = \langle d\rangle$, $\mathcal{L}_9 = \langle b\rangle$, $\mathcal{L}_{10} = \langle C\rangle$ and

$\mathcal{L}_{11} = \langle D, B\rangle$. For $n = 4$ we get the $\mathbb{Z}_{12}$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle c\rangle$, $\mathcal{L}_3 = \langle B, D\rangle$, $\mathcal{L}_4 = \langle a\rangle$, $\mathcal{L}_5 = \langle b\rangle$, $\mathcal{L}_6 = \langle F, C\rangle$, $\mathcal{L}_7 = \langle g\rangle$, $\mathcal{L}_8 = \langle d\rangle$, $\mathcal{L}_9 = \langle A, G\rangle$ and $\mathcal{L}_{11} = \langle f\rangle$. For $n = 2$ we obtain a $\mathbb{Z}_6$-grading which has previously been obtained. In case $(g_1) \cap (g_2) \neq 0$, we have $(g_2) \subset (g_1)$ and therefore g_1 is a generator of $\mathbb{Z}_k$. We can suppose without loss of generality that $g_1 = \bar{1}$. Besides $k = 3n$ and $g_2 = \bar{n}$ or $g_2 = 2\bar{n}$. In the first case the epimorphism f is given by $f(x, y) = \bar{x} + n\bar{y}$. For $n > 3$ the epimorphism f acts in the way

$$\begin{array}{lcc@{\qquad\qquad}lcc}
(1,0) & \mapsto & \bar{1} & (-1,0) & \mapsto & \overline{3n-1} \\
(0,1) & \mapsto & \bar{n} & (0,-1) & \mapsto & \overline{2n} \\
(1,1) & \mapsto & \overline{n+1} & (-1,-1) & \mapsto & \overline{2n-1} \\
(1,2) & \mapsto & \overline{2n+1} & (-1,-2) & \mapsto & \overline{n-1} \\
(1,3) & \mapsto & \bar{1} & (-1,-3) & \mapsto & \overline{3n-1} \\
(2,3) & \mapsto & \bar{2} & (-2,-3) & \mapsto & \overline{3n-2},
\end{array}$$

and we can order $1 < 2 < n-1 < n, n+1, 2n-1, 2n, 2n+1, 3n-2, 3n-1$ in a strictly increasing series. Thus the induced grading is $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{\bar{1}} = \langle A, G\rangle$, $\mathcal{L}_{\bar{2}} = \langle F\rangle$, $\mathcal{L}_{\bar{n}-\bar{1}} = \langle g\rangle$, $\mathcal{L}_{\bar{n}} = \langle a\rangle$, $\mathcal{L}_{\bar{n}+\bar{1}} = \langle c\rangle$, $\mathcal{L}_{2\bar{n}-\bar{1}} = \langle f\rangle$, $\mathcal{L}_{2\bar{n}} = \langle d\rangle$, $\mathcal{L}_{2\bar{n}+\bar{1}} = \langle b\rangle$, $\mathcal{L}_{3\bar{n}-\bar{2}} = \langle C\rangle$, $\mathcal{L}_{3\bar{n}-\bar{1}} = \langle D, B\rangle$. All the gradings with $n \geq 4$ are equivalent, getting for $n = 4$ just the first described $\mathbb{Z}_{12}$-grading. For $n = 3$ we get the $\mathbb{Z}_9$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, G\rangle$, $\mathcal{L}_2 = \langle F, g\rangle$, $\mathcal{L}_3 = \langle a\rangle$, $\mathcal{L}_4 = \langle c\rangle$, $\mathcal{L}_5 = \langle f\rangle$, $\mathcal{L}_6 = \langle d\rangle$, $\mathcal{L}_7 = \langle b, C\rangle$, $\mathcal{L}_8 = \langle D, B\rangle$. For $n = 2$ we get the $\mathbb{Z}_6$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, G, g\rangle$, $\mathcal{L}_2 = \langle F, a\rangle$, $\mathcal{L}_3 = \langle c, f\rangle$, $\mathcal{L}_4 = \langle d, C\rangle$ and $\mathcal{L}_5 = \langle b, B, D\rangle$. For $n = 1$ we get a $\mathbb{Z}_3$-grading previously obtained. The second possibility for f is $f(x, y) = \bar{x} + 2n\bar{y}$, but this does not provide new gradings up to equivalence.

