FINITARY QUADRATIC JORDAN ALGEBRAS

ANTONIO FERNÁNDEZ LÓPEZ

Abstract. Let $\mathbb{F}$ be a field of arbitrary characteristic. A quadratic Jordan $\mathbb{F}$-algebra J will be called finitary if U_aJ is finite dimensional for any element $a \in J$. We determine the structure of nondegenerate finitary quadratic Jordan $\mathbb{F}$-algebras.

Introduction

In this paper we deal with *quadratic Jordan algebras* (definition below) over a field $\mathbb{F}$ of arbitrary characteristic. Any associative $\mathbb{F}$-algebra A gives rise to a Lie algebra $A^{(-)}$ with Lie product $[a, b] = ab - ba$, and a quadratic Jordan algebra $A^{(+)}$ with quadratic mappings $a \mapsto a^2$ and $a \mapsto U_a$, $U_ab = aba$ for all $a, b \in A$.

Let X be a vector space over $\mathbb{F}$. Denote by $\mathcal{F}(X)$ the associative algebra of finite-rank linear mappings of the vector space X. A Lie $\mathbb{F}$-algebra L is said to be *finitary* if it is isomorphic to a subalgebra of the Lie algebra $\mathfrak{f}gl(X) := \mathcal{F}(X)^{(-)}$ for some vector space X over $\mathbb{F}$. Infinite-dimensional simple finitary Lie algebras over a field of characteristic 0 were classified by Baranov [2]. Later, Baranov and Strade [3] classified infinite-dimensional simple finitary Lie algebras over an algebraically closed field of characteristic not 2 or 3

By analogy with the Lie case, by a *finitary associative* $\mathbb{F}$*-algebra* we mean any subalgebra A of $\mathcal{F}(X)$ for some vector space X over $\mathbb{F}$.

Following the same analogy, we could define a finitary quadratic Jordan $\mathbb{F}$-algebra as a subalgebra J of $\mathcal{F}(X)^{(+)}$ for some vector space X over $\mathbb{F}$. But such a definition would rule out all exceptional quadratic Jordan algebras. Yet being unsuitable, the above definition still helps us to find the correct one. A quadratic Jordan $\mathbb{F}$-algebra J will be called *finitary* if the inner ideal U_aJ is finite-dimensional for every element $a \in J$. As will be seen, any subalgebra of a quadratic Jordan algebra $\mathcal{F}(X)^{(+)}$ is finitary according to this definition.

Key words and phrases. quadratic Jordan algebras, finitary algebras, socle.

Recall that a quadratic Jordan algebra J is said to be nondegenerate if $U_a J = 0$ implies $a = 0$. If $J = A^{(+)}$ for some associative algebra A, then J is nondegenerate if and only if A is semiprime.

The main result of this paper is a structure theorem for nondegenerate finitary quadratic Jordan algebras over a field $\mathbb{F}$ of arbitrary characteristic. Among other results, we will prove that any infinite dimensional central simple finitary quadratic Jordan algebra comes from a central simple finitary associative algebra (with or without involution), so we begin by studying finitary associative algebras.

1. Finitary associative algebras

Throughout this section $\mathbb{F}$ will be denote a field and A an associative $\mathbb{F}$-algebra. It is convenient to introduce some notation.

1.1. Following [7], let $(X, Y, \langle\cdot,\cdot\rangle)$ be a pair of dual vector spaces over a division $\mathbb{F}$-algebra Δ, where X is a left vector space, Y is a right vector space and $\langle\cdot,\cdot\rangle : X \times Y \to \Delta$ is a nondegenerate bilinear form. A linear mapping $a : X \to X$ is *adjointable* if there exists $a^{\#} : Y \to Y$ such that $\langle xa, y\rangle = \langle x, a^{\#}y\rangle$ for all $x \in X$, $y \in Y$. Notice that we write the mappings of a left vector space on the right (thus composing them from left to right), and the mappings on a right vector space on the left (thus composing them from right to left). We denote by $\mathcal{L}_Y(X)$ the associative $\mathbb{F}$-algebra of all adjointable linear mappings of X, and by $\mathcal{F}_Y(X)$ the ideal of those linear mappings having finite rank. The algebras $\mathcal{F}_Y(X)$ are precisely those simple algebras containing minimal one-sided ideals [7].

