Some forms of exceptional Lie algebras

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Abstract

Some forms of Lie algebras of types E_6, E_7, and E_8 are constructed using the exterior cube of a rank 9 finitely generated projective module.

1 Introduction

Let $\mathcal{G}(\mathbb{C})$ be a simple Lie algebra over $\mathbb{C}$ of type X_l and let $\mathcal{G}(\mathbb{Z})$ be the $\mathbb{Z}$-span of a Chevalley basis of $\mathcal{G}(\mathbb{C})$. We say that a Lie algebra $\mathcal{G}$ over a unitary commutative ring k is a *form* of X_l if there is a faithfully flat, commutative, unital k-algebra F with $\mathcal{G}_F \cong \mathcal{G}(\mathbb{Z})_F$ where $\mathcal{G}_F = \mathcal{G} \otimes_k F$ as a F-module. The main purpose of this paper is the construction of some forms of E_6, E_7, and E_8 using the exterior cube of a rank 9 finitely generated projective module. In §2, we develop the necessary exterior algebra and localization machinery. In §3, we construct a Lie algebra from the exterior cube of a rank 9 finitely generated projective module, and then give a twisted version of the construction. In §4, we show that the Lie algebras are forms of E_8 and identify some subalgebras which are forms of E_6 and E_7.

2 Preliminary results

Let k be a unitary commutative ring. Throughout, we require that a k-module M be unital; i.e., $1x = x$ for $x \in M$. Let $M^* = Hom_k(M, k)$, the dual module. Recall that a k-module M is *projective* if M is a direct summand of a free module ([B88],II.2.2). Moreover, M is a finitely generated projective module if and only if M is a direct summand of a free module of finite rank

([B88],II.2.2). Let M and N be finitely generated projective modules. Then M^* and $M \otimes N$ are also finitely generated projective (([B88],II.2.6,II.3.7), and we may identify M with M^{**} where $m(\phi) = \phi(m)$ for $m \in M$ and $\phi \in M^*$ ([B88],II.2.7). Moreover, the linear map

$$M \otimes M^* \to End(M)$$

with $m \otimes \phi \to m\phi$ where $(m\phi)(m') = \phi(m')m$ is bijective ([B88],II.4.2). Thus, we can define the *trace* function tr on $End(M)$ as the unique linear map with $tr(m\phi) = \phi(m)$. Since

$$tr((m\phi)(m'\phi')) = \phi'(m)\phi(m'),$$

we see that $tr(\alpha\beta) = tr(\beta\alpha)$ for $\alpha, \beta \in End(M)$. Letting $gl(M) = End(M)$ with Lie product $[\alpha, \beta] = \alpha\beta - \beta\alpha$, we see

$$[gl(M), gl(M)] \subset sl(M) := \{\alpha \in gl(M) : tr(\alpha) = 0\},$$

so $sl(M)$ is an ideal in $gl(M)$.

Let k-alg denote the category of commutative unital k-algebras. If $K \in k$-alg and M, N are k-modules, let $M_K = M \otimes_k K$ as a K-module. If M is a finitely generated projective k-module, then

$$\begin{aligned} (M \otimes_k N)_K &\cong M_K \otimes_K N_K, \\ (M^*)_K &\cong (M_K)^*, \\ gl(M)_K &\cong gl(M_K) \end{aligned}$$

via canonical isomorphisms ([B88],II.5.1,II.5.4).

If $\mathfrak{p}$ is a prime ideal of k, let $k_\mathfrak{p} = (k \setminus \mathfrak{p})^{-1} k$ be the *localization* of k at $\mathfrak{p}$ and $M_\mathfrak{p} = M_{k_\mathfrak{p}}$ be the *localization* of M at $\mathfrak{p}$ ([B89],II). If M is finitely generated projective, then $M_\mathfrak{p}$ is a free $k_\mathfrak{p}$-module of finite rank ([B89],II.5.2). If $M_\mathfrak{p}$ has rank n for all prime ideals $\mathfrak{p}$ of k, we say M has *rank* n. In this case, M_K has rank n for all $K \in k$-alg ([B89],II.5.3). Moreover, if M, N are finitely generated projective modules and $\alpha \in Hom(M, N)$, then α is injective (respectively, surjective, bijective, zero) if and only if $\alpha_\mathfrak{p} = \alpha \otimes Id_{k_\mathfrak{p}} \in Hom(M_\mathfrak{p}, N_\mathfrak{p})$ is injective (respectively, surjective, bijective, zero) for each prime ideal $\mathfrak{p}$ ([B89], II.3.3). This allows the transfer of multilinear identities using localization as follows: if $M_1, \ldots, M_l, N$ are finite generated projective modules and

$$\mu : M_1 \times \cdots \times M_l \to N$$

is a k-multilinear map, then for $K \in k$-alg there is a unique K-multilinear map

$$\mu_K : M_{1K} \times \cdots \times M_{lK} \to N_K$$

with

$$\mu_K(m_1 \otimes 1, \ldots, m_l \otimes 1) = \mu(m_1, \ldots, m_l) \otimes 1. \tag{1}$$

We claim $\mu_{\mathfrak{p}} = 0$ for each prime ideal $\mathfrak{p}$ implies $\mu = 0$. Indeed, $M_1 \otimes \cdots \otimes M_l$ is finitely generated projective and μ induces a linear map

$$\tilde{\mu} : M_1 \otimes \cdots \otimes M_l \to N$$

with each $(\tilde{\mu})_{\mathfrak{p}} = \widetilde{(\mu_{\mathfrak{p}})} = 0$, so $\tilde{\mu} = 0$ and $\mu = 0$.

Recall $F \in k$-alg is *faithfully flat* provided a sequence $M' \to M \to M''$ is exact if and only if the induced sequence $M'_F \to M_F \to M''_F$ is exact. We shall need the following example of a faithfully flat algebra. Recall a quadratic form q on M is *nonsingular* if $a \to q(a, \ )$ is an isomorphism $M \to M^*$ where

$$q(a,b) := q(a+b) - q(a) - q(b).$$

We say that $K \in k$-alg is a *quadratic étale algebra* if K is a finitely generated projective k-module of rank 2 with a nonsingular quadratic form n admitting composition; i.e.,

$$n(ab) = n(a)n(b).$$

We did not find a suitable reference for the following result, so we include a proof communicated to us by H. Petersson.

Proposition 1 *If K is a quadratic étale algebra over k, then K is faithfully flat and $K_K \cong K \oplus K$.*

Proof. For each maximal ideal m of k, K_m is a nonzero free k_m-module, and hence faithfully flat ([B89], II.3.1). Thus, K is faithfully flat over k ([B89], II.3.4). Let $t(a) = n(a,1)$ and $\bar{a} = t(a)1 - a$, for $a \in K$. We claim $\eta : K_K \to K \oplus K$ with $\eta(a \otimes b) = ab \oplus \bar{a}b$ is a K-algebra isomorphism. Using localization, it suffices to assume that k is a field. In this case, it is well-known that K is commutative, $n(1) = 1$, $a \to \bar{a}$ is an involution, and $a^{-1} = n(a)^{-1}\bar{a}$, if $n(a) \neq 0$. Thus, η is a homomorphism of K-algebras with involution where $K \oplus K$ has the exchange involution. By dimensions, it suffices to show η is surjective. Let $1, u$ be a k-basis of K. We see

$$\begin{aligned} n(\bar{u} - u) &= n(t(u)1 - 2u) \\ &= 4n(u) - t(u)^2 \\ &= \det \begin{bmatrix} n(1,1) & n(1,u) \\ n(u,1) & n(u,u) \end{bmatrix} \neq 0 \end{aligned}$$

since n is nonsingular, so $\bar{u} - u$ is invertible. Now $\eta(u \otimes 1 - 1 \otimes u) = 0 \oplus (\bar{u} - u)$, so $\eta(K_K)$ contains $0 \oplus 1, 1 \oplus 0 = \overline{0 \oplus 1}$, and hence $K \oplus K$. ■

