CLASSIFICATION OF CONTRACTIVELY COMPLEMENTED HILBERTIAN OPERATOR SPACES

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Abstract. We construct some separable infinite dimensional homogeneous Hilbertian operator spaces $H_\infty^{m,R}$ and $H_\infty^{m,L}$, which generalize the row and column spaces R and C (the case $m = 0$). We show that separable infinite-dimensional Hilbertian JC^*-triples are completely isometric to an element of the set of (infinite) intersections of these spaces . This set includes the operator spaces R, C, $R \cap C$, and the space Φ spanned by creation operators on the full anti-symmetric Fock space. In fact, we show that $H_\infty^{m,R}$ (resp. $H_\infty^{m,L}$) is completely isometric to the space of creation (resp. annihilation) operators on the m (resp. $m+1$) anti-symmetric tensors of the Hilbert space. Together with the finite-dimensional case studied in [14], this gives a full operator space classification of all rank-one JC^*-triples in terms of creation and annihilation operator spaces.

We use the above to show that all contractive projections on a C*-algebra A with infinite dimensional Hilbertian range are "expansions" (which we define precisely) of normal contractive projections from A^{**} onto a Hilbertian space which is completely isometric to R, C, $R \cap C$, or Φ. This generalizes the well known result, first proved for $B(H)$ by Robertson in [17], that all Hilbertian operator spaces that are *completely* contractively complemented in a C*-algebra are completely isometric to R or C. We also compute various completely bounded Banach-Mazur distances between these spaces, or Φ.

1. Preliminaries

The goals of the present paper are to classify all infinite dimensional rank 1 JC*-triples up to complete isometry (Theorem 1 in section 2) and then use that result to give a suitable "classification" of all Hilbertian operator spaces which are contractively complemented in a C*-algebra or normally contractively complemented in a W*-algebra (Theorems 2 and 3 in section 3). In particular, we show that these spaces are "essentially" R, C, $R \cap C$, or Φ modulo a "degenerate" piece.

In Theorem 4 in section 4 we compute various completely bounded Banach-Mazur distances between these JC*-triples. In Theorem 5 in section 5, we show that all of these JC*-triples in the separable infinite dimensional and finite dimensional cases can be represented completely isometrically as creation and annihilation operator spaces on pieces of the anti-symmetric Fock space.

In the rest of this section, we give some background on operator space theory and on JC^*-triples.

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1.1. **Operator spaces.** Operator space theory is a non-commutative or quantized theory of Banach spaces. By definition, an operator space is a Banach space together with an isometric linear embedding into $B(H)$, the bounded linear operators on a complex Hilbert space. While the objects are obviously the Banach spaces themselves, the more interesting aspects concern the morphisms, namely, the completely bounded maps. These are defined by considering an operator space as a subspace X of $B(H)$. Its *operator space structure* is then given by the sequence of norms on the set of matrices $M_n(X)$ with entries from X, determined by the identification $M_n(X) \subset M_n(B(H)) = B(H \oplus H \oplus \cdots \oplus H)$. A linear mapping $\varphi : X \to Y$ between two operator spaces is *completely bounded* if the induced mappings $\varphi_n : M_n(X) \to M_n(Y)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ satisfy $\|\varphi\|_{\mathrm{cb}} := \sup_n \|\varphi_n\| < \infty$.

Operator space theory has its origins in the work of Stinespring in the 1950s, and Arveson in the 1960s. Many tools were developed in the 1970s and 1980s by a number of operator algebraists, and an abstract framework was developed in 1988 in the thesis of Ruan. All definitions, notation, and results used in this paper can be found in recent accounts of the subject, namely (in chronological order) [6],[15],[16],[3]. Let us just recall that a completely bounded map is a *complete isomorphism* if its inverse exists and is completely bounded. Two operator spaces are *completely isometric* if there is a linear isomorphism T between them with $\|T\|_{\mathrm{cb}} = \|T^{-1}\|_{\mathrm{cb}} = 1$. We call T a *complete isometry* in this case. Other important types of morphisms in this category are complete contractions ($\|\varphi\|_{\mathrm{cb}} \le 1$) and complete semi-isometries (:= isometric complete contraction).

Examples of completely bounded maps are the restriction to a subspace of a C^*-algebra of a *-homomorphism and multiplication by an fixed element. It is a fact that every completely bounded map is essentially a product of these two examples, [16, Th. 1.6]. The space $CB(X,Y)$ of completely bounded maps between operator spaces X and Y is a Banach space with the completely bounded norm $\|\cdot\|_{\mathrm{cb}}$.

Analogous to the Banach-Mazur distance for Banach spaces, the class of all operator spaces can be made into a metric space by using the logarithm of the *completely bounded* Banach-Mazur distance:

$$\mathrm{d}_{\mathrm{cb}}(E,F) = \inf\{\|u\|_{\mathrm{cb}} \cdot \|u^{-1}\|_{\mathrm{cb}} \ ; \ u : E \to F \text{ complete isomorphism}\}.$$

Two important examples of Hilbertian operator spaces (:= operator spaces isometric to Hilbert space) are the row and column spaces R, C, and their finite-dimensional versions R_n, C_n. These are defined as follows. In the matrix representation for $B(\ell_2)$, *column Hilbert space* $C := \overline{\mathrm{sp}}\{e_{i1} : i \ge 1\}$ and *row Hilbert space* $R := \overline{\mathrm{sp}}\{e_{1j} : j \ge 1\}$. Their finite dimensional versions are $C_n = \mathrm{sp}\{e_{i1} : 1 \le i \le n\}$ and $R_n = \mathrm{sp}\{e_{1j} : 1 \le j \le n\}$. Here of course e_{ij} is the operator defined by the matrix with a 1 in the (i,j)-entry and zeros elsewhere. Although R and C are Banach isometric, they are not completely isomorphic ($\mathrm{d}_{\mathrm{cb}}(R,C) = \infty$); and R_n and C_n, while completely isomorphic, are not completely isometric. In fact, it is known that $\mathrm{d}_{\mathrm{cb}}(R_n, C_n) = n$.

R, C, R_n, C_n are examples of *homogeneous* operator spaces, that is, operator spaces E for which $\forall u : E \to E$, $\|u\|_{\mathrm{cb}} = \|u\|$. Another important example of an Hilbertian homogeneous operator space is $\Phi(I)$. The space $\Phi(I)$ is defined by $\Phi(I) = \overline{\mathrm{sp}}\{V_i : i \in I\}$, where the V_i are bounded operators on a Hilbert space satisfying the canonical anti-commutation relations. In some special cases, the notations $\Phi_n := \Phi(\{1,2,\ldots,n\})$, and $\Phi = \Phi(\{1,2,\ldots\})$ are used. For more properties of this space and related constructs, see [16, 9.3].

Two more examples of homogeneous operator spaces are $\min(E)$, $\max(E)$, where E is any Banach space. For any such E, the operator space structure of $\min(E)$ is defined by the embedding of E into the continuous functions on the unit ball of E^* in the weak*-topology, namely, $\|(a_{ij})\|_{M_n(\min(E))} = \sup_{\xi\in B_{E^*}} \|(\xi(a_{ij}))\|_{M_n}$. The operator space structure of $\max(E)$ is given by

$$\|(a_{ij})\|_{M_n(\max(E))} = \sup\{\|(u(a_{ij}))\|_{M_n(B(H_u))} : u : E \to B(H_u),\ \|u\| \le 1\}.$$

More generally, if F and G are operator spaces, then in $F \xrightarrow{u} \min(E)$, $\|u\|_{\mathrm{cb}} = \|u\|$, and in $\max(E) \xrightarrow{v} G$, $\|v\|_{\mathrm{cb}} = \|v\|$. The notations $\min(E)$ and $\max(E)$ are justified by the fact that for any Banach space E, the identity map on E is completely contractive in $\max(E) \to E \to \min(E)$.

By analogy with the classical Banach spaces $\ell_p,\ c_0,\ L_p,\ C(K)$ (as well as their "second generation", Orlicz, Sobolev, Hardy, Disc algebra, Schatten p-classes), we can consider the (Hilbertian) operator spaces $R,\ C,\ \min(\ell_2),\ \max(\ell_2),\ OH,\ \Phi$, as well as their finite dimensional versions $R_n,\ C_n,\ \min(\ell_2^n),\ \max(\ell_2^n),\ OH_n,\ \Phi_n$, as "classical operator spaces". Among these spaces, only the spaces R, C, and Φ play important roles in this paper. (For the definition and properties of the space called OH, see [16, Chapter 7].) The classical operator spaces are mutually completely non-isomorphic. If E_n, F_n are n-dimensional versions, then $\mathrm{d}_{\mathrm{cb}}(E_n, F_n) \to \infty$, [16, Ch. 10].

We propose to add to this list of classical operator spaces the Hilbertian operator spaces $H_\infty^{m,R}$ and $H_\infty^{m,L}$ constructed here, as well as their finite-dimensional versions H_n^k studied in [13] and [14]. Like the space Φ, the spaces $H_\infty^{m,R}$, $H_\infty^{m,L}$ and H_n^k can be represented up to complete isometry as spaces of creation operators or annihilation operators on anti-symmetric Fock spaces ([14, Lemma 2.1] and Theorem 5 below).

Let us recall from [13, Sections 6,7] the construction of the spaces H_n^k, $1 \le k \le n$. Let I denote a subset of $\{1,2,\ldots,n\}$ of cardinality $|I| = k-1$. The number of such I is $q := \binom{n}{k-1}$. Let J denote a subset of $\{1,2,\ldots,n\}$ of cardinality $|J| = n-k$. The number of such J is $p := \binom{n}{n-k}$. We assume that each $I = \{i_1,\ldots,i_{k-1}\}$ is such that $i_1 < \cdots < i_{k-1}$, and that if $J = \{j_1,\ldots,j_{n-k}\}$, then $j_1 < \cdots < j_{n-k}$.

The space H_n^k is the linear span of matrices $b_i^{n,k}$, $1 \le i \le n$, given by

$$b_i^{n,k} = \sum_{I\cap J=\emptyset,(I\cup J)^c=\{i\}} \epsilon(I,i,J)e_{J,I},$$

where $e_{J,I} = e_J \otimes e_I = e_J e_I^t \in M_{p,q}(\mathbf{C}) = B(\mathbf{C}^q, \mathbf{C}^p)$, and $\epsilon(I,i,J)$ is the signature of the permutation taking $(i_1,\ldots,i_{k-1},i,j_1,\ldots,j_{n-k})$ to $(1,\ldots,n)$. Since the $b_i^{n,k}$ are the image under a triple isomorphism (actually ternary isomorphism) of a rectangular grid in a JW^*-triple of rank one, they form an orthonormal basis for H_n^k (cf. [13, subsection 5.3 and section 7]).

The following definition from [16, 2.7] plays a key role in this paper. If $E_0 \subset B(H_0)$ and $E_1 \subset B(H_1)$ are operator spaces whose underlying Banach spaces form a compatible pair in the sense of interpolation theory, then the Banach space $E_0 \cap E_1$ (with the norm $\|x\|_{E_0\cap E_1} = \max(\|x\|_{E_0}, \|x\|_{E_1})$) equipped with the operator space structure given by the embedding $E_0 \cap E_1 \ni x \mapsto (x,x) \in E_0 \oplus E_1 \subset B(H_0 \oplus H_1)$ is called the *intersection* of E_0 and E_1 and is denoted by $E_0 \cap E_1$. We note, for examples, that $\cap_{k=1}^{n} H_n^k = \Phi_n$ ([14]) and the space $R \cap C$ is defined relative to the embedding of C into itself and R into C given by the transpose map ([16, p. 184]). The definition of intersection extends easily to arbitrary families of compatible operator spaces (cf. Theorem 1 below).

Lemma 1.1. Let H be an Hilbertian operator space, and suppose that every finite dimensional subspace of H is homogeneous. Then H itself is homogeneous.

Proof. Let ϕ be any unitary operator on H. According to the first statement of [16, Prop.9.2.1], it suffices to prove that ϕ is a complete isometry.

Let F be any finite dimensional subspace of H and let G be the subspace spanned by $F \cup \phi(F)$. By the second statement of [16, Prop.9.2.1], F and $\phi(F)$, being of the same dimension as subspaces of the homogeneous space G, are completely isometric, and $\phi|F$ is a complete isometry.