Any left vector space X over Δ gives rise to the canonical pair $(X, X^*, \langle\cdot,\cdot\rangle)$, where X^* stands for the dual of X. Thus, according to the notation above, $\mathcal{F}(X) = \mathcal{F}_{X^*}(X)$.

Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a pair of dual vector spaces over Δ. For $x \in X$, $y \in Y$, write $y \otimes x$ to denote the adjointable linear mapping defined by

$$x'(y \otimes x) = \langle x', y\rangle x \text{ for } x' \in X$$

with adjoint $(y \otimes x)^{\#} y' = y\langle x, y'\rangle$.

The following two results can be easily verified. .

1.2. $(y \otimes x)a = y \otimes xa$ for all mapping $a : X \to X$ and $a(y \otimes x) = a^{\#}y \otimes x$ for all adjointable a.

1.3. Every $a \in \mathcal{F}_Y(X)$ can be expressed as $a = \sum y_i \otimes x_i$ where both sets $\{y_i\} \subset Y$ and $\{x_i\} \subset X$ are linearly independent, which just means that $\mathcal{F}_Y(X)$ is isomorphic as an $\mathbb{F}$-vector space to the tensor product $Y \otimes_\Delta X$. Actually, $\mathcal{F}_Y(X)$ is isomorphic as an $\mathbb{F}$-algebra to $Y \otimes_\Delta X$ under the product

$$(y \otimes x)(z \otimes w) = y\langle x, z\rangle \otimes w = y \otimes \langle x, z\rangle w.$$

Definition 1.4. By a *finitary associative* $\mathbb{F}$*-algebra* we mean any subalgebra A of $\mathcal{F}(X)$ for some vector space X over $\mathbb{F}$.

Lemma 1.5. *If A is finitary, then aAa is finite dimensional for any $a \in A$.*

Proof. Let $A \leq \mathcal{F}(X)$, with X being a vector space over $\mathbb{F}$. It follows from (1.2) and (1.3) that for any $a \in A$, $aAa \subset a\mathcal{F}(X)a \cong a(X^* \otimes_\mathbb{F} X)a = a^\# X^* \otimes_\mathbb{F} Xa$ is finite dimensional, with $\dim_\mathbb{F} aAa \leq \text{rank}(a)^2$. □

Theorem 1.6. *Let A be an associative $\mathbb{F}$-algebra.*

(1) *If A is semiprime, then A is finitary if and only if for any element $a \in A$ the subspace aAa is finite dimensional.*
(2) *A is semiprime and finitary if and only if it is isomorphic to a direct sum of simple finitary $\mathbb{F}$-algebras.*
(e) *A is simple and finitary if and only it is isomorphic to some $\mathcal{F}_Y(X)$, where $(X, Y, \langle\cdot,\cdot\rangle)$ is a pair of dual vector spaces over a finite-dimensional division $\mathbb{F}$-algebra Δ, equivalently, A is simple and contains an idempotent e such that eAe is a finite-dimensional division $\mathbb{F}$-algebra.*

Proof. By Lemma 1.5, if A is finitary then for any $a \in A$, $\dim_\mathbb{F} aAa < \infty$.

Suppose now that A is semiprime and $\dim_\mathbb{F} aAa < \infty$ for any $a \in A$. By [5, Teorem 2.3], the element a belongs to the socle of A, so, by socle theory for semiprime rings (see [7] or [4]), $A = \bigoplus M_i$ is a direct sum of ideals each of which is a simple algebra with minimal one-sided ideals. Moreover, for any division idempotent $e \in M_i$ the division algebra $eM_ie = eAe$ is finite dimensional. Now it follows from the structure theorem for simple associative rings with minimal one sided ideals (see [4, Theorems 4.3.7 and 4.3.8]) that M_i is isomorphic to some $\mathcal{F}_Y(X)$, where $(X, Y, \langle\cdot,\cdot\rangle)$ is a pair of dual vector spaces over a finite-dimensional division $\mathbb{F}$-algebra Δ. For any $a \in \mathcal{F}_Y(X)$, $\dim_\mathbb{F} Xa = (\dim_\Delta Xa)(\dim_\mathbb{F} \Delta) < \infty$, so $\mathcal{F}_Y(X) \leq \mathcal{F}(X_\mathbb{F})$ is a simple finitary associative $\mathbb{F}$-algebra, equivalently, A is simple and contains an idempotent e such that eAe is a finite-dimensional division $\mathbb{F}$-algebra. Finally, it is easy to see that any direct sum of simple finitary associative $\mathbb{F}$-algebras is a semiprime finitary associative $\mathbb{F}$-algebra. This completes the proof of the theorem. □