We now recall some facts about exterior algebras. For more details see [B88]. Let M be a k-module and form the exterior algebra $\Lambda(M)$ with the standard $\mathbb{Z}$-grading

$$\Lambda(M) = \sum_{i \geq 0} \Lambda_i(M),$$

and write $|x| = i$, if $x \in \Lambda_i(M)$. For simplicity of notation, we write the product in $\Lambda(M)$ as xy rather than the usual $x \wedge y$. We have $\Lambda(M)_K \cong \Lambda(M_K)$ via a canonical isomorphism ([B88],III.7.5). If M is finitely generated projective, then so is $\Lambda(M)$ ([B88],III.7.8). If $\alpha \in Hom(M,N)$, then α extends uniquely to a graded algebra homomorphism $\theta_\alpha : \Lambda(M) \to \Lambda(N)$. Also, if $\alpha \in gl(M)$, then there is a unique extension of α to a derivation D_α of $\Lambda(M)$. Thus, $\Lambda(M)$ is a module for the Lie algebra $gl(M)$ via $(\alpha, x) \to D_\alpha(x)$. Similarly, if $\phi \in M^*$, then there is a unique extension of ϕ to an anti-derivation (or odd super derivation) Δ_ϕ of $\Lambda(M)$. Recall Δ is an *anti-derivation* if

$$\Delta(xy) = \Delta(x)y + (-1)^{|x|}x\Delta(y)$$

if x is homogeneous. One can show by induction on i that

$$\Delta_\phi(\Lambda_i(M)) \subset \Lambda_{i-1}(M), \tag{2}$$

where $\Lambda_l(M) = 0$ for $l < 0$, and $\Delta_\phi^2 = 0$. Thus, the universal property for $\Lambda(M^*)$ shows that $\phi \to \Delta_\phi$ extends to a homomorphism $\Delta : \Lambda(M^*)$ into $End_k(\Lambda(M))$, so we can view $\Lambda(M)$ as a left module for the associative algebra $\Lambda(M^*)$ with $\xi \cdot x = \Delta_\xi(x)$ for $\xi \in \Lambda(M^*)$, $x \in \Lambda(M)$. Using (2), we see

$$\Lambda_i(M^*) \cdot \Lambda_j(M) \subset \Lambda_{j-i}(M).$$

Let M be a finitely generated projective k-module. Since $M^{**} = M$, we can reverse the roles of M and M^* and see that $\Lambda(M^*)$ is a left module for $\Lambda(M)$ via $x \cdot \xi$. Also, we can identify $\Lambda_i(M^*)$ with $\Lambda_i(M)^*$ where $\xi(x) = \xi \cdot x$ for $\xi \in \Lambda_i(M^*)$, $x \in \Lambda_i(M)$ ([B88],III.11.5).

For $\alpha \in Hom(M,N)$, let $\alpha^* \in Hom(N^*, M^*)$ with $\alpha^*(\phi) = \phi\alpha$ for $\phi \in N^*$. Thus, $\alpha \to -\alpha^*$ is a Lie algebra homomorphism $gl(M) \to gl(M^*)$ and $\Lambda(M^*)$ is a module for $gl(M)$ via $(\alpha, \xi) \to D_{-\alpha^*}(\xi)$.

Lemma 2 *Let $l \leq n$ and let $S \subset S_n$ be such that $\sigma \to \sigma |_{\{1,\ldots,l\}}$ is a bijection of S with the set of all injections*

$$\{1, \ldots, l\} \to \{1, \ldots, n\}.$$

For $\phi_i \in M^*, m_j \in M$, we have

$$(\phi_l\phi_{l-1}\cdots\phi_1) \cdot (m_1 m_2 \cdots m_n) = \sum_{\sigma \in S} (-1)^\sigma \phi_1(m_{\sigma 1}) \cdots \phi_l(m_{\sigma l}) m_{\sigma(l+1)} \cdots m_{\sigma n}.$$

Proof. Applying $\Delta_{\phi_l} \cdots \Delta_{\phi_1}$ to $m_1 m_2 \cdots m_n$, we get terms

$$\pm\phi_1(m_{i_1}) \cdots \phi_l(m_{i_l}) m_{i_{l+1}} \cdots m_{i_n}$$

with the sign depending only on $i_1, \ldots, i_n$. There is a unique $\sigma \in S$ with $\sigma(j) = i_j$ for $1 \leq j \leq l$. After suitably rearranging the factors of $m_{i_{l+1}} \cdots m_{i_n}$, we can assume $i_j = \sigma(j)$ for all j. Thus,

$$(\phi_l\phi_{l-1}\cdots\phi_1) \cdot (m_1 m_2 \cdots m_n) = \sum_{\sigma \in S} \varepsilon_\sigma \phi_1(m_{\sigma 1}) \cdots \phi_l(m_{\sigma l}) m_{\sigma(l+1)} \cdots m_{\sigma(n)}$$

for some $\varepsilon_\sigma = \pm 1$, depending only on σ In particular, if $m_1, \ldots, m_n$ is the basis of a vector space V over a field of characteristic not 2 and $\phi_i \in V^*$ with $\phi_i(m_j) = \delta_{ij}$, then for $\tau \in S$, we have

$$\begin{aligned} m_{l+1} \cdots m_n &= (\phi_l \cdots \phi_1) \cdot (m_1 \cdots m_n) \\ &= (-1)^\tau (\phi_l \cdots \phi_1) \cdot (m_{\tau^{-1}1} \cdots m_{\tau^{-1}n}) \\ &= (-1)^\tau \sum_{\sigma \in S} \varepsilon_\sigma \phi_1(m_{\tau^{-1}\sigma 1}) \cdots \phi_l(m_{\tau^{-1}\sigma l}) m_{\tau^{-1}\sigma(l+1)} \cdots m_{\tau^{-1}\sigma(n)} \\ &= (-1)^\tau \varepsilon_\tau m_{l+1} \cdots m_n \end{aligned}$$

and $\varepsilon_\tau = (-1)^\tau$. ■

We remark that if $l = 1$ in Lemma 2, we can take $S = C_n$, the cyclic group generated by the permutation $(1, \ldots, n)$.

If $\alpha \in gl(M)$ and $\phi \in M^*$, then $[D_\alpha, \Delta_\phi]$ is an antiderivation with

$$[D_\alpha, \Delta_\phi](m) = D_\alpha(\phi(m)) - \phi(\alpha m) = \Delta_{-\alpha^*(\phi)}(m),$$

for $m \in M$ Thus, $[D_\alpha, \Delta_\phi] = \Delta_{-\alpha^*(\phi)} = \Delta_{D_{-\alpha^*}(\phi)}$. Since Δ is a homomorphism, we have

$$[D_\alpha, \Delta_\xi] = \Delta_{D_{-\alpha^*}(\xi)}$$

for all $\xi \in \Lambda(M^*)$, so

$$D_\alpha(\xi \cdot x) = D_{-\alpha^*}(\xi) \cdot x + \xi \cdot D_\alpha(x), \tag{3}$$

for all $x \in \Lambda(M)$.

Let M be finitely generated projective. For $x \in \Lambda_l(M), \xi \in \Lambda_l(M^*)$, define $e(x, \xi) \in End(M)$ by

$$e(x, \xi)(m) = (m \cdot \xi) \cdot x \in \Lambda_{l-1}(M^*) \cdot \Lambda_l(M) \subset M$$

for $m \in M$. We also have $e(\xi, x) \in End(M^*)$.

Lemma 3 *Let M be a finitely generated projective module, and let $x, y, z \in \Lambda_l(M)$, $\xi \in \Lambda_l(M^*)$, and $\mu \in \Lambda_{3l}(M^*)$. We have*

(i) $x \cdot \xi = \xi \cdot x$,

(ii) $e(x, \xi)^* = e(\xi, x)$,

(iii) if $\phi_1, \ldots, \phi_l \in M^*$, then

$$D_{e(x, \phi_1 \cdots \phi_l)} = \sum_{\sigma \in C_l} (-1)^\sigma ((\phi_{\sigma 2} \cdots \phi_{\sigma l}) \cdot x) \Delta_{\phi_{\sigma 1}},$$

where C_l is the cyclic group generated by the permutation $(1, \ldots, l)$,

(iv) $tr(e(x, \xi)) = l\xi \cdot x$,

(v) $e(xyz, \mu) = \sum_{x,y,z \circlearrowleft} e(x, (yz) \cdot \mu)$, where the sum is over all cyclic permutations of x, y, z,

(vi) if $l = 3$, then $\xi \cdot (xy) = (\xi \cdot x)y - D_{e(x,\xi)}y + D_{e(y,\xi)}x - (\xi \cdot y)x$.

Proof. Using Lemma 2, we have

$$\begin{aligned}(\phi_l\phi_{l-1}\cdots\phi_1)\cdot(m_1m_2\cdots m_l) &= \sum_{\sigma\in S_l}(-1)^\sigma\phi_1(m_{\sigma 1})\cdots\phi_l(m_{\sigma l}) \\ &= \sum_{\sigma\in S_l}(-1)^\sigma m_{\sigma 1}(\phi_1)\cdots m_{\sigma l}(\phi_l) \\ &= \sum_{\sigma\in S_l}(-1)^\sigma m_1(\phi_{\sigma 1})\cdots m_l(\phi_{\sigma l}) \\ &= (m_1m_2\cdots m_l)\cdot(\phi_l\phi_{l-1}\cdots\phi_1)\end{aligned}$$

for $m_i \in M, \phi_i \in M^*$, showing (i). For $\phi \in M^*, m \in M$, we have

$$\begin{aligned}(e(x,\xi)^*(\phi))(m) &= \phi(e(x,\xi)(m)) \\ &= \phi\cdot((m\cdot\xi)\cdot x) = (\phi(m\cdot\xi))\cdot x \\ &= (-1)^{l-1}((m\cdot\xi)\phi)\cdot x = (-1)^{l-1}(m\cdot\xi)\cdot(\phi\cdot x) \\ &= (-1)^{l-1}(\phi\cdot x)\cdot(m\cdot\xi) = m\cdot((\phi\cdot x)\cdot\xi) \\ &= (e(\xi,x)(\phi))(m)\end{aligned}$$

showing (ii).