Now let $[x_{ij}] \in M_n(H)$. Then $\{x_{ij}, \phi(x_{ij}) : 1 \le i, j \le n\}$ spans a finite dimensional subspace F of H, and

$$\|\phi_n([x_{ij}])\|_{M_n(H)} = \|\phi_n([x_{ij}])\|_{M_n(F)} = \|[x_{ij}]\|_{M_n(F)} = \|[x_{ij}]\|_{M_n(H)}. \quad \square$$

1.2. **Rank one JC^*-triples.** A JC^*-triple is a norm closed complex linear subspace of $B(H, K)$ (equivalently, of a C^*-algebra) which is closed under the operation $a \mapsto aa^*a$. JC^*-triples were defined and studied (using the name J^*-algebra) as a generalization of C^*-algebras by Harris [10] in connection with function theory on infinite dimensional bounded symmetric domains. By a polarization identity, any JC^*-triple is closed under the triple product

$$(a, b, c) \mapsto \{abc\} := \frac{1}{2}(ab^*c + cb^*a), \tag{1}$$

under which it becomes a Jordan triple system. A linear map which preserves the triple product (1) will be called a *triple homomorphism*. Cartan factors are examples of JC^*-triples, as are C^*-algebras, and Jordan C^*-algebras. Cartan factors are defined for example in [13, Section 1]. We shall only make use of Cartan factors of type 1, that is, spaces of the form $B(H, K)$ where H and K are complex Hilbert spaces.

A special case of a JC^*-triple is a *ternary algebra*, that is, a subspace of $B(H, K)$ closed under the *ternary product* $(a, b, c) \mapsto ab^*c$. A *ternary homomorphism* is a linear map ϕ satisfying $\phi(ab^*c) = \phi(a)\phi(b)^*\phi(c)$. These spaces are also called ternary rings of operators and abbreviated TRO. They have been studied both concretely in [11] and abstractly in [21]. Given a TRO M, its left (resp. right) linking C^*-algebra is defined to be the norm closed span of the elements ab^* (resp. a^*b) with $a, b \in M$. Ternary isomorphic TROs have isomorphic left and right linking algebras.

TROs have come to play a key role in operator space theory, serving as the algebraic model in the category. Recall that the algebraic models for the categories of order-unit spaces, operator systems, and Banach spaces, are respectively Jordan C^*-algebras, C^*-algebras, and JB^*-triples. Indeed, for TROs, a ternary isomorphism is the same as a complete isometry.

If v is a partial isometry in a JC^*-triple $M \subset B(H, K)$, then the projections $l = vv^* \in B(K)$ and $r = v^*v \in B(H)$ give rise to (Peirce) projections $P_k(v) : M \to M$, $k = 2, 1, 0$ as follows; for $x \in M$,

$$P_2(v)x = lxr \quad , \quad P_1(v)x = lx(1-r) + (1-l)xr \quad , \quad P_0(v)x = (1-l)x(1-r).$$

The projections $P_k(v)$ are contractive, and their ranges, called Peirce spaces and denoted by $M_k(v)$, are JC^*-subtriples of M satisfying $M = M_2(v) \oplus M_1(v) \oplus M_0(v)$.

A partial isometry v is said to be *minimal* in M if $M_2(v) = \mathbf{C}v$. This is equivalent to v not being the sum of two non-zero orthogonal partial isometries. Recall that two partial isometries v and w (or any two Hilbert space operators) are orthogonal if $v^*w = vw^* = 0$. Orthogonality of partial isometries v and w is equivalent to $v \in M_0(w)$ and will be denoted by $v \perp w$. Each finite dimensional JC^*-triple is the linear span of its minimal partial isometries. More generally, a JC^*-triple is defined to be *atomic* if it is the weak closure of the span of its minimal partial isometries. In this case, it has a predual and is called a *JW^*-triple*. The *rank* of a JC^*-triple is the maximum number of mutually orthogonal minimal partial isometries. For example, the rank of the Cartan factor $B(H, K)$ of type 1 is the minimum of the dimensions of H and K; and the rank of the Cartan factor of type 4 (spin factor) is 2.

In a JC*-triple, there is a natural ordering on partial isometries. We write $v \leq w$ if $vw^*v = v$; this is equivalent to $vv^* \leq ww^*$ and $v^*v \leq w^*w$. Moreover, if $v \leq w$, then there exists a partial isometry v' orthogonal to v with $w = v + v'$.

Another relation between two partial isometries that we shall need is defined in terms of the Peirce spaces as follows. Two partial isometries v and w are said to be *collinear* if $v \in M_1(w)$ and $w \in M_1(v)$, notation $v \top w$. Let u, v, w be partial isometries. The following is part of [13, Lemma 5.4], and is referred to as *"hopping"*: If v and w are each collinear with u, then $uu^*vw^* = vw^*uu^*$ and $u^*uv^*w = v^*wu^*u$. If u, v, w are mutually collinear partial isometries, then $\{uvw\} = 0$.

JC^*-triples of arbitrary dimension occur naturally in functional analysis and in holomorphy. A special case of a theorem of Friedman and Russo [8, Theorem 2] states that if P is a contractive projection on a C^*-algebra A, then there is a linear isometry of the range $P(A)$ of P onto a JC^*-subtriple of A^{**}. A special case of a theorem of Kaup [12] gives a bijective correspondence between Cartan factors and irreducible bounded symmetric domains in complex Banach spaces.

Contractive projections play a ubiquitous role in the structure theory of the abstract analog of JC^*-triples (called JB^*-triples). Of use to us will be both of the following two conditional expectation formulas for a contractive projection P on a JC*-triple M (which are valid for JB^*-triples) ([7, Corollary 1]):

$$P\{Px, Py, Pz\} = P\{Px, Py, z\} = P\{Px, y, Pz\}, \quad (x, y, z \in M). \tag{2}$$

By a special case of [4, Cor.,p.308], every JW^*-triple of rank one is isometric to a Hilbert space and every maximal collinear family of partial isometries corresponds to an orthonormal basis. Conversely, every Hilbert space with the abstract triple product $\{xyz\} := ((x|y)z + (z|y)x)/2$ can be realized as a JC^*-triple of rank one in which every orthonormal basis forms a maximal family of mutually collinear minimal partial isometries.

2. Operator space structure of Hilbertian JC*-triples

2.1. Hilbertian JC*-triples: The spaces $H_\infty^{m,R}$ and $H_\infty^{m,L}$. The general setting for the next two sections will be the following: Y is a JC^*-subtriple of $B(H)$ which is Hilbertian in the operator space structure arising from $B(H)$, and $\{u_i : i \in \Omega\}$ is an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y. Note that the u_i are each minimal in Y, but not necessarily minimal in any JC^*-triple containing Y.

We let T and A denote the TRO and the C^*-algebra respectively generated by Y. For any subset $G \subset \Omega$, $(uu^*)_G := \prod_{i\in G} u_i u_i^*$ and $(u^*u)_G := \prod_{i\in G} u_i^* u_i$. The elements $(uu^*)_G$ and $(u^*u)_G$ lie in the weak closure of A and more generally in the left and right linking von Neumann algebras of T.

In the following lemma, parts (a) and (a$'$) justify the definitions of the integers m_R and m_L in parts (b) and (b$'$).

Lemma 2.1. Let Y be an Hilbertian operator space which is a JC^*-subtriple of $B(H)$ and let $\{u_i : i \in \Omega\}$ be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y.

- **(a):** If $(uu^*)_{\Omega-F} = 0$ for some finite set $F \subset \Omega$, then, $(uu^*)_{\Omega-G} = 0$ for every finite set G with the same cardinality as F.
- **(a$'$):** If $(u^*u)_{\Omega-F} = 0$ for some finite set $F \subset \Omega$, then, $(u^*u)_{\Omega-G} = 0$ for every finite set G with the same cardinality as F.
- **(b):** Assume $(uu^*)_{\Omega-F} \neq 0$ for some finite set F. Let m_R be the smallest nonnegative integer with $(uu^*)_{\Omega-F} \neq 0$ for every F with cardinality m_R. Define $p_R = \sum_{|F|=m_R} (uu^*)_{\Omega-F}$. Then the maps $y \mapsto p_R y$ and $y \mapsto (1-p_R)y$ are completely contractive triple isomorphisms of Y onto rank one subtriples of the weak closure of T in $B(H)$. Moreover, $p_R Y \perp (1-p_R)Y$.
- **(b$'$):** Assume $(u^*u)_{\Omega-F} \neq 0$ for some finite set F. Let m_L be the smallest nonnegative integer with $(u^*u)_{\Omega-F} \neq 0$ for every F with cardinality m_L. Define $p_L = \sum_{|F|=m_L} (u^*u)_{\Omega-F}$. Then the maps $y \mapsto y p_L$ and $y \mapsto y(1-p_L)$ are completely contractive triple isomorphisms of Y onto rank one subtriples of the weak closure of T in $B(H)$. Moreover, $Y p_L \perp Y(1-p_L)$.
- **(c):** In case (b), let $w_i = p_R u_i$ and let m_R' be the smallest nonnegative integer with $(ww^*)_{\Omega-F} \neq 0$ for all F with cardinality m_R'. Then m_R' exists, and $m_R' = m_R$. Furthermore, $(w^*w)_G \neq 0$ for $|G| = m_R + 1$ and $(w^*w)_G = 0$ for $|G| = m_R + 2$. Thus, if we define k_R to be the smallest integer k such that $(w^*w)_G \neq 0$ for $|G| = k$, then $k_R = m_R + 1$.
- **(c$'$):** In case (b$'$), let $w_i = u_i p_L$ and let m_L' be the smallest nonnegative integer with $(w^*w)_{\Omega-F} \neq 0$ for all F with cardinality m_L'. Then m_L' exists, and $m_L' = m_L$. Furthermore, $(ww^*)_G \neq 0$ for $|G| = m_L + 1$ and $(ww^*)_G = 0$ for $|G| = m_L + 2$. Thus, if we define k_L to be the smallest integer k such that $(ww^*)_G \neq 0$ for $|G| = k$, then $k_L = m_L + 1$.

Proof. The proofs of (a) and (a$'$) are identical to the proof given in [13, Lemma 5.8]. The fact that the set $\Omega - F$ is infinite has no effect on the proof in [13].

The proofs of (b) and (b$'$) are identical to the proof given in [13, Lemma 5.9]. The facts that the set $\Omega - F$ is infinite and that the sums defining the projections p_R and p_L are infinite have no effect on the proof in [13].

We now prove (c), the proof of (c′) being entirely similar. For any finite set $F \subset \Omega$,

$$\begin{aligned}(ww^*)_{\Omega-F} &= \prod_{i\in\Omega-F}\left(\sum_{|G|=m_R}(uu^*)_{\Omega-G}\right)u_iu_i^*\left(\sum_{|H|=m_R}(uu^*)_{\Omega-H}\right)\\ &= \prod_{i\in\Omega-F}\left(\sum_{|G|=m_R,i\in\Omega-G}(uu^*)_{\Omega-G}\right) = \sum_{G\subset F,|G|=m_R}(uu^*)_{\Omega-G}.\end{aligned}$$

From this it follows that $(ww^*)_{\Omega-F} = 0$ if $|F| < m_R$ and that $(ww^*)_{\Omega-F} = (uu^*)_{\Omega-F} \neq 0$ if $|F| = m_R$. This proves that $m'_R = m_R$, that is $(ww^*)_{\Omega-F} = 0 \Leftrightarrow |F| < m_R$.

Now let $|F| = r$ and for convenience, suppose that $F = \{1, 2, \ldots, r\}$. Then

$$(w^*w)_F = (w^*w)_{\{1,2,\ldots,r\}} = \sum u_1^*(uu^*)_{\Omega-F_1}u_1u_2^*(uu^*)_{\Omega-F_2}u_2u_3^*\cdots u_r^*(uu^*)_{\Omega-F_r}u_r,$$

where the sum is over all $|F_j| = m_R, j \in \Omega - F_j, F - \{j\} \subset F_j$ (by "hopping"), and $j = 1, 2, \ldots r$. Every term in this sum is zero if $r - 1 > m_R$, that is $r \geq m_R + 2$. Further, if $r = m_R + 1$, there is only one term, namely $x :=$

$$\begin{aligned}(w^*w)_{\{1,2,\ldots,m+1\}} &= u_1^*(uu^*)_{\Omega-\{2,3,\ldots,m+1\}}u_1u_2^*(uu^*)_{\Omega-\{1,3,4,\ldots,m+1\}}u_2u_3^* \times \cdots \times\\ &\quad (uu^*)_{\Omega-\{1,2,\ldots,m-1,m+1\}}u_mu_{m+1}^*(uu^*)_{\Omega-\{1,2,\ldots,m\}}u_{m+1}.\end{aligned}$$

which by a sequence of "hoppings" becomes

$$x = (u^*u)_{\{1,2,\ldots,m\}}u_{m+1}^*(uu^*)_{\Omega-\{1,2,\ldots,m+1\}}u_{m+1}.$$

In turn, using the collinearity of the u_k, this becomes

$$x = u_{m+1}^*(uu^*)_{\Omega-\{1,2,\ldots,m+1\}}u_{m+1}.$$

Thus, if $x = 0$, then $0 = u_{m+1}xu_{m+1}^* = (uu^*)_{\Omega-\{1,2,\ldots,m\}}$, a contradiction. □

Our goal for the remainder of this section is to give a completely isometric representation for the spaces p_RY and Yp_L in parts (b) and (b′) of Lemma 2.1. This will be achieved via a coordinatization procedure which we now describe.