- $f(1, 0) = f(1, 2)$, hence $2f(0, 1) = 0$. Then g_2 has order 2 and $k = 2n$ for certain integer n which can be taken greater than 1. If $(g_1) \cap (g_2) = 0$, f can be seen as the epimorphism $f\colon \mathbb{Z}^2 \to \mathbb{Z}_n \times \mathbb{Z}_2$ such that $f(1, 0) = (\bar{1}, 0)$ and $f(0, 1) = (0, \bar{1})$. Moreover, n is prime to 2 and it can be easily checked that the induced grading for $n = 5$ is the $\mathbb{Z}_{10}$-grading given by $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle G, c\rangle$, $\mathcal{L}_3 = \langle C\rangle$, $\mathcal{L}_4 = \langle D, g\rangle$, $\mathcal{L}_5 = \langle a, d\rangle$, $\mathcal{L}_6 = \langle A, b\rangle$, $\mathcal{L}_7 = \langle F\rangle$ and $\mathcal{L}_9 = \langle f, B\rangle$. This is equivalent to a $\mathbb{Z}_8$-grading. For $n \geq 5$ the obtained gradings are also equivalent to this one. Now the unique $n < 5$ prime to 2 is $n = 3$ but the $\mathbb{Z}_6$-grading induced by this f has previously appeared. If $(g_1) \cap (g_2) \neq 0$, since $(g_2) \cong \mathbb{Z}_2$ we have $(g_2) \subset (g_1)$ so that $(g_1) = \mathbb{Z}_k$ and g_1 is a generator of $\mathbb{Z}_{2n}$. As in previous cases we can take $g_1 = \bar{1}$ and $g_2 = \bar{n}$. Thus $f\colon \mathbb{Z}^2 \to \mathbb{Z}_{2n}$ is given by $f(x, y) = \bar{x} + n\bar{y}$, and the epimorphism f acts in the way

$$\begin{array}{lcc@{\qquad\qquad}lcc}
(1,0) & \mapsto & \bar{1} & (-1,0) & \mapsto & \overline{2n-1} \\
(0,1) & \mapsto & \bar{n} & (0,-1) & \mapsto & \overline{n} \\
(1,1) & \mapsto & \overline{n+1} & (-1,-1) & \mapsto & \overline{n-1} \\
(1,2) & \mapsto & \overline{1} & (-1,-2) & \mapsto & \overline{2n-1} \\
(1,3) & \mapsto & \overline{1+n} & (-1,-3) & \mapsto & \overline{n-1} \\
(2,3) & \mapsto & \overline{2+n} & (-2,-3) & \mapsto & \overline{n-2}.
\end{array}$$

Thus for $n > 3$ the induced gradings are equivalent to the $\mathbb{Z}_8$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, b\rangle$, $\mathcal{L}_2 = \langle C\rangle$, $\mathcal{L}_3 = \langle B, f\rangle$, $\mathcal{L}_4 = \langle a, d\rangle$, $\mathcal{L}_5 = \langle c, G\rangle$, $\mathcal{L}_6 = \langle F\rangle$ and $\mathcal{L}_7 = \langle D, g\rangle$. This $\mathbb{Z}_8$-grading is equivalent to the $\mathbb{Z}_{10}$-grading obtained above for $n \geq 5$. For $n = 1, 2, 3$ the obtained gradings have already appeared in previous cases.