If $m \in M, \phi \in M^*$ it is easy to see that $m\Delta_\phi : x \to m(\phi\cdot x)$ is a derivation of $\Lambda(M)$, so $m\Delta_\phi = D_{m\phi}$. By Lemma 2, we have

$$\begin{aligned}e(x,\phi_1\cdots\phi_l)m &= (m\cdot(\phi_1\cdots\phi_l))\cdot x \\ &= \sum_{\sigma\in C_l}(-1)^\sigma((m\cdot\phi_{\sigma 1})(\phi_{\sigma 2}\cdots\phi_{\sigma l}))\cdot x \\ &= \sum_{\sigma\in C_l}(-1)^\sigma((\phi_{\sigma 2}\cdots\phi_{\sigma l})\cdot x)\Delta_{\phi_{\sigma 1}}(m),\end{aligned}$$

for $m \in M$, and (iii) follows. Also,

$$\begin{aligned}tr(e(x,\phi_1\cdots\phi_l)) &= \sum_{\sigma\in C_l}(-1)^\sigma\phi_{\sigma 1}((\phi_{\sigma 2}\cdots\phi_{\sigma l})\cdot x) \\ &= \sum_{\sigma\in C_l}(-1)^\sigma\phi_{\sigma 1}\cdot((\phi_{\sigma 2}\cdots\phi_{\sigma l})\cdot x) \\ &= \sum_{\sigma\in C_l}(-1)^\sigma(\phi_{\sigma 1}\phi_{\sigma 2}\cdots\phi_{\sigma l})\cdot x \\ &= l(\phi_1\cdots\phi_l)\cdot x,\end{aligned}$$

showing (iv). For (v), we see

$$\phi\cdot(xyz) = (\phi\cdot x)yz + (-1)^l x(\phi\cdot y)z + xy(\phi\cdot z) = \sum_{x,y,z\circlearrowleft}(\phi\cdot x)yz,$$

for $\phi \in M^*$, so

$$\begin{aligned} e(\mu, xyz)\phi &= (\sum_{x,y,z\circlearrowleft} (\phi \cdot x)yz) \cdot \mu \\ &= \sum_{x,y,z\circlearrowleft} (\phi \cdot x) \cdot ((yz) \cdot \mu) \\ &= \sum_{x,y,z\circlearrowleft} e((yz) \cdot \mu, x)\phi. \end{aligned}$$

Thus, $e(\mu, xyz) = \sum_{x,y,z\circlearrowleft} e((yz) \cdot \mu, x)$, and (v) follows from (ii). Finally, if $\xi = \phi_1\phi_2\phi_3$, then

$$\begin{aligned} \xi \cdot (xy) &= (\xi \cdot x)y - \sum_{\sigma \in C_3} (-1)^\sigma ((\phi_{\sigma 1}\phi_{\sigma 2}) \cdot x)(\phi_{\sigma 3} \cdot y) \\ &\quad + \sum_{\sigma \in C_3} (-1)^\sigma (\phi_{\sigma 1} \cdot x)((\phi_{\sigma 2}\phi_{\sigma 3}) \cdot y) - x(\xi \cdot y) \\ &= (\xi \cdot x)y - D_{e(x,\xi)}y + D_{e(y,\xi)}x - (\xi \cdot y)x, \end{aligned}$$

showing (vi). ■

Lemma 4 *Let M be a finitely generated projective module of rank n.*
(i) $(x \cdot \mu) \cdot u = (\mu \cdot u)x$, for $x \in \Lambda(M)$, $u \in \Lambda_n(M)$, $\mu \in \Lambda_n(M^*)$.
(ii) The following are equivalent:
(a) there exist $u \in \Lambda_n(M)$ and $\mu \in \Lambda_n(M^*)$ with $\mu \cdot u = 1$,
(b) $\Lambda_n(M)$ is free of rank 1.
(iii) $D_\alpha(u) = tr(\alpha)u$ for $\alpha \in gl(M)$, $u \in \Lambda_n(M)$.

Proof. We first show (i) in case M is a free module of rank n. Since $\Lambda_n(M)$ is free of rank 1, we may assume $x = m_l \cdots m_1$, $u = m_n \cdots m_1$, and $\mu = \phi_1 \cdots \phi_n$ where $m_1, \ldots, m_n$ is a basis for M and $\phi_1, \ldots, \phi_n$ is the dual basis of M^*; i.e., $\phi_i(m_j) = \delta_{ij}$. We have

$$\begin{aligned} ((m_l \cdots m_1) \cdot (\phi_1 \cdots \phi_n)) \cdot (m_n \cdots m_1) &= (\phi_{l+1} \cdots \phi_n) \cdot (m_n \cdots m_1) \\ &= m_l \cdots m_1 \\ &= ((\phi_1 \cdots \phi_n) \cdot (m_n \cdots m_1))m_l \cdots m_1, \end{aligned}$$

showing (i) in this case. To show the general case, we observe that $\Lambda(M)$, $\Lambda_n(M^*)$, and $\Lambda_n(M)$ are finitely generated projective, and that we can identify $\Lambda_l(M)_\mathfrak{p}$ with $\Lambda_l(M_\mathfrak{p})$. Since the trilinear identity (i) holds for the free $k_\mathfrak{p}$-module $M_\mathfrak{p}$ of rank n for each $\mathfrak{p}$, it holds for M.

If (a) holds, then $q = (q \cdot \mu) \cdot u = (\mu \cdot q)u$ for $q \in \Lambda_n(M)$ by (i). Thus, $q \to \mu \cdot q$ is a linear map $\Lambda_n(M) \to k$ with inverse $a \to au$, and (b) holds. Conversely, if $\mu : \Lambda_n(M) \to k$ is an isomorphism, then $\mu \in \Lambda_n(M)^* = \Lambda_n(M^*)$ and $\mu \cdot u = \mu(u) = 1$, so (a) holds, showing (ii). Let

$$\lambda : gl(M) \otimes \Lambda_n(M) \to \Lambda_n(M)$$

be the linear map with $\lambda(\alpha \otimes u) = D_\alpha(u) - tr(\alpha)u$. Since (iii) holds for free modules, $\lambda_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of k, so $\lambda = 0$ and (iii) holds. ■

We remark that if condition (ii)(a) in Lemma 4 holds, then $\{u\}$ is a basis for $\Lambda_n(M)$, $\{\mu\}$ is a basis for $\Lambda_n(M^*)$, and μ is uniquely determined by u.

3 Constructions of Lie algebras

Let M be a finitely generated projective module of rank 9 and suppose there exist $u \in \Lambda_9(M)$ and $\mu \in \Lambda_9(M^*)$ with $\mu \cdot u = 1$. The Lie algebra $gl(M)$ acts on $\Lambda_3(M)$ via $\rho_M : \alpha \to D_\alpha \mid_{\Lambda_3(M)}$. Clearly, $\widetilde{gl}(M) := \rho_M(gl(M)) + kId_{\Lambda_3(M)}$ is a Lie algebra. Since $\rho_M(Id_M) = 3Id_{\Lambda_3(M)}$, we see that $\widetilde{gl}(M) = \rho_M(gl(M))$ if $\frac{1}{3} \in k$. Suppose $\beta \in gl(\Lambda_3(M))$ extends to a derivation d_β of the subalgebra

$$\Lambda_{(3)}(M) := k \oplus \Lambda_3(M) \oplus \Lambda_6(M) \oplus \Lambda_9(M)$$

of $\Lambda(M)$. Since β uniquely determines d_β, we can define $T(\beta) = \mu \cdot d_\beta(u)$. If $\alpha \in gl(M)$, then $\rho_M(\alpha)$ and $Id_{\Lambda_3(M)}$ extend to derivations of $\Lambda_{(3)}(M)$ with $d_{\rho_M(\alpha)} = D_\alpha \mid_{\Lambda_{(3)}(M)}$ and $d_{Id_{\Lambda_3(M)}}(x) = rx$ for $x \in \Lambda_{3r}(M)$. Thus, each $\beta \in \widetilde{gl}(M)$ extends to a derivation d_β of $\Lambda_{(3)}(M)$, and we have defined a linear map $T : \widetilde{gl}(M) \to k$ with $T(\rho_M(\alpha)) = tr(\alpha)$ by Lemma 4(iii) and $T(Id_{\Lambda_3(M)}) = 3$. Set $\widetilde{sl}(M) = \{\beta \in \widetilde{gl}(M) : T(\beta) = 0\}$, so $\widetilde{sl}(M) = \rho_M(sl(M))$ if $\frac{1}{3} \in k$. Note that

$$[\widetilde{gl}(M), \widetilde{gl}(M)] \subset \rho_M([gl(M), gl(M)]) \subset \rho_M(sl(M)) \subset \widetilde{sl}(M),$$

so $\widetilde{sl}(M)$ is an ideal of $\widetilde{gl}(M)$. Note that $\widetilde{gl}(M)$ is a Lie algebra of linear transformations of $\Lambda_3(M)$ with the contragredient action on $\Lambda_3(M)^* = \Lambda_3(M^*)$. In particular, (3) shows

$$\rho_M(\alpha)^* = D_{\alpha^*} \mid_{\Lambda_3(M^*)} = \rho_{M^*}(\alpha^*) \text{ for } \alpha \in gl(M). \tag{4}$$

Theorem 5 *Let M be a finitely generated projective module of rank 9 and suppose there exist $u \in \Lambda_9(M)$ and $\mu \in \Lambda_9(M^*)$ with $\mu \cdot u = 1$. Then*

$$\mathcal{G}(M, u) = \widetilde{sl}(M) \oplus \Lambda_3(M) \oplus \Lambda_3(M^*)$$

is a Lie algebra with skew symmetric product given by

$$\begin{aligned}
[\alpha, \beta] &= \alpha\beta - \beta\alpha, \\
[\alpha, x] &= \alpha(x), \ [\alpha, \xi] = -\alpha^*(\xi), \\
[x, y] &= (xy) \cdot \mu, \ [\xi, \psi] = (\xi\psi) \cdot u, \\
[x, \xi] &= \delta(x, \xi) := \rho(e(x, y)) - (x \cdot \xi) Id_{\Lambda_3(M)}
\end{aligned}$$

for $\alpha, \beta \in \widetilde{sl}(M)$, $x, y \in \Lambda_3(M)$, and $\xi, \psi \in \Lambda_3(M^*)$.