Corollary 1.7. *Let R be a simple associative ring. Then R is finitary, as an algebra over its centroid, if and only if R satisfies a generalized polynomial identity.*

Proof. Let $\mathbb{F}$ denote the centroid of R, which is a field. By the theorem above, R is finitary as an algebra over $\mathbb{F}$ if and only if it contains an idempotent e such that eRe is a finite dimensional division $\mathbb{F}$-algebra, equivalently, by [4, Theorem 6.1.6], the simple ring R satisfies a generalized polynomial identity. □

2. General facts on quadratic Jordan algebras

In what follows Φ will denote a ring of scalars, that is, a commutative associative ring with 1. Following the exposition given in [12], we begin by reminding the reader of the definition of quadratic Jordan algebra.

2.1. Definitions. A *unital quadratic Jordan algebra* $J = (J, U, 1)$ over an arbitrary ring of scalars Φ consists of a Φ-module J, a distinguished element $1 \in J$, and a quadratic map $U : J \to \mathrm{End}_\Phi(J)$ such that if we denote the linearization of U by

$$V_{x,y}(z) = \{x, y, z\} = U_{x,z}(y) \quad (U_{x,z} = U_{x+z} - U_x - U_z)$$

then

(QJ1) $U_1 = \mathrm{Id}$
(QJ2) $U_x V_{y,x} = V_{x,y} U_x = U_{U(x)y,x}$
(QJ3) $U_{U(x)y} = U_x U_y U_x$

hold in all (free) scalar extensions; equivalently, all linearizations of these identities hold in J itself.

A *quadratic Jordan algebra* is just a Φ-submodule $J = (J, U; 2)$ of some unital Jordan algebra closed under the products $U_x y$ and the square

$$x^2 = U_x 1,$$

in which case J imbeds in the *unital hull.*

$$\hat{J} = \Phi 1 \oplus J :$$

$$U_{\alpha 1 \oplus x}(\beta 1 \oplus y) = \alpha^2 \beta \oplus (\alpha^2 y + 2\alpha\beta x + \alpha x \circ y + \beta x^2 + U_x y),$$

where we denote the linearization of the square by

$$V_x(y) = x \circ y = U_{x,y} 1 \quad (= (x + y)^2 - x^2 - y^2).$$

If $\frac{1}{2} \in \Phi$ we can characterize these algebras axiomatically as the *linear Jordan algebras* with product $x \cdot y$ satisfying

$$x \cdot y = y \cdot x, \quad (x^2 \cdot y) \cdot x = x^2 \cdot (x \cdot y)$$

Every quadratic Jordan algebra J gives rise to a *Jordan pair* $V = (J, J)$ where $Q^\sigma_x y = U_x y$ for $\sigma = \pm$ (see [8]). Hence every notion defined by Jordan pairs makes sense for quadratic Jordan algebras.

In what follows, by a Jordan algebra we will mean a quadratic Jordan algebra.

2.2. Examples. Most Jordan algebras come from associative algebras. Any associative Φ-algebra A, with product denoted by juxtaposition, yields a Jordan algebra $A^{(+)}$ via

$$U_x y = xyx, \quad x^2 = xx, \quad \{x, y, z\} = xyz + zyx, \quad x \circ y = xy + yx$$

(which is unital if A is). A Jordan algebra J is *special* if it is isomorphic to a subalgebra of some $A^{(+)}$; otherwise, J is called *exceptional.* An importan example of special Jordan algebras is a *hermitian algebra*

$$H(A, *) = \{x \in A : x^* = x\} \leq A^{(+)}$$

of self-adjoint elements in an associative algebra A with involution $*$. More generally, we must consider *ample hermitian algebras*

$$H_0 = H_0(A, *) \leq H(A, *)$$

such that $aH_0a^* \subset H_0$ for all $a \in A$ and all traces $a + a^*$ and norms aa^* lie in H_0. If $\frac{1}{2} \in \Phi$ then the only ample subspace is the whole $H(A, *)$.