- $f(1,0) = f(0,1)$, so that $g_1 = g_2$ and the epimorphism f can be identified with $f\colon \mathbb{Z}^2 \to \mathbb{Z}_k$ such that $f(x,y) = \bar{x} + \bar{y}$. For $k > 10$ the induced grading is equivalent to the $\mathbb{Z}_{11}$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, a\rangle$, $\mathcal{L}_2 = \langle c\rangle$, $\mathcal{L}_3 = \langle b\rangle$, $\mathcal{L}_4 = \langle G\rangle$, $\mathcal{L}_5 = \langle F\rangle$, $\mathcal{L}_6 = \langle C\rangle$, $\mathcal{L}_7 = \langle B\rangle$, $\mathcal{L}_8 = \langle g\rangle$, $\mathcal{L}_9 = \langle f\rangle$ and $\mathcal{L}_{10} = \langle D, d\rangle$. This grading is equivalent to one of the $\mathbb{Z}$-gradings previously found. For $k \leq 10$ we only obtain new gradings in the following cases. For $k = 10$ we get the $\mathbb{Z}_{10}$-grading: $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, a\rangle$, $\mathcal{L}_2 = \langle c\rangle$, $\mathcal{L}_3 = \langle b\rangle$, $\mathcal{L}_4 = \langle G\rangle$, $\mathcal{L}_5 = \langle C, F\rangle$, $\mathcal{L}_6 = \langle B\rangle$, $\mathcal{L}_7 = \langle g\rangle$, $\mathcal{L}_8 = \langle f\rangle$ and $\mathcal{L}_9 = \langle D, d\rangle$. For $k = 8$ we get the $\mathbb{Z}_8$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, a\rangle$, $\mathcal{L}_2 = \langle c\rangle$, $\mathcal{L}_3 = \langle C, b\rangle$, $\mathcal{L}_4 = \langle B, G\rangle$, $\mathcal{L}_5 = \langle F, g\rangle$, $\mathcal{L}_6 = \langle f\rangle$ and $\mathcal{L}_7 = \langle D, d\rangle$. For $k = 7$ we get the $\mathbb{Z}_7$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, a\rangle$, $\mathcal{L}_2 = \langle C, c\rangle$, $\mathcal{L}_3 = \langle B, b\rangle$, $\mathcal{L}_4 = \langle G, g\rangle$, $\mathcal{L}_5 = \langle F, f\rangle$ and $\mathcal{L}_6 = \langle D, d\rangle$.
- $f(1,0) = f(-1,0)$, hence $2f(1,0) = 0$, that is, g_1 has order 2. As in previous cases we have possibilities depending on the fact that the intersection $(g_1) \cap (g_2)$ is zero or not. In the first case f is identified with the map $f\colon \mathbb{Z}^2 \to \mathbb{Z}_2 \times \mathbb{Z}_n$ such that $f(1,0) = (\bar{1}, \bar{0})$ and $f(0,1) = (\bar{0}, \bar{1})$. The integer n is necessarily odd and for $n > 6$ we get a grading equivalent to the $\mathbb{Z}_{14}$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle c\rangle$, $\mathcal{L}_3 = \langle G\rangle$, $\mathcal{L}_4 = \langle C\rangle$, $\mathcal{L}_5 = \langle g\rangle$, $\mathcal{L}_6 = \langle d\rangle$, $\mathcal{L}_7 = \langle A, D\rangle$, $\mathcal{L}_8 = \langle a\rangle$, $\mathcal{L}_9 = \langle b\rangle$, $\mathcal{L}_{10} = \langle F\rangle$, $\mathcal{L}_{11} = \langle B\rangle$ and $\mathcal{L}_{13} = \langle f\rangle$. For $n = 5$ we get a $\mathbb{Z}_{10}$-grading equivalent to the one in Theorem 2. For $n = 3$ we get one of the $\mathbb{Z}_6$-gradings in Theorem 2. In case $(g_1) \cap (g_2) \neq 0$, we can suppose that $f\colon \mathbb{Z}^2 \to \mathbb{Z}_k$ is of the form $f(x,y) = n\bar{x} + \bar{y}$ where $k = 2n$ for some $n > 1$. For $n > 6$ the obtained grading is equivalent to the $\mathbb{Z}_{14}$-grading above. For $n = 6$ we get a $\mathbb{Z}_{12}$-grading which has previously appeared. For $n = 5$ we get the $\mathbb{Z}_{10}$-grading appearing in Theorem 2. For $n = 4$ we get one of the $\mathbb{Z}_8$-gradings in the theorem, and for $n = 3$, one of the $\mathbb{Z}_6$-gradings in the same theorem.
- $f(1,0) = f(-1,-1)$, which implies $f(2,1) = 0$ and $2g_1 + g_2 = 0$. Hence $(g_2) \subset (g_1)$ and $\mathbb{Z}_k = (g_1)$. We can take $g_1 = \bar{1}$ and $f(x,y) = \bar{x} - 2\bar{y}$. Thus the obtained gradings for $k > 10$ are equivalent to a $\mathbb{Z}$-grading previously described. The obtained gradings for $k \leq 10$ are easily (though tediously) seen that have already been described, and appear in Theorem 2.
- $f(1,0) = f(-1,-2)$, which implies $2f(1,1) = 0$. Letting the Weyl group act, this possibility reduces to $2f(0,1) = 0$, which has been previously studied. We can argue similarly in case $f(1,0) = f(-2,-3)$. The other possibilities lead to the relation $0 \in f(\mathfrak{S})$.

If we consider now $f(P) = f(Q)$ with $P, Q \in \mathfrak{m}$ and $0 \notin f(\mathfrak{S})$, the only possibility to study is $f(0,1) = f(0,-1)$, which implies $2f(0,1) = 0$. But this has also been previously studied.

Up to the moment we have found 20 gradings by cyclic groups up to equivalence. These are the 20 first gradings in Theorem 2.

4.3. **Noncyclic toral gradings.** Now we deal with toral gradings which are not equivalent to any cyclic grading. These come from epimorphisms $f\colon \mathbb{Z}^2 \to G$ where G is the grading group. By standard results in basic algebra, the group G is isomorphic to $\mathbb{Z}^2$, $\mathbb{Z}\times\mathbb{Z}_n$ $(n>1)$, or $\mathbb{Z}_n\times\mathbb{Z}_m$ with $n|m$. The following result restricts the possibilities for the direct factors of G.

Lemma 3. *Let $f\colon \mathbb{Z}^2 \to G$ be a group epimorphism such that the induced G-grading in $\mathfrak{g}_2$ is not equivalent to a grading by a cyclic group. Then there exists $\omega \in W$ (the Weyl group of $\mathfrak{g}_2$ acting on $\mathbb{Z}^2$) such that the set $\mathcal{S} := \{f\omega(1,0), f\omega(0,1)\}$ has either an element of order 2 or an element of order 3.*

Proof. First we prove that there is no $P \in \mathfrak{S}$ such that $f(P) = 0$. Otherwise there is some $\omega \in W$ such that either $\omega(1,0) = P$ or $\omega(0,1) = P$. As $\hat f := f\omega$ is an epimorphism, $\{\hat f(1,0), \hat f(0,1)\}$ is a system of generators of G. But one of the elements in the previous set is null so that G would be cyclic contradicting our hypothesis. As a consequence f does not vanish on $\mathfrak{S}$. If there are 12 different elements in $f(\mathfrak{S})$ (also nonzero), the grading induced by f is the Cartan one, which is a cyclic grading. Thus, there are different elements $P, Q \in \mathfrak{S}$ such that $f(P) = f(Q)$. Now, if some of the elements P or Q is in $\mathfrak{M}$, letting the Weyl group act we can suppose that this element is $(1,0)$. In this way we have an equality $f(1,0) = f(Q)$ for some $Q \in \mathfrak{S}$. Letting Q range over $\mathfrak{S}$ we obtain different equations. The possibilities which do not contradict that f does not vanish on $\mathfrak{S}$ are:

- $f(1,0) = f(1,3)$, which implies $3f(0,1) = 0$, hence $\mathcal{S}$ has an element of order 3.
- $f(1,0) = f(1,2)$, which implies $2f(0,1) = 0$, hence $\mathcal{S}$ has an element of order 2.
- $f(1,0) = f(0,1)$, this is contradictory since in this case G would be cyclic.
- $f(1,0) = f(-1,0)$, which implies $2f(1,0) = 0$, hence $\mathcal{S}$ has an element of order 2.
- $f(1,0) = f(-1,-1)$, which implies $0 = f(2,1) = 2f(1,0) + f(0,1)$. This is also contradictory since in this case $f(0,1) = -2f(1,0)$ and the group would be cyclic.
- $f(1,0) = f(-1,-2)$, which implies $2f(1,1) = 0$, so for some $\omega \in W$ we have $2f\omega(0,1) = 0$ and $\mathcal{S}$ has an element of order 2.
- $f(1,0) = f(-2,-3)$, which implies $3f(1,1) = 0$, so arguing as in the previous case $\mathcal{S}$ has an element of order 3.

If none of the elements P, Q is in $\mathfrak{M}$, then both are in $\mathfrak{m}$ and letting W act, we have an equation $f(0,1) = f(Q)$ where Q ranges on $\mathfrak{m}$. Then, arguing as before we find that $\mathcal{S}$ has some order 2 element. $\square$

Let G be an abelian noncyclic group with a system of generators $\{g_1, g_2\}$ such that $pg_1 = 0$ for a prime integer p. Then $G \cong \mathbb{Z}_p \times H$ for some cyclic group H. Indeed, the element g_1 has order p, so it generates a subgroup (g_1) isomorphic to $\mathbb{Z}_p$. Therefore (g_1) has no nonzero proper subgroup. Thus $(g_1)\cap(g_2) = 0$, since on the contrary $0 \neq (g_1)\cap(g_2) \subset (g_1)$, which implies $(g_1) \subset (g_2)$ and the group G would be cyclic. So $G = (g_1)\oplus(g_2) \cong \mathbb{Z}_p \times H$ where $H = (g_2)$ is cyclic.

Corollary 2. *If $f\colon \mathbb{Z}^2 \to G$ provides a grading of $\mathfrak{g}_2$ which is not equivalent to a cyclic grading, then either $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ or $G \cong \mathbb{Z}_3 \times \mathbb{Z}_{3k}$ for some $k \geq 1$.*

Proof. We know that G contains a system of generators $g_1 = f(0,1)$ and $g_2 = f(0,1)$ such that some of these elements has order 2 or 3. By the previous observation we have $G \cong \mathbb{Z}_p \times H$ where $p \in \{2,3\}$, $\mathbb{Z}_p \cong (g_1)$ and $H \cong (g_2)$ or $\mathbb{Z}_p \cong (g_2)$ and $H \cong (g_1)$. We have to prove that $H \not\cong \mathbb{Z}$. If $H \cong \mathbb{Z}$ define

$$m_0 = \max\{b \in \mathbb{Z}^+ : \exists \bar{a} \in \mathbb{Z}_p \text{ with either } (\bar{a}, b) \in f(\mathfrak{S}) \text{ or } (\bar{a}, -b) \in f(\mathfrak{S})\}.$$

Let us take now any $m \geq 2m_0+1$ and prime to p. Define the projection epimorphism $\Pi : \mathbb{Z}_p \times \mathbb{Z} \to \mathbb{Z}_p \times \mathbb{Z}_m \cong \mathbb{Z}_{pm}$ such that $\Pi(\bar{a}, b) = (\bar{a}, \bar{b})$. The grading given by $\Pi f\colon \mathbb{Z}^2 \to \mathbb{Z}_{pm}$ is equivalent to the given by f alone since the restriction of Π to $f(\mathfrak{S} \cup \{(0,0)\})$ is injective. $\square$

Even more important than the previous corollary is the fact that the epimorphism $f\colon \mathbb{Z}^2 \to \mathbb{Z}_p \times \mathbb{Z}_{pk}$ (with $p \in \{2,3\}$) can be chosen such that either $f(1,0) = (\bar{1}, 0)$ and $f(0,1) = (0, \bar{1})$ or $f(1,0) = (0, \bar{1})$ and $f(0,1) = (\bar{1}, 0)$. To finish this section we study first the case $f(x,y) = (\bar{x}, \bar{y})$:

- $p = 2$. For $k > 3$ the grading provided by f turns out to be cyclic and equivalent to the $\mathbb{Z}_{14}$-grading in Theorem 2. For $k = 3$ we find the $\mathbb{Z}_2 \times \mathbb{Z}_6$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{1,0} = \langle A, D\rangle$, $\mathcal{L}_{0,1} = \langle a\rangle$, $\mathcal{L}_{1,1} = \langle c\rangle$, $\mathcal{L}_{1,2} = \langle b\rangle$, $\mathcal{L}_{1,3} = \langle G, B\rangle$, $\mathcal{L}_{0,5} = \langle d\rangle$, $\mathcal{L}_{1,4} = \langle g\rangle$, $\mathcal{L}_{1,5} = \langle f\rangle$, $\mathcal{L}_{0,3} = \langle F, C\rangle$. For $k = 2$ we get the $\mathbb{Z}_2 \times \mathbb{Z}_4$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{1,0} = \langle A, D\rangle$, $\mathcal{L}_{0,1} = \langle a, C\rangle$, $\mathcal{L}_{1,1} = \langle c, B\rangle$, $\mathcal{L}_{1,2} = \langle b, g\rangle$, $\mathcal{L}_{1,3} = \langle G, f\rangle$ and $\mathcal{L}_{0,3} = \langle F, d\rangle$. Finally, for $k = 1$ we have the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{1,0} = \langle A, D, b, g\rangle$, $\mathcal{L}_{0,1} = \langle a, d, F, C\rangle$ and $\mathcal{L}_{1,1} = \langle c, f, G, B\rangle$.
- $p = 3$. For $k > 2$ all the induced gradings are equivalent to the Cartan grading, which is cyclic. For $k = 2$ we get one of the $\mathbb{Z}_{12}$-gradings in Theorem 2. For $k = 1$ we get the $\mathbb{Z}_3 \times \mathbb{Z}_3$-grading given by $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{1,0} = \langle A, G, C\rangle$, $\mathcal{L}_{0,1} = \langle a\rangle$, $\mathcal{L}_{1,1} = \langle c\rangle$, $\mathcal{L}_{1,2} = \langle b\rangle$, $\mathcal{L}_{2,0} = \langle F, D, B\rangle$, $\mathcal{L}_{0,2} = \langle d\rangle$, $\mathcal{L}_{2,2} = \langle f\rangle$ and $\mathcal{L}_{2,1} = \langle g\rangle$.

The case $f(x,y) = (\bar{y}, \bar{x})$ is similar to the previous one and does not provide new gradings other than the already found.

4.4. **Nontoral gradings.** Suppose a nontoral grading $\mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ of $\mathfrak{g}_2$. Then it comes through the transferring mechanism from a grading

$$C = \oplus_{g\in G} C_g \tag{9}$$

of $C = \mathbb{O}_\mathbb{C}$ which is of course nontoral. But the unique nontoral grading on C up to equivalence is the $\mathbb{Z}_2^3$-grading (9) in subsection 3.2. Therefore the grading (9) is equivalent to (9) of 3.2 whose universal grading group is $\mathbb{Z}_2^3$. Since $|G| \geq |\{g \in G : C_g \neq 0\}| = 8$, the epimorphism $f\colon \mathbb{Z}_2^3 \to G$ is necessarily an isomorphism and it turns out that the grading (9) is isomorphic to (9) of 3.2. As a consequence our original nontoral grading on $\mathfrak{g}_2$ is isomorphic to the obtained by transferring (9) of 3.2 to $\mathfrak{g}_2$. So what we must do is to compute the induced grading on $\mathfrak{g}_2$ obtained from the grading (9) of 3.2. This is produced by the automorphisms $\{t_{1,-1}, t_{-1,1}, f\}$ where f is as in (7) with $\alpha = \beta = 1$. Therefore the grading we are searching for is the induced in $\mathfrak{g}_2$ by the automorphisms $\{s_{1,-1}, s_{-1,1}, \mathrm{Ad}(f)\}$. This is easily computed by making a simultaneous diagonalization of $\mathfrak{g}_2$ relative to the three commuting semisimple automorphisms. The resulting grading is the last one in Theorem 2.

Summarizing the results in this section we claim:

Theorem 2. *Up to equivalence, the G-gradings on $\mathfrak{g}_2$ are the following:*

(1) $G = \mathbb{Z}$, $\mathcal{L}_{-2} = \langle C\rangle$, $\mathcal{L}_{-1} = \langle D, f, g, B\rangle$, $\mathcal{L}_0 = \mathfrak{h} + \langle a, d\rangle$, $\mathcal{L}_1 = \langle A, c, b, G\rangle$ *and* $\mathcal{L}_2 = \langle F\rangle$.

(2) $G = \mathbb{Z}$, $\mathcal{L}_{-3} = \langle B, C\rangle$, $\mathcal{L}_{-2} = \langle g\rangle$, $\mathcal{L}_{-1} = \langle d, f\rangle$, $\mathcal{L}_0 = \mathfrak{h} + \langle A, D\rangle$, $\mathcal{L}_1 = \langle a, c\rangle$, $\mathcal{L}_2 = \langle b\rangle$ *and* $\mathcal{L}_3 = \langle G, F\rangle$.