Proof. We recall that Lemma 4(ii) shows that μ is uniquely determined by u. Also, Lemma 3(iv) shows that $\delta(x,y) \in \widetilde{sl}(M)$. It suffices to check the Jacobi identity

$$J(z_1,z_2,z_3) = [[z_1 z_2]z_3] + [[z_2 z_3]z_1] + [[z_3 z_1]z_2] = 0$$

for $z_i \in \widetilde{sl}(M) \cup \Lambda_3(M) \cup \Lambda_3(M^*)$. Moreover, since the product is skew-symmetric,

$$J(z_1,z_2,z_3) = 0 \text{ implies } J(z_{\pi 1}, z_{\pi 2}, z_{\pi 3}) = 0$$

for any $\pi \in S_3$. Since $\widetilde{sl}(M)$ is a Lie algebra of linear transformations of $\Lambda_3(M)$ with the contragredient action on $\Lambda_3(M)^* = \Lambda_3(M^*)$, the Jacobi identity holds if two or more of z_i are in $\widetilde{sl}(M)$. Interchanging the roles of M and M^*, if necessary, we are left with the following cases with $\alpha \in \widetilde{sl}(M)$, $x,y,z \in \Lambda_3(M)$, $\xi \in \Lambda_3(M^*)$:

Case 1: $J(\alpha, x, \xi)$. We know that $gl(M)$ acts as derivations of $\Lambda(M)$ via $\gamma \to D_\gamma$, and as derivations of $\Lambda(M^*)$ via $\gamma \to -D_{\gamma^*}$. Also, these actions are derivations of the products $\Lambda(M^*) \cdot \Lambda(M)$ and $\Lambda(M) \cdot \Lambda(M^*)$ by (3). Thus, $gl(M)$ acts as derivations of the triple product

$$\delta(x,\xi)(y) = D_{e(x,\xi)}(y) - (x \cdot \xi)y.$$

Now $End(\Lambda_3(M))$ acts on $\Lambda_3(M^*)$ via $\alpha \to -\alpha^*$. Since $\rho_M(\gamma)^* = D_{\gamma^*}\mid_{\Lambda_3(M^*)}$ for $\gamma \in gl(M)$, we see that $\rho_M(gl(M))$ also acts as derivations of $\delta(x,\xi)(y)$. Clearly, $Id_{\Lambda_3(M)}$ acts as derivations of the triple product, so $[\alpha, \delta(x,\xi)] = \delta(\alpha x, \xi) + \delta(x, -\alpha^* \xi)$, showing case 1.

Case 2: $J(\alpha, x, y)$. As above, $\widetilde{sl}(M)$ acts as derivations of $\mu \cdot u = 1$ and $(xy) \cdot \mu$. Thus,

$$0 = (d_{-\alpha^*}\mu) \cdot u + \mu \cdot (d_\alpha u) = (d_{-\alpha^*}\mu) \cdot u,$$

so $d_{-\alpha^*}\mu = 0$, and

$$\alpha((xy) \cdot \mu) = ((\alpha x)y) \cdot \mu + (x(\alpha y)) \cdot \mu + (xy) \cdot (d_{-\alpha^*}\mu),$$

so $[\alpha[x,y]] = [\alpha x, y] + [x, \alpha y]$.

Case 3 : $J(x,y,\xi)$. We see by Lemma 3(vi) that

$$\begin{aligned} [[x,y],\xi] &= (((xy)\cdot\mu)\xi)\cdot u = -(\xi((xy)\cdot\mu))\cdot u \\ &= -\xi \cdot (((xy)\cdot\mu)\cdot u) = -\xi \cdot (xy) \\ &= -(\xi \cdot x)y + D_{e(x,\xi)}y - D_{e(y,\xi)}x + (\xi \cdot y)x \\ &= \delta(x,\xi)(y) - \delta(y,\xi)(x) \\ &= [[x,\xi],y] - [[y,\xi],x]. \end{aligned}$$

Case 4: $J(x,y,z)$. We have

$$\begin{aligned} [[x,y],z] &= -\delta(z,(xy)\cdot\mu) = -\rho_M(e(z,(xy)\cdot\mu)) + z \cdot ((xy)\cdot\mu) Id_{\Lambda_3(M)} \\ &= -\rho_M(e(z,(xy)\cdot\mu)) + ((xyz)\cdot\mu) Id_{\Lambda_3(M)}. \end{aligned}$$

Also, by Lemma 3(v) and Lemma 4(i),

$$\sum_{x,y,z\circlearrowleft} e(x,(yz)\cdot\mu) = e(xyz,\mu) = ((xyz)\cdot\mu)Id_M.$$

Thus,

$$\sum_{x,y,z\circlearrowleft} [[x,y],z] = -((xyz)\cdot\mu)\rho_M(Id_M) + 3((xyz)\cdot\mu)Id_{\Lambda_3(M)} = 0.$$

■

Suppose $\omega : M \to N$ is a σ-semilinear homomorphism where σ is an automorphism of k. Extending the definition for linear maps, we define the σ^{-1}-semilinear map $\omega^* : N^* \to M^*$ with $\omega^*(\phi) = \sigma^{-1}\phi\omega$. Let θ_ω be the unique extension of ω to a σ-semilinear homomorphism $\Lambda(M) \to \Lambda(N)$. Note $\theta_\omega(a) = \sigma(a)$ for $a \in k$.

Lemma 6 *Let M, u be as in Theorem 5. The map*

$$\alpha \oplus x \oplus \xi \to -\alpha^* \oplus \xi \oplus x \tag{5}$$

for $\alpha \in \widetilde{sl}(M)$, $x \in \Lambda_3(M)$, $\xi \in \Lambda_3(M^*)$ is an isomorphism $\mathcal{G}(M,u) \to \mathcal{G}(M^*,\mu)$. If $\omega : M \to N$ is a σ-semilinear isomorphism, then

$$\alpha \oplus x \oplus \xi \to \theta_\omega\alpha\theta_\omega^{-1} \oplus \theta_\omega x \oplus \theta_{\omega^{*-1}}\xi \tag{6}$$

for $\alpha \in \widetilde{sl}(M)$, $x \in \Lambda_3(M)$, $\xi \in \Lambda_3(M^*)$ is a σ-semilinear isomorphism $\mathcal{G}(M,u) \to \mathcal{G}(N,\theta_\omega u,)$.

Proof. Using (4) and Lemma 3, we see $\delta(x,\xi)^* = \delta(\xi,x)$. It is then clear that (5) is an isomorphism.

The Lie product on $\mathcal{G}(M,u)$ is completely determined by the graded products on $\Lambda(M)$ and $\Lambda(M^*)$, the actions of $\Lambda(M^*)$ on $\Lambda(M)$ and $\Lambda(M)$ on $\Lambda(M^*)$, the actions $\beta \to \rho_M(\beta) = D_\beta \mid_{\Lambda_3(M)}$ and $\beta \to -\rho_M(\beta)^*$of $gl(M)$ on $\Lambda_3(M)$ and $\Lambda_3(M^*)$, and the elements $u \in \Lambda_9(M)$, $\mu \in \Lambda(M^*)$. Thus, if $\eta : \Lambda(M) \to \Lambda(N)$ and $\eta' : \Lambda(M^*) \to \Lambda(N^*)$ are graded ring isomorphisms and $\breve{\eta} : gl(M) \to gl(N)$ is a Lie ring isomorphism with

$$\begin{aligned} \eta(\xi\cdot x) &= \eta'(\xi)\cdot\eta(x), & (7)\\ \eta'(x\cdot\xi) &= \eta(x)\cdot\eta'(\xi), & (8)\\ \rho_N(\breve{\eta}(\beta)) &= \eta\rho_M(\beta)\eta^{-1}, & (9)\\ \rho_N(\breve{\eta}(\beta))^* &= \eta'\rho_M(\beta)^*\eta'^{-1}, & (10) \end{aligned}$$

for $x \in \Lambda_3(M)$, $\xi \in \Lambda_3(M^*)$, and $\beta \in gl(M)$, then

$$\alpha \oplus x \oplus \xi \to \eta\alpha\eta^{-1} \oplus \eta x \oplus \eta'\xi$$

is a Lie ring isomorphism $\mathcal{G}(M,u) \to \mathcal{G}(N,\eta u)$. Now let $\eta = \theta_\omega$, $\eta' = \theta_{\omega^{*-1}}$, and $\breve{\eta}(\beta) = \omega\beta\omega^{-1}$. We can rewrite (7) as

$$\theta_\omega \Delta_\xi \theta_\omega^{-1} = \Delta_{\theta_{\omega^{*-1}}(\xi)}. \tag{11}$$