In the following, let us restrict to the special case that Y is a Hilbertian JC*-triple which satisfies the properties of p_RY in Lemma 2.1 part (c). For notational convenience, let $m = m_R$. Thus $(u^*u)_G \neq 0$ for $|G| \leq m + 1$ and $(u^*u)_G = 0$ for $|G| \geq m + 2$.

Analogous to [13, Def. 6.1], we are going to define elements which are indexed by an arbitrary pair of subsets I, J of Ω satisfying

$$|\Omega - I| = m + 1, \quad |J| = m. \tag{3}$$

Here and throughout the rest of this paper, $|F|$ denotes the cardinality of the finite set F.

The set $I \cap J$ is finite, and if $|I \cap J| = s \geq 0$, then $|(I \cup J)^c| = s + 1$. Let us write $I \cap J = \{d_1, \ldots, d_s\}$ and $(I \cup J)^c = \{c_1, \ldots, c_{s+1}\}$, and let us agree (for the moment) that there is a natural linear ordering on Ω such that $c_1 < c_2 < \cdots < c_{s+1}$ and $d_1 < d_2 < \cdots < d_s$.

With the above notation, we define

$$u_{IJ} = u_{I,J} = (uu^*)_{I-J}u_{c_1}u_{d_1}^*u_{c_2}u_{d_2}^*\cdots u_{c_s}u_{d_s}^*u_{c_{s+1}}(u^*u)_{J-I}. \tag{4}$$

Note that in general $I - J$ is infinite and $J - I$ is finite so that $u_{I,J}$ lies in the weak closure of T.

In the special case of (4) where $I \cap J = \emptyset$, we have $s = 0$ and $u_{I,J}$ has the form

$$u_{I,J} = (uu^*)_I u_c (u^*u)_J,$$

where $I \cup J \cup \{c\} = \Omega$ is a partition of Ω. As in [13], we call such an element a "one", and denote it also by $u_{I,c,J}$.

The proof of the following lemma, which is the analog of [13, Lemma 6.6], is complicated by the fact that the sets I are infinite if Ω is infinite.

Lemma 2.2. Let Y be an Hilbertian operator space which is a JC^*-subtriple of $B(H)$ and let $\{u_i : i \in \Omega\}$ be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y. Assume that Y satisfies the properties of $p_R Y$ in Lemma 2.1 part (c) with $m = m_R$, that is, $(u^*u)_G \neq 0$ for $|G| \leq m+1$ and $(u^*u)_G = 0$ for $|G| \geq m+2$. For any $c \in \Omega$,

$$u_c = \sum_{I,J} u_{I,J} = \sum_{I,J} u_{I,c,J} \tag{5}$$

where the sum is taken over all disjoint I, J satisfying (3) and not containing c, and converges weakly in the weak closure of T.

Proof. The proof of [13, Prop. 6.3] remains valid in our context insofar as $\{u_{I,J}\}$ is a collection of pairwise orthogonal partial isometries in the weak closure of the ternary envelope T of Y. Since $u_c^* u_c$ commutes with $(u^*u)_J$, $u_{I,J}^* u_{I,J} u_c^* u_c = u_{I,J}^* u_{I,J}$, so that $u_{I,J}^* u_{I,J} \leq u_c^* u_c$ and similarly $u_{I,J} u_{I,J}^* \leq u_c u_c^*$ so that $\sum u_{I,J} \leq u_c$. To prove (5), we proceed as follows.

Let us write $u_c = v_c + w_c$, where $w_c = \sum_{I,c,J} I u_c J$ and for example, $I u_c J$ is shorthand for $u_{I,c,J} = (uu^*)_I u_c (u^*u)_J$, and v_c is a partial isometry orthogonal to w_c. We shall show that $v_c = 0$. From the simple facts that $I u_c J = u_c J$ and $w_c J = I u_c J$, it follows that $v_c J = 0$. Similarly, $I v_c = 0$. From this, it follows that for each pair $i \neq j$, $v_i \in M_1(u_j)$. Indeed,

$$\begin{aligned} v_i + w_i &= u_i = u_i u_j^* u_j + u_j u_j^* u_i \\ &= v_i u_j^* u_j + w_i u_j^* u_j + u_j u_j^* v_i + u_j u_j^* w_i \\ &= v_i u_j^* u_j + \sum_{j \in J} u_i J + u_j u_j^* v_i + \sum_{j \notin J} u_i J \\ &= v_i u_j^* u_j + u_j u_j^* v_i + w_i. \end{aligned}$$

Hence $v_i = v_i u_j^* u_j + u_j u_j^* v_i$, so $v_i \in M_1(u_j)$.

Next, we observe that v_i is orthogonal to w_j for every i and j. For $i = j$ this is clear by definition. For $i \neq j$, we have $v_i(J u_j^* I) = (v_i J) u_j^* I = 0$ and similarly $J u_j^* I v_i = 0$ so that $v_i w_j^* = w_j^* v_i = 0$.

It now follows that $v_i \top v_j$ for $i \neq j$. Indeed,

$$v_j = v_j u_i^* u_i + u_i u_i^* v_j = v_j v_i^* v_i + v_j w_i^* w_i + v_i v_i^* v_j + w_i w_i^* v_j = v_j v_i^* v_i + v_i v_i^* v_j.$$

Let us adopt the notation J_v for $(v^*v)_J = \prod_{j \in J} v_j^* v_j$. (What we previously denoted by J would now be denoted by $J_u = (u^*u)_J$.) We know that $v_i J_v = v_i J_u = 0$. Suppose that $v_i J'_v \neq 0$ for some $J' \subset J$. We will show that $J' = \emptyset$. In the first place, $v_i J'_v = I'_v v_i J'_v$, since letting $I' = \Omega - (J' \cup \{i\}) = \{i_\alpha : \alpha \in \Lambda\}$ say,

$$v_i J'_v = v_i (v^*v)_{J'} = (v_{i_\alpha} v_{i_\alpha}^* v_i + v_i v_{i_\alpha}^* v_{i_\alpha})(v^*v)_{J'} = v_{i_\alpha} v_{i_\alpha}^* v_i (v^*v)_{J'} = \cdots = I'_v v_i J'_v.$$

In the second place, by the orthogonality of v_j and w_j,

$$I'_u u_j = (uu^*)_{I'} u_j = [\prod_{i\in I'}(v_i + w_i)(v_i^* + w_i^*)](v_j + w_j) = I'_v v_j + I'_w w_j.$$

Hence, $I'_v v_j = I'_u u_j - I'_w w_j$ and each term is zero because, as with the $\{u_j\}$, the $\{w_j\}$ satisfy $(ww^*)_{I'} = 0$ if $|\Omega - I'| < m$. This contradiction shows that $J' = \emptyset$, and therefore either $v_j v_i^* = 0$ for all i, j or $v_j = 0$ for all j. In the latter case, there is nothing to prove. In the former case, since $v_j \top v_i$,

$$v_j = v_i v_i^* v_j + v_j v_i^* v_i = (\prod_{i\in\Omega-\{j\}} v_i v_i^*) v_j = (\prod_{i\in\Omega-\{j\}} u_i u_i^*) u_j + (\prod_{i\in\Omega-\{j\}} w_i w_i^*) w_j = 0$$

as required. □

We shall now assume that our set Ω is countable and for convenience set $\Omega = \mathbf{N} = \{1, 2, 3, \ldots\}$ with its natural order. Note that in this case, the number of possible sets I in (3) is $\aleph_0$ and the number of such J is also $\aleph_0$.

Again as in [13], we assign a *signature* to each "one" $u_{I,k,J}$ as follows: Let the elements of I be $i_1 < i_2 < \cdots < i_p < \cdots$ and the elements of J be $j_1 < j_2 < \cdots < j_m$, where p is chosen such that $\max\{k, j_m\} < i_p$. Then $\epsilon(I, k, J)$ is defined to be the signature of the permutation taking the $(p + m + 1)$-tuple $(i_1, \ldots, i_p, k, j_1, \ldots, j_m)$ onto $(1, 2, \ldots, p+m+1)$. This is clearly independent of p as long as $\max\{k, j_m\} < i_p$.

The proof of [13, Lemma 6.7] shows that every element $u_{I,J}$ decomposes uniquely into a product of "ones." The signature $\epsilon(I, J)$ (also denoted by $\epsilon(IJ)$) of $u_{I,J}$ is defined to be the product of the signatures of the factors in this decomposition. Then the proof of [13, Prop. 6.10] shows that the family $\{\epsilon(IJ)u_{I,J}\}$ forms a rectangular grid which satisfies the extra property

$$\epsilon(IJ)u_{IJ}[\epsilon(IJ')u_{IJ'}]^*\epsilon(I'J')u_{I'J'} = \epsilon(I'J)u_{I'J}. \tag{6}$$

It follows as in [13] that the map $\epsilon(IJ)u_{IJ} \to E_{JI}$ is a ternary isomorphism (and hence complete isometry) from the norm closure of $\mathrm{sp}_C\, u_{IJ}$ to the norm closure of $\mathrm{sp}_C\,\{E_{JI}\}$, where E_{JI} denotes an elementary matrix, whose rows and columns are indexed by the sets J and I, with a 1 in the (J, I)-position. By [4, Lemma 1.14], this map can be extended to a ternary isomorphism from the w*-closure of $\mathrm{sp}_C\, u_{IJ}$ onto the Cartan factor of type I consisting of all $\aleph_0$ by $\aleph_0$ complex matrices which act as bounded operators on ℓ_2. By restriction to Y and (5), Y is completely isometric to a subtriple $\tilde{Y}$, of this Cartan factor of type 1.

Definition 1. We shall denote the space $\tilde{Y}$ above by $H_\infty^{m,R}$. An entirely symmetric argument (with J infinite and I finite) under the assumption that Y satisfies the conditions of Yp_L in Lemma 2.1 part (c) with $m = m_L$ defines the space $H_\infty^{m,L}$.

Explicitly,

$$H_\infty^{m,R} = \overline{\mathrm{sp}}_C\,\{b_i^m = \sum_{I\cap J=\emptyset,(I\cup J)^c=\{i\},|J|=m} \epsilon(I, i, J)e_{J,I} : i \in \mathbf{N}\}.$$

and

$$H_\infty^{m,L} = \overline{\mathrm{sp}}_C\,\{\sum_{I\cap J=\emptyset,(I\cup J)^c=\{i\},|I|=m} \epsilon(I, i, J)e_{J,I} : i \in \mathbf{N}\},$$

with $\epsilon(I, i, J)$ defined in the obvious analogous way with I finite instead of J.

Corollary 4.1 below shows that these spaces are all distinct from each other and from Φ. This discussion has proved the following lemma.

Lemma 2.3. The spaces $p_R Y$ and $Y p_L$ in Lemma 2.1 parts (c) and (c′) are completely isometric to $H_\infty^{m_R,R}$ and $H_\infty^{m_L,L}$, respectively.

Remark 2.4. It is immediate from [13, Cor. 5.3] that $H_\infty^{0,R} = C$ and $H_\infty^{0,L} = R$. Also note that $H_\infty^{m,R}$ and $H_\infty^{m,L}$ are homogeneous Hilbertian operator spaces by Lemma 1.1 and [14, Theorem 1].

2.2. **The coordinatization of Hilbertian JC*-triples.** Let Y satisfy the hypothesis of Lemma 2.1. Our analysis will consider the following three mutually exhaustive and (by Corollary 4.1(b)) mutually exclusive possibilities (in each case, the set F is allowed to be empty):

Case 1: $(uu^*)_{\Omega-F} \neq 0$ for some finite set $F \subset \Omega$;
Case 2: $(u^*u)_{\Omega-F} \neq 0$ for some finite set $F \subset \Omega$;
Case 3: $(uu^*)_{\Omega-F} = (u^*u)_{\Omega-F} = 0$ for all finite subsets F of Ω.

We will first address cases 1 and 2.

Proposition 2.5. Let Y be a separable infinite dimensional Hilbertian operator space which is a JC^*-subtriple of $B(H)$ and let $\{u_i : i \in \Omega\}$ ($\Omega = \mathbf{N}$) be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y.