Another important example is a (unital Jordan) *Clifford algebra*, which lives in the associative clifford algebra $C(Q, X, 1)$ of a quadratic form Q with base point 1 on a vector space X over a field $\mathbb{F}$:

$$J = J(Q, X, 1) \leq C(Q, X, 1)^{(+)} : \quad U_x y = Q(x, \bar{y})x - Q(x)\bar{y},$$

($\bar{y} = Q(y, 1)1 - y$, for $x, y \in X$).

More generally, we can consider an *outer ideal* I of $J(Q, X, 1)$ containing 1, $U(J)I + \{J, J, I\} \subset I$. If the characteristic of $\mathbb{F}$ is not 2, then an outer ideal is an ideal.

A Jordan algebra is *i-special* if it satisfies all the identities of special Jordan algebras (equivalently, is a homomorphic image of a special Jordan algebra). A Jordan algebra is *i-exceptional* if it is not i-special. The basic i-exceptional Jordan algebras are the 27-dimensional *Albert algebras* (see [10]).

2.3. Alternating algebras. Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a pair of dual vector spaces over a division Φ-algebra Δ. If the Φ-algebra $\mathcal{F}_Y(X)$ has an involution $*$, then Δ has an involution $-$, X has either a nondegenerate hermitian (and nonalternating) inner product or a nondegenerate alternating inner product, denoted in both cases by $\langle\cdot,\cdot\rangle$, with the involution $*$ being the adjoint $\#$ with respect to the inner product. Notice that $(X, \langle\cdot,\cdot\rangle)$ yields a pair of dual vector spaces $(X, X, \langle\cdot,\cdot\rangle)$ where the second X is regarded as right vector space over Δ by defining $x \cdot \alpha = \bar{\alpha}x$ ($x \in X$, $\alpha \in \Delta$.

Suppose that $(X, \langle\cdot,\cdot\rangle)$ is alternating over a field $\mathbb{F}$. Following [11, p. 460], a linear mapping $a \in \mathcal{F}_X(X)$ is called an *alternating mapping* if $\langle x, xa\rangle = 0$ for each $x \in X$. We write $\mathcal{A}lt(X, \langle\cdot,\cdot\rangle)$ to denote the set of all alternating mappings. As observed in [6], if the characteristic of $\mathbb{F}$ is not 2, then $\mathcal{A}lt(X, \langle\cdot,\cdot\rangle) = H(\mathcal{F}_X(X), *)$.

2.4. Homotopes, isotopes and local algebras. Definitions and notation of this paragraph are taking from [6, 8].

For any element b in a Jordan algebra J we obtain a new Jordan algebra structure $J^{(b)}$ in the Φ-submodule J called the *b-homotope* of J by taking the operations

$$U_x^{(b)} = U_x U_b, \quad x^{(2,b)} = U_x b.$$

If J is unital and b is invertible, then $J^{(b)}$ is a unital Jordan algebra called the *b-isotope* of J, with b^{-1} as its unit element.

Isotopes of nonunital Jordan algebras J are defined via a unital Jordan algebra K containing J as a subalgebra and an invertible element $b \in K$ satisfying

$$U_b J = J, \quad U_J b \subset J.$$

For instance, the above conditions are satisfied if J is an ideal of K; in particular, if K is the unital hull of J.

Let b be an arbitrary element of J. Then the Φ-submodule $\mathrm{Ker}_J(b)$ of J defined by

$$\mathrm{Ker}_J(b) = \{z \in J : U_b z = U_b U_z b = 0\}$$

is an ideal of the Jordan algebra $J^{(b)}$ and the quotient algebra $J/Ker_J(b)$ is called the *local algebra of J at b* and denoted by J_b. If $\frac{1}{2} \in \Phi$, then condition $U_b U_z b = 0$ is superfluous.

2.5. Inner ideals and socle. An *inner ideal* of a Jordan algebra J is a Φ-submodule B of J such that $U_b J \subset B$ for any $b \in J$. Any element $x \in J$ yields the *principal inner ideal determined by x*, $U_x J$, and the *inner ideal generated by x*, $\Phi x + U_x J$. These inner ideals coincide if and only if x is a von Neumann regular element of J. The *socle* $\mathrm{Soc}(J)$ of a Jordan algebra J is defined as the sum of its minimal inner ideals.