Since both sides of (11) are multiplicative in ξ, we can assume $\xi \in M^*$. In that case, both sides are antiderivations of $\Lambda(N)$, so it suffices to apply both sides to $\theta_\omega(M) = N$. We have

$$\begin{aligned} \Delta_{\theta_{\omega^{*-1}}(\xi)}\theta_\omega(m) &= \omega^{*-1}(\xi)(\omega(m)) = (\sigma\xi\omega^{-1})(\omega(m)) \\ &= \sigma\xi(m) = \theta_\omega(\xi(m)) = \theta_\omega\Delta_\xi(m), \end{aligned}$$

and (7) follows. Reversing the roles of M and M^* gives (8). If $\beta \in gl(M)$, then $\theta_\omega D_\beta \theta_\omega^{-1} = D_{\omega\beta\omega^{-1}}$ since they are derivations agreeing on $\theta_\omega(M) = N$. This shows (9). Finally,

$$\begin{aligned} \rho_N(\omega\beta\omega^{-1})^* &= \rho_{N^*}((\omega\beta\omega^{-1})^*) = \rho_{N^*}(\omega^{*-1}\beta^*\omega^*) \\ &= \theta_{\omega^{*-1}}\rho_M(\beta)^*\theta_{\omega^*}, \end{aligned}$$

showing (10). Thus, the σ-semilinear map (6) is a Lie isomorphism. ■

Let K be a unital commutative ring with involution $a \to \bar{a}$ and let k be the subring of fixed elements. Let M be a finite generated projective K-module.of rank 9 with a nonsingular hermitian form h; i.e., $\eta : m \to h(m,)$ is a semilinear isomorphism $M \to M^*$. Define the semilinear involution τ on $gl(M)$ by $h(m, \alpha n) = h(\tau(\alpha)m, n)$; i.e., $\tau(\alpha) = \eta^{-1}\alpha^*\eta$. Let

$$\begin{aligned} u(M,h) &= \{\alpha \in gl(M) : \tau(\alpha) = -\alpha\}, \\ su(M,h) &= u(M,h) \cap sl(M), \\ sk(K) &= \{a \in K : \bar{a} = -a\}, \\ \tilde{u}(M,h) &= \rho_M(u(M,h)) + sk(K)Id_{\Lambda_3(M)}. \end{aligned}$$

Clearly, $\tilde{u}(M,h)$ is a subalgebra of $\widetilde{gl}(M)$. Note, $sk(K)Id_M \subset u(M,h)$, so $\tilde{u}(M,h) = \rho_M(u(M,h))$ if $\frac{1}{3} \in K$. Finally, set

$$\widetilde{su}(M,h) = \tilde{u}(M,h) \cap \widetilde{sl}(M).$$

We also set $x \cdot y = \theta_\eta(x) \cdot y$ for $x, y \in \Lambda(M)$ and $\delta(x,y) = \delta(x, \theta_\eta(y))$ for $x, y \in \Lambda_3(M)$.

Theorem 7 *Let K be a unital commutative ring with involution $a \to \bar{a}$ and let k be the subring of fixed elements. Let M be a finite generated projective K-module.of rank 9 with a nonsingular hermitian form h. If $u \in \Lambda_9(M)$ with $u \cdot u = 1$ and $\mu = \theta_\eta(u)$, then*

$$\zeta(\alpha \oplus x \oplus \xi) = -\theta_\eta^{-1}\alpha^*\theta_\eta \oplus \theta_\eta^{-1}(\xi) \oplus \theta_\eta(x)$$

for $\alpha \in \widetilde{sl}(M)$*,* $x \in \Lambda_3(M)$*,* $\xi \in \Lambda_3(M^*)$ *is a semi-linear automorphism of* $\mathcal{G}(M,u)$*. Moreover,* $\alpha \oplus x \oplus \theta_\eta(x) \to \alpha \oplus x$ *is an isomorphism of the Lie algebra* $\mathcal{G}(\zeta)$ *over* k *of fixed points of* ζ *to*

$$\mathcal{G}(M,h,u) = \widetilde{su}(M,h) \oplus \Lambda_3(M)$$

with skew-symmetric product given by

$$\begin{aligned} [\alpha,\beta] &= \alpha\beta - \beta\alpha, \\ [\alpha,x] &= \alpha x, \\ [x,y] &= (\delta(x,y) - \delta(y,x)) \oplus (xy)\cdot u \end{aligned}$$

for $\alpha, \beta \in \widetilde{su}(M,h)$*,* $x, y \in \Lambda_3(M)$*.*

Proof. Since h is hermitian, it is easy to see that $\eta^* = \eta$ and $(\theta_\eta \alpha \theta_\eta^{-1})^* = \theta_\eta^{-1}\alpha^*\theta_\eta$. Thus, ζ is the product of the semilinear isomorphism $\mathcal{G}(M,u) \to \mathcal{G}(M^*,\mu)$ given by (6) with $N = M^*$ and $\omega = \eta$ and the inverse of the isomorphism (5). Since

$$\begin{aligned} \theta_\eta^{-1}\rho_M(\alpha)^*\theta_\eta &= \rho_M(\eta^{-1}\alpha^*\eta) = \rho_M(\tau(\alpha)), \\ \theta_\eta^{-1}(aId^*_{\Lambda_3(M)})\theta_\eta &= \bar{a}Id_{\Lambda_3(M)}, \end{aligned}$$

we see that the Lie algebra $\mathcal{G}(\zeta)$ of fixed points of ζ is

$$\mathcal{G}(\zeta) = \{\alpha \oplus x \oplus \theta_\eta(x) : \alpha \in \widetilde{su}(M,h), x \in \Lambda_3(M)\}.$$

The $\widetilde{su}(M,h)$ component of $[x \oplus \theta_\eta(x), y \oplus \theta_\eta(y)]$ is

$$[x, \theta_\eta(y)] - [y, \theta_\eta(x)] = \delta(x,y) - \delta(y,x),$$

while the $\Lambda_3(M)$ component is

$$\begin{aligned} [\theta_\eta(x), \theta_\eta(y)] &= (\theta_\eta(x)\theta_\eta(y))\cdot u = \theta_\eta(xy)\cdot u \\ &= (xy)\cdot u. \end{aligned}$$

Thus, $\alpha \oplus x \oplus \theta_\eta(x) \to \alpha \oplus x$ is an isomorphism of $\mathcal{G}(\zeta)$ with $\mathcal{G}(M,h,u)$. ■

4 Forms of exceptional Lie algebras

Lemma 8 *If* $F \in k$*-alg is faithfully flat, then there are canonical isomorphisms*

$$\begin{aligned} \mathcal{G}(M,u)_F &\cong \mathcal{G}(M_F, u_F), \\ \mathcal{G}(M,h,u)_F &\cong \mathcal{G}(M_F, h_F, u_F), \end{aligned}$$

where u_F *is the image of* $u \otimes 1$ *in the canonical isomorphism* $\Lambda_9(M)_F \to \Lambda_9(M_F)$ *and* h_F *is the extension of the* k*-bilinear map* h *given by (1).*

Proof. Since M is finitely generated projective, we have seen that there are canonical isomorphisms

$$\begin{aligned} \Lambda_3(M)_K &\cong \Lambda_3(M_K), && (12)\\ \Lambda_3(M^*)_K &\cong \Lambda_3(M_K^*), && (13)\\ gl(M)_K &\cong gl(M_K), && (14) \end{aligned}$$

for $K \in k$-alg. Moreover, $(\rho(gl(M)))_K \cong \rho_K(gl(M_K))$ for $\rho : \alpha \to D_\alpha \mid_{\Lambda_3(M)}$, so

$$\widetilde{gl}(M)_K \cong \widetilde{gl}(M_K).$$

If $F \in k$-alg is faithfully flat, then the exact sequence

$$\widetilde{sl}(M) \to \widetilde{gl}(M) \xrightarrow{T} k$$

implies that

$$\widetilde{sl}(M)_F \to \widetilde{gl}(M)_F \xrightarrow{T_F} F$$

is exact. Thus, $\widetilde{sl}(M)_F = \ker(T_F) \cong \widetilde{sl}(M_F)$. Similarly, $\widetilde{su}(M,h)$ is the kernel of the map $\alpha \to (\alpha + \tau(\alpha)) \oplus T(\alpha)$, so $\widetilde{su}(M,h)_F \cong \widetilde{su}(M_F, h_F)$. The canonical isomorphisms of the lemma are now obvious. ■

Suppose $K = k_+ \oplus k_-$ where k_σ is an isomorphic copy of k via $a \to a_\sigma$ and $\bar{a}_\sigma = a_{-\sigma}$ for $\sigma = \pm$. We shall identify $a \in k$ with $a_+ \oplus a_- \in K$, and write $M_\sigma = 1_\sigma M$ and $m_\sigma = 1_\sigma m$ where M is a K-module and $m \in M$.