(a): Suppose there exists a finite subset F of Ω such that $(uu^*)_{\Omega-F} \neq 0$. Then Y is completely isometric to an intersection $Y_1 \cap Y_2$ such that Y_1 is completely isometric to a space $H_\infty^{m,R}$ (that is, in the notation of Lemma 2.1, $m'_R = m \geq 1$ and $k'_R = m+1$), or C, and Y_2 is a Hilbertian JC^*-triple.

(b): Suppose there exists a finite subset F such that $(u^*u)_{\Omega-F} \neq 0$. Then Y is completely isometric to an intersection $Y_1 \cap Y_2$ such that Y_1 is completely isometric to a space $H_\infty^{m,L}$ (that is, $m'_L = m \geq 1$ and $k'_L = m+1$), or R, and Y_2 is an Hilbertian JC^*-triple.

Proof. (a) follows from Lemma 2.1 (b) and (c), the coordinatization procedure outlined in subsection 2.1, and Lemma 2.3. (b) follows by symmetry using Lemma 2.1 (b′) and (c′), and Lemma 2.3. □

It is worth emphasizing that the space $H_\infty^{m,R}$ (resp. $H_\infty^{m,L}$) is determined up to complete isometry among Hilbertian JC*-triples by the condition

$$(uu^*)_{\Omega-F} = 0 \Leftrightarrow |F| < m, \quad (u^*u)_G = 0 \Leftrightarrow |G| > m+1 \tag{7}$$

(resp.

$$(u^*u)_{\Omega-F} = 0 \Leftrightarrow |F| < m, \quad (uu^*)_G = 0 \Leftrightarrow |G| > m+1),$$

and that $H_\infty^{0,R} = C$ and $H_\infty^{0,L} = R$.

Remark 2.6. Recall from [13, p. 2245] that i_R (resp. i_L) is the largest i such that $(uu^*)_J \neq 0$ (resp. $(u^*u)J \neq 0$) for any J with $|J| = i$. For the spaces H_n^k from [13], we have $i_R = k$ and $i_L = n-k+1$ so that $i_R + i_L = n+1$. We may therefore think of the condition (7) as "$i_R = \infty - m$, $i_L = m+1$", so that "$i_R + i_L = \infty + 1$".

To handle the remaining case 3, we shall need the following lemma.

Lemma 2.7. Let Y be a separable infinite dimensional Hilbertian operator space which is a JC^*-subtriple of a C^*-algebra A and let $\{u_i : i \in \Omega\}$ be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y.

Let S and T be finite subsets of Ω and let $k \in \Omega-(S\cup T)$. If $(uu^*)_S u_k (u^*u)_T = 0$, then $(uu^*)_{S'} u_{k'} (u^*u)_{T'} = 0$ for all sets S', T' with $|S'| = |S|$, $|T'| = |T|$ and for all $k' \in \Omega - (S' \cup T')$.

Proof. It suffices to prove this with (S,k,T) replaced in turn by $(S\cup\{l\}-\{j\},k,T)$ (with $l\notin S$ and $j\in S$); by (S,l,T) (with $l\neq k$); and by $(S,k,T\cup\{l\}-\{i\})$ (with $l\notin T$ and $i\in T$).

In the first case,

$$\begin{aligned}
u_l u_l^*(uu^*)_{S-j}u_k(u^*u)_T &= (u_j u_j^* u_l + u_l u_j^* u_j)u_l^*(uu^*)_{S-j}u_k(u^*u)_T\\
&= 0 + u_l u_j^* u_j u_l^*(uu^*)_{S-j}u_k(u^*u)_T\\
&= u_l u_j^*(uu^*)_{S-j}u_j u_l^* u_k(u^*u)_T \text{ (by hopping)}\\
&= -u_l u_j^*(uu^*)_{S-j}u_k u_l^* u_j(u^*u)_T\\
&= -u_l u_j^*(uu^*)_{S-j}u_k(u^*u)_T u_l^* u_j = 0.
\end{aligned}$$

By symmetry, $(uu^*)_S u_k(u^*u)_{T-i}u_l^* u_l = 0$, proving the second case.

Finally,

$$\begin{aligned}
(uu^*)_S u_l(u^*u)_T &= (uu^*)_S(u_l u_k^* u_k + u_k u_k^* u_l)(u^*u)_T\\
&= (uu^*)_S u_l u_k^* u_k(u^*u)_T + (uu^*)_S u_k u_k^* u_l(u^*u)_T\\
&= u_l u_k^*(uu^*)_S u_k(u^*u)_T + (uu^*)_S u_k(u^*u)_T u_k^* u_l = 0. \quad \square
\end{aligned}$$

We can now handle the final case 3.

Proposition 2.8. Let Y be a separable infinite dimensional Hilbertian JC*-triple and let $\{u_i : i\in\Omega\}$ be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y. Suppose that for all finite subsets $G\subset\Omega$, $(uu^*)_{\Omega-G}=0$ and $(u^*u)_{\Omega-G}=0$. Then Y is completely isometric to Φ.

Proof. We show first that all finite products $(uu^*)_F u_i(u^*u)_G$ with $F, G, \{i\}$ pairwise disjoint (and F, G not both empty), are not zero. Suppose, on the contrary, that $(uu^*)_F u_i(u^*u)_G = 0$ for some F, G, i. If F and G are both non-empty, pick a subset $F'\subset F$ of maximal cardinality such that $(uu^*)_{F'}u_i(u^*u)_G \neq 0$ (F' could be empty). Then by repeated use of collinearity and passing to the limit, we arrive at $(uu^*)_{F'}u_i(u^*u)_G = (uu^*)_{F'}u_i(u^*u)_{\Omega-(\{i\}\cup F')} = 0$, a contradiction. So either $F=\emptyset$ and $u_i(u^*u)_G = 0$, or $G=\emptyset$ and $(u^*u)_F u_i = 0$. In the first case, picking a subset $G'\subset G$ of maximal cardinality such that $u_i(u^*u)_{G'}\neq 0$, then by collinearity $u_i(u^*u)_{G'} = (uu^*)_{\Omega-(\{i\}\cup G')}u_i(u^*u)_{G'} = 0$, a contradiction, and similarly in the second case. We have now shown that all finite products $(uu^*)_F u_i(u^*u)_G$ with $F, G, \{i\}$ pairwise disjoint, are not zero.

Now consider the space $Y_n := \mathrm{sp}\{u_1,\ldots,u_n\}$. By [13, Th. 3(b)], Y_n is completely isometric to a space $H_n^{k_1}\cap\cdots\cap H_n^{k_m}$, where $n\geq k_1>\cdots>k_m\geq 1$. We claim that $m=n$ and $k_j = n-j+1$ for $j=1,\ldots,n$. By way of contradiction, suppose that there is a k, $1\leq k\leq n$ such that the space H_n^k is not among the spaces $H_n^{k_j}$, $1\leq j\leq m$. Let $\psi : x\mapsto (x^{(k_1)},\ldots,x^{(k_m)})$ denote the ternary isomorphism of the ternary envelope of Y_n whose restriction to Y_n implements the complete isometry of Y_n with $H_n^{k_1}\cap\cdots\cap H_n^{k_m}$, and consider the element $x := (uu^*)_{\{1,\ldots,k-1\}}u_k(u^*u)_{\{k+1,\ldots,n\}}$. As shown above, $x\neq 0$. However, $x^{(k_j)} = 0$ for each j, a contradiction. To see that $x^{(k_j)}=0$, suppose first that $k_j<k$. Since ψ is a ternary isomorphism, $x^{(k_j)} = (u^{(k_j)}u^{(k_j)*})_{\{1,\ldots,k-1\}}u_k^{(k_j)}(u^{(k_j)*}u^{(k_j)})_{\{k+1,\ldots,n\}} = 0$ since $(u^{(k_j)}u^{(k_j)*})_{\{1,\ldots,k-1\}}u_k^{(k_j)}$ is zero in $H_n^{k_j}$. Similarly, if $k_j>k$, then $n-k_j+1 < n-k+1$, $u_k^{(k_j)}(u^{(k_j)*}u^{(k_j)})_{\{k+1,\ldots,n\}} = 0$ so that $x^{(k_j)}=0$ in this case as well.

We now have for each n that, completely isometrically, $Y_n = \cap_{k=1}^n H_n^k$ and the latter space is completely isometric to Φ_n by [14, Lemma 2.1]. Since $Y = \overline{\cup Y_n}$ and $\Phi = \overline{\cup \Phi_n}$, it follows that $Y = \Phi$ completely isometrically. □

We come now to the first main result of this paper.

Theorem 1. Let Y be a separable infinite dimensional Hilbertian operator space which is concretely represented as a JC^*-triple. Then Y is completely isometric to one of the following spaces, where each Y_i is completely isometric to one of the spaces $H_\infty^{m_i,R}$ or $H_\infty^{m_i,L}$.

- Φ.
- $Z \cap Y_1 \cap \cdots \cap Y_n$, where $n \geq 1$ and Z is Φ or is absent.
- $Z \cap Y_1 \cap \cdots \cap Y_n \cap \cdots$, where Z is Φ or is absent.

Proof. Let $\{u_i\}$ be an orthonormal basis for Y consisting of a maximal family of mutually collinear minimal partial isometries. By Proposition 2.8 and Proposition 2.5, either Y is completely isometric to Φ, in which case the theorem is proved, or Y is completely isometric to an intersection $Y_1 \cap Z$, where Y_1 is completely isometric to either $H_\infty^{m,R}$ or $H_\infty^{k,L}$, with $m, k \geq 0$. It follows by induction, in the case that Y is not completely isometric to Φ, that $Y = Y_1 \cap Y_2 \cap \cdots \cap Y_n \cap Z_n$, where $n \geq 1$, each Y_i is completely isometric to one of the spaces $H_\infty^{m_i,R}$ or $H_\infty^{k_i,L}$, and either Z_n is completely isometric to the space Φ, in which case the theorem is proved, or Z_n can be further decomposed as $Y_{n+1} \cap W$, where Y_{n+1} is completely isometric to either $H_\infty^{m_{n+1},R}$ or $H_\infty^{k_{n+1},L}$.

It remains to consider the case in which no Z_n is completely isometric to Φ. By the constructions in the proof of Proposition 2.5, Z_n is obtained by multiplying Y on the left and right by a sequence of projections of the form $1 - p_{i,R}$ or $1 - p_{j,L}$ (see Lemma 2.1). The resulting products of projections converge strongly and it follows that Y is the intersection of an infinite sequence Y_i and a space Z, which has the property that all products $(ww^*)_{\Omega - F}$ and $(w^*w)_{\Omega - F}$ are zero, for all finite sets F. An appeal to Proposition 2.8 now shows that Z is completely isometric to Φ, completing the proof. □

Remark 2.9. By the argument in [13, p. 2259], the sequences m_i and k_i are strictly increasing. This fact is not needed in the preceding proof.

Remark 2.10. In section 5, we show that the spaces $H_\infty^{m,R}$ and $H_\infty^{k,L}$ are completely isometric to spaces of creation and annihilation operators on pieces of the anti-symmetric Fock space. Hence all rank 1 JC*-triples are really spaces of creation and annihilation operators.

We close this section with a well known lemma about Hilbertian TRO's. Recall that TROs are operator subspaces of a C*-algebra which are closed under the product xy^*z, and are fundamental in operator space theory. Indeed, every operator space has both a canonical injective envelope [18] and a canonical "Shilov boundary" [2] which are TROs. A proof of the following lemma can be found in [19], which classifies all W*-TRO's up to complete isometry. We include a quick alternate proof from the point of view of this section.

Lemma 2.11. If X is a Hilbertian TRO, then X is completely isometric to R or C

Proof. Let $\{u_j\}$ be an orthonormal basis consisting of mutually collinear minimal partial isometries in X. For a fixed $i \neq j$, since $u_i u_i^* u_j$ is a partial isometry in X,

$u_iu_i^*u_j = P_2(u_j)(u_iu_i^*u_j)$ is either equal to $e^{i\theta}u_j$ or 0. If the latter case holds, then by the calculation in [13, Lemma 5.1], $u_iu_i^*u_j = 0$ for all $i \neq j$, and X is ternary isomorphic and thus completely isometric to C. On the other hand, if $u_iu_i^*u_j = e^{i\theta}u_j$, then by collinearity, $e^{i\theta} = 1$, $u_ju_i^*u_i = 0$, and again by [13, Lemma 5.1], X is completely isometric to R. □

3. Contractively complemented Hilbertian operator spaces

Suppose that a Hilbert space H is complemented in a C*-algebra A via a contractive projection P. Let L be a contractive linear map from H into A with the properties that $L(H) \perp H$ and $P(L(H)) = 0$. Then the space $K = \{h + L(h) : h \in H\}$ is clearly contractively complemented by $P + LP$. From this it follows that a classification of contractively complemented Hilbertian operator spaces is hopeless without some qualifications.