A Jordan algebra J is *nondegenerate* if $U_xJ = 0$ implies $x = 0$. Simple Jordan algebras are nondegenerate (see [1]). If J is nondegenerate, then $\mathrm{Soc}(J)$ is a direct sum of ideals each of which is a simple Jordan algebra coinciding with its socle (see [9] or [15]) Moreover, an element $x \in J$ lies in the socle if and only if $\Phi x + U_xJ$ has descending chain condition on principal inner ideals [9, Theorem 1].

Simple Jordan algebras with minimal inner ideals were classified in [6] as a refinement of the more general structure theorem of K. McCrimmon and E. Zelmanov for simple Jordan algebras (see [12]).

Theorem 2.6. [6, Theorem 2] *Every simple Jordan algebra containing minimal inner ideals is up to isotopy one of the following:*

(i) *A Jordan algebra of Clifford or Albert type,*
(ii) *a Jordan algebra $\mathcal{F}_Y(X)^{(+)}$, where $(X, Y, \langle\cdot,\cdot\rangle)$ is a pair of dual vector spaces over a division Φ-algebra Δ,*
(iii) *an ample hermitian algebra $H_0(\mathcal{F}_X(X), *)$ relative to a nondegenerate hermitian (and nonalternating) inner product space $(X, \langle\cdot,\cdot\rangle)$ over a division Φ-algebra Δ with involution, and where $*$ is the adjoint involution, or*
(iv) *$\mathcal{A}lt(X, \langle\cdot,\cdot\rangle)$ for a nondegenerate alternating inner product space $(X, \langle\cdot,\cdot\rangle)$ over a field $\mathbb{F}$ which is a Φ-algebra.*

3. finitary Jordan algebras

By analogy with the associative and Lie cases, we could define a finitary (quadratic) Jordan $\mathbb{F}$-algebra J as a subalgebra of the Jordan algebra $\mathcal{F}(X)^{(+)}$ for some vector space X over $\mathbb{F}$. But such a definition would exclude the exceptional Jordan algebras.

Definition 3.1. A Jordan $\mathbb{F}$-algebra J is *finitary* if the principal inner ideal U_aJ is finite dimensional for each element $a \in J$.

Lemma 3.2. *If $J \leq \mathcal{F}(X)^{(+)}$ for some $\mathbb{F}$-vector space X, then J is finitary.*

Proof. The same proof of Lemma 1.5 works. For any $a \in J$, $U_aJ \subset U_a\mathcal{F}(X) = a\mathcal{F}(X)a$. □

Remarks 3.3. Notice that the converse of Lemma 3.2 does not hold. Consider a 27-dimensional Albert algebra. Notice also that, according to our definition, a unital

Jordan algebra is finitary if and only if it is finite dimensional; in particular, a Clifford algebra is finitary if and only if it is finite dimensional.

Our aim is to classify nondegenerate finitary (quadratic) Jordan algebras J over an arbitrary field $\mathbb{F}$. The theorem below reduces the study to the case that J is simple.

Theorem 3.4. *Let J be a Jordan $\mathbb{F}$-algebra. Then J is finitary and nondegenerate if and only if it is a direct sum of simple finitary Jordan algebras.*

Proof. Let $J = \bigoplus J_i$ be a direct sum of ideals each of which is a simple finitary Jordan $\mathbb{F}$-algebras. But simple Jordan algebras are nondegenerate [1] and it is easy to check that direct sums preserve nondegeneracy, so J is nondegenerate. Now let $a \in J$. Then $a = a_{i_1} + \cdots a_{i_r}$ where each summand a_{i_k} belongs to some J_{i_k}. Since the ideals J_i are mutually orthogonal, we have that $U_a J = U(a_{i_1})J_{i_1} \oplus \cdots \oplus U(a_{i_r})J_{i_r}$ is finite dimensional, which proves that J is finitary.