Lemma 9 *If M, h, u and ζ are as in Theorem 7 for $K = k_+ \oplus k_-$, then*

$$\alpha \oplus x \to \alpha_+ \oplus x_+ \oplus \theta_\eta(x_-)$$

is an isomorphism of $\mathcal{G}(M,h,u)$ with $\mathcal{G}(M_+, u_+)$.

Proof. Clearly, $\mathcal{G}(M,u) = \mathcal{G}(M,u)_+ \oplus \mathcal{G}(M,u)_-$ as Lie algebras over K. Moreover, since ζ is semilinear, ζ interchanges $\mathcal{G}(M,u)_+$ with $\mathcal{G}(M,u)_-$, so

$$\mathcal{G}(\zeta) = \{z + \zeta(z) : z \in \mathcal{G}(M,u)_+\}.$$

Thus, $z \to z_+$ is a Lie algebra isomorphism $\mathcal{G}(\zeta) \to \mathcal{G}(M,u)_+$ over k. Using Theorem 7, we see that

$$\alpha \oplus x \to (\alpha \oplus x \oplus \theta_\eta(x))_+ = \alpha_+ \oplus x_+ \oplus \theta_\eta(x_-)$$

is a Lie algebra isomorphism $\mathcal{G}(M,h,u) \to \mathcal{G}(M,u)_+$ over k. On the other hand, $M_{k_+} = 1_+ \otimes M_+$ can be identified with M_+ as k_+-modules. Thus,

$$\mathcal{G}(M,u)_+ = \mathcal{G}(M,u)_{k_+} = \mathcal{G}(M_{k_+}, 1_+ \otimes u)_+ = \mathcal{G}(M_+, u_+)$$

as Lie algebra over k_+ and hence over k. ■

Suppose M, u, μ are as in Theorem 5 and M is free over k. Let $B = \{m_1, \ldots, m_9\}$ be a basis for M and $\phi_1, \ldots, \phi_9$ the dual basis of M^*; i.e., $\phi_i(m_j) = \delta_{ij}$. For

$$S = \{i_1 < \cdots < i_l\} \subset \{1, \ldots, 9\},$$

let

$$\begin{aligned} m_S &= m_{i_1} \cdots m_{i_l}, \\ \phi_S &= \phi_{i_l} \cdots \phi_{i_1}, \end{aligned}$$

so $\{m_S : |S| = l\}$ and $\{\phi_S : |S| = l\}$ are dual bases for $\Lambda_l(M)$ and $\Lambda_l(M^*)$. Set $u_B = m_{\{1,\ldots,9\}}$ and $\mu_B = \phi_{\{1,\ldots,9\}}$. Since $u = au_B$, $\mu = b\mu_B$ and $1 = \mu \cdot u = ab$, so a and b are invertible, we may replace m_1 by am_1 and ϕ_1 by $b\phi_1$ to assume that $u = u_B$ and $\mu = \mu_B$. Now $e_{ij} := e(m_i, \phi_j)$, $1 \leq i, j \leq 9$ is a basis for $gl(M)$ and the matrix of e_{ij} relative to the basis for M is just the usual matrix unit. Let

$$\begin{aligned} h_1 &= \rho(e_{11} + e_{22} + e_{33}) - Id_{\Lambda_3(M)}, \\ h_i &= \rho(e_{ii} - e_{i-1,i-1}) \text{ for } 2 \leq i \leq 8. \end{aligned}$$

Lemma 10 *If M is a free module with basis $B = \{m_1, \ldots, m_9\}$, then*

$$\tilde{B} = \{h_i : 1 \leq i \leq 8\} \cup \{\rho(e_{ij}) : i \neq j\}$$

is a basis for $\widetilde{sl}(M)$ and

$$\hat{B} = \tilde{B} \cup \{m_S : |S| = 3\} \cup \{\phi_S : |S| = 3\}$$

is a basis for $\mathcal{G}(M, u_B)$. Thus, $\mathcal{G}(M, u_B)_K$ is canonically isomorphic to $\mathcal{G}(M_K, u_{B\otimes 1}.)$ for any $K \in k$-alg.

Proof. First, note $T(h_1) = 3-3 = 0$, so $h_1 \in \widetilde{sl}(M)$. Suppose $\alpha = \sum_{i,j} a_{ij}e_{ij} \in gl(M)$ and $b \in k$ with $\rho(\alpha) + bId_{\Lambda_3(M)} = 0$. If $i \neq j$, choose k, s with i, j, k, s distinct. We see that $\beta = \rho(e_{ij})$ is the only element among $\rho(e_{pq}), Id_{\Lambda_3(M)}$ with $\beta(m_j m_k m_s)$ having a nonzero coefficient of $m_i m_k m_s$. Thus, $a_{ij} = 0$ for $i \neq j$. Also,

$$\rho(\alpha)m_i m_j m_k = \sum_{p=1}^{9} a_{pp}\rho(e_{pp})m_i m_j m_k = (a_{ii} + a_{jj} + a_{kk})m_i m_j m_k,$$

so $a_{ii} + a_{jj} + a_{kk} = -b$ for distinct i, j, k. Thus, $a_{ii} = a$ and $b = -3a$ for $a = a_{11}$. Now suppose

$$\sum_{i=1}^{8} c_i h_i + \sum_{1\leq i\neq j\leq 9} c_{ij}\rho(e_{ij}) = 0.$$

Letting

$$\begin{aligned}\alpha &= c_1(e_{11}+e_{22}+e_{33}) + \sum_{i=2}^{8} c_i(e_{ii}-e_{i-1,i-1}) + \sum_{1\le i\ne j\le 9} c_{ij}e_{ij} \\ &= \sum_{i,j} a_{ij}e_{ij},\end{aligned}$$

we have $\rho(\alpha) - c_1 Id_{\Lambda_3(M)} = 0$. Thus, $c_{ij} = a_{ij} = 0$ for $i \neq j$. Also, $a_{99} = 0$, so all $a_{ii} = 0$ and $c_1 = -3a_{11} = 0$. Moreover, $\sum_{i=2}^{8} c_i(e_{ii} - e_{i-1,i-1}) = 0$ forces all $c_i = 0$. Thus, $\tilde{B}$ is independent. To show that it spans $\widetilde{sl}(M)$, suppose $\alpha = \sum_{i,j} a_{ij}e_{ij}$ and $x = \rho(\alpha) + bId_{\Lambda_3(M)} \in \widetilde{sl}(M)$; i.e., $tr(\alpha) + 3b = 0$. After subtracting $a_{99}(\rho(Id_M) - 3Id_{\Lambda_3(M)}) = 0$, we may assume $a_{99} = 0$. Subtracting $a_{ij}\rho(e_{ij})$ for $i \neq j$ and $-bh_1$, we can also assume $a_{ij} = 0$ for $i \neq j$ and $b = 0$. Thus, $tr(\alpha) = 0$ and $\rho(\alpha)$ is in the span of $h_2, \ldots, h_8$. Thus, $\tilde{B}$ is a basis for $\widetilde{sl}(M)$, and hence $\hat{B}$ is a basis for $\mathcal{G}(M, u_B)$.

Now $B \otimes 1 := \{m \otimes 1 : m \in B\}$ is a basis for M_K and $\hat{B} \otimes 1$ is a basis for $\mathcal{G}(M, u_B)_K$. The natural bijection between $\hat{B} \otimes 1$ and the basis $\widehat{B \otimes 1}$ of $\mathcal{G}(M_K, u_{B\otimes 1})$ induces a canonical isomorphism $\mathcal{G}(M, u_B)_K \to \mathcal{G}(M_K, u_{B\otimes 1})$. ∎

We remark that the rank of $\mathcal{G}(M, u)$ is $8 + 9\cdot 8 + \binom{9}{3} + \binom{9}{3} = 80 + 2\cdot 84 = 248$.

Theorem 11 *Let $\mathbb{C}^9$ be the complex vector space of dimension 9 with standard basis C. Then $\mathcal{G}(\mathbb{C}^9, u_C)$ is a simple Lie algebra of type E_8 and $\hat{C}$ is a Chevalley basis.*

Proof. Let $M = \mathbb{C}^9$, $C = \{m_1, \ldots, m_9\}$, $u = u_C$, and $\mu = \mu_C$. Since $\frac{1}{3} \in \mathbb{C}$, $\rho : sl(M) \to \widetilde{sl}(M)$ is an isomorphism. Now $\widetilde{sl}(M)$, $\Lambda_3(M)$, and $\Lambda_3(M^*)$ are nonisomorphic irreducible $\widetilde{sl}(M)$-modules, so they are the only irreducible $\widetilde{sl}(M)$-modules in $\mathcal{G}(M, u)$. Thus, if I is a nonzero ideal of $\mathcal{G}(M, u)$, then complete reducibility shows that I contains at least one of these submodules. Moreover,

$$\begin{aligned}0 &\neq [\widetilde{sl}(M), \Lambda_3(M)] \subset \Lambda_3(M), \\ 0 &\neq [\widetilde{sl}(M), \Lambda_3(M^*)] \subset \Lambda_3(M^*), \\ 0 &\neq [\Lambda_3(M), \Lambda_3(M^*)] \subset \widetilde{sl}(M),\end{aligned}$$

so I contains each of these submodules. Thus, $\mathcal{G}(M, u)$ is simple. Let $\mathcal{H}$ be the trace 0 diagonal maps of M relative to the given basis, so $\mathcal{H}$ is a Cartan subalgebra of $sl(M)$, and $\tilde{\mathcal{H}} = \rho(\mathcal{H})$ is a Cartan subalgebra of $\widetilde{sl}(M)$. Since $h_1 = \rho(e_{11} + e_{22} + e_{33} - \frac{1}{3}Id_M)$, we see h_i, $1 \le i \le 8$ is a basis for $\tilde{\mathcal{H}}$. The centralizer of $\tilde{\mathcal{H}}$ in $\mathcal{G}(M, u)$ is contained in $\widetilde{sl}(M)$ and is hence $\tilde{\mathcal{H}}$. Thus, $\tilde{\mathcal{H}}$ is a Cartan subalgebra of $\mathcal{G}(M, u)$. Let $\varepsilon_i \in \tilde{\mathcal{H}}^*$ with $\varepsilon_i(h) = a_i$ where $\rho^{-1}(h) = diag(a_1, \ldots, a_9) \in \mathcal{H}$, as a diagonal matrix. Clearly, $\sum_{i=1}^{9} \varepsilon_i = 0$.