3.1. **Expansions of contractive projections.** The following definitions are crucial.

Definition 2. Consider a triple $\{K, A, P\}$ consisting of a Hilbertian operator space K, a C*-algebra A, and a contractive projection P from A onto K. If there exists a Hilbertian subspace H of A which is contractively complemented by a projection Q and a contractive linear map L from H into A such that $P = Q + LQ$, $L(H) \perp H$ and $Q(L(H)) = 0$, we say that$\{K, A, P\}$ is an **expansion** of $\{H, A, Q\}$. (Note that this implies that $K = \{h + L(h) : h \in H\}$.)

The following is immediate.

Lemma 3.1. If $\{K, A, P\}$ is an expansion of $\{H, A, Q\}$ then $Q|_K$ is a completely contractive isometry from K onto H

Suppose $X \subset A$ is a contractively complemented Hilbertian operator subspace by a projection Q. Further suppose that Y is a Hilbertian operator subspace of A which is isometric to X and which is orthogonal to X and lies in $\ker(Q)$. Then $\{x+Lx : x \in X\}$ is contractively complemented in A by the projection $P = Q+LQ$, where L is any isometry from X onto Y. It is clear that $\{x + Lx : x \in X\}$ is an expansion of X. Thus one cannot hope to classify contractively complemented Hilbertian operator spaces up to complete isometry. However, we will show in this section that all contractively complemented Hilbert spaces are expansions of a"minimal" 1-complemented Hilbert space which is a JC*-triple.

Definition 3. The **support partial isometry** of a non-zero element ψ of the predual A_* of a JW*-triple A is the smallest element of the set of partial isometries v such that $\psi(v) = \|\psi\|$, and is denoted by v_ψ. For each non-empty subset G of A_*, the **support space** $s(G)$ of G is the smallest weak*-closed subspace of A containing the support partial isometries of all elements of G.

The existence and uniqueness of the support partial isometry was proved and exploited in the more general case of a JB^*-triple (in which case the partial isometries are replaced by their abstract analog, the tripotents) in [9]. One of its important properties is that of "faithfulness": if a non-zero partial isometry w satisfies $w \leq v_\psi$, then $\psi(w) > 0$.

We now give two examples of expansions which naturally occur and are relevant to our work.

Example 1. From [14, Theorem 2], if P is a contractive projection on a C^*-algebra A, with $X := P(A)$ which is isometric to a Hilbert space, then there are projections $p, q \in A^{**}$, such that, $X = P^{**}A^{**} = \{pxq+(1-p)x(1-q) : x \in X\}$. The space pXq is exactly the norm closed span of the support partial isometries of the elements of P^*A^* (see [8] for the construction). The map $\mathbf{E_0} : x \mapsto pxq$ is an isometry of X onto a JC^*-subtriple $\mathbf{E_0}X$ of A^{**}, $\mathbf{E_0}P^{**}$ is a normal contractive projection on A^{**} with range $\mathbf{E_0}X$ and clearly $pXq \perp (1-p)X(1-q)$. It follows that

$$\{X, A^{**}, P^{**}\} \text{ is an expansion of } \{\mathbf{E_0}X, A^{**}, \mathbf{E_0}P^{**}\}.$$

Specifically, let $L : \mathbf{E_0}X \to A^{**}$ be the map $pxq \mapsto (1-p)x(1-q)$. Then $P(A) = P^{**}A^{**} = \{pxq + (1-p)x(1-q) : x \in P(A)\}$ and $P^{**} = \mathbf{E_0}P^{**} + L\mathbf{E_0}P^{**}$, since if $a \in A^{**}$, there is $x \in A$ with $a = Px = p(Px)q + (1-p)Px(1-q)$ and $\mathbf{E_0}P^{**}a+L\mathbf{E_0}P^{**}a = \mathbf{E_0}Px+L\mathbf{E_0}Px = p(Px)q+(1-p)Px(1-q)$. Finally, if $x \in A$, then $L\mathbf{E_0}Px = (1-p)Px(1-q)$ and $\mathbf{E_0}P^{**}L\mathbf{E_0}Px = \mathbf{E_0}P^{**}((1-p)Px(1-q)) = 0$ by [14, Theorem 2(e)].

Definition 4. The triple $\{\mathbf{E_0}X, A^{**}, \mathbf{E_0}P^{**}\}$ (or simply $\mathbf{E_0}X$) will be called the **support** of $\{X, A^{**}, P^{**}\}$. It is also called the **enveloping support** of $\{X, A, P\}$.

Example 2. It follows from [5], that, for a normal contractive projection P from a von Neumann algebra (or JW^*-triple) A onto a Hilbert space X, there is a similar projection $\mathbf{E}$ on A such that

$$\{X, A, P\} \text{ is an expansion of } \{\mathbf{E}A, A, \mathbf{E}\}$$

and $\mathbf{E}A$ is the norm closure of the span of support partial isometries of elements of P_*A_*.

Indeed, as set forth in [5, Lemma 3.2], $P(A) \subset s(P_*(A_*)) \oplus s(P_*(A_*))^\perp \subset A$, and $\mathbf{E} : A \to A$ is a normal contractive projection onto $s(P_*(A_*))$ given by $\mathbf{E} = \phi \circ P$ where $\phi : P(A) \to s(P_*(A_*))$ is the restriction of the M-projection of $s(P_*(A_*)) \oplus s(P_*(A_*))^\perp$ onto $s(P_*(A_*))$. (Although we will not use these facts, ϕ is a triple isomorphism from $P(A)$ with the triple product $\{xyz\}_{P(A)} := P\{xyz\}$ onto the JW^*-subtriple $s(P_*(A_*))$ of A, and ϕ^{-1} coincides with P on $s(P_*(A_*))$.

The map $L : \mathbf{E}(A) \to \mathbf{E}(A)^\perp$ in this case is given by $L = \phi^\perp \circ \phi^{-1} = \phi^\perp \circ P$, where $\phi^\perp : P(A) \to s(P_*(A_*))^\perp$ is the restriction of the M-projection of $s(P_*(A_*))\oplus s(P_*(A_*))^\perp$ onto $s(P_*(A_*))^\perp$. Then for $h \in s(P_*(A_*))$, say $h = \phi \circ P(x)$ for some $x \in A$, $h + Lh = \phi(Px) + \phi^\perp(Px) = Px$ so that $P(A) = \{h + Lh : h \in s(P_*(A_*))\}$. Furthermore, for $x \in A$, $\mathbf{E}x + L\mathbf{E}x = \phi(Px) + \phi^\perp(Px) = Px$. It is obvious that $L\mathbf{E}(A) \perp \mathbf{E}(A)$. Finally, for $x \in A$, $\mathbf{E}(L\mathbf{E}x) = \phi \circ P(\phi^\perp(Px)) = 0$ since $Px = PPx = P\phi Px + P\phi^\perp Px$ and $\phi \circ Px = (\phi \circ P)^2x + \phi \circ P(\phi^\perp(Px))$.

Definition 5. By analogy with Example 1, we will call $\{\mathbf{E}A, A, \mathbf{E}\}$ (or simply $\mathbf{E}A$) the **support** of $\{X, A, P\}$ in this case. If $\{X, A, P\}$ is not the expansion of any tuple other than itself, we say that $\{X, A, P\}$ is **essential** and that X is **essentially normally complemented** in A.

A concrete instance of Example 2 is the projection of $B(H)$ onto R (or C). It is easy to see that R and C are essentially normally complemented in $B(H)$, as is $R \cap C$ in $B(H \oplus H)$. (See the paragraph preceding Theorem 3 belowd.)

Remark 3.2. If $\{P(A), A, P\}$ is as in Example 1, then $\{P^{**}(A^{**}), A^{**}, P^{**}\}$ is as in Example 2, and the enveloping support of P is the same as the support of P^{**}, since both $\mathbf{E_0}P(A)$ and $\mathbf{E}(A^{**})$ coincide with the norm closed linear span of A^{**} generated by $s(P^*(A^*))$.

Proposition 3.3. Suppose X is Hilbertian and complemented in a von Neumann algebra A by a normal contractive projection P. Then $\{X, A, P\}$ is essential if and only if it equals its support.

Proof. Suppose $\{X, A, P\}$ equals its support and is the expansion of $\{Y, A, Q\}$ given by a contractive map L. For each partial isometry $v \in X$, $v = w + z$ where w and z are orthogonal partial isometries, $w = Qv$, $QL = 0$, $z = L(w)$and $P = Q + LQ$. Suppose v is the support partial isometry of $\psi \in P_*A_*$. Then

$$\psi(v) = \psi(Pv) = \psi((Q + LQ)(v)) = \psi((Q + QLQ)(v)) = \psi(Qv) = \psi(w),$$

and hence $w = v$, $L = 0$ and $\{X, A, P\}$=$\{Y, A, Q\}$. The converse is immediate. $\square$

3.2. **Operator space structure of 1-complemented Hilbert spaces.** As noted at the beginning of the previous subsection, we cannot classify 1-complemented Hilbert spaces up to complete isometry. However, in Theorem 2 below, we are able to give a classification up to "trivial" expansion.

We assume in what follows that P is a normal contractive projection on a von Neumann algebra A, whose range $Y = P(A)$ is a JC^*-subtriple of A of rank one, and $\{u_i\}$ is an orthonormal basis for Y consisting of a maximal family of minimal (in Y) collinear partial isometries. We shall assume for convenience that Y is infinite dimensional and separable. In Theorem 2 below, we shall also be able to handle the case of a contractive projection on a C^*-algebra.

We know from Theorem 1 that Y is completely isometric to an intersection of operator spaces $\tilde{Y} = \mathcal{R} \cap \mathcal{L} \cap \Phi$, where $\mathcal{R} = \cap_i H_\infty^{r_i}$ and $\mathcal{L} = \cap_k H_\infty^{l_k}$. Some of these spaces may be missing, and for short we have written $H_\infty^{r_i} = H_\infty^{r_i,R}$ and $H_\infty^{l_k} = H_\infty^{l_k,L}$.

As shown in section 2, the weak*-ternary envelope of Y in A is generated by the partial isometries $\{u_{I,J}\}$ and is ternary isomorphic, hence completely isometric, to a Cartan factor M of type I which is generated by the matrix units $\{E_{J,I}\}$. We may therefore assume that P is defined on M and has range $\tilde{Y} = \mathcal{R} \cap \mathcal{L} \cap \Phi$, which is a JC^*-subtriple of M. We shall identify Y with $\tilde{Y} = \mathcal{R} \cap \mathcal{L} \cap \Phi$, and the weak*-ternary envelope of Y with M.

Note that by the definition of intersection, if the operator space structures of $H_\infty^{r_i}, H_\infty^{l_k}, \Phi$ come from embeddings $H_\infty^{r_i} \subset B(H^{r_i}), H_\infty^{l_k} \subset B(H^{l_k}), \Phi \subset B(H^\Phi)$, then

$$[\cap_i H_\infty^{r_i}] \cap [\cap_k H_\infty^{l_k}] \cap \Phi \subset M \subset B([\oplus H^{r_i}] \oplus [\oplus H^{l_k}] \oplus H^\Phi).$$

Lemma 3.4. If $u_j = \sum_i u_j^{r_i} + \sum_k u_j^{l_k} + u_j^\Phi$ is the decomposition of u_j into orthogonal partial isometries of $[\cap_i H_\infty^{r_i}] \cap [\cap_k H_\infty^{l_k}] \cap \Phi$, and if $P(u_j^{r_i}) = 0$ for some i (resp. $P(u_j^{l_k}) = 0$ for some k), then $u_j^{r_i} = 0$ (resp. $u_j^{l_k} = 0$).