(3) $G = \mathbb{Z}$, $\mathcal{L}_{-5} = \langle C\rangle$, $\mathcal{L}_{-4} = \langle B\rangle$, $\mathcal{L}_{-3} = \langle g\rangle$, $\mathcal{L}_{-2} = \langle f\rangle$, $\mathcal{L}_{-1} = \langle d, D\rangle$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle a, A\rangle$, $\mathcal{L}_2 = \langle c\rangle$, $\mathcal{L}_3 = \langle b\rangle$, $\mathcal{L}_4 = \langle G\rangle$, $\mathcal{L}_5 = \langle F\rangle$.

(4) $G = \mathbb{Z}$, $\mathcal{L}_{-8} = \langle C\rangle$, $\mathcal{L}_{-7} = \langle B\rangle$, $\mathcal{L}_{-5} = \langle g\rangle$, $\mathcal{L}_{-3} = \langle f\rangle$, $\mathcal{L}_{-2} = \langle d\rangle$, $\mathcal{L}_{-1} = \langle D\rangle$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A\rangle$, $\mathcal{L}_2 = \langle a\rangle$, $\mathcal{L}_3 = \langle c\rangle$, $\mathcal{L}_5 = \langle b\rangle$, $\mathcal{L}_7 = \langle G\rangle$, $\mathcal{L}_8 = \langle F\rangle$ *(this is the Cartan grading).*

(5) $G = \mathbb{Z}_2$, $\mathcal{L}_0 = \mathfrak{h} + \langle a, d, C, F\rangle$ *and* $\mathcal{L}_1 = \langle D, B, f, g, A, G, b, c\rangle$.

(6) $G = \mathbb{Z}_3$, $\mathcal{L}_0 = \mathfrak{h} + \langle d, a\rangle$, $\mathcal{L}_1 = \langle A, G, C, b, c\rangle$ *and* $\mathcal{L}_2 = \langle D, B, F, f, g\rangle$.

(7) $G = \mathbb{Z}_3$, $\mathcal{L}_0 = \mathfrak{h} + \langle A, D, G, F, B, C\rangle$, $\mathcal{L}_1 = \langle a, c, g\rangle$ *and* $\mathcal{L}_2 = \langle d, f, b\rangle$.

(8) $G = \mathbb{Z}_4$, $\mathcal{L}_0 = \mathfrak{h} + \langle d, a\rangle$, $\mathcal{L}_1 = \langle A, G, b, c\rangle$, $\mathcal{L}_2 = \langle F, C\rangle$ *and* $\mathcal{L}_3 = \langle D, B, f, g\rangle$.

(9) $G = \mathbb{Z}_4$, $\mathcal{L}_0 = \mathfrak{h} + \langle A, D\rangle$, $\mathcal{L}_1 = \langle a, c, B, C\rangle$, $\mathcal{L}_2 = \langle b, g\rangle$ *and* $\mathcal{L}_3 = \langle d, f, G, F\rangle$.

(10) $G = \mathbb{Z}_5$, $\mathcal{L}_0 = \mathfrak{h} + \langle A, D\rangle$, $\mathcal{L}_1 = \langle a, c\rangle$, $\mathcal{L}_2 = \langle b, B, C\rangle$, $\mathcal{L}_3 = \langle g, G, F\rangle$ *and* $\mathcal{L}_4 = \langle d, f\rangle$.

(11) $G = \mathbb{Z}_6$, $\mathcal{L}_0 = \mathfrak{h} + \langle A, D\rangle$, $\mathcal{L}_1 = \langle a, c\rangle$, $\mathcal{L}_2 = \langle b\rangle$, $\mathcal{L}_3 = \langle G, F, B, C\rangle$, $\mathcal{L}_4 = \langle g\rangle$ *and* $\mathcal{L}_5 = \langle d, f\rangle$.

(12) $G = \mathbb{Z}_6$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, G, g\rangle$, $\mathcal{L}_2 = \langle F, a\rangle$, $\mathcal{L}_3 = \langle c, f\rangle$, $\mathcal{L}_4 = \langle d, C\rangle$ *and* $\mathcal{L}_5 = \langle b, B, D\rangle$.

(13) $G = \mathbb{Z}_7$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, a\rangle$, $\mathcal{L}_2 = \langle C, c\rangle$, $\mathcal{L}_3 = \langle B, b\rangle$, $\mathcal{L}_4 = \langle G, g\rangle$, $\mathcal{L}_5 = \langle F, f\rangle$ *and* $\mathcal{L}_6 = \langle D, d\rangle$.

(14) $G = \mathbb{Z}_8$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, a\rangle$, $\mathcal{L}_2 = \langle c\rangle$, $\mathcal{L}_3 = \langle C, b\rangle$, $\mathcal{L}_4 = \langle B, G\rangle$, $\mathcal{L}_5 = \langle F, g\rangle$, $\mathcal{L}_6 = \langle f\rangle$ *and* $\mathcal{L}_7 = \langle D, d\rangle$.

(15) $G = \mathbb{Z}_8$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, b\rangle$, $\mathcal{L}_2 = \langle C\rangle$, $\mathcal{L}_3 = \langle B, f\rangle$, $\mathcal{L}_4 = \langle a, d\rangle$, $\mathcal{L}_5 = \langle c, G\rangle$, $\mathcal{L}_6 = \langle F\rangle$ *and* $\mathcal{L}_7 = \langle D, g\rangle$.