We see that the roots Σ of $\tilde{\mathcal{H}}$ for $\mathcal{G}(M,u)$ are all $\varepsilon_i - \varepsilon_j$ for $i \neq j$ (in $\widetilde{sl}(M)$) and all $\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)$ for distinct i,j,k (in $\Lambda_3(M)$ and $\Lambda_3(M^*)$). Let $\alpha_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ and $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ for $2 \leq i \leq 8$. Now $\Pi = \{\alpha_1, \ldots, \alpha_8\}$ is a basis of $\tilde{\mathcal{H}}^*$. Moreover, an examination of the α_j-string through α_i shows that Π is a fundamental system of roots with Dynkin diagram E_8 with $\alpha_2, \ldots, \alpha_8$ forming a diagram of type A_7 and α_1 connected to α_4. Hence, $\mathcal{G}(M,u)$ is a Lie algebra of type E_8. To show that $\hat{C}$ is a Chevalley basis, we need to show ([H72], p. 147)

(a) for each root α, there is $x_\alpha \in \hat{C} \cap \mathcal{G}(M,u)_\alpha$,

(b) $[x_\alpha, x_{-\alpha}] = h_\alpha$ with $[h_\alpha, x_\alpha] = 2x_\alpha$,

(c) $h_{\alpha_i} = h_i$,

(d) the linear map with $x_\alpha \to -x_{-\alpha}$, $h_i \to -h_i$ is an automorphism of $\mathcal{G}(M,u)$.

Clearly, $x_\alpha = \rho(e_{ij})$ for $\alpha = \varepsilon_i - \varepsilon_j$, $x_\alpha = m_S$ and $x_{-\alpha} = \phi_S$ for $\alpha = \varepsilon_i + \varepsilon_j + \varepsilon_k$ and $S = \{i < j < k\}$ satisfies (a). Now $[[e_{ij}, e_{ji}], e_{ij}] = [e_{ii} - e_{jj}, e_{ij}] = 2e_{ij}$, so (b) holds for $\alpha = \varepsilon_i - \varepsilon_j$ and (c) holds for $i \neq 1$. Lemma 3(v) with $l = 1$ shows

$$\begin{aligned} e(m_S, \phi_S) &= e(m_i m_j m_k, \phi_k \phi_j \phi_i) = \sum_{i,j,k\circlearrowleft} e(m_i, (m_j m_k) \cdot (\phi_k \phi_j \phi_i)) \\ &= e_{ii} + e_{jj} + e_{kk}. \end{aligned}$$

Thus,

$$\begin{aligned} [m_S, \phi_S] &= \rho(e(m_S, \phi_S) - \frac{1}{3}(m_S \cdot \phi_S) Id_M) \\ &= \rho(e_{ii} + e_{jj} + e_{kk} - \frac{1}{3} Id_M), \end{aligned}$$

so (b) holds for $\alpha = \pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)$ and (c) holds for $i = 1$. Finally, let $\mathbb{C}$ have the trivial involution and let h be the symmetric bilinear form on M with $h(m_i, m_j) = \delta_{ij}$. Thus, η as in Theorem 7 has $\eta(m_i) = \phi_i$. Now $\theta_\eta(m_C) = \phi_1 \cdots \phi_9 = \phi_9 \cdots \phi_1 = \phi_C$, and we have an automorphism ζ given by Theorem 7. Since $\theta_\eta^{-1} \rho(\beta)^* \theta_\eta = \rho(\tau(\beta))$ for $\beta \in sl(M)$ where $\tau(e_{ij}) = e_{ji}$, we see that $\zeta(h_i) = -h_i$, and $\zeta(x_\alpha) = -x_{-\alpha}$ for $\alpha = \varepsilon_i - \varepsilon_j$. Also, $\zeta(x_\alpha) = \theta_\eta(m_S) = \phi_i \phi_j \phi_k = -\phi_S = -x_{-\alpha}$ for $\alpha = \varepsilon_i + \varepsilon_j + \varepsilon_k$ and $S = \{i < j < k\}$. Thus, (d) holds and $\hat{C}$ is a Chevalley basis. ∎

Let $\mathcal{G}(\mathbb{C})$ be a simple Lie algebra over $\mathbb{C}$ of type X_l and let $\mathcal{G}(\mathbb{Z})$ be the $\mathbb{Z}$-span of a Chevalley basis of $\mathcal{G}(\mathbb{C})$. Up to isomorphism, $\mathcal{G}(\mathbb{Z})$ is independent of the choice of Chevalley basis ([H72], p. 150, Exercise 5). Set $\mathcal{G}(k) = \mathcal{G}(\mathbb{Z})_k$. We say that a Lie algebra $\mathcal{G}$ over k is a *split form* of X_l if $\mathcal{G} \cong \mathcal{G}(k)$ and that $\mathcal{G}$ is a *form* of X_l if $\mathcal{G}_F \cong \mathcal{G}(F)$ for some faithfully flat $F \in k$-alg. If $F \in k$-alg and $E \in F$-alg are faithfully flat, then $E \in k$-alg is faithfully flat. Thus, if $\mathcal{G}_F$ is a form of X_l for some faithfully flat $F \in k$-alg, then $\mathcal{G}$ is a form of X_l.

Corollary 12 *The Lie algebra $\mathcal{G}(M,u)$ in Theorem 5 is a form of E_8 and is a split form if M is free. If K is a quadratic étale k-algebra, then the Lie algebra $\mathcal{G}(M,h,u)$ in Theorem 7 is a form of E_8.*

Proof. If $\hat{C}$ is the Chevalley basis of $\mathcal{G}(\mathbb{C}^9, u_C)$ given by Theorem 11, we can identify C with the standard basis of $\mathbb{Z}^9$ and $\hat{C}$ with the corresponding basis for $\mathcal{G}(\mathbb{Z}^9, u_C)$. In particular, $\mathcal{G}(\mathbb{Z}^9, u_C) = \mathcal{G}(\mathbb{Z})$, the $\mathbb{Z}$-span $\hat{C}$. If M, u are as in Theorem 5 with M free, we can choose a basis B for M with $u = u_B$ and $\mu = \mu_B$. The isomorphism $M \to \mathbb{Z}_k^9 \cong k^9$ taking B to $C \otimes 1$ induces an isomorphism $\mathcal{G}(M, u_B) \to \mathcal{G}(\mathbb{Z}_k^9, u_{C\otimes 1})$. Since

$$\mathcal{G}(k) = \mathcal{G}(\mathbb{Z})_k = \mathcal{G}(\mathbb{Z}^9, u_C)_k \cong \mathcal{G}(\mathbb{Z}_k^9, u_{C\otimes 1}),$$

by Lemma 10, we see that $\mathcal{G}(M, u)$ is a split form if M is free. For the general case, we know there is a faithfully flat $F \in k$-alg with M_F a free k_F-module of rank 9 ([B89], II.5, Exercise 8). By Lemma 8 and the result for free M, we see

$$\mathcal{G}(M, u)_F \cong \mathcal{G}(M_F, u_F) \cong \mathcal{G}(F)$$

and $\mathcal{G}(M, u)$ is a form of E_8.

For M, h, u as in Theorem 7 with K a quadratic étale k-algebra, we know by Proposition 1 that K is faithfully flat and $K_K \cong K \oplus K$. Thus,

$$\mathcal{G}(M, h, u)_K \cong \mathcal{G}(M_K, h_K, u_K) \cong \mathcal{G}((M_K)_+, (u_K)_+) \tag{15}$$

by Lemmas 8 and 9, so $\mathcal{G}(M, h, u)_K$ and hence $\mathcal{G}(M, h, u)$ are forms of E_8. ∎

Theorem 13 *Let M, u, μ be as in Theorem 5*

(i) If $M = M_1 \oplus M_2$ with M_1 of rank 3 and M_2 of rank 6, then

$$\mathcal{G}(M_1, M_2, u) = [M_1\Lambda_2(M_2), M_1^*\Lambda_2(M_2^*)] \oplus M_1\Lambda_2(M_2) \oplus M_1^*\Lambda_2(M_2^*)$$

is a Lie subalgebra of $\mathcal{G}(M, u)$ and a form of E_7.