Proof. u_j is the support partial isometry of some functional $\psi_j \in Y_* = P_*(A_*)$. By the faithfulness of ψ_j on its support, $\psi_j(u_j^{r_i}) = \psi_j(P(u_j^{r_i})) = 0$ implies, since $u_j^{r_i} \leq u_j$, that $u_j^{r_i} = 0$. Similarly for $u_j^{l_k}$. $\square$

We again adopt the more compact notation Iu_iJ, used in the proof of Lemma 2.2, for the "one" $(uu^*)_I u_i (u^*u)_J$. We note next that for $j \neq i$, $\{u_j, Iu_iJ, u_j\} =$

$u_j(Iu_iJ)^*u_j = u_jJu_i^*Iu_j = (J\cup\{j\}u_i^*(I\cup\{j\}) = 0$ since either $j\notin I$ or $j\notin J$. By a conditional expectation formula in (2),

$$0 = P(\{u_j, Iu_iJ, u_j\}) = \{u_j, P(Iu_iJ), u_j\}.$$

Since every element of Y is in the closed linear span of the u_j, we may write $P(Iu_iJ) = \sum_k \lambda_k^{i,J} u_k$ and thus $0 = \sum_k \overline{\lambda_k^{i,J}}\{u_ju_ku_j\} = \overline{\lambda_j^{i,J}}u_j$. We conclude that $\lambda_j^{i,J} = 0$ for $j\neq i$ and hence $P(Iu_iJ) = \lambda_{i,J}u_i$ for each "one" Iu_iJ, where we have written $\lambda_{i,J}$ for $\lambda_i^{i,J}$.

Now suppose that i is fixed and $k\neq i$, say $k\in J$. Then

$$\begin{aligned} 2\{u_k, u_i, Iu_iJ\} &= u_ku_i^*Iu_iJ + Iu_iJu_i^*u_k = u_ku_i^*Iu_iJ \text{ (as } k\in J) \\ &= Iu_ku_i^*u_iJ \text{ (by "hopping")} \\ &= Iu_k((J-\{k\})\cup\{i\}). \end{aligned}$$

Thus by another conditional expectation formula in (2),

$$\begin{aligned} \lambda_{k,(J-\{k\})\cup\{i\}}u_k &= P(Iu_k((J-\{k\})\cup\{i\})) \\ &= P(2\{u_k, u_i, Iu_iL\}) \\ &= 2\{u_k, u_i, P(Iu_iJ)\} \\ &= 2\lambda_{i,J}\{u_ku_iu_i\} = \lambda_{i,J}u_k. \end{aligned}$$

Thus $\lambda_{i,J} = \lambda_{k,(J-\{k\})\cup\{i\}}$ and so $\lambda_{i,J} = \lambda$ is independent of i, J such that $i\notin J$ and $|J| = m$. Similarly, it can be shown that $\lambda_{i,J} = \lambda_{k,J}$ for any $i\notin J, k\notin J$.

We have now shown that there is a complex number $\lambda = \lambda_m$ such that $P(Iu_jJ) = \lambda u_j$, for all partitions $I\cup\{j\}\cup J$ of Ω with $|J| = m$.

Now, since $P(\sum_{|J|=m} Iu_iJ) = \sum_{|J|=m} P(Iu_iJ) = \sum_{|J|=m}\lambda_m u_i$, we must have $\lambda_m = 0$ and $P(\sum_{|J|=m} Iu_iJ)) = 0$ unless $r_i = 0$. Thus $P(u_i^{r_i}) = 0$ unless $m = 0$. Similarly, $P(u_i^{l_k}) = 0$ unless $l_k = 0$. By Lemma 3.4, $u_j^{r_i} = u_j^{l_k} = 0$ for $r_i\neq 0$, and $l_k\neq 0$. By Theorem 1, $P(A)$ is an intersection of at most the three spaces R, C, Φ. Together with Examples 1 and 2, Remark 3.2, and Proposition 3.3 in subsection 3.1, this proves the second main theorem of this paper.

Theorem 2. Suppose Y is a separable infinite dimensional Hilbertian operator space which is contractively complemented (resp. normally contractively complemented) in a C^*-algebra A (resp. W*-algebra A) by a projection P. Then,

- **(a):** $\{Y, A^{**}, P^{**}\}$ (resp. $\{Y, A, P\}$) is an expansion of its support $\{H, A^{**}, Q\}$ (resp. $\{H, A^{**}, Q\}$, which is essential)
- **(b):** H is contractively complemented in A^{**} (resp. A) by Q and is completely isometric to either R, C, $R\cap C$, or Φ.

This theorem says that, in A^{**}, Y is the diagonal of a contractively complemented space H which is completely isometric to R, C, $R\cap C$ or Φ and an orthogonal degenerate space K which is in the kernel of P. As pointed out at the beginning of section 3.1, this is the best possible classification.

By [20], the range Y of a completely contractive projection on a C*-algebra is a TRO. By Lemma 2.11 it follows that, if Y is Hilbertian, Y is completely isometric to R or C. This gives an alternate proof of the result of Robertson [17], stated here for completely contractive projections on a C^*-algebra.

Although Theorem 2 is only a classification modulo expansions, the following Lemma shows that it is the correct analogue for contractively complemented Hilbert spaces.

Lemma 3.5. Suppose that $\{Y, A, P\}$ is an expansion of $\{H, A, Q\}$ and that P is a completely contractive projection. Then Y is completely isometric to H.

Proof. By definition of expansion, in A^{**}, Y coincides with $\{h + L(h) : h \in H\}$, $Q + LQ = P$, $L(H) \perp H$ and $Q(L(H)) = 0$. Thus, $P^{**}|_H$ is a complete contraction from H onto Y with completely contractive inverse $Q|_Y$. Hence, Y is completely isometric to H. □

3.3. **An essential contractive projection onto Φ.** As noted earlier, the spaces R, C and $R \cap C$ are each essentially normally contractively complemented in a von Neumann algebra. We now proceed to show that the same holds for Φ.

We begin by taking a closer look at the contractive projection $P = P_n^k$ of the ternary envelope $T = T(H_n^k) = M_{p_k,q_k}(\mathbf{C})$ of H_n^k, onto H_n^k. This projection and the space H_n^k were first constructed in [1] and rediscovered in [13]. By [13, Cor. 7.3],

$$P_n^k x = \frac{1}{\binom{n-1}{k-1}} \sum_{i=1}^{n} \mathrm{tr}\,(x u_i^*) u_i.$$

Consistent with the identification of Y with $\tilde{Y}$ in the previous subsection, we let u_i denote the image of the orthonormal basis u_i, of a finite dimensional JC^*-triple of rank one. Thus, $u_i = \sum \epsilon(I, i, J) E_{J,I}$.

Lemma 3.6. The action of $P = P_n^k$ is as follows: if $x \in T$ is not a "one", then $Px = 0$. If $x = \epsilon(I, i, J)E_{J,I}$ is a "one", then $P(\epsilon(I, i, J)E_{J,I}) = \frac{1}{\binom{n-1}{k-1}} u_i$.

Proof. Suppose first that $x = \epsilon(I, J)E_{J,I} \in T$ is not a "one", that is, $I \cap J \neq \emptyset$. Then

$$\begin{aligned} x u_i^* &= \epsilon(I, J) E_{J,I} \sum_{I',J'} \epsilon(I', i, J') E_{I',J'} \\ &= \epsilon(I, J) \sum_{I',J'} \epsilon(I', i, J') E_{J,I} E_{I',J'} \\ &= \epsilon(I, J) \sum_{J'} \epsilon(I, i, J') E_{J,J'}. \end{aligned}$$

Since $J' \cap I = \emptyset$ and $J \cap I \neq \emptyset$, J' is never equal to J and so $\mathrm{tr}\,(x u_i^*) = 0$ and $Px = 0$.

Suppose now that $x = \epsilon(I, i, J)E_{J,I} \in T$ is a "one". Then for $1 \leq j \leq n$, $u_j = \sum_{I' \cap J' = \emptyset} \epsilon(I', j, J') E_{J',I'}$, and as above $x u_j^* = \epsilon(I, i, J) \sum_{J'} \epsilon(I, i, J') E_{J,J'}$. Thus, $\mathrm{tr}\,(x u_j^*) = 1$ if $j = i$ and $\mathrm{tr}\,(x u_j^*) = 0$ if $j \neq i$. It follows that

$$P(\epsilon(I, i, J)E_{J,I}) = \frac{1}{\binom{n-1}{k-1}} \sum_j \mathrm{tr}\,(x u_j^*) u_j = \frac{1}{\binom{n-1}{k-1}} u_i. \quad \square$$

We proceed to construct a contractive projection defined on a TRO A which has range Φ. Since every TRO is the corner of a C^*-algebra, we will have constructed a projection on a C^*-algebra with range Φ. Now, let u_i be an orthonormal basis for the Hilbertian operator space Φ and let $H_n = \mathrm{sp}\{u_1, \ldots, u_n\}$. As noted in the proof of Proposition 2.8, $H_n = \Phi_n$ is completely isometric to the intersection $\cap_{i=1}^n H_n^i \subset \oplus_{k=1}^n T(H_n^i) =$ the ternary envelope $T(H_n)$ of H_n in A.

We construct a contractive projection P^n on $T(H_n)$ with range H_n as follows. For $x = \oplus_{i=1}^n x_i \in T(H_n)$, write $x = \sum_{i=1}^n (0 \oplus \cdots \oplus x_i \oplus \cdots \oplus 0)$, ($x_i$ is in the i^{th}-position). Then define

$$P^n(x) = \sum_{i=1}^n P^n(0 \oplus \cdots \oplus x_i \oplus \cdots \oplus 0) := \frac{1}{n}\sum_{i=1}^n (P_n^i(x_i), \ldots, P_n^i(x_i)).$$

Note that since $(P_n^i(x_i), \ldots, P_n^i(x_i))$ belongs to $H_n = \cap_{i=1}^n H_n^i$, we shall sometimes write it as $((P_n^i(x_i))^1, \ldots, (P_n^i(x_i))^n)$ and view $(P_n^i(x_i))^j$ as an element of H_n^j.

With $u_k = (u_k, \ldots, u_k) = (u_k^1, \ldots, u_k^n) = \sum_i (0, \ldots, u_k^i, \ldots, 0)$, we have

$$\begin{aligned} P^n(u_k) &= \sum_i P^n((0, \ldots, u_k^i, \ldots, 0)) \\ &= \frac{1}{n}\sum_i (P_n^i(u_k^i), \ldots, P_n^i(u_k^i)) \\ &= \sum_i (u_k^i, \ldots, u_k^i)/n \\ &= \sum_i (u_k^1, \ldots, u_k^n)/n \\ &= (u_k^1, \ldots, u_k^n) = u_k. \end{aligned}$$

By Lemma 3.6, P^n is zero an any non-“one”, so the range of P^n is H_n. To calculate the action of P^n on “ones”, let $x = Iu_kJ$ be such and write $x = \oplus x_i = \oplus I^i u_k^i J^i$, where $x_i \in H_n^i$. We claim that

$$P^n(x) = \frac{u_k}{n\binom{n-1}{i-1}}, \tag{8}$$

where $|I| = i - 1$. Let us illustrate this first in a specific example: Let $n = 3$, $x = u_2u_2^*u_1u_3^*u_3 = x_1 \oplus x_2 \oplus x_3 \in H_3^1 \cap H_3^2 \cap H_3 s$, so that $x_1 = 0$, $x_3 = 0$, and $i = 2$. By Lemma 3.6 again,

$$\begin{aligned} P^3(x) &= P^3(x_1 \oplus 0 \oplus 0) + P^3(0 \oplus x_2 \oplus 0) + P^3(0 \oplus 0 \oplus x_3) \\ &= \frac{1}{3}[(P_1^3(x_1), P_1^3(x_1), P_1^3(x_1)) + (P_2^3(x_2), P_2^3(x_2), P_2^3(x_2)) \\ &+ (P_3^3(x_3), P_3^3(x_3), P_3^3(x_3))] \\ &= \frac{1}{3}[(0,0,0) + (\frac{1}{2}u_1^2, \frac{1}{2}u_1^2, \frac{1}{2}u_1^2) + (0,0,0)] \\ &= \frac{1}{3}\frac{1}{2}(u_1^2, u_1^2, u_1^2) = \frac{1}{6}u_1. \end{aligned}$$

In general, for $x = \oplus x_i$ as above,

$$P^n(x) = (1/n)[\sum (P_n^i(x_i), P_n^i(x_i), P_n^i(x_i))] = (1/n)[\frac{1}{\binom{n-1}{i-1}}(u_k 1, \ldots, u_k^n)],$$

as required for (8).

Lemma 3.7. Under the embedding $T(H_n) = \oplus_{i=1}^n T(H_n^i) \subset \oplus_{i=1}^{n+1} T(H_n^i) = T(H_{n+1})$, given by $x_1 \oplus \cdots \oplus x_n \mapsto x_1 \oplus \cdots \oplus x_n \oplus 0$, we have $P^{n+1}|T(H_n) = P^n$.