(16) $G = \mathbb{Z}_9$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, G\rangle$, $\mathcal{L}_2 = \langle F, g\rangle$, $\mathcal{L}_3 = \langle a\rangle$, $\mathcal{L}_4 = \langle c\rangle$, $\mathcal{L}_5 = \langle f\rangle$, $\mathcal{L}_6 = \langle d\rangle$, $\mathcal{L}_7 = \langle b, C\rangle$ *and* $\mathcal{L}_8 = \langle D, B\rangle$.

(17) $G = \mathbb{Z}_{10}$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle A, a\rangle$, $\mathcal{L}_2 = \langle c\rangle$, $\mathcal{L}_3 = \langle b\rangle$, $\mathcal{L}_4 = \langle G\rangle$, $\mathcal{L}_5 = \langle C, F\rangle$, $\mathcal{L}_6 = \langle B\rangle$, $\mathcal{L}_7 = \langle g\rangle$, $\mathcal{L}_8 = \langle f\rangle$ *and* $\mathcal{L}_9 = \langle D, d\rangle$.

(18) $G = \mathbb{Z}_{12}$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{\bar{1}} = \langle A, G\rangle$, $\mathcal{L}_{\bar{2}} = \langle F\rangle$, $\mathcal{L}_3 = \langle g\rangle$, $\mathcal{L}_4 = \langle a\rangle$, $\mathcal{L}_5 = \langle c\rangle$, $\mathcal{L}_7 = \langle f\rangle$, $\mathcal{L}_8 = \langle d\rangle$, $\mathcal{L}_9 = \langle b\rangle$, $\mathcal{L}_{10} = \langle C\rangle$ *and* $\mathcal{L}_{11} = \langle D, B\rangle$.

(19) $G = \mathbb{Z}_{12}$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle c\rangle$, $\mathcal{L}_3 = \langle B, D\rangle$, $\mathcal{L}_4 = \langle a\rangle$, $\mathcal{L}_5 = \langle b\rangle$, $\mathcal{L}_6 = \langle F, C\rangle$, $\mathcal{L}_7 = \langle g\rangle$, $\mathcal{L}_8 = \langle d\rangle$, $\mathcal{L}_9 = \langle A, G\rangle$ *and* $\mathcal{L}_{11} = \langle f\rangle$.

(20) $G = \mathbb{Z}_{14}$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_1 = \langle c\rangle$, $\mathcal{L}_3 = \langle G\rangle$, $\mathcal{L}_4 = \langle C\rangle$, $\mathcal{L}_5 = \langle g\rangle$, $\mathcal{L}_6 = \langle d\rangle$, $\mathcal{L}_7 = \langle A, D\rangle$, $\mathcal{L}_8 = \langle a\rangle$, $\mathcal{L}_9 = \langle b\rangle$, $\mathcal{L}_{10} = \langle F\rangle$, $\mathcal{L}_{11} = \langle B\rangle$ *and* $\mathcal{L}_{13} = \langle f\rangle$.

(21) $G = \mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{1,0} = \langle A, D\rangle$, $\mathcal{L}_{0,1} = \langle a\rangle$, $\mathcal{L}_{1,1} = \langle c\rangle$, $\mathcal{L}_{1,2} = \langle b\rangle$, $\mathcal{L}_{1,3} = \langle G, B\rangle$, $\mathcal{L}_{0,5} = \langle d\rangle$, $\mathcal{L}_{1,4} = \langle g\rangle$, $\mathcal{L}_{1,5} = \langle f\rangle$, $\mathcal{L}_{0,3} = \langle F, C\rangle$.

(22) $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{1,0} = \langle A, D\rangle$, $\mathcal{L}_{0,1} = \langle a, C\rangle$, $\mathcal{L}_{1,1} = \langle c, B\rangle$, $\mathcal{L}_{1,2} = \langle b, g\rangle$, $\mathcal{L}_{1,3} = \langle G, f\rangle$ *and* $\mathcal{L}_{0,3} = \langle F, d\rangle$.

(23) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{1,0} = \langle A, D, b, g\rangle$, $\mathcal{L}_{0,1} = \langle a, d, F, C\rangle$ *and* $\mathcal{L}_{1,1} = \langle c, f, G, B\rangle$.

(24) $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_{1,0} = \langle A, G, C\rangle$, $\mathcal{L}_{0,1} = \langle a\rangle$, $\mathcal{L}_{1,1} = \langle c\rangle$, $\mathcal{L}_{1,2} = \langle b\rangle$, $\mathcal{L}_{2,0} = \langle F, D, B\rangle$, $\mathcal{L}_{0,2} = \langle d\rangle$, $\mathcal{L}_{2,2} = \langle f\rangle$ *and* $\mathcal{L}_{2,1} = \langle g\rangle$.