(ii) $M = M_1 \oplus M_2 \oplus M_3$ with each M_i of rank 3, then

$$\mathcal{G}(M_1, M_2, M_3, u) = [M_1M_2M_3, M_1^*M_2^*M_3^*] \oplus M_1M_2M_3 \oplus M_1^*M_2^*M_3^*$$

is a Lie subalgebra of $\mathcal{G}(M, u)$ and a form of E_6.

Let M, h, u as in Theorem 7 with K a quadratic étale k-algebra. Set $d(x, y) = \delta(x, y) - \delta(y, x)$ for $x, y \in \Lambda_3(M)$.

(iii) If $M = M_1 \perp M_2$ with M_1 of rank 3 and M_2 of rank 6, then

$$\mathcal{G}(M_1, M_2, h, u) = d(M_1\Lambda_2(M_2), M_1\Lambda_2(M_2)) \oplus M_1\Lambda_2(M_2)$$

is a Lie subalgebra of $\mathcal{G}(M, h, u)$ and a form of E_7.

(iv) $M = M_1 \perp M_2 \perp M_3$ with each M_i of rank 3, then

$$\mathcal{G}(M_1, M_2, M_3, h, u) = d(M_1M_2M_3, M_1M_2M_3) \oplus M_1M_2M_3$$

is a Lie subalgebra of $\mathcal{G}(M, h, u)$ and a form of E_6.

Proof. We show that $\mathcal{G}(M_1, M_2, M_3, u)$ is a subalgebra, and the other cases can be handled similarly. Since $M_i \cdot M_j^* = 0$ for $i \neq j$, we see

$$\begin{aligned} & ((M_1M_2M_3)(M_1M_2M_3)) \cdot \Lambda_9(M) \\ = \;& ((M_1M_2M_3)(M_1M_2M_3)) \cdot \Lambda_3(M_1^*)\Lambda_3(M_2^*)\Lambda_3(M_3^*) \\ \subset \;& M_1^*M_2^*M_3^*. \end{aligned}$$

Thus,

$$[M_1M_2M_3, M_1M_2M_3] \subset M_1^*M_2^*M_3^*$$

and similarly

$$[M_1^*M_2^*M_3^*, M_1^*M_2^*M_3^*] \subset M_1M_2M_3.$$

Also,

$$(M_iM_j) \cdot (M_1^*M_2^*M_3^*) \subset M_k^*$$

for $\{i, j, k\} = \{1, 2, 3\}$. Thus,

$$e(M_1M_2M_3, M_1^*M_2^*M_3^*) \subset \sum_{i=1}^{3} e(M_i, M_i^*)$$

by Lemma 3(v). Since $\rho(e(M_i, M_i^*))$ stabilizes $M_1M_2M_3$ and $\rho(e(M_i, M_i^*))^*$ stabilizes $M_1^*M_2^*M_3^*$, we see $\mathcal{G}(M_1, M_2, M_3, u)$ is a subalgebra.

Since $\mathcal{G}(M_1, M_2, h, u)$ is the subalgebra generated by $M_1\Lambda_2(M_2)$ and $\mathcal{G}(M_1, M_2, M_3, h, u)$ is the subalgebra generated by $M_1M_2M_3$, we can use the isomorphism (15) to reduce cases (iii) and (iv) to cases (i) and (ii). In cases (i) or (ii), there is a faithfully flat $F \in k$-alg with each M_{iF} free of rank 3 or 6. We can choose a basis $B = \{m_1, \ldots, m_9\}$ for M_F with $1 \otimes u = u_B$ and $1 \otimes \mu = \mu_B$ which is compatible with the direct sum decomposition; i.e., $M_{1F} = span_F(m_1, m_2, m_3)$ and $M_{2F} = span_F(m_4, \ldots, m_9)$ or $M_{iF} = span_F(m_{3i-2}, m_{3i-1}, m_{3i})$. The isomorphism $\mathcal{G}(M, u)_F \cong \mathcal{G}(\mathbb{Z}^9, u_C)_F$ allows us to reduce to the cases

$$\begin{aligned} M &= \mathbb{Z}^9 = \mathbb{Z}^{(1,3)} \oplus \mathbb{Z}^{(4,9)}, \\ M &= \mathbb{Z}^9 = \mathbb{Z}^{(1,3)} \oplus \mathbb{Z}^{(4,6)} \oplus \mathbb{Z}^{(7,9)} \end{aligned}$$

where $\mathbb{Z}^{(i,j)} = span_{\mathbb{Z}}(m_i, \ldots, m_j)$ for $1 \leq i \leq j \leq 9$ and $C = \{m_1, \ldots, m_9\}$ is the standard basis for $\mathbb{Z}^9$.

Let $\mathcal{G} = \mathcal{G}(\mathbb{C}^9, u_C)$ as in Theorem 11. Let

$$\begin{aligned} \beta_i &= \alpha_i = \varepsilon_i - \varepsilon_{i-1} \text{ for } i = 2, 3, 5, 6, 7, \\ \beta_1 &= \alpha_9 = \varepsilon_9 - \varepsilon_8, \\ \beta_4 &= \varepsilon_2 + \varepsilon_4 + \varepsilon_8, \\ \beta_8 &= \varepsilon_4 + \varepsilon_5 + \varepsilon_6. \end{aligned}$$

As before, by checking the β_j-string through β_i, we see that $\tilde{\Pi} = \{\beta_1, \ldots, \beta_8\}$ is a fundamental system of roots with Dynkin diagram E_8 with $\beta_2, \ldots, \beta_8$ forming a diagram of type A_7 and β_1 connected to β_4. Moreover, replacing h_i in $\hat{C}$ by $\tilde{h}_i = h_{\beta_i}$, we get a Chevalley basis $\tilde{C}$. Let

$$\begin{aligned} h' &= \rho(diag(-2,-2,-2,1,1,1,1,1,1)), \\ h'' &= \rho(diag(1,1,1,-1,-1,-1,0,0,0)). \end{aligned}$$

Since

$$\begin{aligned} \beta_i(h') &= 0 \text{ for } 1 \le i \le 7, \\ \beta_8(h') &= 3, \\ \beta_i(h'') &= 0 \text{ for } 1 \le i \le 6, \\ \beta_7(h'') &= 1, \end{aligned}$$

we see that

$$\Sigma' = \{\alpha \in \Sigma : \alpha(h') = 0\}$$

is a root system of type E_7 and

$$\Sigma'' = \{\alpha \in \Sigma : \alpha(h') = \alpha(h'') = 0\}$$

is a root system of type E_6. Moreover, the subalgebra $\mathcal{G}'$ generated by all $\mathcal{G}_\alpha$ with $\alpha \in \Sigma'$ is a complex simple Lie algebra of type E_7 with Chevalley basis $\tilde{C} \cap \mathcal{G}'$ and the subalgebra $\mathcal{G}''$ generated by all $\mathcal{G}_\alpha$ with $\alpha \in \Sigma''$ is a complex simple Lie algebra of type E_6 with Chevalley basis $\tilde{C} \cap \mathcal{G}''$. We see

$$\begin{aligned} \Sigma' = &\{\varepsilon_i - \varepsilon_j : 1 \le i \ne j \le 3 \text{ or } 4 \le i \ne j \le 9\} \\ \cup \; &\{\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k) : 1 \le i \le 3 \text{ and } 4 \le j \ne k \le 9\}, \\ \Sigma'' = &\{\varepsilon_i - \varepsilon_j : 3l-2 \le i \ne j \le 3l \text{ for } l = 1,2, \text{ or } 3\} \\ \cup \; &\{\pm(\varepsilon_{i_1} + \varepsilon_{i_2} + \varepsilon_{i_3}) : 3l-2 \le i_l \le 3l\}. \end{aligned}$$

Since $[m_i m_k m_l, \phi_l \phi_k \phi_j] = \rho(e_{ij})$ where $C = \{m_1, \ldots, m_9\}$, we see that the $\mathbb{Z}$-span of $\tilde{C} \cap \mathcal{G}'$ is generated as a $\mathbb{Z}$-algebra by

$$\tilde{C} \cap (\mathbb{Z}^{(1,3)} \Lambda_2(\mathbb{Z}^{(4,9)}) \cup \mathbb{Z}^{(1,3)*} \Lambda_2(\mathbb{Z}^{(4,9)*}))$$

while the $\mathbb{Z}$-span of $\tilde{C} \cap \mathcal{G}''$ is generated as a $\mathbb{Z}$-algebra by

$$\tilde{C} \cap (\mathbb{Z}^{(1,3)} \mathbb{Z}^{(4,6)} \mathbb{Z}^{(7,9)} \cup \mathbb{Z}^{(1,3)*} \mathbb{Z}^{(4,6)*} \mathbb{Z}^{(7,9)*}).$$

In other words, $\mathcal{G}(\mathbb{Z}^{(1,3)}, \mathbb{Z}^{(4,9)}, u_C)$ is the $\mathbb{Z}$-span of $\tilde{C} \cap \mathcal{G}'$ and

$$\mathcal{G}(\mathbb{Z}^{(1,3)}, \mathbb{Z}^{(4,6)}, \mathbb{Z}^{(7,9)}, u_C)$$

is the $\mathbb{Z}$-span of $\tilde{C} \cap \mathcal{G}''$. ∎

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