Proof. This is obviously true for generators $u_{I,J}$ of $T(H_n)$ which are not "ones" since all of the P^k_n and P^k_{n+1} vanish on them. On the other hand, if Iu_kJ is a "one" in $T(H_n)$, then by collinearity $u_k = u_k u^*_{n+1} u_{n+1} + u_{n+1} u^*_{n+1} u_k$, and by (8),

$$\begin{aligned} P^{n+1}(Iu_kJ) &= P^{n+1}((I \cup \{n+1\})u_kJ + Iu_k(J \cup \{n+1\})) \\ &= \frac{1}{n+1}\frac{1}{\binom{n}{i}}u_k + \frac{1}{n+1}\frac{1}{\binom{n}{i-1}}u_k \\ &= \frac{1}{n}\frac{1}{\binom{n-1}{i-1}}u_k = P^n(Iu_kJ). \quad \square \end{aligned}$$

Lemma 3.7 enables the definition of a contractive projection P on a TRO A which is the norm closure in $\oplus_{i=1}^\infty M_{p_i,q_i}(\mathbf{C})$ of $\cup_{n=1}^\infty T(H_n)$ with $P(A) = \Phi$. As noted earlier, we can assume that A is a C^*-algebra. By Example 1, $\{\Phi, A^{**}, P^{**}\}$ is an expansion of $\{\mathbf{E}_0\Phi, A^{**}, \mathbf{E}_0 P^{**}\}$, so $\mathbf{E}_0 P^{**}(A^{**}) = \mathbf{E}_0\Phi$. Thus $\mathbf{E}_0\Phi$ is a normally contractively complemented JC^*-subtriple of A^{**}. By Theorem 2, $\mathbf{E}_0\Phi$ is completely isometric to one of $R, C, R\cap C, \Phi$, which we shall write as $R \cap C \cap \Phi$, with the understanding that one or two terms in this intersection may be missing. We claim in fact that R and C are both missing.

Lemma 3.8. The support $\mathbf{E}_0\Phi$ of $\{\Phi, A^{**}, P^{**}\}$ for the above construction is completely isometric to Φ.

Proof. Because of (8), for any partition $\{i_1, i_2, \ldots\} \cup \{k\} \cup \{j_1, j_2, \ldots, j_m\}$ of $\{1, 2, 3, \ldots\}$,

$$P^{**}(Iu_kJ) = \lim_{n\to\infty} P^{n+m+1}(\{i_1,\ldots,i_n\}u_kJ) = \lim_{n\to\infty}\frac{1}{n+m+1}\frac{1}{\binom{n+m}{n}}u_k = 0.$$

Thus, writing $\mathbf{E}_0 u_j = \mathbf{E}_0 u^C_j + \mathbf{E}_0 u^R_j + \mathbf{E}_0 u^\Phi_j$ as in the notation of Lemma 3.4 and using Lemma 2.2, $P^{**}\mathbf{E}_0(u^C_j) = P^{**}\mathbf{E}_0(u^R_j) = 0$ and thus $P^{**}(\mathbf{E_0}u_j) = P^{**}(\mathbf{E_0}u^\Phi_j)$. Recall that $\mathbf{E_0}(u_j)$ is the support partial isometry of a norm 1 element ψ in P^*A^*. Since $\psi(\mathbf{E_0}u_j) = \psi(P^{**}\mathbf{E_0}u_j) = \psi(P^{**}\mathbf{E_0}u^\Phi_j) = \psi(\mathbf{E_0}u^\Phi_j)$, it follows that $\mathbf{E_0}(u_j) = \mathbf{E_0}u^\Phi_j$, so that $\mathbf{E}_0(u^C_j) = \mathbf{E}_0(u^R_j) = 0$, proving that $\mathbf{E}_0\Phi$ is completely isometric to Φ. $\square$

Since R, C and $R\cap C$ are trivially contractively complemented in $B(H)$ as spans of finite rank operators in such a way that they clearly equal their support spaces, this proves that each of the spaces occurring in (b) of Theorem 2 are essentially contractively complemented.

Theorem 3. The operator spaces $R, C, R\cap C$, and Φ are each essentially normally contractively complemented in a von Neumann algebra.

4. Completely bounded Banach-Mazur distance

Since all of the Hilbertian operator spaces under consideration in this paper are homogeneous (by Lemma 1.1 and [14, Theorem 1]), the completely bounded distances can be computed by simply computing $\|\psi\|_{\rm cb}\|\psi^{-1}\|_{\rm cb}$ for any fixed unitary operator between the two Hilbert spaces, [22, Theorem 3.1].

Theorem 4. For any $m, k \geq 1$,

(a): $d_{\rm cb}(C, H^{m,R}_\infty) = \sqrt{m+1}$.

(b): $d_{\rm cb}(H^{k,L}_\infty, H^{m,R}_\infty) = \infty$.

(c): $d_{\rm cb}(\Phi, H_\infty^{m,R}) = \infty$.

Proof. We first prove (a). Let $\{u_i\}$ (resp. $\{v_i\}$) be any orthonormal basis for C (resp. $H_\infty^{m,R}$), and let ψ be the isometry that takes u_i to v_i. For each $n > m+1$, let $\tilde{H}_n^1 = \mathrm{sp}\,\{u_1, \ldots, u_n\}$, $\tilde{H}_{n,R} = \mathrm{sp}\,\{v_1, \ldots, v_n\}$, and $\psi^{(n)} = \psi|\tilde{H}_n^1$. Note that for $\tilde{H}_n^1$, we have $i_R = 1$ and $i_L = n$ (see Remark 2.6) so that by [14, Cor. 5.3], $\tilde{H}_n^1$ is completely isometric to column space $C_n = H_n^1$. Because of this, in what follows, we will write H_n^1 for $\tilde{H}_n^1$. The space $\tilde{H}_{n,R}$ has $i_R = m+1 < n$ and $i_L = n$, and by [13, Th. 3(b)], is completely isometric to an intersection $H_n^{k_1} \cap \cdots \cap H_n^{k_r}$, where $m+1 = k_1 > k_2 > \cdots > k_r$. Now, for any p,

$$\|(\psi^{(n)})_p\| = \sup_{0 \neq x \in M_p(H_n^1)} \frac{\|\psi_p(x)\|_{M_p(\tilde{H}_n)}}{\|x\|_{M_p(H_n^1)}}.$$

Let us write $x = [x_{ij}]$ with $x_{ij} \in H_n^1$ and $y = [y_{ij}] = \psi_p(x)$, with $y_{ij} = \psi(x_{ij}) = (y_{ij}^{k_1}, \ldots, y_{ij}^{k_r}) \in \tilde{H}_n$ where $y_{ij}^{k_l} \in H_n^{k_l}$.

Now $M_p(\tilde{H}_n) \subset M_p(H_n^{k_1}) \oplus \cdots \oplus M_p(H_n^{k_r})$, and $M_p(H_n^1) \ni [x_{ij}] \mapsto [y_{ij}^{k_l}] \in M_p(H_n^{k_l})$ has norm $\sqrt{k_l}$ by [14, Lemma 3.1]. Thus $y = [y_{ij}^{k_1}] \oplus \cdots \oplus [y_{ij}^{k_r}]$ and

$$\begin{aligned} \|y\|_{M_p(\tilde{H}_n)} &= \max_{1 \le l \le r} \|[y_{ij}^{k_l}]\|_{M_p(H_n^{(k_l)})} \\ &\le \max_{1 \le l \le r} \sqrt{k_l}\|x\|_{M_p(H_n^1)} \\ &= \sqrt{m+1}\|x\|_{M_p(H_n^1)}. \end{aligned}$$

Thus $\|\psi : H_n^1 \to \tilde{H}_n\|_{\rm cb} \le \sqrt{m+1}$, and by a simple approximation argument based on the fact that $H_\infty^{m,R}$ (resp. C) is the norm closure of the increasing union of the $\tilde{H}_{n,R}$ (resp. H_n^1), it follows that $\|\psi\|_{\rm cb} \le \sqrt{m+1}$. Moreover, equality holds. Indeed, by the proof of [14, Lemma 3.1], there exists, for each $n \ge 1$, an element $(h_{n1}^m, \ldots, h_{nn}^m) \in M_{1,n}(H_n^1)$, such that $\|(h_{n1}^m, \ldots, h_{nn}^m)\|_{M_{1,n}(H_n^1)} = 1$ and $\|(\psi(h_{n1}^m), \ldots, \psi(h_{nn}^m))\|_{M_{1,n}(H_n^m)} = \sqrt{m+1}$. Then with $x_n := \begin{bmatrix} h_{n1}^m, \ldots, h_{nn}^m & 0 \\ 0 & 0 \end{bmatrix} \in M_p(C)$ and $y_n = \psi_p(x_n)$, we have $\|x_n\|_{M_p(C)} = 1$ and $\|y_n\|_{M_p(H_\infty^{m,R})} = \sqrt{m+1}$, so that $\|\psi_p\| = \sqrt{m+1}$ and $\|\psi\|_{\rm cb} = \sqrt{m+1}$.

We next show that $\|\psi^{-1}\|_{\rm cb} = 1$, which will complete the proof of (a). Let $y = [y_{ij}] \in M_p(\tilde{H}_n)$ and $x = [x_{ij}] = (\psi^{-1})_p(y) \in M_p(H_n^1)$ so that $x_{ij} = \psi^{-1}(y_{ij})$. Then for any $1 \le l \le r$, by [14, Lemma 3.1] and for sufficiently large p,

$$\begin{aligned} \|\psi^{-1} : \tilde{H}_n \to H_n^1\|_{\rm cb} &= \|(\psi^{-1})_p : M_p(\tilde{H}_n) \to M_p(H_n^1)\| \\ &= \sup_{y \neq 0} \frac{\|(\psi^{-1})_p y\|_{M_p(H_n^1)}}{\|y\|_{M_p(\tilde{H}_n)}} \\ &\le \sup_{y \neq 0} \frac{\sqrt{\frac{n}{n-k_l+1}}\|y\|_{M_p(H_n^{k_l})}}{\max_{1 \le q \le r} \|[y_{ij}^{k_q}]\|_{M_p(H_n^{k_q})}} \\ &\le \sqrt{\frac{n}{n-k_l+1}} \le \sqrt{\frac{n}{n-m+1}} \le 1. \end{aligned}$$

Again, by the proof of [14, Lemma 3.1], for each $n \ge 1$, there exists an element $(h_{n1}^m, \ldots, h_{nn}^m)^t \in M_{n,1}(H_n^m)$, such that $\|(h_{n1}^m, \ldots, h_{nn}^m)^t\|_{M_{n,1}(H_n^m)} = 1$ and

$\|(\psi^{-1}(h^m_{n1}),\dots,\psi^{-1}(h^m_{nn}))^t\|_{M_{n,1}(H^1_n)} = \sqrt{\frac{n}{n-m+1}}$. Then with

$$y_n := \begin{bmatrix} (h^m_{n1},\dots,h^m_{nn})^t & 0 \\ 0 & 0 \end{bmatrix} \in M_p(H^{m,R}_\infty)$$

and $x_n = (\psi_p)^{-1}(y_n)$, we have $\|y_n\|_{M_p(H^{m,R}_\infty)} = 1$ and $\|x_n\|_{M_p(C)} = \sqrt{\frac{n}{n-m+1}}$. Hence $\|\psi^{-1}\|_{\mathrm{cb}} = 1$ and this proves (a).

We now prove (b). Let $\{u_i\}$ (resp. $\{v_i\}$) be any orthonormal basis for $H^{k,L}_\infty$ (resp. $H^{m,R}_\infty$), and let ψ be the isometry that takes u_i to v_i. For each $n > \max(k+1, m)$, let $\tilde H_{n,L} = \mathrm{sp}\,\{u_1,\dots,u_n\}$, $\tilde H_{n,R} = \mathrm{sp}\,\{v_1,\dots,v_n\}$, and $\psi^{(n)} = \psi|\tilde H_{n,L}$. Note that for $\tilde H_{n,L}$, we have $i_R = n$ and $i_L = k+1$ so that by [13, Th. 3(b)], $\tilde H_{n,L}$ is completely isometric to an intersection $H^{j_1}_n \cap \cdots \cap H^{j_s}_n$, where $n = j_1 > j_2 > \cdots > j_s$. Similarly for $\tilde H_{n,R}$, we have $i_R = m+1$ and $i_L = n$ so that by [13, Th. 3(b)], $\tilde H_{n,R}$ is completely isometric to an intersection $H^{k_1}_n \cap \cdots \cap H^{k_r}_n$, where $m+1 = k_1 > k_2 > \cdots > k_r$.