Suppose conversely that J is nondegenerate with $U_a J$ being finite dimensional for any element $a \in J$. It follows from Loos' elemental characterization of the socle [9, Theorem 1] that J coincides with its socle and therefore, by [9, Theorem 2], J is a direct sum of ideals each of which is a simple Jordan algebra, clearly finitary. This completes the proof of the theorem. □

Theorem 3.5. *Every simple finitary Jordan $\mathbb{F}$-algebra is up to isotopy one of the following:*

(i) *a 27-dimensional Albert algebra over a finite field extension $\mathbb{K}$ of $\mathbb{F}$,*

(ii) *a finite-dimensional Clifford Jordan algebra J, defined by a nondegenerate quadratic form on a vector space over a finite field extension $\mathbb{K}$ of $\mathbb{F}$, or an outer ideal I of J containing 1.*

(iii) *an ample hermitian algebra $H_0(\mathcal{F}_X(X), *)$ relative to a nondegenerate hermitian (and nonalternating) inner product space $(X, \langle\cdot,\cdot\rangle)$ over a finite-dimensional division $\mathbb{F}$-algebra Δ with involution, and where $*$ is the adjoint involution, or*

(iv) *$\mathcal{A}lt(X, \langle\cdot,\cdot\rangle)$ for a nondegenerate alternating inner product space $(X, \langle\cdot,\cdot\rangle)$ over a finite field extension $\mathbb{K}$ of $\mathbb{F}$.*

Proof. Each of the Jordan algebras listed above is simple and finitary. Suppose then that J is a simple finitary Jordan $\mathbb{F}$-algebra. As proved in Theorem 3.4, J coincides with its socle. Thus we only need to look and the list of simple Jordan algebras given in Theorem 2.6 and identify the finitary ones. □

Remark 3.6. Finitary Jordan algebras are closely related to Jordan algebras with PI-elements, as defined and studied by F. Montaner in [13, 14]. In terms of local algebras (see 2.4), our definition of a finitary Jordan algebra J can rephrased by saying that the local algebra J_a is finite dimensional for any element $a \in J$, while, according to [13], an element $a \in J$ is PI if the local algebra J_a satisfies a polynomial identity. Notice that by [13, Proposition 2.6], simple unital Jordan algebras with a nonzero PI-element have finite capacity, and in the particular case of a Jordan algebra $J = A^{(+)}$, where A is a central simple associative algebra, J is finitary if and only it contains a nonzero PI-element.

Acknowledgement, I am indebted to my colleague Professor Esperanza Sánchez for the careful reading of the manuscript.

References

[1] José A. Anquela, Teresa Cortés and Efim Zelmanov, Local nilpotence of the McCrimmon radical of a Jordan system, (to appear).
[2] A. A. Baranov, Finitary simple Lie algebras, *J. Algebra* **219** (1999), 299-329.
[3] A. A. Baranov and H. Strade, Finitary Lie algebras, *J. Algebra* **254** (2002), 173-211.
[4] K.I. Beidar, W.S. Martindale and A.V. Mikhalev, *Rings with Generalized Identities.* Pure and Applied Mathematics; Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.
[5] M. Brešar and D. Eremita, The lower socle and finite rank elementary operators, *Comm. Algebra* **31** (3) (2003), 1485-1497.
[6] A. Fernández López and E. García Rus, Inner ideals in quadratic Jordan algebras of infinite capacity, *Int. J. Math. Game Theory Algebra* **9** (1999), 35-54.
[7] N. Jacobson, *Structure of Rings.* Amer. Math. Soc. Colloquium Publ., Providence, 1964.
[8] O. Loos, *Jordan Pairs*, Vol. 460, Springer, Berin-Heidelberg-New York, 1975.
[9] O. Loos, On the socle of a Jordan pair, *Collect. Math.* **40** (1989), 109-125.
[10] K. McCrimmon, The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras, *Trans. AMS* **139** (1969), 495-510.
[11] K. McCrimmon, Inner ideals in quadratic Jordan algebras, *Trans. AMS* **159** (1971), 445-468.
[12] K. McCrimmon and E. I. Zelmanov, The structure of strongly prime quadratic Jordan algebras, *Adv. in Math.* **69** (1988), 133-222.
[13] F. Montaner, Local PI-theory of Jordan systems, *J. Algebra* **216** (1999), 302-327.
[14] F. Montaner, Local PI-theory of Jordan systems II, *J. Algebra* **241** (2001), 473-514.
[15] J. M. Osborn and M. L. Racine, Jordan rings with nonzero socle, *Trans. AMS* **251** (1979), 375-387.

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Spain
E-mail address: emalfer@uma.es