(25) $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *with* $\mathcal{L}_0 = 0$, $\mathcal{L}_{0,0,1} = \mathfrak{h}$, $\mathcal{L}_{0,1,0} = \langle c + f, B + G\rangle$, $\mathcal{L}_{1,0,0} = \langle a + d, C + F\rangle$, $\mathcal{L}_{1,1,0} = \langle A + D, b + g\rangle$, $\mathcal{L}_{1,0,1} = \langle -a + d, C - F\rangle$, $\mathcal{L}_{0,1,1} = \langle -c + f, B - G\rangle$, $\mathcal{L}_{1,1,1} = \langle -A + D, -b + g\rangle$.

The unique fine gradings are (4) and (25). This last one is the only nontoral grading. The gradings (21)-(25) are not equivalent to any cyclic grading.

Remark. Each cyclic grading on $\mathfrak{g}_2$ must be defined by a triplet of integer numbers in the way explained in the introduction. Indeed, the triplets corresponding to the gradings (1) to (20) are, respectively, (1,2,0), (0,1,2), (1,2,2), (1,2,4), (0,1,0), (1,1,0), (0,0,1), (2,1,0), (1,0,1), (0,1,1), (3,0,1), (1,1,1), (2,1,1), (3,1,1), (1,2,1), (1,1,2), (3,2,1), (1,1,3), (3,3,1) and (3,4,1), up to equivalence.

Acknowledgements We would like to thank the referee for his exhaustive and deep review of this work. We also thank to A. Elduque for warning us about the mistake in [18].

References

[1] A. V. Alekseevski. *Finite commutative Jordan subgroups of complex simple Lie groups.* Functional Analysis Appl. 8 (1974), 277-279.

[2] Y. A. Bahturin and I. P. Shestakov. *Gradings on simple Jordan algebras and their relation of the gradings of simple associative algebras.* Communications in Algebra, 29(9), 4095-4102 (2001).

[3] Y. A. Bahturin and M. V. Zaicev. *Group gradings on matrix algebras.* Canad. Math. Bull. 45 (2002), 499-508.

[4] Y. A. Bahturin, I. P. Shestakov and M. V. Zaicev. *Gradings on simple Jordan and Lie algebras.* Journal of Algebra 283, (2005) 849-868.

[5] A. Elduque. *Gradings on Octonions.* Journal of Algebra 207, 342-354 (1998).

[6] A. Elduque. *A Lie grading which is not a semigroup grading.* arXiv.org, math.RA/0512618

[7] M. Havlícek, J. Patera and E. Pelatonova. *Automorphisms of fine gradings of* $sl(n, \mathbb{C})$ *associated with the generalized Pauli matrices.* J. Math. Phys. 43 (2002), 1083-1094.

[8] M. Havlícek, J. Patera and E. Pelatonova. *On Lie gradings II.* Linear Algebra and its Applications 277 (1998) 97-125.

[9] M. Havlícek, J. Patera and E. Pelatonova. *On Lie gradings III. Gradings of the real forms of classical Lie algebras (dedicated to the memory of H. Zassenhaus).* Linear Algebra and its Applications 314 (2000) 1-47.

[10] M. Havlícek, J. Patera and E. Pelatonova and J. Tolar. *On fine gradings and their symmetries.* Czechoslovak J. Phys. 51 (2001), 383-391.

[11] J. Patera, E. Pelatonova and M. Svobodova. *The eight fine gradings of* $sl(4, \mathbb{C})$ *and* $o(6, \mathbb{C})$. J. Phys. 43 (2002), 6353-6378.

[12] W. H. Hesselink. *Special and Pure Gradings of Lie Algebras.* Math. Z. 179, 135-149 (1982).

[13] J. E. Humphreys. *Linear Algebraic Groups.* Graduate Texts In Mathematics 21. Springer-Verlag. New York Heidelberg Berlin. 1975.

[14] N. Jacobson. *Basic Algebra I.* W. H. Freeman and Company. New York, 1985.

[15] N. Jacobson. *Lie algebras.* Dover Publications, Inc., New York, 1979.

[16] V. G. Kac. *Graded Lie algebras and symmetric spaces.* Functional Anal. i Prilozhen 2 (1968), 93-94.

[17] A. L. Onishchnik, E. B. Vinberg (Editors). *Lie Groups and Lie Algebras III, Encyclopaedia of Mathematical Sciences, Vol. 41.* Springer-Verlag. Berlin 1991.

[18] J. Patera and H. Zassenhaus. *On Lie gradings. I.* Linear Algebra and its Applications. 112, 87-159 (1989).

[19] R. D. Schafer. *An introduction to nonassociative algebras.* Academic Press. New York, San Francisco, London. 1966.

[20] E. B. Vinberg. *Classification of homogeneous nilpotent elements of a semisimple graded Lie algebra*. Soviet Math. Dokl., 16 (1975), 1517-1520.
[21] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov and A.I. Shirshov, *Rings that are nearly associative,* Academic Press, New York. 1982.

Cristina Draper Fontanals: Departamento de Matemática Aplicada, Campus de El Ejido, S/N, 29071 Málaga, Spain.

Cándido Martín González: Departamento de Álgebra, Geometría y Topología, Campus de Teatinos, S/N. Facultad de Ciencias, Ap. 59, 29080 Málaga, Spain.