Now, for any p, with $x = [x_{ij}] = (\psi)^{-1}_p(y)$,

$$\begin{aligned} \|((\psi^{(n)})^{-1})_p\| &= \sup_{0\neq y\in M_p(\tilde H_{n,R})} \frac{\|\psi_p^{-1}(y)\|_{M_p(\tilde H_{n,L})}}{\|y\|_{M_p(\tilde H_{n,R})}} \\ &= \sup_{0\neq y\in M_p(\tilde H_{n,R})} \frac{\max_{1\le q\le s} \|[x^{j_q}_{ij}]\|_{M_p(H^{j_q}_n)}}{\max_{1\le l\le r} \|[y^{k_l}_{ij}]\|_{M_p(H^{k_l}_n)}}, \end{aligned}$$

which, for suitable choices of y, as above, is greater than

$$\frac{\max(\sqrt{j_1},\dots,\sqrt{j_s})}{\max(\sqrt{k_1},\dots,\sqrt{k_r})} = \frac{\sqrt{n}}{\sqrt{m+1}}.$$

Thus, $\|\psi^{-1}\|_{\mathrm{cb}} \ge \|(\psi^{(n)})^{-1}\|_{\mathrm{cb}} \ge \|((\psi^{(n)})^{-1})_p\| \ge \frac{\sqrt{n}}{\sqrt{m+1}} \to \infty$. This proves (b).

Finally, we prove (c). Let $\{u_i\}$ (resp. $\{v_i\}$) be any orthonormal basis for Φ (resp. $H^{m,R}_\infty$, and let ψ be the isometry that takes u_i to v_i. For each $n > m$, let $\tilde H_{n,R} = \mathrm{sp}\,\{u_1,\dots,u_n\}$, $\tilde H_{n,\Phi} = \mathrm{sp}\,\{v_1,\dots,v_n\}$, and $\psi^{(n)} = \psi|\tilde H_{n,\Phi}$. Note that for $\tilde H_{n,R}$, we have $i_R = m+1$ and $i_L = n$ so that by [13, Th. 3(b)], $\tilde H_{n,R}$ is completely isometric to an intersection $H^{j_1}_n \cap \cdots \cap H^{j_s}_n$, where $m+1 = j_1 > j_2 > \cdots > j_s$. For $\tilde H_{n,\Phi}$, we have $i_R = n$ and $i_L = n$ so that by [13, Th. 3(b)], $\tilde H_{n,\Phi}$ is completely isometric to an intersection $H^{k_1}_n \cap \cdots \cap H^{k_r}_n$, where $n = k_1 > k_2 > \cdots > k_r$ (in fact, as shown in the proof of Proposition 2.8, $r = n$ and $k_j = n - j + 1$ but we do not need this fact).

Now, for any p, with $x = [x_{ij}] \in M_p(\tilde H_{n,\Phi})$, $y = [y_{ij}] = \psi_p(x)$,

$$\begin{aligned} \|(\psi^{(n)})_p\| &= \sup_{0\neq x\in M_p(\tilde H_{n,R})} \frac{\|\psi_p(x)\|_{M_p(\tilde H_{n,R})}}{\|x\|_{M_p(\tilde H_{n,\Phi})}} \\ &= \sup_{0\neq x\in M_p(\tilde H_{n,R})} \frac{\max_{1\le l\le r} \|[y^{k_l}_{ij}]\|_{M_p(H^{k_l}_n)}}{\max_{1\le q\le s} \|[x^{j_q}_{ij}]\|_{M_p(H^{j_s}_n)}}, \end{aligned}$$

which, for suitable choices of x, as above, is $\ge \frac{\max(\sqrt{k_1},\dots,\sqrt{k_r})}{\max(\sqrt{j_1},\dots,\sqrt{j_s})} = \frac{\sqrt{n}}{\sqrt{m+1}}$, showing $\|\psi\|_{\mathrm{cb}} = \infty$ and proving (c). □

Corollary 4.1. For $m, k \ge 0$,

(a): $H_\infty^{m,R}$ and $H_\infty^{k,R}$ are completely isomorphic but not completely isometric if $m \neq k$.
(b): $H_\infty^{k,L}$ and $H_\infty^{m,R}$ are not completely isomorphic.
(c): Φ and $H_\infty^{m,R}$ are not completely isomorphic.

Similar arguments yield the following distances as well as their corresponding consequences:

- $d_{\text{cb}}(R, H_\infty^{k,L}) = \sqrt{k+1}$.
- $d_{\text{cb}}(R, H_\infty^{m,R}) = d_{\text{cb}}(C, H_\infty^{m,L}) = d_{\text{cb}}(\Phi, H_\infty^{m,L}) = \infty$

5. Representation on the Fock space

We begin by recalling the construction of the spaces $H_\infty^{m,R}$; see subsection 2.1. Let I denote a subset of Ω with $|\Omega - I| = m+1$. and let J denote a subset of Ω of cardinality $|J| = m$. We assume that each $I = \{i_1, i_2, \ldots\}$ is such that $i_1 < i_2 < \cdots$, and that the collection of all such subsets I is ordered lexicographically. Similarly, if $J = \{j_1, \ldots, j_m\}$, then $j_1 < \cdots < j_m$ and the collection of all such subsets J is ordered lexicographically.

We shall use the notation e_i to denote the column vector with a 1 in the i^{th} position and zeros elsewhere. Thus $e_1, e_2, \ldots$ denotes the canonical basis of column vectors for separable column space C. More generally, e_I denotes the basis vector for ℓ_2 consisting of a 1 in the "I^{th}" position.

The space $H_\infty^{m,R}$ is the closed linear span of matrices b_i^m, $i \in \Omega$, given by

$$b_i^m = \sum_{I\cap J=\emptyset,(I\cup J)^c=\{i\},|J|=m} \epsilon(I,i,J) e_{J,I},$$

where $e_{J,I} = e_J \otimes e_I = e_J e_I^t \in M_{\aleph_0,\aleph_0}(\mathbf{C}) = B(\ell_2)$, and $\epsilon(I,i,J)$ is the signature of the permutation defined for disjoint I, J in subsection 2.1.

Let H be any separable infinite dimensional Hilbert space. For any $h \in H$, let C_h^m denote the wedge (or creation) operator from $\wedge^m H$ to $\wedge^{m+1} H$ given by

$$C_h^m(h_1 \wedge \cdots \wedge h_m) = h \wedge h_1 \wedge \cdots \wedge h_m.$$

The space of creation operators $\overline{\text{sp}}\{C_{e_i}^m\}$ will be denoted by $\mathcal{C}^m$. Its operator space structure is given by its embedding in $B(\wedge^m H, \wedge^{m+1} H)$.

It will be convenient to identify the space $\wedge^k H$ with $\ell_2(\{J \subset \Omega : |J| = k\})$ or with $\ell_2(\{I \subset \Omega : |\Omega - I| = k\})$.

Define the unitary operators V_k and W_k on $\wedge^k H$ by

$$V_k : \ell_2(\{I \subset \Omega : |\Omega - I| = k\}) \to \ell_2(\{K \subset \Omega : |K| = k\})$$

and

$$W_k : \ell_2(\{J \subset \Omega : |J| = k\}) \to \ell_2(\{J \subset \Omega : |J| = k\})$$

as follows:

- $V_k(e_I) = e_{\mathbf{N}-I}$; More specifically, $V_k(e_I) = e_{j_1} \wedge \cdots \wedge e_{j_k}$ where $\mathbf{N} - I = \{j_1 < \cdots < j_k\}$.
- $W_k(e_I) = \epsilon(i,I)\epsilon(I,i,J)e_I$ where $I \cup \{i\} \cup J = \mathbf{N}$ is a disjoint union.

It is easy to see, as in [14, section 2], that the definition of W_k is independent of the choice of i. Indeed, if p is chosen so that $i_p > \max\{i, j_{k-1}\}$, then $\epsilon(I,i,J) = (-1)^p \epsilon(i, i_1, \ldots, i_p, j_1, \ldots, j_{k-1}) = (-1)^p \epsilon(i, i_1, \ldots, i_p)\epsilon(\{i\}\cup\{i_1, \ldots, i_p\}, j_1, \ldots, j_{k-1})$ and therefore for any $i' \neq i$, $\epsilon(i,I)\epsilon(I,i,J) = \epsilon(i',I')\epsilon(I',i',J)$.

Lemma 5.1. $H_\infty^{m,R}$ is completely isometric to $\mathcal{C}^m$.

Proof. With $b_i^m = \sum \epsilon(I,i,J)e_{J,I}$ we have

$$b_i^m W_m(e_{I_0}) = b_i^m((\epsilon(i,I_0)\epsilon(I_0,i,J_0)e_{I_0}) = \epsilon(i,I_0)e_{J_0},$$

and

$$V_{m+1}C_{e_i}^m(e_{I_0}) = V_{m+1}(\epsilon(i,I_0)e_{\{i\}\cup I_0}) = \epsilon(i,I_0)e_{J_0}. \quad \square$$

Since the anti-creation operator space A^m is simply the adjoint of the creation operator space C^m, by construction it is clear that

Lemma 5.2. $H_\infty^{m,L}$ is completely isometric to the space of anti-creation operators $\mathcal{A}^m$.

By Lemma 5.1, Lemma 5.2, Theorem 1, and [14, Lemma 2.1] we now have

Theorem 5. Every n-dimensional or infinite dimensional separable Hilbertian JC*-triple is completely isometric to an intersection over a set of values of k of the spaces of creation and annihilation operators on k-fold antisymmetric tensors of the Hilbert space.

References

1. Jonathan Arazy and Yaakov Friedman, *Contractive projections in C_1 and C_∞*, Mem. Amer. Math. Soc. **13** (1978), no 200.
2. David Blecher, *The Shilov boundary of an operator space and the characterization theorems.*, J. Funct. Anal. 182 (2001), no. 2, 280–343.
3. David P. Blecher and Christian Le Merdy, Operator algebras and their modules—an operator space approach, Clarendon Press, Oxford 2004.
4. Truong Dang and Yaakov Friedman, *Classification of JBW^*-triple factors and applications*, Math. Scand. **61** (1987), 292–330.
5. C. Martin Edwards, Remo V. Hügli, and Gottfried T. Rüttimann, *A geometric characterization of structural projections on a JBW^*-triple*, J. Funct. Anal. **202** (2003), 174–194.
6. Edward Effros and Zhong-Jin Ruan, Operator Spaces, Oxford University Press, 2000.
7. Yaakov Friedman and Bernard Russo, *Conditonal expectation without order*, Pac. J. Math. 115 (1984), 351–360.
8. Yaakov Friedman and Bernard Russo, *Solution of the Contractive Projection Problem*, J. Funct. Anal. **60** (1985), 56-79.
9. Yaakov Friedman and Bernard Russo, *Structure of the predual of a JBW^*-triple*, J. Reine Angew. Math. **356** (1985), 67–89.
10. Lawrence A. Harris, *Bounded symmetric domains in infinite dimensional spaces*, In: Hayden, T. L. ,Suffridge,T. J. (eds.) Infinite dimensional holomorphy. Proceedings, 1973 (Lecture Notes in Mathematics **364**,pp. 13–40), Springer 1974.
11. Magnus R. Hestenes, *A ternary algebra with applications to matrices and linear transformations*, Arch. Rational Mech. Anal. **11** (1962), 138–194.
12. Wilhelm Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Zeit. **183** (1983), 503–529.
13. Matthew Neal and Bernard Russo, *Contractive projections and operator spaces*, Trans. Amer. Math. Soc. **355** (2003), 2223–2362.
14. Matthew Neal and Bernard Russo, *Representation of contractively complemented Hilbertian operator spaces on the Fock space*, Proc. Amer. Math. Soc. (to appear).
15. Vern Paulson, Completely bounded maps and operator algebras, Cambridge Studies in Advanced Mathematics 78, Cambridge University Press, Cambridge, 2002.
16. Gilles Pisier, Introduction to Operator Space Theory, Cambridge University Press 2003.
17. A. Guyan Robertson, *Injective matricial Hilbert spaces*, Math. Proc.Cambridge Philos. Soc. **110** (1991), no.1, 183-190.
18. Zhong-Jin Ruan, *Injectivity of operator spaces.*, Trans. Amer. Math. Soc. 315 (1989), no. 1, 89–104.

19. Zhong-Jin Ruan, *Type decomposition and the rectangular AFD property for W^*-TRO's.*, Canad. J. Math. 56 (2004), no. 4, 843–870.
20. M. A. Youngson, *Completely contractive projections on C*-algebras*, Quart. J. Oxford Ser. (2) **34** (1983), 507-511.
21. Heinrich Zettl, *A characterization of ternary rings of operators*, Adv. Math. **48** (1983), 117–143.
22. Chun Zhang, *Completely bounded Banach-Mazur distance*, Proc. Edinburgh Math. Soc. **40** (1997),247–260.

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